Central Charges and Vacuum Moduli of 2d $\mathcal{N} = (0, 4)$ Theories from Class $\mathcal{S}$

Wei Cui,[a] Junkang Huang,[b] Zi-Xiao Huang,[b] Satoshi Nawata,[b] Shutong Zhuang[b]

[a] *Beijing Institute of Mathematical Sciences and Applications (BIMSA) Huaibei Town, Huairou District, Beijing 101408, China*

[b] *Department of Physics and Center for Field Theory & Particle Physics, Fudan University, 20005, Songhu Road, 200438 Shanghai, China*

E-mail: snawata@gmail.com

Abstract: To be written.

Contents

1 Introduction

The 6d $\mathcal{N} = (2, 0)$ superconformal field theory (SCFT), which describes the effective worldvolume theory on M5-branes, occupies a distinctive position since it represents the maximally supersymmetric conformal theory in the highest possible spacetime dimension [1]. Despite the absence of a Lagrangian description, it governs a remarkably rich structure that can be probed indirectly through its compactifications, protected observables, and associated moduli spaces.

A powerful method to study the 6d $\mathcal{N} = (2, 0)$ of ADE type, labeled by a simply-laced Lie algebra G, is to compactify it on lower-dimensional manifolds, with a partial topological twist to preserve supersymmetry. The resulting theories, typically denoted $\mathcal{T}_G[M]$ for a compactification manifold M, not only provide explicit constructions of supersymmetric QFTs in various dimensions but also reveal intriguing relations between supersymmetric theory $\mathcal{T}_G[M]$ and the geometry and topology of M. Furthermore, this perspective sheds new light on the structure of moduli spaces, dualities, and protected operators of $\mathcal{T}_G[M]$.

A celebrated example is obtained by compactifying the 6d $\mathcal{N} = (2, 0)$ SCFT on a punctured Riemann surface $C_{g_1,n}$, generating a large class of 4d $\mathcal{N} = 2$ SCFTs known as class $\mathcal{S}$ theories [2, 3]. Over the years, the study of class $\mathcal{S}$ theories has led to remarkable developments and produced profound results in understanding physical and mathematical structures of 4d $\mathcal{N} = 2$ theories.

Given these developments, a natural next step is to compactify class $\mathcal{S}$ theories further. By the dimensional reduction of a 4d $\mathcal{N} = 2$ SCFT $\mathcal{T}_G[C_{g_1,n}]$ on another Riemann surface C_{g_2}, we obtain a 2d $\mathcal{N} = (0,4)$ theory when the $U(1)_{C_{g_2}}$ holonomy of C_{g_2} is twisted by the $U(1)_r$ subgroup of the 4d R-symmetry $SU(2)_R \times U(1)_r$ [4–6].

The case $C_{g_2} = S^2$ has been analyzed in [7, 8]. There it was shown that the resulting $(0,4)$ theory flows to a non-linear sigma model on the hypermultiplet moduli space (the Higgs branch), equipped with a nontrivial left-moving bundle. For genus $g_1 > 0$, the Cartan subgroup of gauge group is unbroken at a generic point of the Higgs branch, so a $U(1)^{r_G g_1}$ gauge symmetry survives [9], where r_G is the rank of G. In two dimensions, such unbroken abelian gauge sectors are gapped in the infrared [10]. This phenomenon leads to subtle infrared structures and renders the computation of central charges nontrivial. In particular, although 't Hooft anomaly coefficients of 2d theories descending from 6d can often be extracted by integrating the 6d anomaly polynomial over the compactification manifold [11, 12], the naive integration for central charges fails because of the infrared gapped abelian gauge sectors.

The purpose of this paper is to systematically analyze the infrared structure of the 2d $\mathcal{N} = (0,4)$ theory obtained by the dimensional reduction of $\mathcal{T}_G[C_{g_1,n}]$ on C_{g_2}. In particular, we clarify why the standard anomaly-polynomial method does not produce the correct central charges due to infrared gapped abelian sectors. We further propose a set of conjectural formulas for the infrared central charges of the resulting $(0,4)$ theory.

To determine the central charges, we study the structure of the vacuum moduli space. For generic 2d $\mathcal{N} = (0,4)$ theories, the vacuum moduli space decomposes into two principal branches: the Higgs branch and the twisted Higgs branch. The twisted Higgs branch is parametrized solely by vacuum expectation values (VEVs) of twisted hypermultiplets, while the Higgs branch is more intricate and typically supported by simultaneous VEVs of both hypermultiplets and twisted hypermultiplets. Each branch carries its own affine SU(2) R-symmetry current, reflecting distinct infrared realizations of the $(0,4)$ superconformal algebra.

To make the infrared physics more explicit, we analyze these moduli spaces for gauge group $G = SU(2)$ where an explicit Lagrangian description is available. In this setting, the Higgs branch contains a particularly interesting component—the so-called *special Higgs branch*—whose structure depends delicately on the topology of the compactification. By computing the Hilbert series of the special Higgs branches and the twisted Higgs branches, we show detailed features of the vacuum moduli space and provide nontrivial checks of our proposed formulas for the central charges.

2 2d $\mathcal{N} = (0,4)$ theory from class $\mathcal{S}$ on Riemann surfaces

In this section, we study the 2d $(0,4)$ supersymmetric theory obtained by dimensional reduction of the class $\mathcal{S}$ theory $\mathcal{T}_G[C_{g_1,n}]$ on a Riemann surface C_{g_2}. We begin by reviewing the structure of generic $(0,4)$ multiplets, their anomaly coefficients, and the associated R-symmetries, thereby establishing the conventions used throughout the paper.

We then examine the computation of the central charges of the resulting 2d $(0,4)$ theory in the infrared. In particular, we analyze the limitations of the standard anomaly-polynomial integration method when applied to compactifications on manifolds with nontrivial one-cycles. We show that for $g_1 > 0$, infrared-gapped abelian gauge sectors arise, and their contributions are not captured by (a naive integration of) the anomaly polynomial. Understanding the origin of this mismatch, we put forth our conjectural formulas for the infrared central charges of the 2d $(0,4)$ theory.

To test our conjecture, we turn to the reduction of a single M5-brane as the simplest setting in which the relevant twist, field decomposition, and anomaly structure can be analyzed explicitly and transparently.

2.1 Class $\mathcal{S}$ theories and reduction to 2d $\mathcal{N}=(0,4)$ theories

Class $\mathcal{S}$ theories are 4d $\mathcal{N}=2$ theories that arise by compactifying the 6d $\mathcal{N}=(2,0)$ theory of type G on a punctured Riemann surface $C_{g_1,n}$ of genus g_1 with n punctures [2, 3]. Consider the twisted compactification of 6d $\mathcal{N}=(2,0)$ SCFT $\mathcal{T}_G$ on a Riemann surface $C_{g_1,n}$ of genus g_1 with n punctures. To preserve 4d $\mathcal{N}=2$ supersymmetry, one turns on a partial topological twist that embeds the $U(1)_{C_{g_1,n}}$ holonomy of $C_{g_1,n}$ into the $SO(5)_R$ symmetry. After the twist, the resulting four-dimensional theory depends only on the complex structure of $C_{g_1,n}$ and the local data specified at its punctures. We denote this theory by $\mathcal{T}_G[C_{g_1,n}]$, and refer to it as a *class $\mathcal{S}$ theory* of type G.[1]

In this paper, we consider only regular punctures, which are characterized by embeddings

$$\rho : \mathfrak{su}(2) \hookrightarrow \mathfrak{g}, \tag{2.1}$$

or equivalently by nilpotent orbits in $\mathfrak{g}$ [14]. They determine local boundary conditions for fields near each puncture and give rise to flavor symmetries in the resulting 4d theory. The resulting 4d effective theory is an $\mathcal{N}=2$ SCFT with $SU(2)_R \times U(1)_r$ R-symmetry and it does not admit Lagrangian description generically.

Since any Riemann surface can be decomposed into three-punctured spheres, the theory $\mathcal{T}_G[C_{g_1,n}]$ may be assembled by gauging together copies of the *trinion* theories $\mathcal{T}_G[C_{0,3}]$, which serve as fundamental building blocks. The complexified gauge couplings of $\mathcal{T}_G[C_{g_1,n}]$ are identified with the complex structure moduli of the curve $C_{g_1,n}$, and different pair-of-pants decompositions of $C_{g_1,n}$ correspond to different weakly coupled frames of the same 4d theory. Changing the pants decomposition amounts to an action of the mapping class group, which manifests in the 4d theory as generalized S-duality.

For a class $\mathcal{S}$ theory $\mathcal{T}_G[C_{g_1,n}]$ with all maximal punctures, the effective numbers of 4d vector and hypermultiplets can be expressed in terms of group-theoretic data [12]:

$$n_v = \left(\frac{2}{3}h^\vee_G d_G + \frac{r_G}{2}\right)(2g_1-2+n) - \frac{1}{2}\,d_G\, n, \qquad n_h = \frac{2}{3}\,h^\vee_G d_G\,(2g_1-2+n), \tag{2.2}$$

where $h^\vee_G$, r_G, and d_G denote the dual Coxeter number, rank, and dimension of the Lie algebra of G, respectively. For $G=SU(N)$, $h^\vee_G=N,\ r_G=N-1,\ d_G=N^2-1$, the results are

$$\begin{aligned} n_v &= \frac{1}{3}(g_1-1)(N-1)\,[4N(N+1)+3] + \frac{1}{6}N(N-1)\,(4N+1)\,n, \\ n_h &= \frac{2}{3}N(N^2-1)\,[2(g_1-1)+n]\ . \end{aligned} \tag{2.3}$$

Generally [12, 14], for $G=SU(N)$ with general punctures labeled by partitions $\rho^{(i)}, i=1,\cdots,n$ of N:

$$n_v = (g_1-1)\left(\frac{4}{3}h^\vee_G d_G + r_G\right) + \sum_{i=1}^n n_v\left(\rho^{(i)}\right), \quad n_h = (g_1-1)\frac{4}{3}h^\vee_G d_G + \sum_{i=1}^n n_h\left(\rho^{(i)}\right)\ , \tag{2.4}$$

where we have

$$n_v(\rho) = \sum_{k=1}^N (2k-1)\left(k - h_k(\rho^T)\right)\ , \quad n_h(\rho) = n_v(\rho) + \frac{1}{2}\left(\sum_{a=1}^{|\rho^T|}(\rho^T_a)^2 - N\right) \tag{2.5}$$

where $h_k(\rho^T)$ is the row of the k-th box in the Young diagram associated with ρ^T, with rows sorted in non-decreasing order. For example, as in Figure 1, when $\rho=[4,3,1]$, $\rho^T=[3,2,2,1]$ and $h_k=(1,1,1,2,2,3,3,4)$, one has $n_v(\rho)=199,\ n_h(\rho)=204$.

[1]More precisely, extended operators in class $\mathcal{S}$ theories are sensitive to the global structure of the gauge group. For this, one must specify additional data in the form of a maximal isotropic sublattice of $H^1(C_{g_1,n}, Z(G))$ [13]. We will not pursue these subtleties in this paper.

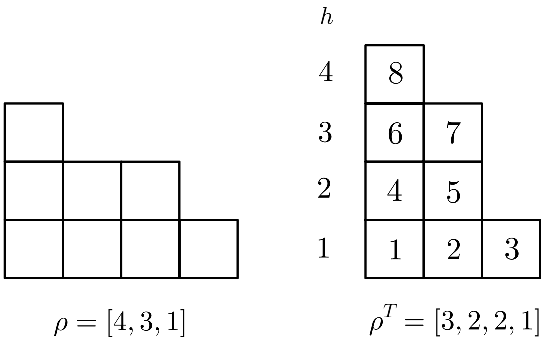

Figure 1. For a puncture labeled by $\rho = [4,3,1]$ when $N = 8$, label the boxes in ρ^T by $k = 1, \cdots, 8$ as on the right, and the corresponding h_k are $(1,1,1,2,2,3,3,4)$, and consequently $n_v(\rho) = 199$, $n_h(\rho) = 204$.

As another example, for a full puncture $\rho = [1^N]$, $\rho^T = [N]$, and $h_k([N]) = 1$ for all k,

$$n_v([1^N]) = \frac{1}{6}N(N-1)(4N+1), \quad n_h([1^N]) = \frac{2}{3}N(N^2-1) , \tag{2.6}$$

and therefore the total numbers of multiplets are given by (2.3).

We now consider the dimensional reduction of the 4d $\mathcal{N} = 2$ class $\mathcal{S}$ theory on C_{g_2}. The holonomy $U(1)_{C_{g_2}}$ of C_{g_2} breaks supersymmetry unless we perform a topological twist. To obtain a 2d theory with $\mathcal{N} = (0,4)$ supersymmetry, we perform the topological twist by taking the diagonal subgroup of the $U(1)_{C_{g_2}}$ holonomy group and $U(1)_r$:

$$U(1)^{\text{tw}}_{C_{g_2}} \;=\; \text{diag}\big(U(1)_{C_{g_2}} \times U(1)_r\big). \tag{2.7}$$

To see this explicitly, recall that under

$$SO(4) \;\to\; SO(2) \times U(1)_{C_{g_2}}, \tag{2.8}$$

the 4d supercharges split into left- and right-moving 2d spinors, carrying $U(1)_{C_{g_2}}$ charge $\pm\frac{1}{2}$ and $U(1)_r$ charge $\pm\frac{1}{2}$. Exactly those supercharges with $q_2 + q_r = 0$ become scalars on C_{g_2} and remain unbroken. This leaves four right-moving supercharges and no left-moving ones, so the resulting 2d theory has $\mathcal{N} = (0,4)$ supersymmetry [7, 15].

2.2 (0,4) multiplets, R-symmetries, and anomalies

The structure of fields and symmetries in 2d $(0,4)$ theories has been systematically analyzed in [16]. A detailed derivation of the $(0,4)$ reduction of 4d $\mathcal{N} = 2$ theories on a Riemann surface C_{g_2} is provided in the appendices of [7, 17]. To be self-contained, we briefly summarize the relevant results here.

The R-symmetry of a 2d $\mathcal{N} = (0,4)$ theory is $SO(4)_R \;\cong\; SU(2)_- \times SU(2)_+$, and the R-charges of the various $(0,4)$ multiplets are listed in Table 1. When the parent 4d theory admits a Lagrangian description, its dimensional reduction on C_{g_2} produces the UV field contents in two dimensions [7, 17]. The resulting 2d field contents are inherited from the zero modes of 4d multiplets and are organized as follows (also see Figure 2):

- 1 4d vector $(U_{4\text{d}}, \Phi_{4\text{d}}) \longrightarrow$ 1 $(0,4)$ vector (U, Θ) + g_2 $(0,4)$ twisted hyper $(\Sigma_j, \tilde{\Sigma}_j)$
- 1 4d hyper $(Q_{4\text{d}}, \tilde{Q}_{4\text{d}}) \longrightarrow$ 1 $(0,4)$ hyper $(Q, \tilde{Q})$ + g_2 $(0,4)$ Fermi $(\Gamma_j, \tilde{\Gamma}_j)$,

where $j = 1, \ldots, g_2$ labels the g_2 copies originating from holomorphic one-forms $H^0(C_{g_2}, K)$ (with the canonical bundle K). The charges carried by 2d fields are summarized in Table 2.

(0,4) multiplets	(0,2) multiplets	Components	$SU(2)_- \times SU(2)_+$
vector	vector $U = (A_\mu, \lambda_-)$ + Fermi $\Theta = (\tilde{\lambda}_-)$	A_μ, λ_-^a	$(\mathbf{1},\mathbf{1}), (\mathbf{2},\mathbf{2})$
twisted hyper	chiral $\Sigma = (\sigma, \lambda_+^\dagger)$ + chiral $\tilde{\Sigma} = (\tilde{\sigma}, \tilde{\lambda}_+)$	σ_a, λ_+^b	$(\mathbf{1},\mathbf{2}), (\mathbf{2},\mathbf{1})$
hypermultiplet	chiral $Q = (q, \psi_+)$ + chiral $\tilde{Q} = (\tilde{q}, \tilde{\psi}_+)$	$q^a, \psi_{+,b}$	$(\mathbf{2},\mathbf{1}), (\mathbf{1},\mathbf{2})$
Fermi	Fermi $\Gamma = (\psi_-)$ + Fermi $\tilde{\Gamma} = (\tilde{\psi}_-)$	ψ_-^a	$(\mathbf{1},\mathbf{1})$

Table 1. Field contents of 2d $(0,4)$ multiplets expressed in terms of $(0,2)$ multiplets, and their charges under $SU(2)_- \times SU(2)_+$ R-symmetry.

