2d supersymmetric theories

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Abstract: To be written.

Contents

Write supersymmetry algebras, supermultiplets, supersymmetric transformations, Lagrangians in a consistent notation.

1 Basics of Two-Dimensional Field Theories

We consider quantum field theories in two-dimensional Minkowski spacetime $\mathbb{R}^{1,1}$, equipped with the standard Lorentzian metric:

$$\eta_{\mu\nu} = \mathrm{diag}(-1, +1), \tag{1.1}$$

so that the line element takes the form

$$ds^2 = -dx^0 dx^0 + dx^1 dx^1. \tag{1.2}$$

It is often convenient to introduce light-cone coordinates,[1]

$$x^{\pm\pm} \equiv x^0 \pm x^1, \qquad \partial_{\pm\pm} \equiv \frac{1}{2}(\partial_0 \pm \partial_1) = \frac{1}{2}\left(\frac{\partial}{\partial x^0} \pm \frac{\partial}{\partial x^1}\right). \tag{1.3}$$

In this basis, a vector field $v \in \mathfrak{X}(\mathbb{R}^{1,1})$ can be expressed as

$$v = v^0\partial_0 + v^1\partial_1 = v^{++}\partial_{++} + v^{--}\partial_{--}, \tag{1.4}$$

with light-cone components defined by

$$v^{\pm\pm} \equiv v^0 \pm v^1. \tag{1.5}$$

Similarly, a 1-form $\omega \in \mathfrak{X}^*(\mathbb{R}^{1,1})$ is written as

$$\omega = \omega_0 dx^0 + \omega_1 dx^1 = \omega_{++}dx^{++} + \omega_{--}dx^{--}, \tag{1.6}$$

with

$$\omega_{\pm\pm} \equiv \frac{1}{2}(\omega_0 \pm \omega_1). \tag{1.7}$$

The metric in light-cone coordinates becomes

$$ds^2 = -dx^{++}dx^{--} = -\frac{1}{2}\left(dx^{++}dx^{--} + dx^{--}dx^{++}\right), \tag{1.8}$$

implying the non-vanishing components of the metric are

$$\eta_{++--} = \eta_{--++} = -\frac{1}{2}. \tag{1.9}$$

As a result, raising and lowering indices proceeds according to the rules:

$$(\cdots)^{\pm\pm} = -2(\cdots)_{\mp\mp}, \qquad (\cdots)_{\pm\pm} = -\frac{1}{2}(\cdots)^{\mp\mp}. \tag{1.10}$$

[1] The double-index $\pm\pm$ notation is preferred over a single $\pm$ for clarity. See, for example, [1].

Two-Dimensional Poincaré Algebra

The proper Lorentz group of two-dimensional Minkowski spacetime $\mathbb{R}^{1,1}$ is $SO(1,1)$, which is an abelian group generated by a single boost operator M corresponding to transformations along the x^1-direction. Since the group is abelian, its representations are labeled simply by the eigenvalues (charges) of M.

For a rapidity parameter γ, a finite Lorentz boost is given by

$$\Lambda(\gamma) = e^{-\mathrm{i}\gamma M} \in SO(1,1),$$

which acts on spacetime coordinates as

$$\Lambda(\gamma) \cdot \begin{pmatrix} x^0 \\ x^1 \end{pmatrix} = \begin{pmatrix} \cosh\gamma & \sinh\gamma \\ \sinh\gamma & \cosh\gamma \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \end{pmatrix}. \tag{1.11}$$

In light-cone coordinates, this transformation becomes diagonal:

$$\Lambda(\gamma) \cdot \begin{pmatrix} x^{++} \\ x^{--} \end{pmatrix} = \begin{pmatrix} e^{\gamma} & 0 \\ 0 & e^{-\gamma} \end{pmatrix} \begin{pmatrix} x^{++} \\ x^{--} \end{pmatrix}. \tag{1.12}$$

This shows that the two light-cone directions transform independently:

$$\Lambda(\gamma) \cdot x^{\pm\pm} = e^{\pm\gamma} x^{\pm\pm}, \qquad \Lambda(\gamma) \cdot x_{\pm\pm} = e^{\mp\gamma} x_{\pm\pm}. \tag{1.13}$$

Let us now define the Hermitian generators of translations along the x^0 and x^1 directions as

$$H \equiv P^0, \qquad P \equiv P^1. \tag{1.14}$$

Then the light-cone combinations of momentum generators are given by

$$P^{\pm\pm} \equiv H \pm P, \qquad \text{so that} \qquad P^{\pm\pm} = -2P_{\mp\mp}. \tag{1.15}$$

Under Lorentz transformations, these generators transform as

$$\Lambda^{-1}(\gamma) P^{\pm\pm} \Lambda(\gamma) = e^{\pm\gamma} P^{\pm\pm}, \tag{1.16}$$

which implies the infinitesimal commutation relations

$$[M, P^{\pm\pm}] = \pm\mathrm{i} P^{\pm\pm}, \qquad [M, P_{\pm\pm}] = \mp\mathrm{i} P_{\pm\pm}. \tag{1.17}$$

The generators $\{H, P, M\}$, together with the nontrivial commutators given in (1.17), define the *Poincaré algebra* in two-dimensional Minkowski spacetime $\mathbb{R}^{1,1}$.

Two-Dimensional Spinors

In two-dimensional Minkowski spacetime $\mathbb{R}^{1,1}$, the minimal spinor is a *Majorana-Weyl spinor* with a single real component (see [2]). Such spinors can be labeled as

$$\psi^{\pm} \equiv \pm\psi_{\mp}, \tag{1.18}$$

where $\psi^+ \equiv \psi_-$ denotes a *left-moving* spinor, and $\psi^- \equiv -\psi_+$ denotes a *right-moving* spinor. Each spinor is real (i.e., Majorana), so the reality condition is

$$\psi_{\pm}^* = \psi_{\pm}\ . \tag{1.19}$$

The Clifford algebra in $\mathbb{R}^{1,1}$ can be represented using Pauli matrices as follows:

$$\Gamma_0 = \mathrm{i}\sigma^1, \qquad \Gamma_1 = \sigma^2, \qquad \Gamma_3 = \sigma^3, \tag{1.20}$$

where Γ_3 serves as the chirality matrix, satisfying

$$\Gamma_3 \psi_\pm = \mp \psi_\pm. \tag{1.21}$$

The Lorentz generator M acting on spinors is given by

$$M = -\frac{\mathrm{i}}{4}\Gamma_{01} = \frac{1}{4}[\sigma^1, \sigma^2] = \frac{\mathrm{i}}{2}\sigma^3. \tag{1.22}$$

Under a Lorentz boost with rapidity γ, spinors transform as

$$\Lambda(\gamma) \cdot \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix} = \exp\left(\frac{\gamma}{2}\sigma^3\right) \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix}, \tag{1.23}$$

which yields the transformation laws

$$\Lambda(\gamma) \cdot \psi^\pm = e^{\pm\frac{\gamma}{2}} \psi^\pm, \qquad \Lambda(\gamma) \cdot \psi_\pm = e^{\mp\frac{\gamma}{2}} \psi_\pm. \tag{1.24}$$

Comparing (1.24) with the transformation law for vectors (1.13), we observe that spinors $\psi^\pm$ carry *half* the Lorentz weight of vectors $v^{\pm\pm}$. This observation justifies the use of double-sign $\pm\pm$ notation for vector components, distinguishing them from the spinor components $\psi^\pm$ with half-integer boost charges. And the Lorentz invariance of a Lagrangian term can be checked easily as having balanced $\pm$ indices.

Go to Euclidean spacetime

The minimal spinor in $\mathbb{R}^{0,2}$ is Weyl, with two real components.

$\mathcal{N} = (\mathcal{N}_-, \mathcal{N}_+)$ Supersymmetry

We adopt the convention that a two-dimensional $\mathcal{N} = (\mathcal{N}_-, \mathcal{N}_+)$ supersymmetric theory contains:

- $\mathcal{N}_-$ left-moving Majorana-Weyl supercharges Q_{A-}, with $A = 1, 2, \ldots, \mathcal{N}_-$,
- $\mathcal{N}_+$ right-moving Majorana-Weyl supercharges Q_{B+}, with $B = 1, 2, \ldots, \mathcal{N}_+$.

