# Branes and Type $C^\vee C_1$ DAHA Representations

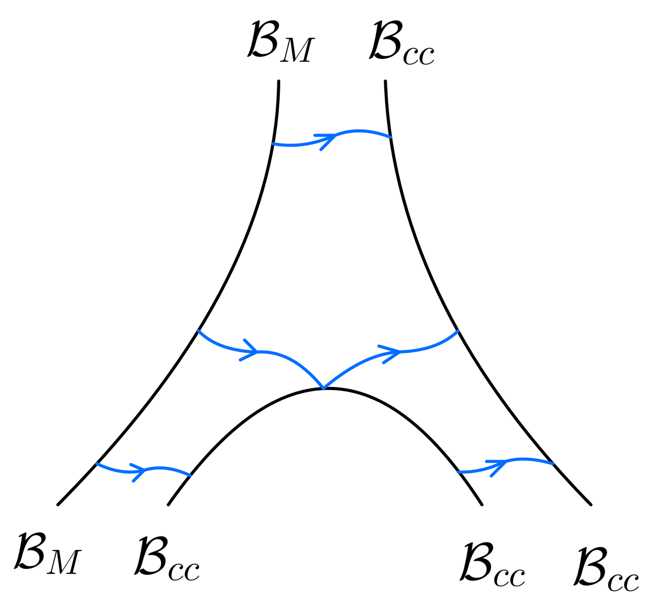
Junkang Huang[1], Satoshi Nawata[1], Yutai Zhang[1], Shutong Zhuang[1]

[1]Department of Physics, Fudan University, Shanghai, China.

## Introduction: Brane quantization

The **brane quantization** scheme[GW09, GKN⁺23] uses 2d topological string to establish a bridge between deformation quantization and geometric quantization:

- Two A-branes are involved for a target space $\mathfrak{X}$ with a holomorphic symplectic form $\Omega$: a unique space-filling **canonical co-isotropic brane** $\mathfrak{B}_{\mathrm{cc}}$ and **Lagrangian branes** $\mathfrak{B}_M$ for Lagrangian submanifolds $M$ w. r. t. $\operatorname{Im}\Omega$.
- It is known that:
$$\operatorname{End}(\mathfrak{B}_{\mathrm{cc}}) \cong O^q(\mathfrak{X}), \quad \operatorname{Hom}(\mathfrak{B}_M, \mathfrak{B}_{\mathrm{cc}}) \cong \mathcal{H}(M, \operatorname{Im}\Omega)$$
where $O^q(\mathfrak{X})$ is the **deformation quantization** of the holomorphic functions over $\mathfrak{X}$, $\mathcal{H}(M, \operatorname{Im}\Omega)$ is the Hilbert space of the **geometric quantization** of $(M, \operatorname{Re}\Omega)$.
- The process of joining of strings provides the natural action of $O^q(\mathfrak{X})$ on $\mathcal{H}(M, \operatorname{Im}\Omega)$:



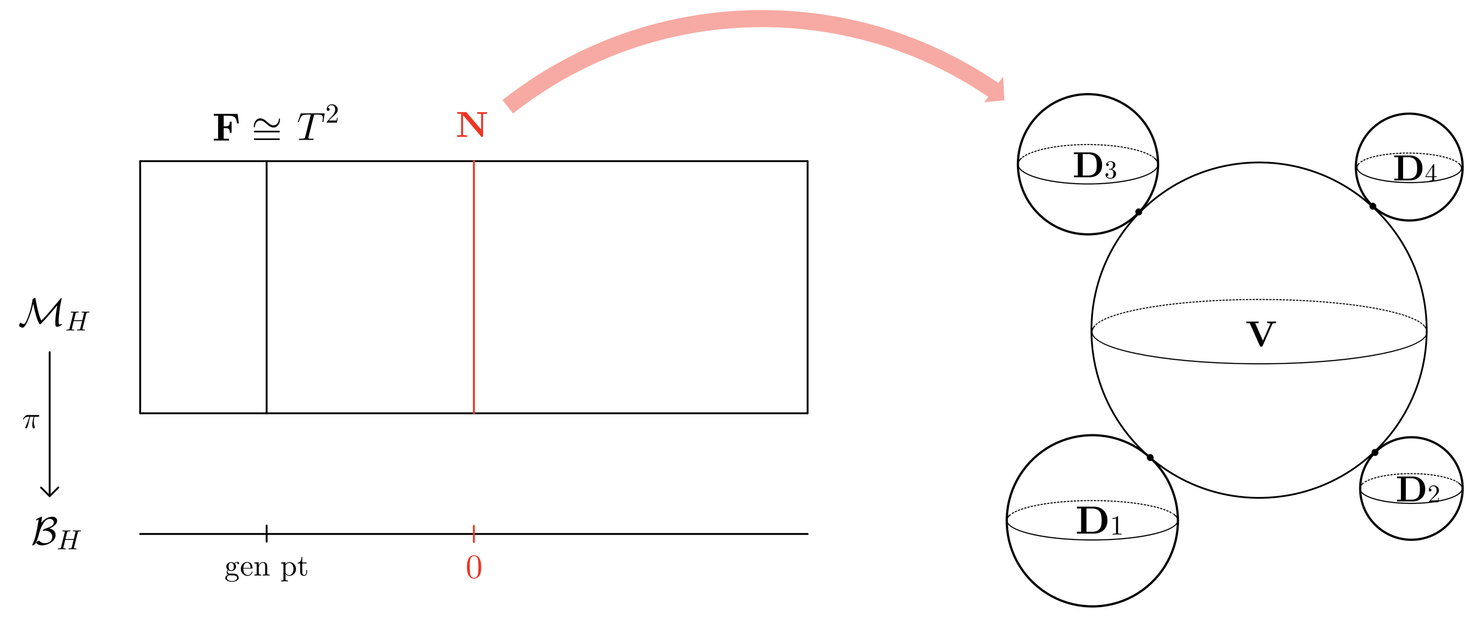

which can be generalized to the **conjectured correspondence**
$$\text{A-Brane}\,(\mathfrak{X}, \omega_{\mathfrak{X}}) \cong \operatorname{Rep}\,(O^q(\mathfrak{X}))\,.$$

In this project, the target space $\mathfrak{X}$ is taken as the Coulomb branch of 4d $\mathcal{N}=2$ $SU(2)$ SCFT compactified on $S^1$, which can be identified with the **Hitchin moduli space** $\mathcal{M}_H$.

## Geometry: A-branes in Hitchin moduli space

$SU(2)$ Hitchin moduli space $\mathcal{M}_H$ over a 4-punctured sphere is a hyper-Kähler manifold of quaternionic dimension 1, with three complex structure $(I, J, K)$.

- In complex structure $I$, $\mathcal{M}_H$ admits a fibration over the $u$-plane, called **Hitchin fibration**. For some particular ramification parameters, the fibration is as:




which has five compact irreducible components $\mathbf{D}_j, \mathbf{V}$ forming a singular fiber at the origin, and generic fibers $\mathbf{F}$ the complex tori.
- In complex structure $J$, $\mathcal{M}_H$ is a **cubic surface**.
$$-xyz + x^2 + y^2 + z^2 + \theta_1 x + \theta_2 y + \theta_3 z + \theta_4 = 0\,,$$
which is known to have 24 lines as its non-compact components. Equipped with Atiyah-Bott symplectic form $\Omega_J$, the holomorphic function on $\mathcal{M}_H$ is endowed with a Poisson bracket, whose deformation quantization is the **spherical double affine Hecke algebra (DAHA)** $S\ddot{H}_{q,\boldsymbol{t}}$.
$$S\ddot{H}_{q,\boldsymbol{t}} \cong \mathcal{O}_J^q(\mathcal{M}_H)\,.$$

## Algebra: Representations of spherical DAHA $S\ddot{H}_{q,\boldsymbol{t}}$

The **spherical DAHA** $S\ddot{H}_{q,\boldsymbol{t}}$ of $C^\vee C_1$ type is a $\mathbb{C}_{q,\boldsymbol{t}}$ algebra generated by $\{x, y, z\}$ with parameters $(q, \boldsymbol{t}) = (q, t_1, t_2, t_3, t_4)$
$$[x, y]_q = (q^{-1} - q)z + (q^{-\frac{1}{2}} - q^{\frac{1}{2}})\theta_3\,,$$
$$[y, z]_q = (q^{-1} - q)x + (q^{-\frac{1}{2}} - q^{\frac{1}{2}})\theta_1\,,$$
$$[z, x]_q = (q^{-1} - q)y + (q^{-\frac{1}{2}} - q^{\frac{1}{2}})\theta_2\,,$$
along with a Casimir element
$$f_q(x, y, z) = -q^{-\frac{1}{2}}xyz + q^{-1}x^2 + qy^2 + q^{-1}z^2 + q^{-\frac{1}{2}}\theta_1 x + q^{\frac{1}{2}}\theta_2 y + q^{-\frac{1}{2}}\theta_3 z + \theta_4 - q - q^{-1} + 2 = 0\,.$$
Here $(\theta_1, \theta_2, \theta_3, \theta_4) = (\chi_{\mathbf{8_V}}, \chi_{\mathbf{8_S}}, \chi_{\mathbf{8_C}}, \chi_{\mathbf{adj}})$ can be identified with the characters of $SO(8)$ representations, with fugacity labeled by $\boldsymbol{t}$. Therefore, the Weyl group $W[D_4]$ is the symmetry of $S\ddot{H}_{q,\boldsymbol{t}}$.
The representations of $S\ddot{H}_{q,\boldsymbol{t}}$ can be realized as the **polynomial representations**:

- Infinite-dimensional representations on the space of Laurent polynomials $\mathcal{P} \equiv \mathbb{C}_{q,\boldsymbol{t}}[X, X^{-1}]^{\mathbb{Z}_2}$ [NS04, Sto03], which admits a basis given by the Askey-Wilson polynomial $\{P_n\}_{n\geq 0}$.
- One can find the raising and lowering operator regarding this basis:
$$\begin{tikzcd} 1 \arrow[r, bend left, "\mathsf{R}_0"] & P_1 \arrow[l, bend left, "\mathsf{L}_1"] \arrow[r, bend left, "\mathsf{R}_1"] & \cdots \arrow[l, bend left, "\mathsf{L}_2"] \arrow[r, bend left, "\mathsf{R}_{j-2}"] & P_{j-1} \arrow[l, bend left, "\mathsf{L}_{j-1}"] \arrow[r, bend left, "\mathsf{R}_{j-1}"] & P_j \arrow[l, bend left, "\mathsf{L}_j"] \arrow[r, bend left, "\mathsf{R}_j"] & \cdots \arrow[l, bend left, "\mathsf{L}_{j+1}"] \end{tikzcd}$$

Imposing certain conditions between parameter $(q, \boldsymbol{t})$, called **shortening conditions**, the lowering operator $\mathsf{L}_n$ will annihilate $P_n$:
$$q^n = t_1^{r_1}t_2^{r_2}t_3^{r_3}t_4^{r_4} =: \boldsymbol{t}^r, \quad r \in \mathsf{R}(D_4) \cup \{(0,0,0,0)\}\,.$$
And then there exists a finite-dimensional representation of dimension $n$, constructed by the quotient of sub-module generated by $P_n$.

## Geometry vs. Algebra: Matching A-branes and Representations

- Taken into account the symmetries of $S\ddot{H}_{q,\boldsymbol{t}}$, we find out that there are 24 distinct **polynomial representations** labeled by the 24 shortest weights of the $D_4$ weight lattice.
$$\mathbb{S}_x = \{-\boldsymbol{t}^r \mid r \in (\mathbf{8}_V)\}, \quad \mathbb{S}_y = \{-\boldsymbol{t}^r \mid r \in (\mathbf{8}_S)\}, \quad \mathbb{S}_z = \{-\boldsymbol{t}^r \mid r \in (\mathbf{8}_C)\}$$
which correspond to the A-branes supported on the 24 lines in the target.
- We explicitly match between the quantization conditions of the compact components to support A-branes and the shortening conditions of the **finite-dimensional representations**:

| shortening condition | $A$-brane |
|---|---|
| $q^m = 1$ | $\mathfrak{B}_{\mathbf{F}}^{(x_m)}$ |
| $q^k = t_1^{-2}$ | $\mathfrak{B}_{\mathbf{V}}$ |
| $q^{\ell_1} = t_1t_2t_3t_4$ | $\mathfrak{B}_{\mathbf{D_1}}$ |
| $q^{\ell_2} = t_1t_2t_3^{-1}t_4^{-1}$ | $\mathfrak{B}_{\mathbf{D_2}}$ |
| $q^{\ell_3} = t_1t_2^{-1}t_3t_4^{-1}$ | $\mathfrak{B}_{\mathbf{D_3}}$ |
| $q^{\ell_4} = t_1t_2^{-1}t_3^{-1}t_4$ | $\mathfrak{B}_{\mathbf{D_4}}$ |

## Conclusions

Following from the prediction of the brane quantization scheme, we matched both nom-compact and compact A-branes inside the $SU(2)$ Hitchin moduli space over a 4-punctured sphere with the infinite-dimensional and finite-dimensional representations of $S\ddot{H}_{q,\boldsymbol{t}}$.
Our results provide a compelling evidence for the proposed equivalence between A-brane category and representation category of$S\ddot{H}_{q,\boldsymbol{t}}$.

## References

[GKN⁺23] S. Gukov, P. Koroteev, S. Nawata, D. Pei, and I. Saberi, *Branes and DAHA representations*, SpringerBriefs in Mathematical Physics (2023) , arXiv:2206.03565 [hep-th].

[GW09] S. Gukov and E. Witten, *Branes and quantization*, Adv. Theor. Math. Phys. **13** (2009) 1, arXiv:0809.0305 [hep-th].

[NS04] M. Noumi and J. V. Stokman, *Askey–Wilson polynomials: an affine Hecke algebra approach*, Laredo Lectures on Orthogonal Polynomials and Special Functions **111** (2004) C144, arXiv:math/0001033 [math.QA].

[Sto03] J. V. Stokman, *Difference fourier transforms for nonreduced root systems*, Selecta Mathematica **9** (2003) 409–494, arXiv:math/0111221 [math.QA].