Superfield	$U(1)_R$	$U(1)_r$	$U(1)_{C_{g_2}}$	$U(1)^{\text{tw}}_{C_{g_2}}$	$U(1)^{\text{tw}'}_{C_{g_2}}$
$U = (A_\mu, \lambda_-)$	$(0, \frac{1}{2})$	$(0, \frac{1}{2})$	$(0, -\frac{1}{2})$	$(0, 0)$	$(0, -\frac{1}{2})$
$\Theta = (\tilde{\lambda}_-)$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$
$\Sigma = (\sigma, \lambda_+^\dagger)$	$(0, -\frac{1}{2})$	$(0, -\frac{1}{2})$	$(1, -\frac{1}{2})$	$(\frac{1}{2}, -\frac{1}{2})$	$(\frac{1}{2}, 0)$
$\tilde{\Sigma} = (\tilde{\sigma}, \tilde{\lambda}_+)$	$(0, -\frac{1}{2})$	$(1, \frac{1}{2})$	$(0, \frac{1}{2})$	$(\frac{1}{2}, \frac{1}{2})$	$(-\frac{1}{2}, 0)$
$\Phi = (q, \psi_+)$	$(\frac{1}{2}, 0)$	$(0, -\frac{1}{2})$	$(0, \frac{1}{2})$	$(0, 0)$	$(0, \frac{1}{2})$
$\tilde{\Phi} = (\tilde{q}, \tilde{\psi}_+)$	$(\frac{1}{2}, 0)$	$(0, -\frac{1}{2})$	$(0, \frac{1}{2})$	$(0, 0)$	$(0, \frac{1}{2})$
$\Gamma = (\psi_-)$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0
$\tilde{\Gamma} = (\tilde{\psi}_-)$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0

Table 2. Charges of the 2d $\mathcal{N} = (0,4)$ superfields obtained by the dimensional reduction of a 4d $\mathcal{N} = 2$ theory on C_{g_2}. For each superfield, we list its charges under the Cartan subgroup $U(1)_R \subset SU(2)_R$ of the 4d $SU(2)_R$ symmetry, the twisted Lorentz symmetry $U(1)^{\text{tw}}_{C_{g_2}}$, and the 4d $U(1)_r$ symmetry. These assignments follow from the $U(1)_r$ twist on C_{g_2} and determine the resulting $(0,4)$ multiplet structure in two dimensions.

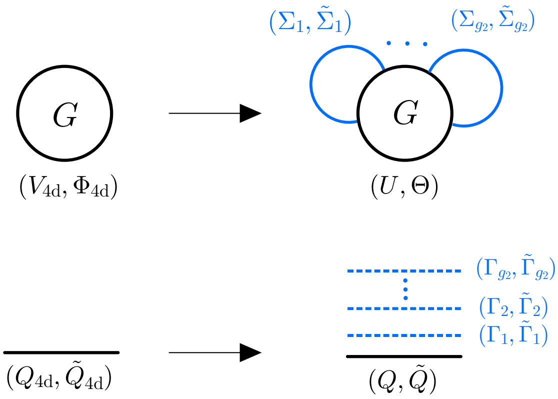

Figure 2. The $\mathcal{N} = (0,4)$ reduction of the 4d $\mathcal{N} = 2$ multiplets on a Riemann surface of genus g_2. For simplicity, in the remaining part, we will use a single solid blue line to represent g_2 twisted hypermultiplets $(\Sigma_j, \tilde{\Sigma}_j)$, and a single dashed blue line to represent g_2 Fermi multiplets Γ_j.

Remark. In this paper, we study the 2d $\mathcal{N} = (0,4)$ theory obtained by the *dimensional reduction* of class $\mathcal{S}$ theories on C_{g_2}, keeping only the zero modes and discarding all Kaluza–Klein (KK) excitations. It is important to distinguish this construction from the genuine *compactification* of the 6d $\mathcal{N} = (2,0)$ theory on the 4-manifold $C_{g_1,n} \times C_{g_2}$. As emphasized in [18, 19], the resulting

2d theory $\mathcal{T}_G[C_{g_1,n} \times C_{g_2}]$ depends crucially on the full KK spectrum and the choice of polarization of the 4-manifold. In particular, its partition function requires summing over the allowed magnetic flux sectors on $C_{g_1,n}$ and C_{g_2} [20, 21]. Therefore, the theory studied in this work is *not* the same as the compactified theory $\mathcal{T}_G[C_{g_1,n} \times C_{g_2}]$.

A straightforward but important consequence is that our 2d theory does *not* enjoy the symmetry exchanging C_{g_1} and C_{g_2} when there is no puncture. Such an exchange symmetry may exist only in the full compactification with KK modes and flux sectors included; in the zero-mode truncation, this symmetry is not expected, and we indeed find its absence.

Central charges of the 2d $\mathcal{N} = (0, 4)$ theory are determined by 't Hooft anomalies. In particular, the right-moving central charge is

$$c_R = 3k_R = 3\,\mathrm{Tr}\,\gamma_3 R^2, \tag{2.9}$$

with k_R the anomaly coefficient of $U(1)$ R-symmetry of 2d $(0, 2)$ subalgebra, and the chiral central charge is determined by the gravitational anomaly via

$$k_g = c_L - c_R = \mathrm{Tr}\,\gamma_3. \tag{2.10}$$

Note that the trace performed over all Weyl fermions. Once we know the Lagrangian, one can calculate the 't Hooft anomalies and central charges.

We now explain how the two R-symmetry factors $SU(2)_-$ and $SU(2)_+$ of the 2d $\mathcal{N} = (0, 4)$ theory arise from the symmetries of the parent 4d class $\mathcal{S}$ theory. Based on the charge assignments of $(0, 4)$ multiplets in Table 2, the two $SU(2)$ factors arise from particular combinations of the 4d R-symmetry and the twisted symmetry along C_{g_2},

$$U(1)_+ = U(1)^{\mathrm{tw}'}_{C_{g_2}} = \mathrm{diag}\left(U(1)_{C_{g_2}} \times U(1)_r^{-1}\right), \qquad SU(2)_- = SU(2)_R, \tag{2.11}$$

where $U(1)_+$ is the Cartan subgroup of $SU(2)_+$.

In the infrared, the theory flows to a non-linear sigma model whose target is a branch of the vacuum moduli space. The resulting IR sigma-model is endowed with a 2d small $\mathcal{N} = (0, 4)$ superconformal algebra with an affine $SU(2)_{\mathrm{IR}}$ R-symmetry at level $c_R/6$. This level equals the quaternionic dimension of the sigma-model target space. Because different branches of the moduli space may realize different IR R-symmetries, the value of c_R generally depends on the identification of IR R-symmetry. A key constraint [10] is that the scalar fields parametrizing the moduli space *cannot* carry charge under the IR R-symmetry. Thus, the IR R-symmetry must be chosen so that all scalars are neutral, while the fermions carry appropriate charges consistent with $(0, 4)$ supersymmetry. There are two natural choices of IR R-symmetry from UV:

Choice 1. Let us first take the IR R-symmetry to be $SU(2)_+$. With this choice, the left-moving fermions $(\lambda_-, \tilde{\lambda}_-)$ in the $(0, 4)$ vector multiplet and the right-moving fermions $(\psi_+, \tilde{\psi}_+)$ in the $(0, 4)$ hypermultiplet carry non-trivial IR R-charge, according to Table 2. Their contributions give

$$c_R^{(0)} = 3 \times 2 \times (n_h - n_v) = 6(n_h - n_v)\ . \tag{2.12}$$

However, this expression is correct only for the sphere reduction $g_1 = g_2 = 0$, with the number of punctures n arbitrary, and it describes the right-moving central charge of non-linear sigma model on the Higgs branch that is parametrized by hypermultiplet scalars. For more general values of (g_1, g_2), this result fails to reproduce the actual IR central charge due to unbroken gauge groups, and further modifications are required, as will be explained in Section 2.4.

Choice 2. Alternatively, we can take $SU(2)_-$ as the IR $\mathcal{N}=(0,4)$ R-symmetry. In this case, the left-moving fermions $(\lambda_-, \tilde{\lambda}_-)$ in the $(0,4)$ vector multiplet and the right-moving fermions $(\lambda_+, \tilde{\lambda}_+)$ in the $(0,4)$ twisted hypermultiplet have non-trivial IR R-charges. The resulting right-moving central charge is

$$c_R^{\text{tw}} = 3 \times 2 \times (g_2 n_v - n_v) = 6(g_2-1)n_v \ , \tag{2.13}$$

For $g_2 \geq 2$, this precisely matches the central charge of non-linear sigma model on the twisted Higgs branch, which is parametrized by twisted hypermultiplet scalars, as we will see in §3.2.2. Although this branch exists for $g_2 = 1$ since the twisted hypermultiplets exist in the theory, additional subtleties arise, and this formula must be modified accordingly.

The gravitational anomaly is given by the difference in number of the left-moving and right-moving fermions:

$$k_g = 2(n_v + g_2 n_h) - 2(n_h + g_2 n_v) = 2(g_2-1)(n_h - n_v) \ . \tag{2.14}$$

Note that for $g_2 = 1$, the 2d gauge theory has $\mathcal{N} = (4,4)$ supersymmetry and the gravitational anomaly vanishes as expected.

Issue on the central charges Consider first the simplest situation in which the the first Riemann surface has no punctures, i.e. $n = 0$. Then, the naive attempt to compute the right-moving central charge as the 't Hooft anomaly $k_{SU(2)_+}$ leads to the expression (2.12), which is

$$c_R^{(0)} = 6(1-g_1) r_G \ . \tag{2.15}$$

However, this expression becomes negative when g_1 is greater than one, which is incompatible with the unitarity of the resulting 2d theory. In addition, when $g_1 = 1$, c_R vanishes although the theory is non-trivial. This already signals that the expression (2.12) is incorrect for a general g_1.

The same issue arises for the 't Hooft anomaly $k_{SU(2)_-}$, and (2.13) vanishes for $g_2 = 1$ although the theory is obtained by the reduction of class $\mathcal{S}$ theory on a torus, which is non-trivial.

In addition, as we will see below, both hypermultiplet and twisted-hypermultiplet scalars mix and generically acquire vacuum expectation values on the (special) Higgs branch. This mixing must be incorporated to obtain the physically correct central charges.

We will give the conjecture on the correct right-moving central charge and explain the discrepancy of the naive one in Section 2.4.

2.3 Central charges from anomaly polynomial and its limitation

Central charges can be read from the anomaly coefficients encoded in the anomaly polynomial. For a 2d theory with at least $\mathcal{N} = (0,2)$ supersymmetry, the anomaly polynomial [6] is given by

$$A_4 = \frac{k_g}{24} p_1(T\mathcal{M}_2) + \frac{c_R}{6} c_1(r_{2d})^2 \tag{2.16}$$

where $p_1(T\mathcal{M}_2)$ is the first Pontryagin class of the tangent bundle to the 2d manifold $\mathcal{M}_2$, $c_1(r_{2d})$ is the first Chern class for the $U(1)_{r_{2d}}$ R-symmetry bundle of $(0,2)$ superalgebra.

From 2d field contents One can calculate the 2d anomaly polynomial directly from the field contents and their charges in Table 1. The result is [2]

$$A_4 = \frac{1}{12}(g_2-1)(n_h-n_v)p_1(T\mathcal{M}_2) - (n_h-n_v)c_2(R_+) + (1-g_2)n_v c_2(R_-) - (1-g_2)(n_h-n_v)c_1(F)^2 \ , \tag{2.17}$$

[2] We omit the flavor symmetry and other possible mixed terms.

where F denotes the background gauge field strength of $U(1)_r$ reduced to 2d, and the anomaly coefficients are given by

$$k_{SU(2)_\pm} = \mathrm{Tr}\gamma_3 R_\pm^2 \ , \quad k_r = \mathrm{Tr}\gamma_3 r^2 \ , \quad k_g = \mathrm{Tr}\gamma_3 \ , \tag{2.18}$$

with the traces over all Weyl fermions. The 't Hooft anomaly and the gravitational anomaly can then be read off as

$$k_{SU(2)_+} = 2(n_h - n_v) \ , \ k_{SU(2)_-} = 2(g_2-1)n_v \ , \ k_r = 2(1-g_2)(n_h-n_v) \ , \ k_g = 2(g_2-1)(n_h-n_v) \ , \tag{2.19}$$

leading to (2.12), (2.13) and (2.14).

From dimensional reduction Starting from the anomaly polynomial A_8 of the 6d $(2,0)$ SCFT, one obtains the anomaly polynomial A_6 of class $\mathcal{S}$ theories by suitably integrating A_8 over the punctured Riemann surface $C_{g_1,n}$. For a resulting 4d $\mathcal{N}=2$ theory with effective numbers of vector and hypermultiplets (n_v, n_h), the anomaly polynomial takes the form [12]

$$A_6 = (n_v - n_h)\left(\frac{c_1(r)^3}{3} - \frac{c_1(r)}{12}p_1(T\mathcal{M}_4)\right) - n_v c_1(r) c_2(R) \ , \tag{2.20}$$

where R and r denote the $SU(2)_R$ and $U(1)_r$ R-symmetry bundles of the 4d $\mathcal{N}=2$ theory.

To obtain the anomaly polynomial of the 2d theory that arises upon compactification on a second Riemann surface C_{g_2}, one further integrates A_6 over C_{g_2}. Preserving $\mathcal{N}=(0,4)$ supersymmetry requires performing a topological twist along $U(1)^{\mathrm{tw}}_{C_{g_2}} = \mathrm{diag}\big(U(1)_{C_{g_2}} \times U(1)_r\big)$ as in (2.7). Under this twist, the 4d characteristic classes decompose into their 2d counterparts as

$$p_1(T\mathcal{M}_4) = p_1(T\mathcal{M}_2), \quad c_1(r) = c_1(F) + \frac{t}{2}, \quad c_2(R) = -c_1(R_-)^2 \ ,$$

where t is the Chern root of the tangent bundle of C_{g_2}, normalized such that $\int_{C_{g_2}} t = 2(1-g_2)$. Integrating (2.20) on C_{g_2} gives the 2d $\mathcal{N}=(0,4)$ anomaly polynomial,

$$A_4 = (g_2 - 1)\left[(n_h - n_v)\left(c_1(F)^2 - \frac{p_1(T\mathcal{M}_2)}{12}\right) - n_v c_1(R_-)^2\right] \ . \tag{2.21}$$

The dimensional-reduction result (2.21) reproduces (2.17) except for the anomaly term of $SU(2)_+$. Although the Cartan subgroup of $SU(2)_+$ is identified with diag $\big(U(1)_{C_{g_2}} \times U(1)_r^{-1}\big)$ (2.11), the characteristic class t of C_{g_2} is completely integrated out during the dimensional reduction. As a result, the $SU(2)_+$ anomaly cannot be detected by integrating the 4d anomaly polynomial, leading to its absence in (2.21).

The general lesson from this analysis is the following. When a 6d $(2,0)$ SCFT is compactified on a 4-manifold with an appropriate topological twist, the standard procedure of integrating its anomaly polynomial A_8 over the 4-manifold does *not* necessarily yield the correct central charge of the resulting 2d theory. Depending on the choice of topological twist and, crucially, on how the superconformal R-symmetry is realized in the infrared, part of the information relevant to the IR central charge may be integrated out along the way.

2.4 Conjectures on central charges

We already discussed the issue on the naive central charge (2.12). Nevertheless, the genuine Higgs branch central charge is expressed by the quaternionic dimension of the Higgs branch. For the 2d theory obtained from $\mathcal{S}$ theory $\mathcal{T}_G[C_{g_1,n}]$ on S^2 reduction, the naive central charge (2.12) should receive a shift to obtain the genuine one [8],

$$c_R^{(0)} \to c_R^{(0)} + 6 g_1 r_G \ . \tag{2.22}$$

The underlying reason is now well understood. As emphasized in [9], for $g_1 > 0$ the Cartan subgroup of the gauge group of the parent 4d theory remains unbroken at a generic point on the Higgs branch. Consequently, an abelian gauge sector $U(1)^{r_G g_1}$ survives in the 2d theory. In two dimensions, however, an abelian vector multiplet is *gapped* in the infrared [10, 22]. Since gapped degrees of freedom do *not* contribute to the anomaly of the IR theory, the contribution from Cartan components of the gauginos must be turned off when computing the right-moving central charge using the full UV degrees of freedoms [7, 8, 23], leading to the shift $6g_1 r_G$. More generally, whenever a subgroup of the gauge group remains unbroken at a generic point on the vacuum moduli space, the naive procedure of integrating the anomaly polynomial over the compactification manifold does *not* yield the physically correct 2d central charges.[3]

We generalize the above discussion to the general reduction of $\mathcal{T}_G[C_{g_1,n}]$ on C_{g_2}. In the IR, there are unbroken $U(1)^{g_1 r_G}$ gauge symmetries, as shown in Section 3.2.1, with $2g_1 r_G$ gauginos and $2g_1 g_2 r_G$ adjoint-valued fermions in the 2d twisted hypermultiplets getting decoupled in IR. The contribution to the right-moving central charge of the decoupled gauginos and fermions in IR should be turned off, and therefore we propose the correct central charge of special Higgs branch CFT to be

$$c_R = c_R^{(0)} + 6g_1(1+g_2)r_G = 6\left(n_h - n_v + g_1(1+g_2)r_G\right) \ . \tag{2.23}$$

In particular, for the case $G = SU(2)$, $r_G = 1$ and $n_h - n_v = 1 - g_1 + n$, so the conjecture becomes

$$c_R = 6(g_1 g_2 + n + 1) \ , \tag{2.24}$$

which will be confirmed by Hilbert series computations in Section 3.2.1. The existence of unbroken gauge groups will also be confirmed in Section 3.2.1.

As another supporting evidence, we study the dimensional reduction of a single M5 brane on $C_{g_1,n} \times C_{g_2}$ in Appendix D. Since the worldvolume theory is abelian, the 2d $\mathcal{N} = (0,4)$ theory after reduction is free. Its central charge can be determined by counting degrees of freedom of bosons and fermions that is positive.