The maximal R-symmetry group for such a supersymmetry algebra is

$$SO(\mathcal{N}_-) \times SO(\mathcal{N}_+), \tag{1.25}$$

which acts on the left- and right-moving supercharges respectively as orthogonal rotations:

$$Q_{A-} \mapsto (R_-)_A{}^C \, Q_{C-}, \qquad Q_{B+} \mapsto (R_+)_B{}^D \, Q_{D+}, \qquad R_\pm \in SO(\mathcal{N}_\pm). \tag{1.26}$$

This symmetry reflects the internal automorphism group of the supersymmetry algebra.

2 $\mathcal{N} = (0,1)$

First, let us consider minimally supersymmetric theories in two dimensions.

Consider adding a single right-moving Majorana-Weyl supercharge Q_+ [3], which is a "square root" of the momentum P_{++}

$$\{Q_+, Q_+\} = -P_{++} = \frac{1}{2}(H - P). \tag{2.1}$$

According to (1.24), the Lorentz transform of Q_+ is

$$\Lambda^{-1}(\gamma) Q_+ \Lambda(\gamma) = e^{-\gamma/2} Q_+, \tag{2.2}$$

and infinitesimally

$$[M, Q_+] = -\frac{\mathrm{i}}{2} Q_+. \tag{2.3}$$

There is no room for a continuous R-symmetry, but a $\mathbb{Z}_2$ R-symmetry is allowed:

$$Q_+ \mapsto -Q_+ \ . \tag{2.4}$$

2.1 $\mathcal{N} = (0,1)$ superfields

The $\mathcal{N} = (0,1)$ superspace is parametrized by $x^{\pm\pm}, \theta^+ \in \mathbb{R}$. The natural derivatives along $x^{\pm\pm}$ and θ^+ are denoted as $\partial_{\pm\pm}$ and ∂_+ respectively [2].

Denote the translation group in this superspace as consisting of

$$G(x^{\pm\pm}, \theta^+) \equiv G(x^{++}, x^{--}, \theta^+) = \exp\left(-\mathrm{i} x P + \theta^+ Q_+\right) \ . \tag{2.5}$$

The multiplication of two elements gives

$$G(y^{\pm\pm}, \xi^+) G(x^{\pm\pm}, \theta^+) = \exp\left(-\mathrm{i}(x+y)P + (\theta^+ + \xi^+)Q_+ + \frac{1}{2}[\xi^+ Q_+, \theta^+ Q_+]\right), \tag{2.6}$$

where

$$[\xi^+ Q_+, \theta^+ Q_+] = -\xi^+ \theta^+ \{Q_+, Q_+\} = \xi^+ \theta^+ P_{++}. \tag{2.7}$$

Therefore, the multiplication rule is

$$G(y^{\pm\pm}, \xi^+) G(x^{\pm\pm}, \theta^+) = G\left(x^{++} + y^{++} + \frac{\mathrm{i}}{2}\xi^+ \theta^+, x^{--} + y^{--}, \theta^+ + \xi^+\right) \ , \tag{2.8}$$

which induces the translation in the superspace as

$$G(y^{\pm\pm}, \xi^+) : \quad (x^{\pm\pm}, \theta^+) \mapsto G(y^{\pm\pm}, \xi^+) \cdot (x^{\pm\pm}, \theta^+) = (x^{++} + y^{++} + \frac{\mathrm{i}}{2}\xi^+ \theta^+, x^{--} + y^{--}, \theta^+ + \xi^+) \ . \tag{2.9}$$

Imitating the four-dimensional case as in [5], an $\mathcal{N} = (0,1)$ superfield $\mathscr{F}(x^{\pm\pm}, \theta^+)$ can then be defined to transform under supersymmetry action as a function:

$$G(y^{\pm\pm}, \xi^+) \cdot \mathscr{F}(x^{\pm\pm}, \theta^+) = \mathscr{F}\left(G(y^{\pm\pm}, \xi^+) \cdot (x^{\pm\pm}, \theta^+)\right). \tag{2.10}$$

[2] This is different from the convention in standard textbooks such as [4], but we think it is more succinct.

Taylor expand the right-hand side and make use of the property $\theta^+\theta^+ = 0$

$$\begin{aligned} \mathrm{RHS} &= \mathscr{F} + \left(y^{++} + \frac{\mathrm{i}}{2}\xi^+\theta^+\right)\partial_{++}\mathscr{F} + y^{--}\partial_{--}\mathscr{F} + \xi^+\partial_+\mathscr{F} \\ &= \mathscr{F} + y\partial\mathscr{F} + \xi^+\left(\partial_+ + \frac{\mathrm{i}}{2}\theta^+\partial_{++}\right)\mathscr{F}. \end{aligned} \tag{2.11}$$

On the other hand, if we take the parameters $y^{\pm\pm}, \xi^+$ to be infinitesimal

$$G(y^{\pm\pm}, \xi^+)\cdot\mathscr{F} = \mathscr{F} - \mathrm{i}yP\cdot\mathscr{F} + \xi^+Q_+\cdot\mathscr{F}. \tag{2.12}$$

Comparing with (2.11), we conclude

$$P_{\pm\pm} = \mathrm{i}\partial_{\pm\pm}, \quad Q_+ = \partial_+ + \frac{\mathrm{i}}{2}\theta^+\partial_{++}. \tag{2.13}$$

If we consider the right action of supersymmetry group on a superfield

$$\mathscr{F}(x^{\pm\pm}, \theta^+)\cdot G(y^{\pm\pm}, \xi^+) = \mathscr{F}\left(x^{++} + y^{++} - \frac{\mathrm{i}}{2}\xi^+\theta^+, x^{--} + y^{--}, \theta^+ + \xi^+\right), \tag{2.14}$$

and do the same infinitesimal expansion, we will obtain different generators

$$P_{\pm\pm} = \mathrm{i}\partial_{\pm\pm}, \quad \mathcal{D}_+ = \partial_+ - \frac{\mathrm{i}}{2}\theta^+\partial_{++}, \tag{2.15}$$

where $\mathcal{D}_+$ is the *super-derivative*. By the associativity of the supersymmetry group (as can be easily verified), $\{\mathcal{D}_+, Q_+\} = 0$ and therefore $\mathcal{D}_+\mathscr{F}$ is a superfield whenever $\mathscr{F}$ is.

Generically, the superfield $\mathscr{F}(x^{\pm\pm}, \theta^+)$ can be Tayler expanded to be

$$\mathscr{F}(x^{\pm\pm}, \theta^+) = f(x^{\pm\pm}) + \theta^+g_+(x^{\pm\pm}), \tag{2.16}$$

for some component fields $f(x)$ and $g_+(x)$, possibly carrying some additional space-time indices.

Denote $\delta \equiv \xi^+Q_+$, the supersymmetry transformations of the components can be obtained as follows

$$\delta\mathscr{F} = \xi^+\left(\partial_+ + \frac{\mathrm{i}}{2}\theta^+\partial_{++}\right)(f + \theta^+g_+) = \xi^+\left(g_+ + \frac{\mathrm{i}}{2}\theta^+\partial_{++}f\right). \tag{2.17}$$

Therefore we conclude that

$$\delta f = \xi^+g_+, \quad \delta g_+ = -\frac{\mathrm{i}}{2}\xi^+\partial_{++}f. \tag{2.18}$$

We are mostly interested in the following cases: (See [6])

$\mathcal{N} = (0,1)$ **scalar multiplet** Set $\mathscr{F}$ to be a scalar Φ

$$\Phi(x^{\pm\pm}, \theta^+) = \varphi(x^{\pm\pm}) + \frac{\mathrm{i}}{2}\theta^+\psi_+(x^{\pm\pm}), \tag{2.19}$$

with components a real scalar $\varphi(x)$ and a right-moving Majorana-Weyl fermion $\psi_+(x)$. The coefficient $\frac{\mathrm{i}}{2}$ is chosen such that the multiplet is real and the terms in the actions have correct normalization. This multiplet is called a *scalar multiplet* of the $\mathcal{N} = (0,1)$ theory.

$\mathcal{N} = (0,1)$ **Fermi multiplet** Set $\mathscr{F}$ to be a left-moving fermion Γ_-

$$\Gamma_-(x^{\pm\pm}, \theta^+) = \psi_-(x^{\pm\pm}) + \theta^+G(x^{\pm\pm}), \tag{2.20}$$

with components a left-moving Majorana-Weyl fermion $\psi_-(x)$ and an auxiliary real scalar $G(x)$. This multiplet is called a *Fermi multiplet* of the $\mathcal{N} = (0,1)$ theory, which is real as well.