From Higgs mechanism To understand the origin of the shift in (2.23) from another perspective, it is useful to re-examine the spectrum from the viewpoint of the Higgs mechanism. Let us first focus on the simplest case, $g_1 = 0$. Turning on generic VEVs for the hypermultiplet scalars, $\langle q \rangle$ and $\langle \tilde{q} \rangle$, completely breaks the gauge group. Consequently, all n_v gauge fields acquire masses, and the associated gauginos must also become massive. However, in two dimensions a single left-moving Weyl fermion cannot acquire a mass on its own: a mass term necessarily pairs a left-moving fermion with a right-moving one. Indeed, a generic fermion mass term in 1+1 dimensions have the form of

$$S = \int d^2x \left(\mathrm{i}\psi_+^\dagger \partial_- \psi_+ + \mathrm{i}\psi_-^\dagger \partial_+ \psi_- \right) - m \left(\psi_-^\dagger \psi_+ + \psi_+^\dagger \psi_- \right) \tag{2.25}$$

which explicitly shows that a massive Dirac fermion requires both chiralities. In the special Higgs branch, the right-moving partners come from the hypermultiplets. Thus, the n_v left-moving gauginos pair up with n_v right-moving fermions from the hypermultiplets, rendering n_v hypermultiplets massive. After integrating out these massive fields, the low-energy theory contains $(n_h - n_v)$ massless hypermultiplets, which reproduces the naive expression (2.12) for the right-moving central charge. Notice that the gauge group is completely broken in this case, and therefore (2.12) gives the correct right-moving central charge.

We now turn to the cases with $g_1 > 0$. As discussed above, generic VEVs of the hypermultiplet scalars no longer completely break the gauge group, and $U(1)^{g_1 r_G}$ subgroup remains unbroken.

[3] Such situations often arise when the 6d $(2,0)$ theory is compactified on a 4-manifold M_4 with nontrivial one-cycles ($b_1(M_4) \neq 0$), leading to residual abelian gauge sectors at a certain 2d vacuum.

Consequently, only $(n_v - g_1 r_G)$ vector multiplets become massive. By the same Higgs mechanism argument, $(n_v - g_1 r_G)$ hypermultiplets become massive, leaving $n_h - (n_v - g_1 r_G)$ massless hypermultiplets in the spectrum.

In addition, the presence of $U(1)^{g_1 r_G}$ gauge sectors implies the existence of massless twisted hypermultiplets. Compactification on the second Riemann surface C_{g_2} contributes g_2 zero modes to each unbroken $U(1)$ factor, giving rise to $g_1 g_2 r_G$ massless twisted hypermultiplets in the effective 2d theory.

Combining the contributions from massless hypermultiplets and twisted hypermultiplets, the deep IR spectrum contains $n_h - (n_v - g_1 r_G) + g_1 g_2 r_G$ bosonic degrees of freedom. These precisely account for the right-moving central charge appearing in our conjecture (2.23). The analysis is summarized in Table 3.

fields	UV scale	VEVs scale	deep IR
$U = (A_\mu, \lambda_-),\ \Theta = (\tilde{\lambda}_-)$	n_v	$g_1 r_G$	0
$\Sigma = (\sigma, \lambda_+^\dagger),\ \tilde{\Sigma} = (\tilde{\sigma}, \tilde{\lambda}_+)$	$g_2 n_v$	$g_1 g_2 r_G$	$g_1 g_2 r_G$
$\Phi = (q, \psi_+),\ \tilde{\Phi} = (\tilde{q}, \tilde{\psi}_+)$	n_h	$n_h - n_v + g_1 r_G$	$n_h - n_v + g_1 r_G$
$\Gamma = (\psi_-),\ \tilde{\Gamma} = (\tilde{\psi}_-)$	$g_2 n_h$	$g_2(n_h - n_v + g_1 r_G)$	$g_2(n_h - n_v + g_1 r_G)$

Table 3. The number of effective fields at three different scales of the 2d theory for the special Higgs branch.

Next, let us move to the twisted Higgs branch. First, consider the case of $g_2 = 1$. The 2d UV theory is a $\mathcal{N} = (4,4)$ gauge theory with Lagrangian identical to the corresponding 4d $\mathcal{N} = 2$ theory. Taking the VEVs of $\langle \sigma \rangle$ and $\langle \tilde{\sigma} \rangle$ leads to the Coulomb branch. Similar to the one of 4d class $\mathcal{S}$ theory, there will be unbroken $U(1)^{n_v/3}$ gauge symmetries corresponding to the Cartan of each gauge nodes and all hypermultiplets are massive. The Coulomb branch CFT is parametrized by scalars from $n_v/3$ massless $\mathcal{N} = (4,4)$ vector multiplets. This gives the right-moving central charge

$$c_R = 6 \times \frac{n_v}{3} = 2 n_v \tag{2.26}$$

matches with the result from Hilbert series in (3.33).

fields	UV scale	VEVs scale	deep IR
$U = (A_\mu, \lambda_-),\ \Theta = (\tilde{\lambda}_-)$	n_v	$n_v/3$	0
$\Sigma = (\sigma, \lambda_+^\dagger),\ \tilde{\Sigma} = (\tilde{\sigma}, \tilde{\lambda}_+)$	n_v	$n_v/3$	$n_v/3$
$\Phi = (q, \psi_+),\ \tilde{\Phi} = (\tilde{q}, \tilde{\psi}_+)$	n_h	0	0
$\Gamma = (\psi_-),\ \tilde{\Gamma} = (\tilde{\psi}_-)$	n_h	0	0

Table 4. The number of effective fields at three different scales of the 2d theory with $g_2 = 1, G = SU(2)$ for the twisted Higgs branch, being similar to the situation of the Coulomb branch of the corresponding 4d class $\mathcal{S}$ theory.

The twisted Higgs branch for $g_2 \geq 2$ has $\mathcal{N} = (0,4)$ supersymmetry. By the analysis in Section 3.2.2, all gauge symmetries are broken. So, all n_v number of gaugino are massive. The right-moving fermions in n_v twisted hypermultiplets will get mass as well. Similar with the $\mathcal{N} = (4,4)$ case, all hypermultiplets are massive. Thus, there are $(g_2 - 1) n_v$ massless twisted hypermultiplets after Higgsing which gives the central charge obtained from the 't Hooft anomaly of $SU(2)_-$ in equation (2.13). The analysis is summarized in Table 5.

fields	UV scale	VEVs scale	deep IR
$U=(A_\mu,\lambda_-),\ \Theta=(\tilde{\lambda}_-)$	n_v	0	0
$\Sigma=(\sigma,\lambda_+^\dagger),\ \tilde{\Sigma}=(\tilde{\sigma},\tilde{\lambda}_+)$	g_2n_v	$(g_2-1)n_v$	$(g_2-1)n_v$
$\Phi=(q,\psi_+),\ \tilde{\Phi}=(\tilde{q},\tilde{\psi}_+)$	n_h	0	0
$\Gamma=(\psi_-),\ \tilde{\Gamma}=(\tilde{\psi}_-)$	g_2n_h	$(g_2-1)n_h$	$(g_2-1)n_h$

Table 5. The number of effective fields at three different scales of the 2d theory for the twisted Higgs branch, when $g_2 \geq 2$.

3 Vacuum moduli spaces for $G = SU(2)$

As a consistency check of our conjecture for the central charges of the 2d $\mathcal{N}=(0,4)$ theories, we explicitly analyze their infrared vacuum structure. In this section, we study the vacuum moduli spaces in the cases of gauge group $G=SU(2)$ where the underlying UV Lagrangian can be written explicitly. In particular, we explicitly compute the *Hilbert series*, which counts the chiral operators (or equivalently holomorphic functions) on the vacuum moduli spaces.

3.1 (0,4) Lagrangian

Now, we derive the effective $(0,4)$ Lagrangian from the dimensional reduction of the 4d theory $\mathcal{T}_{SU(2)}[C_{g_1,n}]$ on C_{g_2}. For notational simplicity, we will omit the gauge group label and write the theory simply as $\mathcal{T}[C_{g_1,n}]$. We will sometimes denote the numbers of gauge nodes and trinions as

$$N_v = \frac{n_v}{3} = 3(g_1-1)+n, \quad N_h = \frac{n_h}{4} = 2(g_1-1)+n \ . \tag{3.1}$$

Upon the dimensional reduction, the 4d superpotential specifically yields $(0,2)$-type couplings with J- and E-terms in two dimensions. Solving the resulting J-, E-equations with D-term constraints defines a hyper-Kähler vacuum moduli space, which we call the *full Higgs branch* of the 2d theory. In general, this vacuum space decomposes into multiple irreducible components. Our goal in what follows is to analyze the structure of this moduli space.

To set the stage, recall the cubic superpotential of the 4d theory $\mathcal{T}_G[C_{g_1,n}]$:

$$\mathcal{L}_{W_{\rm 4d}} = \tilde{Q}_{\rm 4d}\,\Phi_{\rm 4d}\,Q_{\rm 4d} + \text{c.c.} \tag{3.2}$$

Upon reduction on C_{g_2}, this superpotential yields an effective 2d interaction written naturally in terms of $(0,2)$ multiplets [7, 16]:

$$\begin{aligned}\mathcal{L}_{W_{\rm 2d}} &= \tilde{Q}\Theta Q + 2\mathrm{i}\sum_{j=1}^{g_2}\left(\tilde{\Gamma}_j\tilde{\Sigma}_jQ + \tilde{Q}\tilde{\Sigma}_j\Gamma_j\right) + \text{c.c.}\\ &= Q_{abc}Q_{a'b'c'}\Theta^{aa'}\epsilon^{bb'}\epsilon^{cc'} - \mathrm{i}\sum_{j=1}^{g_2}\Gamma_{jabc}Q_{a'b'c'}\tilde{\Sigma}_j^{aa'}\epsilon^{bb'}\epsilon^{cc'} + \text{c.c.}\ ,\end{aligned} \tag{3.3}$$

where $\epsilon = \left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$, and our conventions for the $(0,2)$ multiplets are summarized in Appendix A.

From the superpotential, we can read out the holomorphic J-functions associated to each $(0,2)$ Fermi multiplets,

$$J_{\Theta^{aa'}} = Q_{abc}Q_{a'b'c'}\epsilon^{bb'}\epsilon^{cc'} \ , \quad J_{\Gamma_{jabc}} = -\mathrm{i}Q_{a'b'c'}\tilde{\Sigma}_j^{aa'}\epsilon^{bb'}\epsilon^{cc'} \ . \tag{3.4}$$

The compatible holomorphic E-functions for the Fermi multiplets are

$$E_\Theta = \frac{1}{2}\sum_{j=1}^{g_2}\left[\tilde{\Sigma}_j, \Sigma_j\right] \ , \quad E_{\Gamma_{jabc}} = \mathrm{i}Q_{a'b'c'}\Sigma_j^{aa'}\epsilon^{bb'}\epsilon^{cc'} \ , \tag{3.5}$$

They are subject to the constraints:

$$E \cdot J = E_\Theta J_\Theta + \sum_{j=1}^{g_2} E_{\Gamma_j} J_{\Gamma_j} = 0 \ . \tag{3.6}$$

The supersymmetric vacua are obtained by imposing the vanishing of all holomorphic J- and E-terms, together with the usual D-term constraints. Concretely, the scalar fields q_{abc} in the hypermultiplets Q_{abc} and the scalars $\sigma_j, \tilde{\sigma}_j$ in the twisted hypermultiplets $\Sigma_j, \tilde{\Sigma}_j$ for $j = 1, \ldots, g_2$ must satisfy

$$J_\Theta(q) = J_{\Gamma_j}(q, \sigma, \tilde{\sigma}) = J_{\tilde{\Gamma}_j}(q, \sigma, \tilde{\sigma}) = E_\Theta(\sigma, \tilde{\sigma}) = E_{\Gamma_j}(q, \sigma, \tilde{\sigma}) = E_{\tilde{\Gamma}_j}(q, \sigma, \tilde{\sigma}) = 0 \ . \tag{3.7}$$

After imposing the D-terms as well, it is expected that the solution describes a union of hyper-Kähler cones, providing the full vacuum moduli space of the 2d $\mathcal{N} = (0, 4)$ theory that we analyze as follows.

3.2 Vacuum moduli spaces and Hilbert series

The structure of the Higgs branches of class $\mathcal{S}$ theories of type A_1 has been analyzed in [9] by explicitly calculating the Hilbert series. We perform a similar analysis to the vacuum moduli spaces determined by the J-term and E-term equations (3.7) of the 2d $(0, 4)$ theories. The resulting vacuum moduli space typically decomposes into several irreducible components, each with its own Hilbert series.[4]

For the computation of the Hilbert series, using the equations, we determine the F-flat series $F^\flat_{(g_1,n)}(t, z_a, x_j; g_2)$ associated with the moduli space. Here, t is the fugacity that grades operators, z_a $(a = 1, \ldots, N_v)$ are gauge fugacities, and x_j $(j = 1, \ldots, n)$ are flavor fugacities. The Hilbert series is obtained by projecting the F-flat series onto gauge-invariant combinations using the Molien–Weyl integral [26–28]:

$$G_{(g_1,n)}(t, x_j; g_2) = \oint_{|z_a|=1} \prod_{a=1}^{N_v} d\mu_{SU(2)}(z_a) F^\flat_{(g_1,n)}(t, z_a, x_j; g_2) \ , \tag{3.8}$$

where $d\mu_{SU(2)}(z_a)$ is the Haar measure for the a-th $SU(2)$ gauge group given by

$$\oint_{|z_a|=1} d\mu_{SU(2)}(z_a) = \frac{1}{2\pi \mathrm{i}} \oint_{|z_a|=1} dz_a \frac{1 - z_a^2}{z_a} \ . \tag{3.9}$$

The structure of irreducible components in the moduli space may vary across different duality frames. Nevertheless, there exist two distinguished components whose presence is *frame-independent*. We refer to these as the *special Higgs branch* and the *twisted Higgs branch*, according to the types of scalar fields that parametrize them.[5] The special Higgs branch is parametrized by the scalar fields of hypermultiplets, and—when $g_1 > 0$— also by those of twisted hypermultiplets. In contrast, the twisted Higgs branch is generically parametrized entirely by the scalars arising from twisted hypermultiplets. Precise definitions of these branches will be introduced shortly, and explicit computational examples are provided below and in Appendix C.

The fact that these two components remain invariant across frames provides nontrivial evidence for a generalized S-duality in 2d $\mathcal{N} = (0, 4)$ theories—an analogue of the 4d dualities discussed in [2]. It also naturally extends the duality structure observed in compactifications on S^2 in [7].

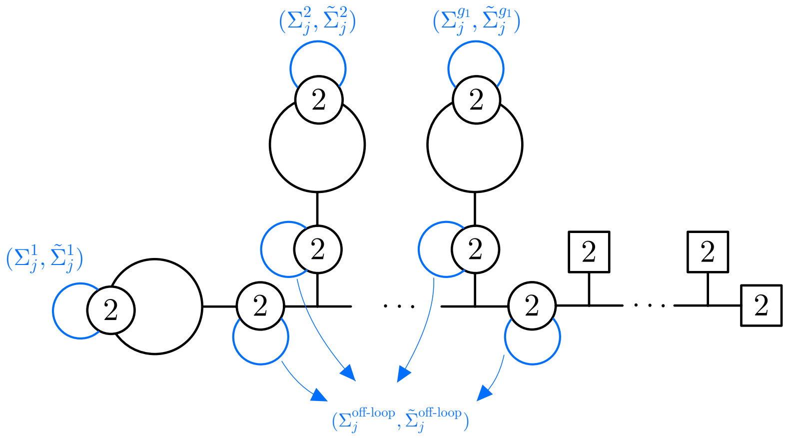

Figure 3. The standard frame of the case with generic (g_1, n). The gauge nodes are of two types: one are on the loops of the quiver, and the others are not. The special Higgs branch requires the scalars of the twisted hypermultiplets $\left(\Sigma_j^{\text{off-loop}}, \tilde{\Sigma}_j^{\text{off-loop}}\right)$ connecting to the nodes of the latter type must vanish.

3.2.1 Special Higgs branch

To define the special Higgs branch, it is convenient to work in a particular duality frame of $\mathcal{T}[C_{g_1,n}]$ on C_{g_2} obtained by gluing the basic building blocks—tadpoles ($C_{1,1}$ on C_{g_2}) and trinions ($C_{0,3}$ on C_{g_2}). We refer to this quiver presentation as the *standard frame*. In this frame, each loop in the quiver diagram (each handle of the Riemann surface) has a single gauge node, as illustrated in Figure 3. The gauge nodes naturally fall into two categories: the loop node from the tadpole, and the node from gluing two building blocks. For a surface of type (g_1, n), the standard frame therefore contains precisely g_1 loop gauge nodes.

Ideal structures In the standard frame, the ideal defining the special Higgs branch is generically generated by three classes of equations,

$$f_{qq} = f_{q\sigma} = f_{\sigma\sigma} = 0 \ , \tag{3.10}$$

where f_{qq} consists of equations involving only the hypermultiplet scalars q and coincides with the familiar four-dimensional F-term relations; $f_{\sigma\sigma}$ involves solely the twisted hypermultiplet scalars $(\sigma, \tilde{\sigma})$; and $f_{q\sigma}$ contains mixed terms coupling q and $(\sigma, \tilde{\sigma})$. The explicit forms of these equations are lengthy and not particularly illuminating, so we omit them here. Instead, we illustrate the structure of $f_{\sigma\sigma}$, which imposes simple and universal constraints determined by the types of gauge nodes to which the twisted hypermultiplets are attached.

Twisted hypermultiplets in the standard frame fall into two categories depending on whether their associated gauge nodes lie on a loop or not. Accordingly, the scalar VEVs on the special Higgs branch are subject to the following conditions:

- The scalars in the twisted hypermultiplets should have vanishing commutator if they are associated with gauge nodes that are *on a loop*,

$$\left[\sigma_i^I, \sigma_j^I\right] = \left[\sigma_i^I, \tilde{\sigma}_j^I\right] = \left[\tilde{\sigma}_i^I, \tilde{\sigma}_j^I\right] = 0 \ , \quad \forall\ i,j = 1, \cdots, g_2,\ I = 1, \cdots, g_1 \ . \tag{3.11}$$

[4] Both the identification of irreducible components and the computation of their Hilbert series can be performed using `Macaulay2` [24]. See also [25] for an illustrative example.