$\mathcal{N}=(0,1)$ **vector multiplet** Define the $\mathcal{N}=(0,1)$ *vector multiplet* as putting together the following two superfields

$$\begin{aligned} V_+(x^{\pm\pm},\theta^+) &= \lambda_+(x^{\pm\pm}) + \theta^+ A_{++}(x^{\pm\pm}), \\ V_{--}(x^{\pm\pm},\theta^+) &= A_{--}(x^{\pm\pm}) + \frac{\mathrm{i}}{2}\theta^+\lambda_-(x^{\pm\pm}), \end{aligned} \tag{2.21}$$

with components $\lambda_\pm(x)$ Majorana-Weyl fermions and $A_{\pm\pm}(x)$ the two components of a vector field. Both V_+ and V_{--} are real multiplets.

Typically in gauge theories, the vector multiplet lives in the adjoint representation of a gauge group G. Let us now determine the gauge transforms of superfields.

Firstly, we may assume that a scalar multiplet Φ transforms as

$$\Phi \mapsto e^{-\mathrm{i}\Omega}\cdot\Phi, \tag{2.22}$$

for some scalar multiplet $\Omega = \alpha + \mathrm{i}\theta^+\beta_+$ valued in the gauge Lie algebra acting on Φ by some representation denoted by $\cdot$. Define the *covariant super-derivatives* as

$$\begin{aligned} \mathscr{D}_+ &\equiv \mathcal{D}_+ - \frac{1}{2}V_+ = \partial_+ - \frac{1}{2}\lambda_+ - \frac{\mathrm{i}}{2}\theta^+\nabla_{++}\ , \\ \mathscr{D}_{--} &\equiv \partial_{--} - \mathrm{i}V_{--} = \nabla_{--} + \frac{1}{2}\theta^+\lambda_-\ , \end{aligned} \tag{2.23}$$

which is the simplest way to complete the ordinary covariant derivatives $\nabla_{\pm\pm}\equiv\partial_{\pm\pm}-\mathrm{i}A_{\pm\pm}$ into supersymmetric multiplets. Then the gauge transforms of vector multiplets should obey

$$\mathscr{D}_+ \mapsto e^{-\mathrm{i}\Omega}\mathscr{D}_+e^{\mathrm{i}\Omega},\quad \mathscr{D}_{--} \mapsto e^{-\mathrm{i}\Omega}\mathscr{D}_{--}e^{\mathrm{i}\Omega}, \tag{2.24}$$

such that $\mathscr{D}_*\Phi$ transforms the same way as Φ. The right-hand sides can be rewritten as

$$\begin{aligned} e^{-\mathrm{i}\Omega}\mathscr{D}_+e^{\mathrm{i}\Omega} &= \mathcal{D}_+ - \frac{1}{2}\left[e^{-\mathrm{i}\Omega}V_+e^{\mathrm{i}\Omega} - 2e^{-\mathrm{i}\Omega}(\mathcal{D}_+e^{\mathrm{i}\Omega})\right], \\ e^{-\mathrm{i}\Omega}\mathscr{D}_{--}e^{\mathrm{i}\Omega} &= \partial_{--} - \mathrm{i}\left[e^{-\mathrm{i}\Omega}V_{--}e^{\mathrm{i}\Omega} + \mathrm{i}e^{-\mathrm{i}\Omega}(\partial_{--}e^{\mathrm{i}\Omega})\right], \end{aligned} \tag{2.25}$$

Therefore we conclude

$$\begin{aligned} V_+ &\mapsto e^{-\mathrm{i}\Omega}V_+e^{\mathrm{i}\Omega} - 2e^{-\mathrm{i}\Omega}(\mathcal{D}_+e^{\mathrm{i}\Omega})\ , \\ V_{--} &\mapsto e^{-\mathrm{i}\Omega}V_{--}e^{\mathrm{i}\Omega} + \mathrm{i}e^{-\mathrm{i}\Omega}(\partial_{--}e^{\mathrm{i}\Omega})\ . \end{aligned} \tag{2.26}$$

In the following, we partially fix the gauge by imposing $\lambda_+=0$, called the *Wess-Zumino gauge*, then the residual gauge transformation has $\beta_+=0$. Then (2.26) can be expanded in components

$$\begin{aligned} V_+ = \theta^+A_{++} &\mapsto \theta^+\left(e^{-\mathrm{i}\alpha}A_{++}e^{\mathrm{i}\alpha} + \mathrm{i}e^{-\mathrm{i}\alpha}(\partial_{++}e^{\mathrm{i}\alpha})\right)\ , \\ V_{--} = A_{--} + \frac{\mathrm{i}}{2}\theta^+\lambda_- &\mapsto e^{-\mathrm{i}\alpha}A_{--}e^{\mathrm{i}\alpha} + \mathrm{i}e^{-\mathrm{i}\alpha}(\partial_{--}e^{\mathrm{i}\alpha}) + \frac{\mathrm{i}}{2}\theta^+e^{-\mathrm{i}\alpha}\lambda_-e^{\mathrm{i}\alpha}\ , \end{aligned} \tag{2.27}$$

which ensures that the Wess-Zumino gauge is preserved by the residual gauge transformations. Therefore we obtain the familiar transformations

$$A_{\pm\pm} \mapsto e^{-\mathrm{i}\alpha}A_{\pm\pm}e^{\mathrm{i}\alpha} + \mathrm{i}e^{-\mathrm{i}\alpha}(\partial_{\pm\pm}e^{\mathrm{i}\alpha}),\quad \lambda_- \mapsto e^{-\mathrm{i}\alpha}\lambda_-e^{\mathrm{i}\alpha}\ , \tag{2.28}$$

and the scalar multiplets transform as

$$\varphi \mapsto e^{-\mathrm{i}\alpha}\cdot\varphi,\quad \psi_+ \mapsto e^{-\mathrm{i}\alpha}\cdot\psi_+\ , \tag{2.29}$$

in Wess-Zumino gauge.

Define the *superfield strength* as the Fermi multiplet

$$\Sigma_- := 2\left[\mathscr{D}_+, \mathscr{D}_{--}\right] = \lambda_- - \theta^+ F \ , \tag{2.30}$$

where F is the ordinary field strength

$$F \equiv \partial_{++} A_{--} - \partial_{--} A_{++} - \mathrm{i}\left[A_{++}, A_{--}\right] \ . \tag{2.31}$$

By definition and the transformations of covariant super-derivatives, the superfield strength transforms in the gauge adjoint representation $F \mapsto e^{-\mathrm{i}\alpha} F e^{\mathrm{i}\alpha}$.

2.2 $\mathcal{N} = (0,1)$ Lagrangian

We are mainly interested in the SQCD (also called the gauged linear sigma model), and non-linear sigma model (NLSM). See [6].

The action should be an integral over the $\mathcal{N} = (0,1)$ superspace, parametrized by $x^\pm, \theta^+$. So the terms in the Lagrangian should be the θ^+-terms of some scalar superfields, i.e. carrying a $+$ superscript or a $-$ subscript.

2d $\mathcal{N} = (0,1)$ SQCD

Let us first consider SQCD. Assume that the gauge theory has: n_s scalar multiplets Φ^i living in the $\mathfrak{R}^\Phi_i$ gauge representations for $i = 1, \cdots, n_s$, n_f Fermi multiplets Γ^i_- in the $\mathfrak{R}^\Gamma_i$ gauge representations for $i = 1, \cdots n_f$ and a vector multiplet (V_+, V_{--}) in the adjoint representation mediating the gauge action.

The only way to construct a gauge invariant kinetic term for a scalar multiplet Φ is $\mathscr{D}_+\Phi\mathscr{D}_{--}\Phi$. The gauge invariant kinetic terms for a Fermi multiplet Γ_- can only be $\Gamma_-\mathscr{D}_+\Gamma_-$, which is real by itself. Since the field strength is a Fermi multiplet, the gauge kinetic term is the same as for Fermi multiplets, up to normalization. We are free to add the interaction terms $\Gamma^i_- J^i(\Phi)$ between the scalar multiplets and the Fermi multiplets by choosing n_f holomorphic functions $\{J^i\}$ of $\Phi^1, \cdots, \Phi^{n_s}$ called the *superpotentials*.

By proper normalization, the action of the SQCD can then be written as[3]

$$S_{\mathrm{SQCD}} = \int dx^2 d\theta^+ \left[8\mathrm{i}\sum_{i=1}^{n_s} \mathscr{D}_+\Phi^i\mathscr{D}_{--}\Phi^i + \sum_{i=1}^{n_f}\left(4\Gamma^i_-\mathscr{D}_+\Gamma^i_- + m_i\Gamma^i_- J^i(\Phi)\right) + \frac{4}{g^2}\mathrm{tr}\left(\Sigma_-\mathscr{D}_+\Sigma_-\right)\right] \ , \tag{2.32}$$

for some mass parameters m_i (assumed to be real) and gauge coupling constant g. To maintain gauge invariance, the superpotential J^i must live in the $\mathfrak{R}^{\Gamma*}_i$ representation dual to $\mathfrak{R}^\Gamma_i$.

In components (omitting the flavour indices i temporarily)

$$\begin{aligned}
8\mathrm{i}\mathscr{D}_+\Phi\mathscr{D}_{--}\Phi &= -4\psi_+\nabla_{--}\varphi + \theta^+\left(4\nabla_{++}\varphi\nabla_{--}\varphi + 2\mathrm{i}\psi_+\nabla_{--}\psi_+ + 2\psi_+\lambda_-\varphi\right) \ , \\
4\Gamma_-\mathscr{D}_+\Gamma_- &= 4\psi_- G + \theta^+\left(2\mathrm{i}\psi_-\nabla_{++}\psi_- + 4G^2\right) \ , \\
\frac{4}{g^2}\mathrm{tr}\left(\Sigma_-\mathscr{D}_+\Sigma_-\right) &= -\frac{4}{g^2}\mathrm{tr}\left(\lambda_- F\right) + \theta^+\frac{1}{g^2}\mathrm{tr}\left(4F^2 + 2\mathrm{i}\lambda_-\nabla_{++}\lambda_-\right) \ , \\
m\Gamma_- J(\Phi) &= m\psi_- J(\varphi) + \theta^+\left(\frac{\mathrm{i}}{2}m\psi_+\psi_-\frac{\partial J}{\partial\varphi}(\varphi) + mGJ(\varphi)\right) \ .
\end{aligned} \tag{2.33}$$

[3] A topological term can be added as well, see [6].

So the action in component form is

$$S_{\text{SQCD}} = \int dx^2 \Bigg[\sum_{i=1}^{n_s} \left(4\nabla_{++}\varphi^i \nabla_{--}\varphi^i + 2\mathrm{i}\psi_+^i \nabla_{--}\psi_+^i + 2\psi_+^i \lambda_- \varphi^i\right) \\ + \sum_{i=1}^{n_f} \left(2\mathrm{i}\psi_-^i \nabla_{++}\psi_-^i + 4(G^i)^2 + \frac{\mathrm{i}}{2} m_i \psi_+^i \psi_-^i \frac{\partial J^i(\varphi)}{\partial \varphi} + m_i G^i J^i(\varphi)\right) \\ + \frac{1}{g^2}\mathrm{tr}\left(4F^2 + 2\mathrm{i}\lambda_- \nabla_{++}\lambda_-\right) \Bigg] . \tag{2.34}$$

The equation of motion of the auxiliary fields G^i gives

$$G^i = -\frac{1}{8} m_i J^i(\varphi) . \tag{2.35}$$

After eliminating G^i, the action becomes

$$S_{\text{SQCD}} = \int dx^2 \Bigg[\sum_{i=1}^{n_s} \left(4\nabla_{++}\varphi^i \nabla_{--}\varphi^i + 2\mathrm{i}\psi_+^i \nabla_{--}\psi_+^i + 2\psi_+^i \lambda_- \varphi^i\right) \\ + \sum_{i=1}^{n_f} \left(2\mathrm{i}\psi_-^i \nabla_{++}\psi_-^i + \frac{\mathrm{i}}{2} m_i \psi_+^i \psi_-^i \frac{\partial J^i(\varphi)}{\partial \varphi} - \frac{1}{16}(m_i J^i(\varphi))^2\right) \\ + \frac{1}{g^2}\mathrm{tr}\left(4F^2 + 2\mathrm{i}\lambda_- \nabla_{++}\lambda_-\right) \Bigg] . \tag{2.36}$$

Making use of ordinary coordinates instead of light-cone coordinates, the kinetic terms become familiar

$$4\nabla_{++}\varphi^i \nabla_{--}\varphi^i = -\nabla^\mu \varphi^i \nabla_\mu \varphi^i, \quad 2\mathrm{i}\psi_\pm^i \nabla_{\mp\mp}\psi_\pm^i = \mathrm{i}\psi_\pm^i \left(\nabla_0 \mp \nabla_1\right)\psi_\pm^i , \tag{2.37}$$

$$\frac{4}{g^2}\mathrm{tr}F^2 \equiv \frac{4}{g^2}\mathrm{tr}\left(F_{++--}F_{++--}\right) = \frac{1}{g^2}\mathrm{tr}\left(F^{--++}F_{++--}\right) = -\frac{1}{2g^2}\mathrm{tr}\left(F^{\mu\nu}F_{\mu\nu}\right) . \tag{2.38}$$

The scalar potential is

$$V(\varphi) = \frac{1}{16}\sum_{i=1}^{n_f}\left(m_i J^i(\varphi)\right)^2 , \tag{2.39}$$

which is non-negative and thus the vacua are exactly the common zeros of the superpotentials $\{J^i\}$.

Finally, we need to make sure that there is no gauge anomaly.

2d $\mathcal{N} = (0, 1)$ sigma model

[7][1]

Central charges, 't Hooft anomaly and gravitational anomaly

SN: 't Hooft anomaly and gravitational anomaly for 2d chiral theories

The gravitational anomaly

$$c_R - c_L = \mathrm{Tr}(\Gamma_3) . \tag{2.40}$$

For chiral theories, it is crucial to pay attention to anomalies. Consider a $(0, 1)$ theory with a global symmetry F described by a simple Lie algebra. The 't Hooft anomaly coefficient k_F associated with this symmetry can be determined by

$$\mathrm{Tr}\,\Gamma_3 f^a f^b = k_F \delta^{ab}, \tag{2.41}$$

where f^a are the generators of F, Γ_3 is the chirality matrix (1.21), and the trace is taken over fermions in the theory considered.

3 $\mathcal{N} = (1,1)$

$\mathcal{N} = (1,1)$ supersymmetry requires two Majorana-Weyl supercharges $Q_\pm$ moving in opposite directions, satisfying

$$\{Q_\pm, Q_\pm\} = -P_{\pm\pm} = \frac{1}{2}(H \mp P) \ , \quad [M, Q_\pm] = \mp\frac{\mathrm{i}}{2}Q_\pm \ . \tag{3.1}$$

All we need is to add the left-moving counterpart to the discussion of (0,1) supersymmetry. There is still no room for a continuous R-symmetry, but a $\mathbb{Z}_2 \times \mathbb{Z}_2$ R-symmetry is allowed, flipping the signs of Q_+ and Q_-.