[5] A systematic analysis of additional irreducible or embedded components is beyond the scope of this work and will be pursued in future investigations.

These are precisely the conditions ensuring that the twisted hypermultiplet scalars parametrize a Cartan subalgebra along the loop directions.

- The scalars in the twisted multiplets should vanish if they are associated with gauge nodes that are *not on a loop*,

$$\sigma_j^{\text{off-loop}} = \tilde{\sigma}_j^{\text{off-loop}} = 0 \ , \quad \forall\, i, j = 1, \cdots, g_2 \ . \tag{3.12}$$

  These fields therefore do not contribute to the parametrization of the special Higgs branch.

Depending on the geometric data, multiple duality frames may arise, e.g., cases with $g_1 = 0$, $n \geq 6$, or $g_1 = 1$, $n \geq 2$. In frames other than the standard one, the defining equations for the special Higgs branch need not take the form of (3.11) and (3.12). Nevertheless, we conjecture that the resulting geometry of the special Higgs branch remains invariant across these different frames. This frame independence can be checked by the agreement of the unrefined Hilbert series computed in each frame. For cases with $g_1 = 0$, the frame independence is inherited from the parent 4d $\mathcal{N} = 2$ theory, as the 2d vacuum equations for the special Higgs branch are exactly the same as that of the 4d theory. For cases with $g_1 = 1$, we verified the frame independence at least for the case of $(g_1, n, g_2) = (1, 2, 1)$ (see (C.9)). Other cases exceed our current available computational capacity, but we conjecture the frame-independence to be true for general cases, and leave the verification in future work.

As observed in [9], the solutions to the hypermultiplet equations $f_{qq} = 0$ leave $U(1)^{g_1}$ gauge group unbroken, where each $U(1)$ corresponds to the an abelian subgroup of an $SU(2)$ gauge node lying on a loop. Similarly, the constraints (3.11) imply that, on the special Higgs branch of the standard quiver, the twisted hypermultiplet scalars σ_i^I associated with loop nodes acquire VEVs valued in a Cartan subalgebra of the corresponding $SU(2)$. Moreover, the $f_{q\sigma}$ constraints require that, for each gauge node on a loop, the VEVs of q and σ take a parallel direction. For example, see (C.5) for the case $(g_1, n) = (1, 1)$. Hence the unbroken $U(1)$ is exactly the Cartan subgroup of the $SU(2)$ on the loop, generated by the VEVs $\sigma_i^I, \tilde{\sigma}_i^I$. Therefore, at the generic point, there are unbroken gauge symmetry $U(1)^{g_1}$. Combining these observations with frame independence, we conclude that there are $U(1)^{g_1}$ unbroken gauge group at the special Higgs branch.[6]

n	**dim**	**Hilbert series**
3	4	$\frac{1}{(1-t)^8}$
4	5	$\frac{(1+t^2)(1+17t^2+48t^4+17t^6+t^8)}{(1-t^2)^{10}}$
5	6	$\frac{1+9t^2+26t^3+41t^4+106t^5+195t^6+234t^7+306t^8+372t^9+\ldots\text{pal}\ldots+t^{18}}{(1-t^2)^6(1-t^3)^6}$
6	7	$\frac{1+11t^2+118t^4+538t^6+1900t^8+4109t^{10}+6901t^{12}+7804t^{14}+\ldots\text{pal}\ldots+t^{28}}{(1-t^2)^7(1-t^4)^7}$
7	8	$\frac{\begin{pmatrix} 1+13t^2+85t^4+120t^5+377t^6+792t^7+1289t^8+2904t^9+4844t^{10}+7736t^{11}+12391t^{12} \\ +16568t^{13}+24151t^{14}+32328t^{15}+38331t^{16}+46416t^{17}+51171t^{18}+55056t^{19}+59094t^{20} \\ +\ldots\text{pal}\ldots+t^{40} \end{pmatrix}}{(1-t^2)^8(1-t^5)^8}$

Table 6. The quaternionic dimensions and unrefined Hilbert series of the special Higgs branches of the case $(g_1, n) = (0, n)$. The results are identical to the Higgs branches of the corresponding 4d theories.

[6] Notice that gauge enhancement can happen at special loci. For example, at the origin, all the scalar fields vanishes, and hence the full gauge group is unbroken.

Superfield	numbers	$U(1)_+$	$U(1)_X$	$U(1)_{\mathrm{IR}}$
$U_C^{\text{on-loop}} = (A_\mu, \lambda_-)$	$g_1 r_G$	$(0, -\frac{1}{2})$	$\frac{1}{2}$	$(\frac{1}{2}, 0)$
$U^{\text{others}} = (A_\mu, \lambda_-)$	$n_v - g_1 r_G$	$(0, -\frac{1}{2})$	0	$(0, -\frac{1}{2})$
$\Theta_C^{\text{on-loop}} = (\tilde{\lambda}_-)$	$g_1 r_G$	$-\frac{1}{2}$	$\frac{1}{2}$	0
$\Theta^{\text{others}} = (\tilde{\lambda}_-)$	$n_v - g_1 r_G$	$-\frac{1}{2}$	0	$-\frac{1}{2}$
$\Sigma_{j,C}^{\text{on-loop}} = (\sigma, \lambda_+^\dagger)$	$g_1 g_2 r_G$	$(\frac{1}{2}, 0)$	$-\frac{1}{2}$	$(0, -\frac{1}{2})$
$\Sigma_j^{\text{others}} = (\sigma, \lambda_+^\dagger)$	$(n_v - g_1 r_G) g_2$	$(\frac{1}{2}, 0)$	0	$(\frac{1}{2}, 0)$
$\tilde{\Sigma}_{j,C}^{\text{on-loop}} = (\tilde{\sigma}, \tilde{\lambda}_+)$	$g_1 g_2 r_G$	$(-\frac{1}{2}, 0)$	$\frac{1}{2}$	$(0, \frac{1}{2})$
$\tilde{\Sigma}_j^{\text{others}} = (\tilde{\sigma}, \tilde{\lambda}_+)$	$(n_v - g_1 r_G) g_2$	$(-\frac{1}{2}, 0)$	0	$(-\frac{1}{2}, 0)$
$\Phi = (q, \psi_+)$	n_h	$(0, \frac{1}{2})$	0	$(0, \frac{1}{2})$
$\tilde{\Phi} = (\tilde{q}, \tilde{\psi}_+)$	n_h	$(0, \frac{1}{2})$	0	$(0, \frac{1}{2})$
$\Gamma_j = (\psi_-)$	$n_h g_2$	0	0	0
$\tilde{\Gamma}_j = (\tilde{\psi}_-)$	$n_h g_2$	0	0	0

Table 7. We conjecture that there is an emergent $SU(2)_X$ symmetry in the IR, and the IR R-symmetry $SU(2)_{\mathrm{IR}} = \mathrm{diag}(SU(2)_+ \times SU(2)_X)$. The charges of fields under the Cartans $U(1)_+ = U(1)^{\mathrm{tw}'}_{C_{g_2}}$, $U(1)_X$ and $U(1)_{\mathrm{IR}}$ are listed. In particular, $U(1)_X$ has non-zero charges only on the Cartan part of the vector multiplets and twisted hypermultiplets on a loop of the quiver. Note that this charge assignment is only for the standard frame as in Figure 3.

Special cases When $g_1 = 0$, the quiver contains no loops. Consequently, all twisted hypermultiplet scalars must vanish on the special Higgs branch. In this situation, the J- and E-term equations of the 2d theory all vanish except for the one that is identical to the F-term equations of the 4d hypermultiplets. Thus, the special Higgs branch is parametrized entirely by the scalars of the 2d hypermultiplets and is *identical* to the Higgs branch of the parent 4d theory studied in [9], with quaternionic dimension $1 + n$. The corresponding unrefined Hilbert series for various values of n are summarized in Table 6.

IR R-symmetry We conjecture that the IR R-symmetry group $SU(2)_{\mathrm{IR}}$ of the CFT on the special Higgs branch is the mix of $SU(2)_+$ with an emergent global symmetry $SU(2)_X$ in the IR,

$$SU(2)_{\mathrm{IR}} = \mathrm{diag}\left(SU(2)_+ \times SU(2)_X\right) \ .$$

$SU(2)_X$ acts on the Cartan part $(U_C^{\text{on-loop}}, \Theta_C^{\text{on-loop}})$ of the vector multiplets and the twisted hypermultiplets $(\Sigma_{j,C}^{\text{on-loop}}, \tilde{\Sigma}_{j,C}^{\text{on-loop}})$ associated with the gauge nodes *on the loops* of the quiver diagram, and acts trivially on the other multiplets. The charges of the fields under the Cartan $U(1)_X$ and $U(1)_{\mathrm{IR}}$ in the standard frame are listed in Table 7. We emphasize that there is no such $SU(2)_X$ at UV, since the superpotential is not invariant under the charge assignment.

If $SU(2)_X$ is a symmetry at UV, the 't Hooft anomaly coefficient of $U(1)_{\mathrm{IR}} = \mathrm{diag}(U(1)^{\mathrm{tw}'}_{C_{g_2}} \times U(1)_X)$ leads to the correct central charge

$$c_R = 3\left[-2 \times (n_v - g_1 r_G) + 2 \times g_1 g_2 r_G + 2 \times n_h\right] = 6\left(n_h - n_v + g_1(1 + g_2) r_G\right) \ , \tag{3.13}$$

matching with (2.23). But since it emerges at the IR, we should use the IR field contents listed in Table 3 to give a valid calculation, which gives identical result

$$c_R = 3\left[2 \times g_1 g_2 r_G + 2 \times (n_h - n_v + g_1 r_G)\right] = 6\left(n_h - n_v + g_1(1 + g_2) r_G\right) \ . \tag{3.14}$$

This IR R-symmetry is also consistent with the requirement that scalars parametrizing the moduli space should be neutral under the IR R-symmetry [10]. From the ideal structure, we see that the special Higgs branch is generically parametrized by the hypermultiplet scalars q and the Cartan part of the twisted hypermultiplet scalars $\sigma_{j,C}^{\text{on-loop}}$ associated with the gauge nodes *on the loops*. As shown Table 7, they are indeed neutral under $SU(2)_{\text{IR}}$. It also preserves the ideal structure, with each term in the vacuum equations homogeneous with respect to the $SU(2)_{\text{IR}}$ charge.

Gluing method For $g_1 \geq 1$, vacuum equations (3.11), (3.12) and (3.7) are complicated, which makes direct computation of the Hilbert series difficult. Instead, we use the *gluing method* to compute the Hilbert series of a larger quiver from that of specialer ones. This method has been applied to the class $\mathcal{S}$ theories [9, 29], and we adapt it here to the 2d $(0,4)$ context.

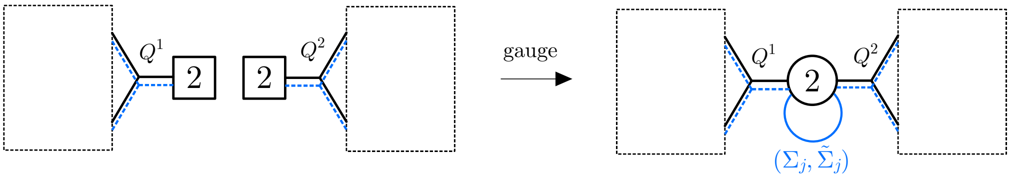

Figure 4. Gluing two quivers. The gluing will introduce 3 new J-term equations of q^1, q^2, and modify the J-term and E-term equations on both sides.

As is standard in the study of class $\mathcal{S}$ theories, a larger quiver can be constructed by gluing two smaller, *disjoint* quivers via gauging a common flavor symmetry (see Figure 4). The J-term and E-term equations of the resulting quiver are inherited from those of the two original components, but the gluing procedure introduces additional constraints that modify the vacuum equations. More precisely, gluing leads to the following effects:

- New twisted hypermultiplet scalars σ_j and $\tilde{\sigma}_j$ $(j = 1, \dots, g_2)$ are introduced at the gauged node.
- Three new J-term equations involving the bifundamental fields q^1 and q^2 appear:
$$q^1_{abc}\, q^2_{a'b'c'}\, (e_A)^{aa'}\, \epsilon^{bb'}\, \epsilon^{cc'} = 0, \qquad \forall\ A = 1,2,3 \tag{3.15}$$
where e_A is the canonical basis of adjoint representation, as defined by (A.8).
- The original J-term and E-term equations in each quiver acquire additional contributions of the form
$$q^I_{a'b'c'}\, \sigma_j^{aa'}\, \epsilon^{bb'}\, \epsilon^{cc'}, \qquad q^I_{a'b'c'}\, \tilde{\sigma}_j^{aa'}\, \epsilon^{bb'}\, \epsilon^{cc'}. \tag{3.16}$$

On the special Higgs branch, however, the fields σ_j and $\tilde{\sigma}_j$ associated with gauge nodes not lying on a loop must vanish. Consequently, for the special Higgs branch of the glued quiver, the only non-trivial new constraints introduced by the gluing procedure are those in (3.15) while the modifications arising from the terms in (3.16) become irrelevant. In this situation, the vacuum equations therefore reduce to those of the two original quivers, supplemented solely by the additional gluing relations.

At the level of Hilbert series, this means that the F-flat generating function of the special Higgs branch of the glued quiver is obtained by

$$F_H^\flat(t,z) = G_{H,1}(t,z) G_{\text{glue}}(t,z) G_{H,2}(t,z)\ , \tag{3.17}$$

where $G_{H,1}(t, x_1)$ and $G_{H,2}(t, x_2)$ are the Hilbert series of the special Higgs branches of the two original quivers, x_1, x_2 are the flavor fugacities for the punctures being glued. The gluing factor implementing the constraints (3.15) is given by

$$G_{\text{glue}}(t,z) = (1-t^2)(1-t^2z^2)(1-t^2z^{-2}) \ . \tag{3.18}$$

which is the same gluing factor in the 4d analysis [9, 29].

The Hilbert series of the special Higgs branch is obtained by integrating over the gauge fugacity:

$$G_{H,(g_1+g_1',n+n'-2)}(t;g_2) = \oint_{|z_a|=1} d\mu_{SU(2)}(z)\, G_{H,(g_1,n)}(t,z;g_2)\, G_{\text{glue}}(t,z)\, G_{H,(g_1',n')}(t,z;g_2), \tag{3.19}$$

where (g_1,n) and (g_1',n') are the geometric data on two sides respectively. Using this gluing method, one can compute the Hilbert series for any configuration (g_1,n). We have checked that the results obtained in this way agree with those derived from explicitly writing and solving the full set of J- and E-term equations; see Appendix C for details. We also confirm frame independence: different quiver frames yield identical Hilbert series, and hence identical Higgs-branch geometries.

From the resulting Hilbert series, we deduce that the quaternionic dimension of the special Higgs branch is

$$\dim_{\mathbb{H}} \mathcal{M}_H[C_{g_1,n}\times C_{g_2}] = g_1g_2+n+1, \tag{3.20}$$

leading to the right-moving central charge

$$c_R = 6(g_1g_2+n+1), \tag{3.21}$$

which agrees with our conjecture (2.24).

Non-palindromic Hilbert series for $g_1\geq 2$ From explicit computations of the Hilbert series, we find an intriguing phenomenon: for cases with $g_1\geq 2$ and $g_2\geq 1$, the numerators of the Hilbert series of the special Higgs branches are *non-palindromic*. Concrete examples for $(g_1,n)=(2,0)$ are presented in Table 11, where we also verify that the corresponding ideal is minimal prime, and hence defines an irreducible and radical variety.

Such non-palindromic Hilbert series violate the Gorenstein property, i.e.,

$$G(t^{-1}) = (-1)^d\, t^a\, G(t) \ , \tag{3.22}$$

for some integers d and a. By Stanley's theorem [30], this implies that the coordinate ring of the special Higgs branch is not Gorenstein (see also [31]). This means that the special Higgs branch for $g_1\geq 2$ and $g_2\geq 1$ is no longer a symplectic singularity [32].

The appearance of non-palindromic Hilbert series suggests that the geometry of the special Higgs branch undergoes a structural change for these values of (g_1,g_2). At present, however, we do not have a clear geometric interpretation of these cases, and understanding the underlying structure remains an interesting open problem.

We further observe that for cases with $n=0$, the Hilbert series of the special Higgs branches are *not* invariant under exchanging the two Riemann surfaces. Thus, the symmetry $g_1\leftrightarrow g_2$ is explicitly broken at the level of the Hilbert series, even though it is not apparent from the dimension formula (3.20). For example, using the gluing method, one finds

$$G_{H,(2,0)}(t;3) \ \neq\ G_{H,(3,0)}(t;2) \ , \tag{3.23}$$

and notably, the numerators of both Hilbert series are non-palindromic.

3.2.2 Twisted Higgs branch

The twisted Higgs branch is parametrized purely by the twisted hypermultiplet scalar fields. It exists for $g_2 \geq 1$ and is characterized by the vacuum equations

$$q^I = 0 \ , \qquad \sum_{j=1}^{g_2} \left[\sigma_j^J, \tilde{\sigma}_j^J\right] = 0 \ , \qquad \forall\ I = 1, \ldots, N_h \ , \quad \forall\ J = 1, \ldots, N_v \ . \tag{3.24}$$

These equations define an ideal whose structure is independent of the choice of duality frame.

For $g_2 = 1$, the vacuum equation reduces to

$$\left[\sigma^J, \tilde{\sigma}^J\right] = 0 \tag{3.25}$$

at each gauge node. Consequently, the twisted hypermultiplet scalars can acquire vacuum expectation values valued in a Cartan subalgebra of $SU(2)$. The resulting twisted Higgs branch is therefore given by

$$\mathcal{M}_{\text{tw}}\left[C_{g_1,n} \times T^2\right] \cong \left(\mathbb{H}/\mathbb{Z}_2\right)^{N_v}, \tag{3.26}$$

where the Weyl group $\mathbb{Z}_2$ of $SU(2)$ acts on the Cartan-valued VEVs. At a generic point on this branch, the Cartan subgroup of each gauge group remains unbroken, and the quaternionic dimension is consistent with the central charge (**??**).

For $g_2 > 1$, although the gauge symmetry is completely broken at a generic point on the twisted Higgs branch, the vacuum equations (3.24) at different gauge nodes are identical and decouple from one another. As a result, the twisted Higgs branch factorizes into a product of identical components,

$$\mathcal{M}_{\text{tw}}[C_{g_1,n} \times C_{g_2}] \cong \mathcal{M}_{g_2}^{N_v} \ , \tag{3.27}$$

where $\mathcal{M}_{g_2}$ denotes the twisted Higgs branch of quaternionic dimension $3(g_2 - 1)$ associated with a single gauge node.

We now compute the Hilbert series of the twisted Higgs branch. Because of the reason described above, both the F-flat series and the Hilbert series factorize into independent contributions from each gauge node. From (3.24), the F-flat series takes the form

$$F^\flat_{\text{tw},(g_1,n)}(t, z_j; g_2) = \prod_{j=1}^{N_v} F^\flat_{\text{tw}}(t, z_j; g_2) \ , \tag{3.28}$$

where z_j are the gauge fugacities, and $F^\flat_{\text{tw}}(t, z; g_2)$ denotes the F-flat series associated with a single gauge node. Explicitly,

$$F^\flat_{\text{tw}}(t, z; g_2) = \begin{cases} \dfrac{2t^3 - t^2\left(z^2 + 1 + z^{-2}\right) + 1}{(1-t)^2\left(1 - tz^{-2}\right)^2\left(1 - tz^2\right)^2} & g_2 = 1 \\[2ex] \dfrac{\left(1-t^2\right)\left(1 - t^2 z^{-2}\right)\left(1 - t^2 z^2\right)}{\left[(1-t)\left(1 - tz^{-2}\right)\left(1 - tz^2\right)\right]^{2g_2}} & g_2 \geq 2 \end{cases} \tag{3.29}$$

The Hilbert series is obtained by integrating over the gauge group at each node,

$$G_{\text{tw},(g_1,n)}(t; g_2) = \prod_{j=1}^{N_v} \oint_{|z_a|=1} d\mu_{SU(2)}(z_j) F^\flat_{\text{tw}}(t, z_j; g_2) = \left(G_{\text{tw}}(t; g_2)\right)^{N_v} \ . \tag{3.30}$$

where $d\mu_{SU(2)}(z)$ denotes the Haar measure of $SU(2)$. The single-node contribution $G_{\text{tw}}(t; g_2)$ is given by

$$G_{\text{tw}}(t; g_2) = \begin{cases} \dfrac{1 + t^2}{(1-t^2)^2} & g_2 = 1 \\[2ex] \dfrac{1 + \cdots}{(1-t^2)^{6g_2 - 6}(1+t^2)^{-1}} & g_2 \geq 2 \ . \end{cases} \tag{3.31}$$

For the torus reduction $g_2 = 1$, the Hilbert series is that of $\mathbb{H}/\mathbb{Z}_2$, confirming the geometry (3.26) of the twisted Higgs branch. Explicit expressions for $G_{\rm tw}(t; g_2)$ with $g_2 = 1, \ldots, 5$ are collected in Table 8.

From the Hilbert series, we see that the quaternionic dimension of the twisted Higgs branch is

$$\dim_{\mathbb{H}} \mathcal{M}_{\rm tw}\left[C_{g_1,n} \times C_{g_2}\right] = \begin{cases} N_v = 3(g_1-1)+n & g_2 = 1 \ , \\ 3(g_2-1)N_v & g_2 \geq 2 \ . \end{cases} \tag{3.32}$$

The central charge is then

$$c_R = 6 \dim_{\mathbb{H}} \mathcal{M}_{\rm tw}\left[C_{g_1,n} \times C_{g_2}\right] = \begin{cases} 2n_v & g_2 = 1 \ , \\ 6(g_2-1)n_v & g_2 \geq 2 \ . \end{cases} \tag{3.33}$$

It matches results (2.13) and the anomaly of the $SU(2)_-$ in (2.19) from IR R-symmetry identification and anomaly polynomial analysis when $g_2 \geq 2$. We will give confirmation on the IR R-symmetry identification from the ideal structures.

g_2	**dim**	**Hilbert series**
1	1	$\frac{1}{(1-t^2)^2(1+t^2)^{-1}}$
2	3	$\frac{1+3t^2+t^4}{(1-t^2)^6(1+t^2)^{-1}}$
3	6	$\frac{1+8t^2+14t^3+22t^4+34t^5+22t^6+14t^7+8t^8+t^{10}}{(1-t^2)^{12}(1+t^2)^{-1}}$
4	9	$\frac{1+17t^2+48t^3+126t^4+320t^5+537t^6+760t^7+894t^8+\ldots\text{pal}\ldots+t^{16}}{(1-t^2)^{18}(1+t^2)^{-1}}$
5	12	$\frac{1+30t^2+110t^3+421t^4+1462t^5+3684t^6+8000t^7+14806t^8+22492t^9+29106t^{10}+31968t^{11}+\ldots\text{pal}\ldots+t^{22}}{(1-t^2)^{24}(1+t^2)^{-1}}$

Table 8. The quatenionic dimensions and Hilbert series of the twisted Higgs branches of the cases with a single gauge node.

An interesting case is $(g_1, n) = (1, 1)$. In this case, when $g_2 = 1$, the twisted Higgs branch becomes a subvariety of the special Higgs branch. , and for $g_2 \geq 2$, the Hilbert series has an additional factor $(1-t)^{-2}$, implying the existence of an additional decoupled free hypermultiplet. Hence an extra $\mathbb{H}$ factor should be multiplied to the twisted Higgs branch.

Note that, given (g_1, n), the number N_v of gauge nodes is independent of the frames chosen, and therefore the Hilbert series of the twisted Higgs branch is automatically frame-independent. Another comment is that, for the cases with $n = 0$, the Hilbert series of the twisted Higgs branches are not symmetric with respect to g_1 and g_2, which implies that the exchange symmetry of two Riemann surfaces is broken. For example, the result for $(g_1, g_2) = (2, 3)$ is given by

$$G_{{\rm tw},(2,0)}(t;3) = \left(G_{\rm tw}(t;3)\right)^3 = \left[\frac{1+8t^2+14t^3+22t^4+34t^5+22t^6+14t^7+8t^8+t^{10}}{(1-t^2)^{12}\left(1+t^2\right)^{-1}}\right]^3 \ , \tag{3.34}$$

while the result for $(g_1, g_2) = (3, 2)$ is given by

$$G_{{\rm tw},(3,0)}(t;2) = \left(G_{\rm tw}(t;2)\right)^6 = \left[\frac{1+3t^2+t^4}{(1-t^2)^6\left(1+t^2\right)^{-1}}\right]^6 \ . \tag{3.35}$$

IR R-symmetry As the ideal structure (3.24) shows, the twisted Higgs branch is parametrized by the scalar σ_i in the twisted hypermultiplets. Recall that the requirements on the IR R-symmetry are:

- All the non-zero scalars should be charged neutral;

- It should preserve the ideal structures;

From Table 1 and Table 2, only $SU(2)_- = SU(2)_R$ satisfies these requirements. Therefore, the IR R-symmetry of this branch is identified as $SU(2)_-$. This is exactly the Choice 2 in 2.2, and the corresponding central charge (2.13) under this R-symmetry choice agrees with the Hilbert series result (3.33) when $g_2 \geq 2$. For $g_2 = 1$, we cannot identify a UV symmetry that yields the correct right-moving central charge $c_R = 2n_v$. We therefore expect an emergent IR R-symmetry, similar to the situation in the special Higgs branch, to mix with $SU(2)_-$. At present, however, we lack a clear identification of this symmetry.

For $g_2 = 1$, there are N_v pairs of fields $(\sigma^J, \tilde{\sigma}^J)$. Each pair satisfies an equation $[\sigma^J, \tilde{\sigma}^J] = 0$, which forces σ^J and $\tilde{\sigma}^J$ to be parallel. As a result, the Cartan subgroup of every $SU(2)$ remains unbroken. Hence, at a generic point, the unbroken gauge group is $U(1)^{N_v}$. At the origin $q = 0$, $\sigma = 0$, $\tilde{\sigma} = 0$, the full gauge group $SU(2)^{N_v}$ is preserved.

For $g_2 \geq 2$, there are $g_2 N_v$ pairs $(\sigma^J_j, \tilde{\sigma}^J_j)$ subject to only N_v equations. At the generic point, these equations do not force the fields to be all parallel, so the gauge group is completely broken. At the origin, where all σ^J_j and $\tilde{\sigma}^J_j$ vanish, the full $SU(2)^{N_v}$ symmetry is restored. At special loci where some sets of fields are non-zero and mutually parallel, the corresponding $U(1)$ factors remain unbroken, leading to an enhancement of the gauge symmetry.

4 Outlook

In this work, we have proposed conjectural formulas for the right-moving central charges of 2d $\mathcal{N} = (0, 4)$ theories arising from the dimensional reduction of 4d class $\mathcal{S}$ theories on a Riemann surface C_{g_2}. For $G = SU(2)$, we performed detailed checks by analyzing the vacuum moduli spaces and computing Hilbert series of both the special Higgs branches and the twisted Higgs branches. These computations confirm our conjectures and reveal several structural features of the resulting theories that merit further investigation.

Higher-rank generalizations. Although we have focused on $G = SU(2)$ for concreteness, it is clearly desirable to test our central-charge conjectures for general ADE types. A natural next step is to extend the Hilbert-series analysis to 2d $(0, 4)$ theories with gauge group $G = SU(N)$, $N > 2$. Higher-rank theories generically do not admit Lagrangian descriptions, and it is expected that the interplay between puncture data, flavor symmetry, and the geometry of the special and twisted Higgs branches may give rise to novel phenomena that do not appear in the $SU(2)$ case.

Magnetic-quiver approach. One promising direction is the *magnetic quiver* program (starting from [33]), which describes hyper-Kähler Higgs branches of supersymmetric theories in terms of Coulomb branches of 3d $\mathcal{N} = 4$ theories. While magnetic quivers have been successfully applied in various dimensions, a systematic formulation for 2d $(0, 4)$ theories has not yet been developed. Establishing such a framework could provide a powerful tool for identifying moduli spaces of higher-rank theories and for interpreting the structure of special and twisted Higgs branches from a unified perspective.

Brane constructions and string dualities. It would also be highly valuable to study the geometry of the special and twisted Higgs branches using brane setups or string dualities. Such geometric realizations may offer a more intrinsic understanding of the moduli spaces and may provide guidance for constructing magnetic quivers for $(0, 4)$ theories.

Adding punctures on C_{g_2}. In this work, we restricted attention to reductions on unpunctured C_{g_2}. Introducing punctures on the second Riemann surface, i.e. considering C_{g_2,n_2}, constitutes an obvious next extension. The resulting 2d $(0, 4)$ theories are expected to exhibit richer flavor symmetries and more complicated Higgs-branch geometries, and it would be interesting to study how the central charges, 't Hooft anomalies, and Hilbert series behave in the presence of such punctures.

Toward a 2d TQFT structure. Compactifications of class $\mathcal{S}$ theories on Riemann surfaces are well known to assemble into a two-dimensional TQFT structure. The frame-independence of the special Higgs branch discovered here strongly suggests the existence of a similar TQFT interpretation on C_{g_2} for the resulting $(0,4)$ theories. Determining whether such a structure exists—and, if so, identifying its algebraic data—would provide a conceptual explanation for many of the patterns observed in this work.

Other topological twists. Finally, one may consider alternative topological twists when reducing $\mathcal{T}_G[C_{g_1,n}]$ on C_{g_2}. The topological twist with $U(1)_R \subset \mathrm{SU}(2)_R$ leads to 2d $(2,2)$ theories, while that with $\mathrm{diag}(U(1)_R \times U(1)_r)$ produces 2d $(0,2)$ theories [15, 17]. For genus $g_1 > 0$, unbroken gauge sectors again appear in the infrared [8, 23], and it would be interesting to revisit the computation of central charges and moduli spaces in these settings. Understanding how different twists modify the IR geometry and the pattern of gapped abelian sectors may also shed light on the origin of our central-charge formulas.

M5-branes on a more general 4-manifold. The compactification of M5-branes on a general 4-manifold [34, 35] remains a largely unexplored direction, particularly when the 4-manifold is not simply connected. In such cases, both the structure of the resulting effective 2d theories and their relation to the Vafa–Witten partition functions of 4-manifolds [36, 37] are poorly understood. A systematic analysis of these compactifications is still missing and calls for further investigation.

Moreover, as already observed in this work and in the literature [38, 39], the *order* in which the compactifications of the M5-branes are performed can affect the resulting physics in 2d when the 4-manifold is a product space. Understanding the origin of this phenomenon, and determining when different orders of compactification lead to equivalent or inequivalent 2d theories, would shed new light on the geometry of 6d theories and the nature of their lower-dimensional reductions.

In summary, the results obtained here represent only the first step in a broader study of 2d theories originating from class $\mathcal{S}$. Moreover, the study of M5-branes on general 4-manifolds represents a challenging and wide-open avenue for future research. We hope that the tools and observations developed in this work will serve as a foundation for further progress in understanding their moduli spaces, dualities, and protected observables.

Acknowledgments

The authors would like to thank Noppadol Mekareeya for his guidance on using Macaulay2 for Hilbert series computations. They are especially grateful to Zhenghao Zhong for his valuable insights and for elucidating the conceptual meaning of the Hilbert series, without which this work would have been impossible. We are also grateful to Jiahao Zheng for his help and suggestions regarding the computational calculations in this work.

This work is supported by the Shanghai Municipal Science and Technology Major Project (No.24ZR1403900).

A Notations

Half-hypermultiplets For generic quivers, it is more convenient to use the half-hypermultiplets to write down the Lagrangian. We use the following basis for the half-hypermultiplets

$$Q_{ab1} \equiv Q_{ab}, \quad Q_{ab2} \equiv \epsilon_{aa'}\epsilon_{bb'}\tilde{Q}^{a'b'} \ , \tag{A.1}$$

where subscripts are $SU(2)$ fundamental and superscripts are $SU(2)$ anti-fundamental. Then the 4d superpotential term from a full-hypermultiplet Q coupled to single $SU(2)$ adjoint multiplet Φ can be written either as

$$\tilde{Q}\Phi Q \equiv \tilde{Q}^{ac}\Phi_a{}^b Q_{bc} \tag{A.2}$$

or equivalently as

$$Q_{abc}Q_{a'b'c'}\Phi^{aa'}\epsilon^{bb'}\epsilon^{cc'} \ , \tag{A.3}$$

up to an overall constant, with $\Phi^{aa'} \equiv \epsilon^{aa''}\Phi_{a''}{}^{a'}$. The equivalence can be seen from

$$\begin{aligned} Q_{abc}Q_{a'b'c'}\Phi^{aa'}\epsilon^{bb'}\epsilon^{cc'} &= Q_{ab1}Q_{a'b'2}\Phi^{aa'}\epsilon^{bb'} - Q_{ab2}Q_{a'b'1}\Phi^{aa'}\epsilon^{bb'} \\ &= \epsilon_{a'a''}\epsilon_{b'b''}\tilde{Q}^{a''b''}\Phi^{a'a}Q_{ab}\epsilon^{bb'} - \epsilon_{aa''}\epsilon_{bb''}\tilde{Q}^{a''b''}\Phi^{aa'}Q_{a'b'}\epsilon^{bb'} \\ &= \epsilon_{a'a''}\tilde{Q}^{a''b}\Phi^{a'a}Q_{ab} + \epsilon_{aa''}\tilde{Q}^{a''b'}\Phi^{aa'}Q_{a'b'} \\ &= -2\tilde{Q}^{ac}\Phi_a{}^b Q_{bc} \ , \end{aligned} \tag{A.4}$$

where in the first equality of the second line, we have used the fact that $\Phi^{aa'}$ is symmetric, as a result of the tracelessness of $\Phi_a{}^{a'}$.