3.1 $\mathcal{N} = (1,1)$ superfields

The $\mathcal{N} = (1,1)$ superspace is parametrized by $x^{\pm\pm}, \theta^\pm \in \mathbb{R}$. The natural derivatives along $x^{\pm\pm}$ and $\theta^\pm$ are denoted as $\partial_{\pm\pm}$ and $\partial_\pm$ respectively. A (1,1) superfield is a field $\mathscr{F}$ over the superspace transforms under the supersymmetry action by

$$G(y^{\pm\pm}, \xi^\pm) \cdot \mathscr{F}(x^{\pm\pm}, \theta^\pm) = \mathscr{F}\left(x^{\pm\pm} + y^{\pm\pm} + \frac{\mathrm{i}}{2}\xi^\pm\theta^\pm, \theta^\pm + \xi^\pm\right). \tag{3.2}$$

The differential operators for supercharges and super-derivatives are

$$Q_\pm = \partial_\pm + \frac{\mathrm{i}}{2}\theta^\pm\partial_{\pm\pm} \ , \quad \mathcal{D}_\pm = \partial_\pm - \frac{\mathrm{i}}{2}\theta^\pm\partial_{\pm\pm} \ . \tag{3.3}$$

Generically, the superfield can be Tayler expanded to be

$$\mathscr{F}(x^{\pm\pm}, \theta^\pm) = f(x^{\pm\pm}) + \theta \cdot g(x^{\pm\pm}) + \theta^+\theta^- h(x^{\pm\pm}) \ , \tag{3.4}$$

for some components $f(x)$, $g_\pm(x)$ and $h(x)$. Denote the supersymmetry variation $\delta = \xi \cdot Q$, then

$$\delta\mathscr{F} = \xi^\pm\left(\partial_\pm + \frac{\mathrm{i}}{2}\theta^\pm\partial_{\pm\pm}\right)(f + \theta^\pm g_\pm + \theta^+\theta^- h) = \xi^\pm\left(g_\pm + \frac{\mathrm{i}}{2}\theta^\pm\partial_{\pm\pm}f\right) \ . \tag{3.5}$$

Therefore we conclude that

$$\delta f = \xi^\pm g_\pm, \quad \delta g_\pm = -\frac{\mathrm{i}}{2}\xi^\pm\partial_{\pm\pm}f \ . \tag{3.6}$$

We are most interested in the following cases:

(1,1) scalar multiplet The superfield

$$\Phi(x^{\pm\pm}, \theta^\pm) = \varphi(x^{\pm\pm}) + \theta^\pm\psi_\pm(x^{\pm\pm}) \ , \tag{3.7}$$

with φ a real scalar, $\psi_\pm$ two Majorana fermions.

One expects a (1,1) scalar can be decomposed into a (0,1) scalar and a (0,1) fermi.

(1,1) vector multiplet One expects a (1,1) vector can be decomposed into a (0,1) vector and a (0,1) scalar

$$\begin{array}{ccccc} \text{(1,1) vector} & = & \text{(0,1) vector} & + & \text{(0,1) scalar,} \\ (A_\pm, \lambda_-, \sigma, \lambda_+) & & (A_\pm, \lambda_-) & & (\sigma, \lambda_+) \end{array} \tag{3.8}$$

However, to get a (2,2) vector$(A_\pm, \lambda_-, \bar{\lambda}_-, \sigma, \bar{\sigma}, \lambda_+, \bar{\lambda}_+)$, one also needs a (1,1) adjoint scalar, which is decomposed into

$$\begin{array}{ccccc} \text{(1,1) adjoint scalar} = & \text{(0,1) scalar} & + & \text{(0,1) fermi,} \\ (\tilde{\sigma}, \tilde{\lambda}_+, \tilde{\lambda}_-, G) & (\tilde{\sigma}, \tilde{\lambda}_+) & & (\tilde{\lambda}_-, G) \end{array} \tag{3.9}$$

which is sensible since the component fermions in (0,1) multiplets are Majorana, and one needs to combine the λ_- (from the (1,1) vector) together with the $\tilde{\lambda}_-$ (from the (1,1) adjoint scalar) to get a complex left-handed gaugino $(\lambda_-, \bar{\lambda}_-)$, as will be seen in the (0,2) vector multiplet.

2d $\mathcal{N} = (1,1)$ Lagrangians

4 $\mathcal{N} = (0,2)$

$\mathcal{N} = (0,2)$ supersymmetry requires two copies of right-moving Majorana-Weyl supercharges $Q_{A+}, A = 1,2$. It is more convenient to combine them into a single complex Weyl spinor

$$\mathcal{Q}_+ \equiv Q_{1+} + \mathrm{i}Q_{2+}, \quad \overline{\mathcal{Q}}_+ \equiv Q_{1+} - \mathrm{i}Q_{2+} \ . \tag{4.1}$$

Each Q_{A+} should satisfies the $\mathcal{N} = (0,1)$ algebra (2.1)

$$\{Q_{A+}, Q_{A+}\} = -P_{++} = \frac{1}{2}(H - P), \quad A = 1,2 \ . \tag{4.2}$$

In terms of complex supercharges

$$\{\mathcal{Q}_+, \overline{\mathcal{Q}}_+\} = -2P_{++} = H - P \ , \quad \{\mathcal{Q}_+, \mathcal{Q}_+\} = \{\overline{\mathcal{Q}}_+, \overline{\mathcal{Q}}_+\} = 0 \ , \tag{4.3}$$

and the Lorentz transformations are

$$[M, \mathcal{Q}_+] = -\frac{\mathrm{i}}{2}\mathcal{Q}_+, \quad [M, \overline{\mathcal{Q}}_+] = -\frac{\mathrm{i}}{2}\overline{\mathcal{Q}}_+. \tag{4.4}$$

There is an $SO(2)_+ \cong U(1)_+$ R-symmetry rotating the two Majorana-Weyl super charges. Equivalently, denote the generator as R_+, the complex supercharges are rotated as

$$e^{-\mathrm{i}\alpha R_+}: \quad \mathcal{Q}_+ \mapsto e^{-\mathrm{i}\alpha}\mathcal{Q}_+, \quad \overline{\mathcal{Q}}_+ \mapsto e^{\mathrm{i}\alpha}\overline{\mathcal{Q}}_+, \tag{4.5}$$

which lead to the commutation relations

$$[R_+, \mathcal{Q}_+] = -\mathcal{Q}_+, \quad [R_+, \overline{\mathcal{Q}}_+] = \overline{\mathcal{Q}}_+ \ , \tag{4.6}$$

i.e. $\mathcal{Q}_+, \overline{\mathcal{Q}}_+$ carry R-charges ∓ 1 respectively.

4.1 $\mathcal{N}=(0,2)$ superfields

The $\mathcal{N}=(0,2)$ superspace, first introduced in [8], is parametrized by $x^{\pm\pm},\theta^{1+},\theta^{2+}$ corresponding to the translations generated by $P_{\pm\pm},Q_{1+},Q_{2+}$, with natural derivatives $\partial_{\pm\pm},\partial_{1+},\partial_{2+}$ respectively. The translational group element is

$$G(x^{\pm\pm},\theta^{1+},\theta^{2+})=\exp\left(-\mathrm{i}xP+\theta^{1+}Q_{1+}+\theta^{2+}Q_{2+}\right). \tag{4.7}$$

In order to use the complex supercharges, we denote

$$\theta^{+}\equiv\frac{1}{2}\left(\theta^{2+}+\mathrm{i}\theta^{1+}\right),\quad \overline{\theta}^{+}\equiv\frac{1}{2}\left(\theta^{2+}-\mathrm{i}\theta^{1+}\right)\ , \tag{4.8}$$

and denote the associated natural derivative $\partial_+,\overline{\partial}_+$. Then

$$G(x^{\pm\pm},\theta^{+},\overline{\theta}^{+})=\exp\left(-\mathrm{i}xP-\mathrm{i}\theta^{+}\mathcal{Q}_{+}+\mathrm{i}\overline{\theta}^{+}\overline{\mathcal{Q}}_{+}\right). \tag{4.9}$$

The multiplication of two elements give

$$\begin{aligned}&G(y^{\pm\pm},\xi^{+},\overline{\xi}^{+})G(x^{\pm\pm},\theta^{+},\overline{\theta}^{+})=\\&\quad\exp\left(-\mathrm{i}(x+y)P-\mathrm{i}(\theta+\xi)^{+}\mathcal{Q}_{+}+\mathrm{i}(\overline{\theta}+\overline{\xi})^{+}\overline{\mathcal{Q}}_{+}-\frac{1}{2}\left[\xi^{+}\mathcal{Q}_{+}-\overline{\xi}^{+}\overline{\mathcal{Q}}_{+},\theta^{+}\mathcal{Q}_{+}-\overline{\theta}^{+}\overline{\mathcal{Q}}_{+}\right]\right),\end{aligned} \tag{4.10}$$

where

$$\begin{aligned}&\left[\xi^{+}\mathcal{Q}_{+}-\overline{\xi}^{+}\overline{\mathcal{Q}}_{+},\theta^{+}\mathcal{Q}_{+}-\overline{\theta}^{+}\overline{\mathcal{Q}}_{+}\right]\\&\qquad=\xi^{+}\overline{\theta}^{+}\{\mathcal{Q}_{+},\overline{\mathcal{Q}}_{+}\}+\overline{\xi}^{+}\theta^{+}\{\overline{\mathcal{Q}}_{+},\mathcal{Q}_{+}\}=-2\left(\xi^{+}\overline{\theta}^{+}+\overline{\xi}^{+}\theta^{+}\right)P_{++}\ .\end{aligned} \tag{4.11}$$