In a 2d $(0,4)$ theory, we can similarly write the 2d J-terms either by the full-multiplets

$$\tilde{Q}^{ac}\tilde{\Sigma}_a{}^b\Gamma_{bc} + \tilde{\Gamma}^{ac}\tilde{\Sigma}_a{}^b Q_{bc} \tag{A.5}$$

or equivalently by the half-multiplets

$$\Gamma_{abc}Q_{a'b'c'}\tilde{\Sigma}^{aa'}\epsilon^{bb'}\epsilon^{cc'} \ , \tag{A.6}$$

up to an overall sign, where $\Gamma, \tilde{\Gamma}$ form a (0,4) Fermi multiplet, $Q, \tilde{Q}$ form a (0,4) hypermultiplet and $\Sigma, \tilde{\Sigma}$ form a (0,4) twisted hypermultiplet. We denote the half-multiplets by

$$Q_{ab1} \equiv Q_{ab}, \quad Q_{ab2} \equiv \epsilon_{aa'}\epsilon_{bb'}\tilde{Q}^{a'b'}, \quad \Gamma_{ab1} \equiv \Gamma_{ab}, \quad \Gamma_{ab2} \equiv \epsilon_{aa'}\epsilon_{bb'}\tilde{\Gamma}^{a'b'} \ . \tag{A.7}$$

Adjoint representations For $SU(2)$ adjoint multiplets such as Σ, we use $\Sigma^{aa'} \equiv \epsilon^{ab}\Sigma_b{}^{a'}$ to make explicit contraction with fundamental multiplets. We also write $\Sigma^{aa'} = \Sigma^A(e_A)^{aa'} \equiv \Sigma^A\epsilon^{ab}(e_A)_b{}^{a'}$, where $A = 1, 2, 3$, when expanding the complex adjoint multiplets with respect to the basis

$$(e_1)_a{}^b = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ (e_2)_a{}^b = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \ (e_3)_a{}^b = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \tag{A.8}$$

which satisfies

$$[e_1, e_2] = e_3 \ , \quad [e_3, e_1] = 2e_1 \ , \quad [e_3, e_2] = -2e_2 \ . \tag{A.9}$$

B 2d (0,4) theory and Higgs mechanism

As is described in [9], for a loop with many nodes (4d hyper) attached on, one can eliminate nodes one by one by Higgs mechanism, or giving the nearby 4d vector mass, so we can simply consider the case of one loop without nodes, where there is an unbroken U(1). The analysis for 2d theory is similar, and one can interpret this procedure as decomposing the standard frame into its building blocks (tadpoles and trinions). We will start with scalar potential and derive the mass matrix for $\sigma, \tilde{\sigma}$ and $q, \tilde{q}$, the latter naturally appears on the stage of 2d discussion.

The Lagarangian is given in [16], and we can work out the scalar potential in the case of $G = SU(2)$ for both fundamental matter and adjoint one.

Fundamental matter Now the scalar potential becomes

$$V = \frac{1}{e^2}\operatorname{tr}\left(\vec{D}_\sigma^2\right) + \frac{1}{2e^2}\vec{D}_q^2 + \sum_i q^\dagger([\sigma_i,\sigma_i^\dagger] + [\tilde{\sigma}_i,\tilde{\sigma}_i^\dagger])q - \tilde{q}([\sigma_i,\sigma_i^\dagger] + [\tilde{\sigma}_i,\tilde{\sigma}_i^\dagger])\tilde{q}^\dagger \\ + \sum_i 2q^\dagger(\sigma_i^\dagger\sigma_i + \tilde{\sigma}_i^\dagger\tilde{\sigma}_i)q + 2\tilde{q}(\sigma_i\sigma_i^\dagger + \tilde{\sigma}_i\tilde{\sigma}_i^\dagger)\tilde{q}^\dagger \tag{B.1}$$

where the doublet ω transforms in $\mathbf{2}$ of $SU(2)^-$, ω' transforms in $\mathbf{2}$ of $SU(2)^+$, $\vec{D}_q^A$ transforms in $\mathbf{3}$ of $SU(2)^-$, $\vec{D}_\sigma$ transforms in $\mathbf{3}$ of $SU(2)^+$:

$$\omega = \begin{pmatrix} q \\ \tilde{q}^\dagger \end{pmatrix}, \quad \omega_i' = \begin{pmatrix} \sigma_i \\ \tilde{\sigma}_i^\dagger \end{pmatrix}, \quad \vec{D}_q^A = e^2\omega^\dagger T^A\vec{\sigma}\omega, \quad \vec{D}_\sigma = \sum_{i=1}^{g_2}[\omega_i'^\dagger, \vec{\sigma}\omega_i'],$$

and one can further integrate it into a nicer form, manifesting the 2d R-symmetry:

$$V = \frac{1}{e^2}\operatorname{tr}\left(\vec{D}_\sigma^2\right) + \frac{1}{2e^2}\vec{D}_q^2 + \omega^\dagger\left(\sum_i\{\omega_i'^\dagger, \mathbf{1}\omega_i'\}\right)\mathbf{1}\omega. \tag{B.2}$$

Take VEV of $q, \tilde{q}$, as has been done by Hanany, and we obtain the mass matrix from derivatives of second line in (B.1):

$$M_\sigma = \frac{\partial^2}{\partial\sigma^A\partial(\sigma^B)^*}V = 2(q^\dagger T^B T^A q + \tilde{q}T^A T^B\tilde{q}^\dagger)$$

which means that when we take the VEV of the scalars of the hypermultiplets, there is no unbroken gauge (or actually massless d.o.f.?). And take VEV of $\sigma, \tilde{\sigma}$ gives

$$M_q = \frac{\partial^2}{\partial q^\alpha\partial(q^\beta)^*}V = 2\sum_i(\tilde{\sigma}_i^\dagger\tilde{\sigma}_i + \sigma_i^\dagger\sigma_i)_{\beta\alpha}$$

which means that when we take the VEV of the scalars of the twisted hyper, the hyper is still massive, and will be integrated out in the deep IR.

Adjoint matter However, things become different when we consider adjoint matter fields, and take into account the $2/e^2$ factor before the matter term, so we introduce $\vec{D}^A_{\mathbf{adj}.q} = [\omega^\dagger, \vec{\sigma}\omega]$, and

$$V = \frac{1}{e^2}\operatorname{tr}\left(\vec{D}_\sigma^2\right) + \frac{1}{e^2}\operatorname{tr}\left(\vec{D}^2_{\mathbf{adj}.q}\right) \\ + \frac{2}{e^2}\operatorname{tr}\left(\sum_i([\tilde{\sigma}_i,\tilde{\sigma}_i^\dagger] + [\sigma_i,\sigma_i^\dagger])([q,q^\dagger] + [\tilde{q},\tilde{q}^\dagger])\right) + \frac{4}{e^2}\sum_i\operatorname{tr}\left(|[\tilde{q},\tilde{\sigma}_i]|^2 + |[\tilde{\sigma}_i,q]|^2 + |[\tilde{q},\sigma_i]|^2 + |[\sigma_i,q]|^2\right)$$

so when we take the VEV of $q, \tilde{q}$, the fourth term gives the mass matrix

$$M_\sigma = M_{\tilde{\sigma}} = \frac{32}{e^2}\left((|q|^2 + |\tilde{q}|^2)\delta^{AB} - (q^*)^A q^B - (\tilde{q}^*)^A\tilde{q}^B\right)$$

which is exactly the same form as [9], so the eigenvalues

$$\begin{aligned}
\lambda_1 &= \frac{32}{e^2}(|q|^2 + |\tilde{q}|^2) \\
\lambda_2 &= \frac{16}{e^2}\left(|q|^2 + |\tilde{q}|^2 + \sqrt{(|q|^2 + |\tilde{q}|^2)^2 - 4\mathcal{F}}\right) \Rightarrow \frac{32}{e^2}(|q|^2 + |\tilde{q}|^2) \\
\lambda_3 &= \frac{16}{e^2}\left(|q|^2 + |\tilde{q}|^2 - \sqrt{(|q|^2 + |\tilde{q}|^2)^2 - 4\mathcal{F}}\right) \Rightarrow 0
\end{aligned}$$

where $\vec{D}_q = 0$, or $|J_\Theta|^2 \sim \mathrm{tr}\big(|[q,\tilde{q}]|^2\big) = 0$, gives

$$\frac{32}{g^2}\left(\tilde{q}^A q^B (q^*)^B (\tilde{q}^*)^A - \tilde{q}^A q^B (q^*)^A (\tilde{q}^*)^B\right) \equiv \frac{32}{g^2}\mathcal{F} = 0 \tag{B.3}$$

This means that when we take the VEV of the scalars of the hypermultiplets, all g_2 twisted hypers are massless, for $\mathcal{T}[C_g]$, there will be $g_1 g_2$ massless d.o.f.

On the other hand, when we take the VEV of the scalars of the twisted hyper

$$M_{\tilde{q}} = M_q = \frac{32}{e^2}\sum_{i=1}^{g_2}\left((|\tilde{\sigma}_i|^2 + |\sigma_i|^2)\delta^{AB} - (\tilde{\sigma}_i^*)^A \tilde{\sigma}_i^B - (\sigma_i^*)^A \sigma_i^B\right)$$

We have $\vec{D}_\sigma = 0$, or $E_\Theta = \sum_i [\sigma_i, \tilde{\sigma}_i] = 0$, but zero eigenvalues occur only when all g_2 pairs of vectors are parallel ($[\sigma_i, \sigma_j] = [\sigma_i, \tilde{\sigma}_j] = [\tilde{\sigma}_i, \tilde{\sigma}_j] = 0$). This is indeed the case of special Higgs branch, so g_1 hypers are kept massless, and will contribute to the central charge.

Normally, for twisted branch, unless $g_2 = 1$, all hypers are massive for a generic g_2, and will be integrated out in the deep IR.

Mass of gaugino and fermi From the Lagrangian, one can extract the Yukawa term for each Fermion. Take VEV of $\langle q\rangle, \langle\tilde{q}\rangle$, we see that the left-handed gauginos from the 2d vector are paired with the right-handed fermions from the 2d hyper, the left-handed fermions from the 2d Fermi are paired with the right-handed fermions from the 2d tw. hyper.

$$\begin{aligned}
&\mathcal{L}_\Phi \supset \sqrt{2}\mathrm{i}(\bar{\psi}_+\bar{\lambda}_- q - \bar{q}\lambda_-\psi_+)\\
&\mathcal{L}_{\tilde{\Phi}} \supset -\sqrt{2}\mathrm{i}\left(\tilde{\psi}_+\lambda_-\bar{\tilde{q}} - \tilde{q}\bar{\lambda}_-\bar{\tilde{\psi}}_+\right)\\
&\mathcal{L}_\Gamma \supset -\sqrt{2}\mathrm{i}\left(\bar{\psi}_-\bar{\lambda}_+ q - \bar{q}\lambda_+\psi_-\right)\\
&\mathcal{L}_{\tilde{\Gamma}} \supset \sqrt{2}\mathrm{i}\left(\tilde{\psi}_-\lambda_+\bar{\tilde{q}} - \tilde{q}\bar{\lambda}_+\bar{\tilde{\psi}}_-\right)\\
&\mathcal{L}_J \supset \sqrt{2}(\tilde{q}\tilde{\lambda}_-\psi_+ - \tilde{\psi}_+\tilde{\lambda}_- q - \tilde{q}\tilde{\lambda}_+\psi_- + \tilde{\psi}_-\tilde{\lambda}_+ q) + \text{h.c.}
\end{aligned} \tag{B.4}$$

Take VEV of $\langle\sigma\rangle, \langle\tilde{\sigma}\rangle$, we see that the left-handed gauginos from the 2d vector are paired with the right-handed fermions from the 2d tw. hyper, the left-handed fermions from the 2d Fermi are paired with the right-handed fermions from the 2d hyper.

$$\begin{aligned}
&\mathcal{L}_\Theta \supset \frac{2}{e^2}\,\mathrm{tr}\Big(-\sqrt{2}\left(\bar{\tilde{\lambda}}_+[\bar{\sigma},\tilde{\lambda}_-] + \bar{\tilde{\lambda}}_-[\sigma,\tilde{\lambda}_+]\right) - \sqrt{2}\mathrm{i}\left(\bar{\tilde{\lambda}}_-[\bar{\lambda}_+,\tilde{\sigma}] - \bar{\tilde{\sigma}}[\lambda_+,\tilde{\lambda}_-]\right)\Big)\\
&\mathcal{L}_\Sigma \supset \frac{2}{e^2}\,\mathrm{tr}\Big(\sqrt{2}\mathrm{i}(\lambda_+[\bar{\lambda}_-,\sigma] - \bar{\sigma}[\lambda_-,\bar{\lambda}_+])\Big)\\
&\mathcal{L}_{\tilde{\Sigma}} \supset \frac{2}{e^2}\,\mathrm{tr}\Big(\sqrt{2}\mathrm{i}([\tilde{\lambda}_+,\lambda_-]\bar{\tilde{\sigma}} - [\tilde{\sigma},\bar{\lambda}_-]\bar{\tilde{\lambda}}_+)\Big)\\
&\mathcal{L}_\Gamma \supset -\sqrt{2}(\bar{\psi}_+\bar{\sigma}\psi_- + \bar{\psi}_-\sigma\psi_+)\\
&\mathcal{L}_{\tilde{\Gamma}} \supset -\sqrt{2}(\tilde{\psi}_+\sigma\bar{\tilde{\psi}}_- + \tilde{\psi}_-\bar{\sigma}\bar{\tilde{\psi}}_+)\\
&\mathcal{L}_J \supset \sqrt{2}(\tilde{\psi}_-\tilde{\sigma}\psi_+ - \tilde{\psi}_+\tilde{\sigma}\psi_-) + \text{h.c.}
\end{aligned} \tag{B.5}$$

We will use bracket to emphasize when the two fermions are paired in the following.

special Higgs branch The special Higgs branch is parametrized by both $q, \tilde{q}$ and $\sigma, \tilde{\sigma}$, so we first take the VEV of scalars in the hypermultiplets, and then take the VEV of scalars in the g_1 looped twisted hypers. The first step gives masses to all the n_v (λ_-, ψ_+) and $g_2 n_v - g_1 g_2 r_G$ (λ_+, ψ_-), since there are $g_1 g_2 r_G$ massless $\sigma, \tilde{\sigma}$. The next step (non-vanishing $\sigma, \tilde{\sigma}$ attached to g_1 looped gauge node) gives mass to hyper/Fermi, keeping $g_1 r_G$ massless d.o.f. to (ψ_+, ψ_-).

Twisted Higgs branch The twisted Higgs branch is parameterized by $\sigma, \tilde{\sigma}$, so we take VEV of scalars in the twisted hypers. We find that for $g_2 = 1$. There will be a unbroken gauge symmetry (?) as found by Hanany. In fact, each pair of $\sigma, \tilde{\sigma}$ attached to all N_v gauge node will give mass to the nearby hyper/Fermi, keeping N_v d.o.f. massless. We show for $g_2 > 1$, there is no unbroken gauge symmetry, i.e. all gauginos are massive.

C Examples of Hilbert series

In this section, we perform explicit calculations of the ideals of Higgs branches and corresponding Hilbert series.

Example: $(g_1, n) = (0, 3)$

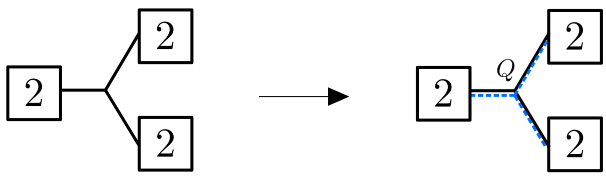

Figure 5. The quiver diagrams of the class $\mathcal{S}$ theory $\mathcal{T}[C_{0,3}]$ and the corresponding 2d $\mathcal{N} = (0,4)$ theory upon the reduction on C_{g_2}. For the 2d theory, the solid black lines represent one fundamental hypermultiplet Q_{abc}. The dashed blue lines represent g_2 fundamental Fermi multiplets $\Gamma_{jabc},\ j = 1, \cdots g_2$.

The 4d and 2d quiver diagrams are shown in Figure 5. There is no J-term and E-term equations due to the absence of the gauge multiplet. Consequently, the Hilbert series is freely generated by the scalars q_{abc} as

$$G_{(0,3)}(t, x_j; g_2) = \frac{1}{1 - t x_1^{\pm 1} x_2^{\pm 1} x_3^{\pm 1}} \ , \tag{C.1}$$

corresponding to the special Higgs branch with quaternionic dimension 4.

Example: $(g_1, n) = (0, 4)$

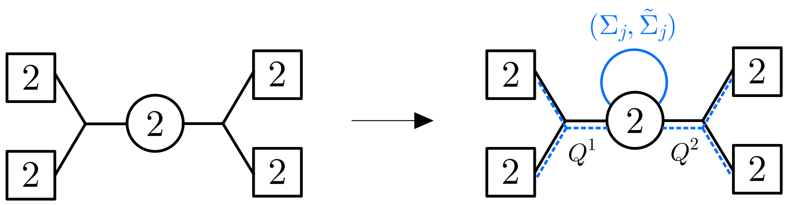

Figure 6. The quiver diagrams of the class $\mathcal{S}$ theory $\mathcal{T}[C_{0,4}]$ and the corresponding 2d $\mathcal{N} = (0,4)$ theory upon the reduction on C_{g_2}. For the 2d theory, the solid black lines represent two fundamental hypermultiplets $Q^I_{abc}, I = 1, 2$. The solid blue line represents g_2 adjoint twisted hypermultiplets $\Sigma_j, \tilde{\Sigma}_j$ and the dashed blue lines represent $2g_2$ fundamental Fermi multiplets $\Gamma^I_{jabc},\ I = 1, 2,\ j = 1, \cdots g_2$.

The 4d quiver and corresponding 2d $\mathcal{N}=(0,4)$ quiver is shown in Figure 6. The J-term and E-term equations are

$$\begin{aligned}\left(q^1_{abc}q^1_{a'b'c'}+q^2_{abc}q^2_{a'b'c'}\right)(e_A)^{aa'}\epsilon^{bb'}\epsilon^{cc'}&=0\ ,\\ q^I_{a'b'c'}\sigma_j^{aa'}\epsilon^{bb'}\epsilon^{cc'}=q^I_{a'b'c'}\tilde{\sigma}_j^{aa'}\epsilon^{bb'}\epsilon^{cc'}&=0\ ,\\ \sum_{j=1}^{g_2}[\sigma_j,\tilde{\sigma}_j]&=0\ ,\end{aligned}\tag{C.2}$$

where e_A is the canonical basis of adjoint representation, as defined by (A.8). The ideal generated by (C.2) has two non-trivial minimal prime ideals corresponding to the special Higgs branch and the twisted Higgs branch, respectively:

- The special Higgs branch, constrained by $\sigma_j=\tilde{\sigma}_j=0$, has quaternionic dimension 5. The Hilbert series is given by

$$G_{H,(0,4)}(t,x_j=1;g_2)=\frac{(1+t^2)(1+17t^2+48t^4+17t^6+t^8)}{(t^2-1)^{10}}\ ,\tag{C.3}$$

  which is identical to the Higgs branch Hilbert series of the 4d theory $\mathcal{T}[C_{0,4}]$. It is also identical to the Hilbert series of the nilpotent orbit of D_4 type corresponding to the partition $(2^2,1^4)$ [40].