Therefore, the multiplication rule is

$$\begin{aligned}&G(y^{\pm\pm},\xi^{+},\overline{\xi}^{+})G(x^{\pm\pm},\theta^{+},\overline{\theta}^{+})\\&\qquad=G\left(x^{++}+y^{++}+\mathrm{i}\xi^{+}\overline{\theta}^{+}+\mathrm{i}\overline{\xi}^{+}\theta^{+},x^{--}+y^{--},\theta^{+}+\xi^{+},\overline{\theta}^{+}+\overline{\xi}^{+}\right).\end{aligned} \tag{4.12}$$

The $\mathcal{N}=(0,2)$ superfields are then defined to transform as

$$\begin{aligned}&G(y^{\pm\pm},\xi^{+},\overline{\xi}^{+})\cdot\mathscr{F}(x^{\pm\pm},\theta^{+},\overline{\theta}^{+})\\&\qquad=\mathscr{F}\left(x^{++}+y^{++}+\mathrm{i}\xi^{+}\overline{\theta}^{+}+\mathrm{i}\overline{\xi}^{+}\theta^{+},\theta^{+}+\xi^{+},\overline{\theta}^{+}+\overline{\xi}^{+}\right).\end{aligned} \tag{4.13}$$

Taylor expansion of the right-hand side gives

$$\mathrm{RHS}=\mathscr{F}+y\partial\mathscr{F}-\mathrm{i}\xi^{+}\left(\mathrm{i}\partial_{+}-\overline{\theta}^{+}\partial_{++}\right)\mathscr{F}+\mathrm{i}\overline{\xi}^{+}\left(-\mathrm{i}\overline{\partial}_{+}+\theta^{+}\partial_{++}\right)\mathscr{F}\ , \tag{4.14}$$

while when $y,\xi,\overline{\xi}$ are infinitesimal, the left-hand side equals

$$\mathrm{LHS}=\mathscr{F}-\mathrm{i}yP\cdot\mathscr{F}-\mathrm{i}\xi^{+}\mathcal{Q}_{+}\cdot\mathscr{F}+\mathrm{i}\overline{\xi}^{+}\overline{\mathcal{Q}}_{+}\cdot\mathscr{F}. \tag{4.15}$$

Comparing two sides, we obtain the differential operators associated with the generators

$$P_{\pm\pm}=\mathrm{i}\partial_{\pm\pm},\quad \mathcal{Q}_{+}=\mathrm{i}\partial_{+}-\overline{\theta}^{+}\partial_{++},\quad \overline{\mathcal{Q}}_{+}=-\mathrm{i}\overline{\partial}_{+}+\theta^{+}\partial_{++}\ . \tag{4.16}$$

And as usual, the right action on the superfields gives the super-derivative

$$\mathcal{D}_+ = \partial_+ - \mathrm{i}\overline{\theta}^+\partial_{++}, \quad \overline{\mathcal{D}}_+ = -\overline{\partial}_+ + \mathrm{i}\theta^+\partial_{++} \ , \tag{4.17}$$

up to an overall i.

Generically, the superfield $\mathscr{F}(x,\theta,\overline{\theta})$ can be directly expanded as

$$\mathscr{F}(x^{\pm\pm},\theta^+,\overline{\theta}^+) = f(x^{\pm\pm}) + \theta^+ g_+(x^{\pm\pm}) - \overline{\theta}^+\overline{g}_+(x^{\pm\pm}) + \theta^+\overline{\theta}^+ h_{++}(x^{\pm\pm}), \tag{4.18}$$

for some component fields $f, g_+, \overline{g}_+, h_{++}$. Under the supersymmetry transformation

$$\delta \equiv -\mathrm{i}\xi^+\mathcal{Q}_+ + \mathrm{i}\overline{\xi}^+\overline{\mathcal{Q}}_+ \ , \tag{4.19}$$

the transformations of the components can be directly calculated to be

$$\begin{aligned} \delta f &= \xi^+ g_+ - \overline{\xi}^+\overline{g}_+ \ , \\ \delta g_+ &= \overline{\xi}^+\left(-\mathrm{i}\partial_{++} f + h_{++}\right) \ , \\ \delta \overline{g}_+ &= \xi^+\left(\mathrm{i}\partial_{++} f + h_{++}\right) \ , \\ \delta h_{++} &= -\mathrm{i}\partial_{++}\left(\xi^+ g_+ + \overline{\xi}^+\overline{g}_+\right) \ , \end{aligned} \tag{4.20}$$

where we see that the top term h_{++} transforms by a total derivatives, thus can be used to construct supersymmetrically invariant terms of a Lagrangian.

We are mainly interested in the following specific cases.

$\mathcal{N}=(0,2)$ **(anti-)chiral multiplet** The superfield Φ satisfying

$$\overline{\mathcal{D}}_+\Phi = 0 \tag{4.21}$$

is called an $\mathcal{N}=(0,2)$ *chiral multiplet*, and the *anti-chiral multiplets* are defined by $\mathcal{D}_+\overline{\Phi} = 0$[4]. Evidently, the conjugate of a chiral multiplet is an anti-chiral multiplet, so we concentrate on the chiral multiplets in the following.

Shift the x^{++} to $y^{++} \equiv x^{++} - \mathrm{i}\theta^+\overline{\theta}^+$. Since $\overline{\mathcal{D}}_+ y^{++} = 0$, therefore, in y-coordinate, $\overline{\mathcal{D}}_+^{(y)} = \overline{\partial}_+^{(y)}$ and thus a chiral multiplet Φ can be expanded as

$$\begin{aligned} \Phi(x^{\pm\pm},\theta^+,\overline{\theta}^+) = \Phi(y^{++},x^{--},\theta^+) &= \varphi(y^{++},x^{--}) + \theta^+\psi_+(y^{++},x^{--}) \\ &= \varphi(x^{\pm\pm}) + \theta^+\psi_+(x^{\pm\pm}) - \mathrm{i}\theta^+\overline{\theta}^+\partial_{++}\varphi(x^{\pm\pm}) \ , \end{aligned} \tag{4.22}$$

for a complex scalar φ and a right-moving complex Weyl spinor ψ_+. Its conjugate is an anti-chiral multiplet:

$$\overline{\Phi}(x^{\pm\pm},\theta^+,\overline{\theta}^+) = \overline{\varphi}(y^{++},x^{--}) - \overline{\theta}^+\overline{\psi}_+(y^{++},x^{--}) = \overline{\varphi}(x^{\pm\pm}) - \overline{\theta}^+\overline{\psi}_+ + \mathrm{i}\theta^+\overline{\theta}^+\partial_{++}\overline{\varphi}(x^{\pm\pm}) \ . \tag{4.23}$$

[4]Note that, for complex Weyl spinors, $\psi_+, \overline{\psi}_+$ are right-moving, $\psi_-, \overline{\psi}_-$ are left-moving. $\psi_\pm$ are chiral, $\overline{\psi}_\pm$ are anti-chiral.

$\mathcal{N} = (0,2)$ **Fermi-multiplet** Just like the $\mathcal{N} = (0,1)$ case, in order to introduce left-moving fermions, we define the $\mathcal{N} = (0,2)$ *(chiral) Fermi multiplet*, denoted as Γ_-, by imposing the chiral condition (4.21) but require Γ_- to be a left-moving spinor instead of a scalar

$$\begin{aligned}\Gamma_-(x^{\pm\pm}, \theta^+, \overline{\theta}^+) = \Gamma_-(y^{++}, x^{--}, \theta^+) &= \psi_-(y^{++}, x^{--}) + \theta^+ G(y^{++}, x^{--}) \\ &= \psi_-(x^{\pm\pm}) + \theta^+ G(x^{\pm\pm}) - \mathrm{i}\theta^+\overline{\theta}^+ \partial_{++}\psi_-(x^{\pm\pm}) \ ,\end{aligned} \tag{4.24}$$

for complex Weyl fermion ψ_- and complex auxiliary scalar G.