- The Hilbert series of the twisted Higgs branch is given by (3.30) at $N_v=1$.

Example: $(g_1,n)=(1,1)$

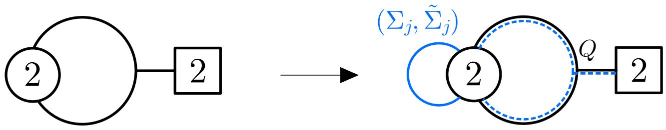

Figure 7. The quiver diagrams of the class $\mathcal{S}$ theory $\mathcal{T}[C_{1,1}]$ and the corresponding 2d $\mathcal{N}=(0,4)$ theory upon the reduction on C_{g_2}. For the 2d theory, the solid black lines represent one fundamental hypermultiplets Q_{abc}. The solid blue line represents g_2 adjoint twisted hypermultiplets $\Sigma_j,\tilde{\Sigma}_j$ and the dashed blue lines represent g_2 fundamental Fermi multiplets Γ_{jabc}, $j=1,\cdots g_2$.

The 4d quiver and corresponding 2d $\mathcal{N}=(0,4)$ quiver diagrams are shown in Figure 7. The J-term and E-term equations are

$$\begin{aligned}q_{abc}q_{a'b'c'}\left[(e_A)^{aa'}\epsilon^{bb'}\epsilon^{cc'}+\epsilon^{aa'}(e_A)^{bb'}\epsilon^{cc'}\right]&=0\ ,\\ q_{a'b'c'}\left(\sigma_j^{aa'}\epsilon^{bb'}\epsilon^{cc'}+\epsilon^{aa'}\sigma_j^{bb'}\epsilon^{cc'}\right)=q_{a'b'c'}\left(\tilde{\sigma}_j^{aa'}\epsilon^{bb'}\epsilon^{cc'}+\epsilon^{aa'}\tilde{\sigma}_j^{bb'}\epsilon^{cc'}\right)&=0\ ,\\ \sum_{j=1}^{g_2}[\sigma_j,\tilde{\sigma}_j]&=0\ .\end{aligned}\tag{C.4}$$

The ideal generated by (C.4) has two non-trivial minimal prime ideals corresponding to the special Higgs branch and the twisted Higgs branch, respectively.

- The special Higgs branch has quaternionic dimension g_2+2, and is constrained by

$$[\sigma_i,\sigma_j]=[\sigma_i,\tilde{\sigma}_j]=[\tilde{\sigma}_i,\tilde{\sigma}_j]=[\sigma_j,q]=[\sigma_j,\tilde{q}]=[\tilde{\sigma}_j,q]=[\tilde{\sigma}_j,\tilde{q}]=0\ ,\quad\forall\, i,j=1,\cdots g_2\ ,\tag{C.5}$$

which implies that $\sigma_j, \tilde{\sigma}_j, q, \tilde{q}$ are parallel to each other. The Hilbert series is given by

$$G_{H,(1,1)}(t, x=1; g_2) = \frac{1}{(1-t)^2} \cdot \frac{1 + (g_2+1)(2g_2+1)t^2 + \cdots}{(1-t^2)^{2g_2+2}} \ , \tag{C.6}$$

where the factor $(1-t)^{-2}$ is the contribution of free hypermultiplets $\epsilon^{ab} q_{abc}$. Except for the factor of free hypermultiplet, this Hilbert series is exactly the Hilbert series for the nilpotent orbits of C_{g_2+1} type corresponding to the partition $(2, 1^{2g_2})$ [40]. The explicit expressions for $g_2 = 0, \ldots, 4$ are listed in Table 9.

g_2	**dim**	**Hilbert series**
0	2	$\frac{1}{(1-t)^2}\frac{1+t^2}{(1-t^2)^2}$
1	3	$\frac{1}{(1-t)^2}\frac{1+6t^2+t^4}{(1-t^2)^4}$
2	4	$\frac{1}{(1-t)^2}\frac{(1+t^2)(t^4+14t^2+1)}{(1-t^2)^6}$
3	5	$\frac{1}{(1-t)^2}\frac{1+28t^2+70t^4+28t^6+t^8}{(1-t^2)^8}$
4	6	$\frac{1}{(1-t)^2}\frac{(1+t^2)(1+44t^2+166t^4+44t^6+t^8)}{(1-t^2)^{10}}$

Table 9. The quaternionic dimensions and unrefined Hilbert series of the special Higgs branches of the case $(g_1, n) = (1, 1)$.

- The twisted Higgs branch appears when $g_2 \geq 2$, with the Hilbert series given by (3.30) at $N_v = 1$, together with an additional factor $(1-t)^{-2}$.

Example: $(g_1, n) = (1, 2)$**, frame 1**

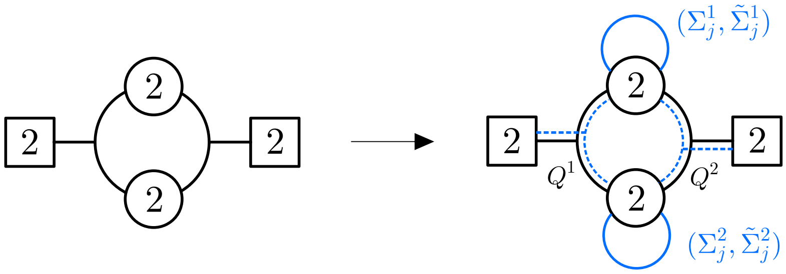

Figure 8. The quiver diagrams of the class $\mathcal{S}$ theory $\mathcal{T}[C_{1,2}]$ and the corresponding 2d $\mathcal{N} = (0,4)$ theory upon the reduction on C_{g_2}, in the first frame. For the 2d theory, the solid black lines represent two fundamental hypermultiplets $Q^I_{abc}, I = 1, 2$. The solid blue line represents $2g_2$ adjoint twisted hypermultiplets $\Sigma^J_j, \tilde{\Sigma}^J_j, J = 1, 2$ and the dashed blue lines represent $2g_2$ fundamental Fermi multiplets Γ^I_{jabc}, $I = 1, 2$, $j = 1, \cdots g_2$.

For the case $(g_1, n) = (1, 2)$, there are two frames of quivers as in Figure 8 and Figure 9. For the case of Figure 8, the J-term and E-term equations are given by

$$\begin{aligned}
\left(q^1_{abc} q^1_{a'b'c'} + q^2_{abc} q^2_{a'b'c'}\right) (e_A)^{aa'} \epsilon^{bb'} \epsilon^{cc'} = \left(q^1_{abc} q^1_{a'b'c'} + q^2_{abc} q^2_{a'b'c'}\right) \epsilon^{aa'} (e_A)^{bb'} \epsilon^{cc'} = 0 \ , \\
q^I_{a'b'c'} \left(\sigma^{1aa'}_j \epsilon^{bb'} \epsilon^{cc'} + \epsilon^{aa'} \sigma^{2bb'}_j \epsilon^{cc'}\right) = q^I_{a'b'c'} \left(\tilde{\sigma}^{1aa'}_j \epsilon^{bb'} \epsilon^{cc'} + \epsilon^{aa'} \tilde{\sigma}^{2bb'}_j \epsilon^{cc'}\right) = 0 \ , \\
\sum_{j=1}^{g_2} \left[\sigma^J_j, \tilde{\sigma}^J_j\right] = 0 \ .
\end{aligned} \tag{C.7}$$

The ideal generated by (C.7) has two non-trivial minimal prime ideals corresponding to the special Higgs branch and the twisted Higgs branch respectively.

g_2	**dim**	**Hilbert series**
0	3	$\frac{1+3t^2+t^4}{(1-t^2)^6(1+t^2)^{-1}}$
1	4	$\frac{1+4t^2+10t^3+4t^4+t^6}{(1-t^2)^8(1+t^2)^{-1}}$
2	5	$\frac{1+9t^2+20t^3+20t^4+20t^5+9t^6+t^8}{(1-t^2)^{10}(1+t^2)^{-1}}$
3	6	$\frac{1+18t^2+30t^3+61t^4+100t^5+61t^6+30t^7+18t^8+t^{10}}{(1-t^2)^{12}(1+t^2)^{-1}}$
4	7	$\frac{1+31t^2+40t^3+155t^4+280t^5+266t^6+280t^7+155t^8+40t^9+31t^{10}+t^{12}}{(1-t^2)^{14}(1+t^2)^{-1}}$

Table 10. The quaternionic dimensions and unrefined Hilbert series of the special Higgs branches of the case $(g_1, n) = (1, 2)$.

- The special Higgs branch has quaternionic dimension $g_2 + 3$, with the Hilbert series given by
$$G_{H,(1,2)}(t, x_j = 1; g_2) = \frac{1 + \cdots}{(1-t^2)^{2g_2+6}(1+t^2)^{-1}} \,. \tag{C.8}$$
The explicit expressions for $g_2 = 0, \ldots, 4$ are listed in Table 10.
- The Hilbert series of the twisted Higgs branch is given by the general result (3.30) at $N_v = 2$.

Example: $(g_1, n) = (1, 2)$**, frame 2 (standard frame)**

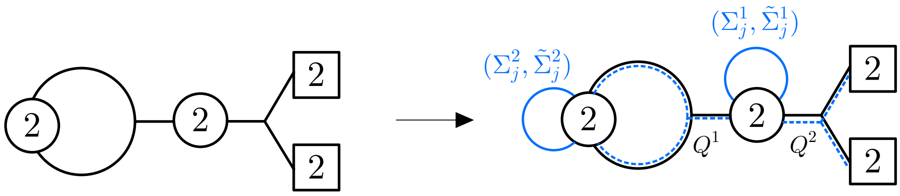

Figure 9. The quiver diagrams of the class $\mathcal{S}$ theory $\mathcal{T}[C_{1,2}]$ and the corresponding 2d $\mathcal{N} = (0, 4)$ theory upon the reduction on C_{g_2}, in the second frame. For the 2d theory, the solid black lines represent two fundamental hypermultiplets $Q^I_{abc}, I = 1, 2$. The solid blue line represents $2g_2$ adjoint twisted hypermultiplets $\Sigma^J_j, \tilde{\Sigma}^J_j, J = 1, 2$ and the dashed blue lines represent $2g_2$ fundamental Fermi multiplets $\Gamma^I_{jabc},\ I = 1, 2,\ j = 1, \cdots g_2$.

For the other frame of the case $(g_1, n) = (1, 2)$ as in Figure 9, the J-term and E-term equations are
$$\begin{aligned}\left(q^1_{abc}q^1_{a'b'c'} + q^2_{abc}q^2_{a'b'c'}\right)(e_A)^{aa'}\epsilon^{bb'}\epsilon^{cc'} = q^1_{abc}q^1_{a'b'c'}\left[\epsilon^{aa'}(e_A)^{bb'}\epsilon^{cc'} + \epsilon^{aa'}\epsilon^{bb'}(e_A)^{cc'}\right] &= 0,\\ q^1_{a'b'c'}\left(\sigma^{1aa'}_j\epsilon^{bb'}\epsilon^{cc'} + \epsilon^{aa'}\sigma^{2bb'}_j\epsilon^{cc'} + \epsilon^{aa'}\epsilon^{2bb'}\sigma^{cc'}_j\right) = q^2_{a'b'c'}\sigma^{1aa'}_j\epsilon^{bb'}\epsilon^{cc'} &= 0,\\ q^1_{a'b'c'}\left(\tilde{\sigma}^{1aa'}_j\epsilon^{bb'}\epsilon^{cc'} + \epsilon^{aa'}\tilde{\sigma}^{2bb'}_j\epsilon^{cc'} + \epsilon^{aa'}\epsilon^{2bb'}\tilde{\sigma}^{cc'}_j\right) = q^2_{a'b'c'}\tilde{\sigma}^{1aa'}_j\epsilon^{bb'}\epsilon^{cc'} &= 0,\\ \sum_{j=1}^{g_2}\left[\sigma^J_j, \tilde{\sigma}^J_j\right] &= 0\,.\end{aligned} \tag{C.9}$$

It turns out that, the full Higgs branch contains a special Higgs branch, a twisted Higgs branch, and several more irreducible components in this frame. Nevertheless, the Hilbert series of the special Higgs branch and twisted Higgs branch are identical to the results in the other frame.

Example: $(g_1, n) = (2, 0)$**, frame 1 (standard frame)**

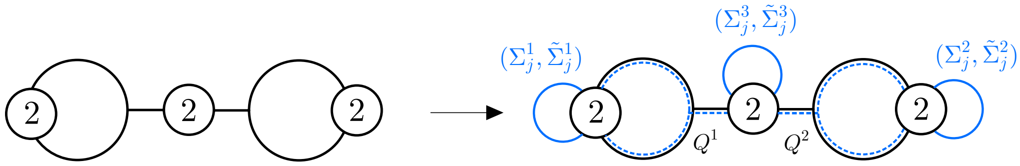

Figure 10. The quiver diagrams of the class $\mathcal{S}$ theory $\mathcal{T}[C_{2,0}]$ and the corresponding 2d $\mathcal{N} = (0,4)$ theory upon the reduction on C_{g_2}, in the "dumbbell" frame. For the 2d theory, the solid black lines represent two fundamental hypermultiplets $Q^I_{abc}, I = 1, 2$. The solid blue line represents $3g_2$ adjoint twisted hypermultiplets $\Sigma_j^J, \tilde{\Sigma}_j^J, J = 1, 2, 3$ and the dashed blue lines represent $2g_2$ fundamental Fermi multiplets $\Gamma^I_{jabc},\ I = 1, 2,\ j = 1, \cdots g_2$.

In this frame, the quiver of the case $(g_1, n) = (2, 0)$ is given by the "dumbbell" quiver as in Figure 10. The J-term and E-term equations are

$$\begin{aligned} q^I_{abc} q^I_{a'b'c'} \left((e_A)^{aa'} \epsilon^{bb'} \epsilon^{cc'} + \epsilon^{aa'} (e_A)^{bb'} \epsilon^{cc'} \right) &= 0 \ , \\ \left(q^1_{abc} q^1_{a'b'c'} + q^2_{abc} q^2_{a'b'c'} \right) \epsilon^{aa'} \epsilon^{bb'} (e_A)^{cc'} &= 0 \ , \\ q^I_{a'b'c'} \left(\sigma_j^{Iaa'} \epsilon^{bb'} \epsilon^{cc'} + \epsilon^{aa'} \sigma_j^{Ibb'} \epsilon^{cc'} + \epsilon^{aa'} \epsilon^{bb'} \sigma_j^{3cc'} \right) &= 0 \ , \\ q^I_{a'b'c'} \left(\tilde{\sigma}_j^{Iaa'} \epsilon^{bb'} \epsilon^{cc'} + \epsilon^{aa'} \tilde{\sigma}_j^{Ibb'} \epsilon^{cc'} + \epsilon^{aa'} \epsilon^{bb'} \tilde{\sigma}_j^{3cc'} \right) &= 0 \ , \\ \sum_{j=1}^{g_2} \left[\sigma_j^J, \tilde{\sigma}_j^J \right] &= 0 \ . \end{aligned} \tag{C.10}$$

From the prime decomposition of the ideal, the full Higgs branch contains a special Higgs branch, a twisted Higgs branch, and several more irreducible components in this frame.

- The special Higgs branch with quaternionic dimension $2g_2+1$ is determined by the equations:

$$\sigma_j^3 = \tilde{\sigma}_j^3 = 0 \ , \quad \left[\sigma_i^J, \sigma_j^J \right] = \left[\sigma_i^J, \tilde{\sigma}_j^J \right] = \left[\tilde{\sigma}_i^J, \tilde{\sigma}_j^J \right] = 0 \ , \tag{C.11}$$

  for all $i, j = 1, \cdots g_2,\ J = 1, 2$. In other words, the scalars in twisted hypermultiplet $\Sigma^3, \tilde{\Sigma}^3$, which is *not on the loops* of the quiver, vanish on the special Higgs branch. The scalars in the twisted hypermultiplet $\Sigma^J, \tilde{\Sigma}^J$, $J = 1, 2$, which is *on the loops* of the quiver, are allowed to have non-zero values. The Hilbert series is given by:

$$G_{H,(2,0)}(t; g_2) = \frac{1 + \cdots}{(1-t^2)^{4g_2+1}(1-t^4)} \ . \tag{C.12}$$

  The full expressions for $g_2 = 0, \ldots, 3$ are listed in Table 11.

- The Hilbert series of the twisted Higgs branch is given by the general result (3.30) at $N_v = 3$.

Example: $(g_1, n) = (2, 0)$**, frame 2**

The other quiver of the case $(g_1, n) = (2, 0)$ is given by the "yinyang" quiver as in Figure 11. The J-term and E-term equations are

g_2	dim	Hilbert series
0	1	$\frac{1+t^4}{(1-t^2)(1-t^4)}$
1	3	$\frac{1+2t^2+8t^3+6t^4+8t^5+10t^6-3t^8}{(1-t^2)^5(1-t^4)}$
2	5	$\frac{1+12t^2+16t^3+55t^4+112t^5+88t^6+112t^7+103t^8+16t^9+12t^{10}-15t^{12}}{(1-t^2)^9(1-t^4)}$
3	7	$\frac{1+30t^2+24t^3+292t^4+440t^5+794t^6+1584t^7+1426t^8+1584t^9+1426t^{10}+440t^{11}+300t^{12}+24t^{13}-138t^{14}-35t^{16}}{(1-t^2)^{13}(1-t^4)}$

Table 11. The quaternionic dimensions and unrefined Hilbert series of the special Higgs branches of the case $(g_1, n) = (2, 0)$. In particular, the numerators are not palindromic for $g_2 \geq 1$.