We usually work with a generalization of the Fermi multiplet by generalizing the chiral condition (4.21) to define a $E-$*Fermi multiplet* by

$$\overline{\mathcal{D}}_+ \Gamma_- = E, \tag{4.25}$$

where E is another chiral multiplet. Suppose E has components

$$E = E + \theta^+ E_+ - \mathrm{i}\theta^+\overline{\theta}^+ \partial_{++} E \ , \tag{4.26}$$

then the solution of Γ_- is

$$\Gamma_- = \psi_- + \theta^+ G - \overline{\theta}^+ E + \theta^+\overline{\theta}^+ (E_+ - \mathrm{i}\partial_{++}\psi_-) \tag{4.27}$$

Typically, $E = E(\Phi)$ is taken as a holomorphic function of the chiral multiplets, and then

$$\begin{aligned} E(\Phi) &= E(\varphi) + \theta^+ \psi_+ \frac{\partial E}{\partial \varphi} - \mathrm{i}\theta^+\overline{\theta}^+ \partial_{++} E(\varphi) \ , \\ \Gamma_- &= \psi_- + \theta^+ G - \overline{\theta}^+ E(\varphi) + \theta^+\overline{\theta}^+ \left(\psi_+ \frac{\partial E}{\partial \varphi} - \mathrm{i}\partial_{++}\psi_-\right) \end{aligned} \tag{4.28}$$

$\mathcal{N} = (0,2)$ **vector multiplet** In order to properly define gauge invariant action, we need to define vector multiplets, living in the gauge adjoint representation. Define the $\mathcal{N} = (0,2)$ *vector multiplet* as two superfields (See [1, 9–11])

$$\begin{aligned} \Psi(x^{\pm\pm}, \theta^+, \overline{\theta}^+) &= v(x^{\pm\pm}) + \theta^+ \zeta_+(x^{\pm\pm}) - \overline{\theta}^+ \overline{\zeta}_+(x^{\pm\pm}) - \theta^+\overline{\theta}^+ A_{++}(x^{\pm\pm}) \ , \\ V_{--}(x^{\pm\pm}, \theta^+, \overline{\theta}^+) &= A_{--}(x^{\pm\pm}) + \theta^+ \overline{\lambda}_-(x^{\pm\pm}) - \overline{\theta}^+ \lambda_-(x^{\pm\pm}) + \theta^+\overline{\theta}^+ D(x^{\pm\pm}) \ , \end{aligned} \tag{4.29}$$

where v, D are real scalars, $\zeta, \overline{\zeta}, \lambda, \overline{\lambda}$ are complex Weyl spinors, and $A_{\pm\pm}$ are the ordinary 2d gauge fields. All these fields live in the adjoint representation of some gauge group G.

We define the covariant super-derivatives as

$$\mathscr{D}_+ \equiv e^{-\Psi} \mathcal{D}_+ e^{\Psi} \ , \quad \overline{\mathscr{D}}_+ \equiv e^{\Psi} \overline{\mathcal{D}}_+ e^{-\Psi} \ , \quad \mathscr{D}_{--} \equiv \partial_{--} - \mathrm{i}V_{--} \ , \tag{4.30}$$

where component fields are defaultly in x-coordinates.

As usual, we define the gauge transforms of chiral multiplets and Fermi multiplets as

$$\Phi \mapsto e^{-\mathrm{i}\Omega} \cdot \Phi \ , \quad \Gamma_- \mapsto e^{-\mathrm{i}\Omega} \cdot \Gamma_- \ , \tag{4.31}$$

for some chiral multiplet

$$\Omega(x^{\pm\pm}, \theta^+, \overline{\theta}^+) = \alpha(x^{\pm\pm}) + \theta^+ \beta_+(x^{\pm\pm}) - \mathrm{i}\theta^+\overline{\theta}^+ \partial_{++}\alpha(x^{\pm\pm}) \ , \tag{4.32}$$

valued in the complexified gauge Lie algebra.

However, to define the gauge transforms of the vector multiplets, instead of the way (2.24), it is more conventional to define by imposing $(e^{-\Psi}\mathscr{D}_+ e^{\Psi})\Phi$ and $(e^{-\Psi}\mathscr{D}_{--}e^{\Psi})\Phi$ to transform the same way as Φ, and $\overline{\Phi}(e^{\Psi}\overline{\mathscr{D}}_+ e^{-\Psi})$ to transform as $\overline{\Phi}$

$$e^{-\Psi}\mathscr{D}_+ e^{\Psi} \mapsto e^{-\mathrm{i}\Omega}(e^{-\Psi}\mathscr{D}_+ e^{\Psi})e^{\mathrm{i}\Omega} , \quad e^{\Psi}\overline{\mathscr{D}}_+ e^{-\Psi} \mapsto e^{-\mathrm{i}\overline{\Omega}}(e^{\Psi}\overline{\mathscr{D}}_+ e^{-\Psi})e^{\mathrm{i}\overline{\Omega}} , \quad e^{-\Psi}\mathscr{D}_{--}e^{\Psi} \mapsto e^{-\mathrm{i}\Omega}(e^{-\Psi}\mathscr{D}_{--}e^{\Psi})e^{\mathrm{i}\Omega} . \tag{4.33}$$

Then it is sufficient to impose

$$e^{2\Psi} \mapsto e^{-\mathrm{i}\overline{\Omega}}e^{2\Psi}e^{\mathrm{i}\Omega} , \quad e^{-\Psi}\mathscr{D}_{--}e^{\Psi} \mapsto e^{-\mathrm{i}\Omega}(e^{-\Psi}\mathscr{D}_{--}e^{\Psi})e^{\mathrm{i}\Omega} . \tag{4.34}$$

It is illuminating to consider the abelian case first, where Ψ transforms by

$$\Psi \mapsto \Psi + \frac{\mathrm{i}}{2}\left(\Omega - \overline{\Omega}\right) , \tag{4.35}$$

which in components means

$$\delta_\Omega \Psi = \frac{\mathrm{i}}{2}\left(\alpha - \overline{\alpha}\right) + \frac{\mathrm{i}}{2}\left(\theta^+\beta_+ + \overline{\theta}^+\overline{\beta}_+\right) + \frac{1}{2}\theta^+\overline{\theta}^+\partial_{++}\left(\alpha + \overline{\alpha}\right) . \tag{4.36}$$

Similar to the 4d case, we can partially fix the gauge, called the *Wess-Zumino gauge*, by imposing

$$v = \zeta_+ = \overline{\zeta}_+ = 0 , \tag{4.37}$$

and therefore

$$\Psi = -\theta^+\overline{\theta}^+ A_{++} . \tag{4.38}$$

Then according to (4.36), the residue gauge transform has $\alpha = \overline{\alpha}$, $\beta_+ = \overline{\beta}_+ = 0$

$$\Omega(x^{\pm\pm}, \theta^+, \overline{\theta}^+) = \alpha(x^{\pm\pm}) - \mathrm{i}\theta^+\overline{\theta}^+\partial_{++}\alpha(x^{\pm\pm}) , \tag{4.39}$$

which is the ordinary gauge transformation and A_{++} transforms by the familiar form

$$A_{++} \mapsto A_{++} - \partial_{++}\alpha . \tag{4.40}$$

As a remark, recall the original gauge transformation (4.32) is valued in the complexified gauge algebra. The Wess-Zumino gauge breaks the complexified gauge group into real gauge algebra.

Back to the non-abelian case, the abelian analysis motivates us to still impose the Wess-Zumino gauge (4.37), and consider the residual gauge transformation (4.39) then we can expand

$$\begin{aligned} e^{2\Psi} &= \exp\left(-2\theta^+\overline{\theta}^+ A_{++}\right) = 1 - 2\theta^+\overline{\theta}^+ A_{++} , \\ e^{\mathrm{i}\Omega} &= \exp\left(\mathrm{i}\alpha + \theta^+\overline{\theta}^+\partial_{++}\alpha\right) = (1 - \mathrm{i}\theta^+\overline{\theta}^+\partial_{++})e^{\mathrm{i}\alpha} . \end{aligned} \tag{4.41}$$

And useful property is that

$$e^{\Psi}e^{\Psi} = e^{2\Psi}. \tag{4.42}$$

Therefore

$$\begin{aligned} e^{-\mathrm{i}\overline{\Omega}}e^{2\Psi}e^{\mathrm{i}\Omega} &= (1 + \mathrm{i}\theta^+\overline{\theta}^+\partial_{++})e^{-\mathrm{i}\alpha}(1 - 2\theta^+\overline{\theta}^+ A_{++})(1 - \mathrm{i}\theta^+\overline{\theta}^+\partial_{++})e^{\mathrm{i}\alpha} \\ &= 1 - 2\theta^+\overline{\theta}^+\left(e^{-\mathrm{i}\alpha}A_{++}e^{\mathrm{i}\alpha} + \mathrm{i}e^{-\mathrm{i}\alpha}\partial_{++}e^{\mathrm{i}\alpha}\right) , \end{aligned} \tag{4.43}$$

which implies

$$A_{++} \mapsto e^{-\mathrm{i}\alpha A_{++}} e^{\mathrm{i}\alpha} + \mathrm{i} e^{-\mathrm{i}\alpha} \partial_{++} e^{\mathrm{i}\alpha} \tag{4.44}$$

as the non-abelian generalization of (4.40), and we see that the Wess-Zumino gauge is indeed preserved under the residual gauge transform.