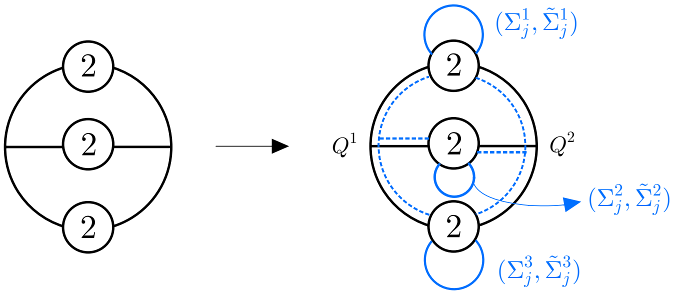

Figure 11. The quiver diagrams of the class $\mathcal{S}$ theory $\mathcal{T}[C_{2,0}]$ and the corresponding 2d $\mathcal{N} = (0,4)$ theory upon the reduction on C_{g_2}, in the "yinyang" frame. For the 2d theory, the solid black lines represent two fundamental hypermultiplets $Q^I_{abc}, I = 1,2$. The solid blue line represents $3g_2$ adjoint twisted hypermultiplets $\Sigma^J_j, \tilde{\Sigma}^J_j, J = 1,2,3$ and the dashed blue lines represent $2g_2$ fundamental Fermi multiplets $\Gamma^I_{jabc},\ I = 1,2,\ j = 1, \cdots g_2$.

D Dimensional reduction of a single M5 brane

We study the dimensional reduction of the worldvolume theory of a single M5-brane on $C_{g_1,n} \times C_{g_2}$, with a partial topological twist chosen to preserve four right-moving supercharges. This is carried out in two steps: first compactifying on $C_{g_1,n}$ to obtain a 4d $\mathcal{N} = 2$ theory, and then further reducing on C_{g_2} to yield a 2d $\mathcal{N} = (0,4)$ theory. Because the original worldvolume theory is abelian, the resulting 2d theory is free. Consequently, its central charge can be determined straightforwardly.

The worldvolume theory of a single M5 brane is described by a free $\mathcal{N} = (2,0)$ tensor multiplet containing a self-dual 2-form field B^+_{MN}, five scalars $T_{I=0,1,\ldots,4}$ and four symplectic Majorana-Weyl fermions $\Psi^+_{\alpha=0,1,2,3}$ transforming as **1**, **5** and **4** under $SO(5)_R$ R-symmetry. We place the theory on $\mathcal{M}_2 \times C_{g_1,n} \times C_{g_2}$. The tangent bundle decomposition reduces the Lorentz group as

$$SO(6)_E \longrightarrow SO(2)_{\mathcal{M}_2} \times U(1)_{C_{g_1,n}} \times U(1)_{C_{g_2}}. \tag{D.1}$$

Punctures correspond to codimension-two defects of the 6d theory and specify flavor symmetries and matter couplings in 4d.

Reduction on $C_{g_1,n}$ First compactify on $C_{g_1,n}$ with the standard class $\mathcal{S}$ twist. Choose a Cartan decomposition of the R-symmetry

$$SO(5)_R \supset U(1)_r \times SU(2)_R. \tag{D.2}$$

The class $\mathcal{S}$ twist mixes $U(1)_{C_{g_1}}$ with the $U(1)_r$ subgroup:

$$U(1)^{\text{tw}}_{C_{g_1}} = \text{diag}\big(U(1)_{C_{g_1}} \times U(1)_r\big). \tag{D.3}$$

With this twist the compactification on C_{g_1} preserves eight supercharges [2], leading to a 4d $\mathcal{N}=2$ theory on $\mathcal{M}_2 \times C_{g_2}$. From the 4d viewpoint, the R-symmetry is $SU(2)_R \times U(1)_r$ while additional flavor symmetries depend on the puncture data.

The 4d $\mathcal{N}=2$ theory from a single M5 brane on $C_{g_1,n}$ is a free theory determined by Kaluza-Klein reduction as follows

- The 2-form field B^+_{MN} transform as **15** under $SO(6)_E$ and singlet under $SO(5)_R$. Since it is self-dual, it is convenient to consider its field strength H which decompose as

$$H = dB + \sum_{k=1}^{g_1}(da_k \wedge \omega_A^k + d\tilde{a}_k \wedge \omega_B^k) + d\varphi_0 \wedge \Omega_{C_{g_1}} \tag{D.4}$$

  where $\Omega_{C_{g_1}}$ is the volume form of C_{g_1} and $\{\omega_A^k\}$ and $\{\omega_B^k\}$ with $k=1,2,\ldots,g_1$ are basis of $H^1(C_{g_1},\mathbb{R})$. After reduction on C_{g_1}, one obtain a 2-form b, a real scalar φ_0 and g_1 gauge fields $a_{k=1,2,\ldots,g_1}$ [7].

- The scalars $T_{I=0,1,\ldots,4}$ transform as

$$\begin{array}{rcl} SO(6)\times SO(5)_R & \to & SU(2)_R \times SU(2)_l \times SU(2)_r \times U(1)^{\mathrm{tw}}_{C_{g_1}} \times U(1)_r, \\ (\mathbf{1},\mathbf{5}) & \to & (\mathbf{3},\mathbf{1},\mathbf{1})_{\mathbf{0},\mathbf{0}} + (\mathbf{1},\mathbf{1},\mathbf{1})_{\pm\mathbf{1},\pm\mathbf{1}}\ , \end{array}$$

  This gives g_1 complex scalar denoted by ϕ_k and three real scalars $\varphi_{1,2,3}$.

- The 6d fermions $\Psi^+_{\alpha=0,1,2,3}$ transform as

$$\begin{array}{rcl} SO(6)\times SO(5)_R & \to & SU(2)_R \times SU(2)_l \times SU(2)_r \times U(1)^{\mathrm{tw}}_{C_{g_1}} \times U(1)_r, \\ (\mathbf{4}^+,\mathbf{4}) & \to & (\mathbf{2},\mathbf{2},\mathbf{1})_{\mathbf{1},\frac{\mathbf{1}}{\mathbf{2}}} + (\mathbf{2},\mathbf{2},\mathbf{1})_{\mathbf{0},-\frac{\mathbf{1}}{\mathbf{2}}} + (\mathbf{2},\mathbf{1},\mathbf{2})_{\mathbf{0},\frac{\mathbf{1}}{\mathbf{2}}} + (\mathbf{2},\mathbf{1},\mathbf{2})_{-\mathbf{1},-\frac{\mathbf{1}}{\mathbf{2}}} \end{array}$$

  From it, we get 1 left-/right-handed fermions $(\psi,\tilde{\psi})$ and g_1 left-/right-handed fermions $(\lambda_k,\tilde{\lambda}_k)$ with $k=1,2,\ldots,g_1$.

- The contribution of punctures on C_{g_1} can be studied by reducing abelian 6d $\mathcal{N}=(2,0)$ theory on cylinder $S^1\times\mathbb{R}^+$. First, compactify it on a circle gives a 5d $\mathcal{N}=2$ SYM theory with gauge group $U(1)$. The puncture contributions to class S theory $\mathcal{T}_G[C_{g_1,n}]$ theory are determined by choices of half-BPS boundary conditions of the 5d $\mathcal{N}=2$ SYM theory. It is equivalent to specify a boundary condition of 4d $\mathcal{N}=4$ SYM theory if reducing it on another circle. The half-BPS boundary conditions of 4d $\mathcal{N}=4$ SYM theory has been studied in [41]. There are two boundary conditions for $G=U(1)$. The Dirichlet one gives a free 3d $\mathcal{N}=4$ (twisted) hypermultiplet and the S-dual Neumann one gives a free vector multiplet [38, 42]. We choose the Dirichlet boundary condition for each puncture when uplifting to 4d contribute n free 4d $\mathcal{N}=2$ hypermultiplet.

These fields can be organized as 4d $\mathcal{N}=2$ supermultiplets. In total, we have

- g_1 $\mathcal{N}=2$ vector multiplets $V_{\mathrm{4d},k}=(a_k,\phi_k,\lambda_k,\tilde{\lambda}_k)$ with $k=1,2,\ldots,g_1$

- $n+1$ $\mathcal{N}=2$ hypermultiplet $H_{\mathrm{4d},i}(q,\tilde{q},\psi,\tilde{\psi})$ with $i=0,1,\ldots,n$. Note $H_{\mathrm{4d},0}=(q,\tilde{q},\psi,\tilde{\psi})$ where we combine the real scalar φ_0 (as singlet of $SU(2)_R$) with the other three real scalars $\varphi_{1,2,3}$ (as triplet of $SU(2)_R$) form two complex scalars transforming as **2** under $SU(2)_R$ denoted by $(q,\tilde{q})$.

[7] Note $\tilde{a}_k$ are the dual of a_k, so we will not consider them in the following.

Reduction on C_{g_2} We now reduce these free 4d $\mathcal{N}=2$ multiplets on C_{g_2}. By a similar Kaluza-Klein reduction [7, 43], the 2d $\mathcal{N}=(0,4)$ theory is a free theory include:

- g_1 vector multiplet and $g_1 g_2$ twisted hypermultiplets
- $n+1$ hypermultiplet and $(n+1)g_2$ Fermi multiplets

The g_1 number of $U(1)$ gauge fields will be gapped in the infrared. The 2d $\mathcal{N}=(0,4)$ theory consists of free scalars and fermions.

The standard central charge of a real scalar is 1 and of a Weyl fermion is $1/2$. So, a $\mathcal{N}=(0,4)$ hypermultiplets or twisted hypermultiplets contribute $(4,6)$ and a $\mathcal{N}=(0,4)$ vector or Fermi multiplets contribute to $(0,2)$ to left/right-moving central charge. From the above field content, we find the central charges of a single M5 brane on $C_{g_1,n}\times C_{g_2}$ are

$$\begin{aligned} c_R &= 6(g_1 g_2 + n + 1) \\ c_L &= 4(g_1 g_2 + n + 1) + 2(g_1 + g_2 + n g_2)\ . \end{aligned} \tag{D.5}$$

The central charge are positive for arbitrary $g_1, g_2 > 0$ and support the conjecture of equation (2.24).

References

[1] W. Nahm, *Supersymmetries and Their Representations*, Nucl. Phys. B **135** (1978) 149.

[2] D. Gaiotto, *$N=2$ dualities*, JHEP **1208** (2012) 034 [0904.2715].

[3] D. Gaiotto, G.W. Moore and A. Neitzke, *Wall-crossing, Hitchin systems, and the WKB approximation*, Adv. Math. **234** (2013) 239 [0907.3987].

[4] M. Bershadsky, A. Johansen, V. Sadov and C. Vafa, *Topological reduction of 4-d SYM to 2-d sigma models*, Nucl. Phys. B **448** (1995) 166 [hep-th/9501096].

[5] A. Kapustin, *Holomorphic reduction of N=2 gauge theories, Wilson-'t Hooft operators, and S-duality*, hep-th/0612119.

[6] F. Benini and N. Bobev, *Two-dimensional SCFTs from wrapped branes and c-extremization*, JHEP **06** (2013) 005 [1302.4451].

[7] P. Putrov, J. Song and W. Yan, *(0,4) dualities*, JHEP **03** (2016) 185 [1505.07110].

[8] S. Nawata, Y. Pan and J. Zheng, *Class S theories on S^2*, Phys. Rev. D **109** (2024) 105015 [2310.07965].

[9] A. Hanany and N. Mekareeya, *Tri-vertices and SU(2)'s*, JHEP **02** (2011) 069 [1012.2119].

[10] E. Witten, *On the conformal field theory of the Higgs branch*, JHEP **07** (1997) 003 [hep-th/9707093].

[11] F. Benini, Y. Tachikawa and B. Wecht, *Sicilian gauge theories and N=1 dualities*, JHEP **01** (2010) 088 [0909.1327].

[12] Y. Tachikawa, *A review of the T_N theory and its cousins*, PTEP **2015** (2015) 11B102 [1504.01481].

[13] O. Aharony, N. Seiberg and Y. Tachikawa, *Reading between the lines of four-dimensional gauge theories*, JHEP **08** (2013) 115 [1305.0318].

[14] O. Chacaltana, J. Distler and Y. Tachikawa, *Nilpotent orbits and codimension-two defects of 6d N=(2,0) theories*, Int. J. Mod. Phys. A **28** (2013) 1340006 [1203.2930].

[15] A. Gadde, S.S. Razamat and B. Willett, *On the reduction of 4d $\mathcal{N}=1$ theories on $\mathbb{S}^2$*, JHEP **11** (2015) 163 [1506.08795].

[16] D. Tong, *The holographic dual of $AdS_3\times S^3\times S^3\times S^1$*, JHEP **04** (2014) 193 [1402.5135].

[17] S. Cecotti, J. Song, C. Vafa and W. Yan, *Superconformal Index, BPS Monodromy and Chiral Algebras*, *JHEP* **11** (2017) 013 [1511.01516].

[18] S. Gukov, P.-S. Hsin and D. Pei, *Generalized global symmetries of $T[M]$ theories. Part I*, *JHEP* **04** (2021) 232 [2010.15890].

[19] S. Gukov, P.-S. Hsin, D. Pei and S. Park, *Generalized Global Symmetries of $T[M]$ Theories: Part II*, 2511.13696.

[20] F. Benini and A. Zaffaroni, *Supersymmetric partition functions on Riemann surfaces*, *Proc. Symp. Pure Math.* **96** (2017) 13 [1605.06120].

[21] C. Closset, H. Kim and B. Willett, *$\mathcal{N}$ = 1 supersymmetric indices and the four-dimensional A-model*, *JHEP* **08** (2017) 090 [1707.05774].

[22] D. Delmastro, J. Gomis and M. Yu, *Infrared phases of 2d QCD*, *JHEP* **02** (2023) 157 [2108.02202].

[23] J. Jiang, S. Nawata and J. Zheng, *2d dualities from 4d*, *SciPost Phys.* **18** (2025) 180 [2407.17350].

[24] D.R. Grayson and M.E. Stillman, "Macaulay2, a software system for research in algebraic geometry." Available at http://www2.macaulay2.com.

[25] S. Cremonesi, G. Ferlito, A. Hanany and N. Mekareeya, *Instanton Operators and the Higgs Branch at Infinite Coupling*, *JHEP* **04** (2017) 042 [1505.06302].

[26] A. Butti, D. Forcella, A. Hanany, D. Vegh and A. Zaffaroni, *Counting Chiral Operators in Quiver Gauge Theories*, *JHEP* **11** (2007) 092 [0705.2771].

[27] A. Hanany and N. Mekareeya, *Counting Gauge Invariant Operators in SQCD with Classical Gauge Groups*, *JHEP* **10** (2008) 012 [0805.3728].

[28] J. Gray, A. Hanany, Y.-H. He, V. Jejjala and N. Mekareeya, *SQCD: A Geometric Apercu*, *JHEP* **05** (2008) 099 [0803.4257].

[29] S. Benvenuti, A. Hanany and N. Mekareeya, *The Hilbert Series of the One Instanton Moduli Space*, *JHEP* **1006** (2010) 100 [1005.3026].

[30] R.P. Stanley, *Hilbert functions of graded algebras*, *Advances in Mathematics* **28** (1978) 57.

[31] A. Hanany and M. Sperling, *Algebraic properties of the monopole formula*, *JHEP* **02** (2017) 023 [1611.07030].

[32] A. Beauville, *Symplectic singularities*, *Inventiones Mathematicae* **139** (2000) 541 [math/9903070].

[33] S. Cabrera, A. Hanany and F. Yagi, *Tropical Geometry and Five Dimensional Higgs Branches at Infinite Coupling*, *JHEP* **01** (2019) 068 [1810.01379].

[34] A. Gadde, S. Gukov and P. Putrov, *Fivebranes and 4-manifolds*, *Prog. Math.* **319** (2016) 155 [1306.4320].

[35] B. Feigin and S. Gukov, *VOA[M_4]*, *J. Math. Phys.* **61** (2020) 012302 [1806.02470].

[36] C. Vafa and E. Witten, *A Strong coupling test of S duality*, *Nucl. Phys. B* **431** (1994) 3 [hep-th/9408074].

[37] R. Dijkgraaf, J.-S. Park and B.J. Schroers, *N=4 supersymmetric Yang-Mills theory on a Kahler surface*, hep-th/9801066.

[38] F. Benini, Y. Tachikawa and D. Xie, *Mirrors of 3d Sicilian theories*, *JHEP* **09** (2010) 063 [1007.0992].

[39] S. Gukov, *Trisecting non-Lagrangian theories*, *JHEP* **11** (2017) 178 [1707.01515].

[40] A. Hanany and R. Kalveks, *Quiver Theories for Moduli Spaces of Classical Group Nilpotent Orbits*, *JHEP* **06** (2016) 130 [1601.04020].

[41] D. Gaiotto and E. Witten, *Supersymmetric Boundary Conditions in N=4 Super Yang-Mills Theory*, *J. Statist. Phys.* **135** (2009) 789 [0804.2902].

[42] D. Gaiotto and E. Witten, *S-Duality of Boundary Conditions In N=4 Super Yang-Mills Theory*, *Adv. Theor. Math. Phys.* **13** (2009) 721 [0807.3720].

[43] A. Kapustin, *Wilson-'t Hooft operators in four-dimensional gauge theories and S-duality*, *Phys. Rev.* **D74** (2006) 025005 [hep-th/0501015].