In Wess-Zumino gauge, the covariant super-derivatives (4.30) can then be expanded as

$$\begin{aligned} &\mathscr{D}_+ = \partial_+ - \mathrm{i}\overline{\theta}^+ \nabla_{++} \ , \quad \overline{\mathscr{D}}_+ = -\overline{\partial}_+ + \mathrm{i}\theta^+ \nabla_{++} \ , \\ &\mathscr{D}_{--} = \nabla_{--} - \mathrm{i}\theta^+ \overline{\lambda}_- + \mathrm{i}\overline{\theta}^+ \lambda_- - \mathrm{i}\theta^+ \overline{\theta}^+ D \ , \end{aligned} \tag{4.45}$$

and the e^{Ψ} conjugations are

$$\begin{aligned} e^{-\Psi} \mathscr{D}_+ e^{\Psi} &= \partial_+ - \mathrm{i}\overline{\theta}^+ \nabla_{++} - \overline{\theta}^+ A_{++} \ , \quad e^{\Psi} \overline{\mathscr{D}}_+ e^{-\Psi} = -\overline{\partial}_+ + \mathrm{i}\theta^+ \nabla_{++} + \theta^+ A_{++} \ , \\ e^{-\Psi} \mathscr{D}_{--} e^{\Psi} &= \nabla_{--} - \mathrm{i}\theta^+ \overline{\lambda}_- + \mathrm{i}\overline{\theta}^+ \lambda_- - \mathrm{i}\theta^+ \overline{\theta}^+ D - \theta^+ \overline{\theta}^+ \nabla_{--} A_{++} \ , \end{aligned} \tag{4.46}$$

Next, let's consider the gauge transform of $\mathscr{D}_{--}$ in (4.34) in Wess-Zumino gauge[5]:

$$\begin{aligned} e^{-\Psi} \mathscr{D}_{--} e^{\Psi} \mapsto\ & e^{-\mathrm{i}\Omega} e^{-\Psi} \mathscr{D}_{--} e^{\Psi} e^{\mathrm{i}\Omega} \\ &= (1 - \mathrm{i}\theta^+ \overline{\theta}^+ \partial_{++}) e^{-\mathrm{i}\alpha} e^{-\Psi} \mathscr{D}_{--} e^{\Psi} (1 - \mathrm{i}\theta^+ \overline{\theta}^+ \partial_{++}) e^{\mathrm{i}\alpha} \\ &= e^{-\mathrm{i}\alpha} \nabla_{--} e^{\mathrm{i}\alpha} - \mathrm{i}\theta^+ e^{-\mathrm{i}\alpha} \lambda_- e^{\mathrm{i}\alpha} + \mathrm{i}\overline{\theta}^+ e^{-\mathrm{i}\alpha} \overline{\lambda}_- e^{\mathrm{i}\alpha} - \mathrm{i}\theta^+ \overline{\theta}^+ e^{-\mathrm{i}\alpha} D e^{\mathrm{i}\alpha} \\ &\quad - \theta^+ \overline{\theta}^+ (e^{-\mathrm{i}\alpha} \nabla_{--} e^{\mathrm{i}\alpha})(e^{-\mathrm{i}\alpha} A_{++} e^{\mathrm{i}\alpha} + \mathrm{i} e^{-\mathrm{i}\alpha} \partial_{++} e^{\mathrm{i}\alpha}) \ , \end{aligned} \tag{4.47}$$

and thus we see that

$$A_{\pm\pm} \mapsto e^{-\mathrm{i}\alpha} A_{\pm\pm} e^{\mathrm{i}\alpha} + \mathrm{i} e^{-\mathrm{i}\alpha} \partial_{\pm\pm} e^{\mathrm{i}\alpha} \ , \tag{4.48}$$

and the other component fields $\lambda_-, \overline{\lambda}_-, D$ transform adjointly.

In gauge theory, it is more convenient to use a covariant version of chiral and Fermi multiplets (See [10]):

$\mathcal{N} = (0,2)$ **covariant multiplets** Replace the super-derivative in the chiral condition (4.21) with covariant super-derivative:

$$\overline{\mathscr{D}}_+ \mathbf{\Phi} = 0 \ . \tag{4.49}$$

We can easily construct a covariant chiral multiplet $\mathbf{\Phi}$ from an ordinary chiral multiplet Φ defined in (4.21) by $\mathbf{\Phi} := e^{\Psi}\Phi$, which evidently satisfies (4.49). Evidently, if Φ has components (4.22), the components of $\mathbf{\Phi}$ is to simply substitute the ∂_{++} in (4.22) by ∇_{++}:

$$\mathbf{\Phi}(x^{\pm\pm}, \theta^+, \overline{\theta}^+) = \varphi(x^{\pm\pm}) + \theta^+ \psi_+(x^{\pm\pm}) - \mathrm{i}\theta^+ \overline{\theta}^+ \nabla_{++} \varphi(x^{\pm\pm}) \ . \tag{4.50}$$

Similarly, we can define a covariant E-Fermi multiplet by

$$\overline{\mathscr{D}}_+ \mathbf{\Gamma}_- = \boldsymbol{E} \ , \tag{4.51}$$

where $\boldsymbol{E} = E + \theta^+ E_+ - 2\mathrm{i}\theta^+ \overline{\theta}^+ \nabla_{++} E$ is some covariant chiral multiplet, and the solution of $\mathbf{\Gamma}_-$ is

$$\mathbf{\Gamma}_- = \psi_- + \theta^+ G - \overline{\theta}^+ E + \theta^+ \overline{\theta}^+ \left(E_+ - \mathrm{i}\nabla_{++} \psi_-\right) \ . \tag{4.52}$$

The covariant version $\mathbf{\Gamma}_-$ can be obtained by simply adding a e^{Ψ} prefactor to both Γ_- and E.

[5]Notice that $(e^{-\mathrm{i}\alpha} \nabla_{--} e^{\mathrm{i}\alpha})(e^{-\mathrm{i}\alpha} A_{++} e^{\mathrm{i}\alpha} + \mathrm{i} e^{-\mathrm{i}\alpha} \partial_{++} e^{\mathrm{i}\alpha}) \neq e^{-\mathrm{i}\alpha} \nabla_{--} (A_{++} e^{\mathrm{i}\alpha} + \mathrm{i}\partial_{++} e^{\mathrm{i}\alpha})$.

$\mathcal{N}=(0,2)$ **superfield strength** Now we can define the *superfield strength* multiplet

$$\boldsymbol{\Upsilon}_- \equiv \left[\overline{\mathscr{D}}_{--}, \overline{\mathscr{D}}_+\right] , \quad \overline{\boldsymbol{\Upsilon}}_- \equiv \left[\mathscr{D}_+, \mathscr{D}_{--}\right] \ . \tag{4.53}$$

In components

$$\begin{aligned} \boldsymbol{\Upsilon}_- &= \mathrm{i}\lambda_- - \theta^+\left(F - \mathrm{i}D\right) + \theta^+\overline{\theta}^+\nabla_{++}\lambda_- \ , \\ \overline{\boldsymbol{\Upsilon}}_- &= -\mathrm{i}\overline{\lambda}_- - \overline{\theta}^+\left(F + \mathrm{i}D\right) + \theta^+\overline{\theta}^+\nabla_{++}\overline{\lambda}_- \ , \end{aligned} \tag{4.54}$$

where $F \equiv \partial_{++}A_{--} - \partial_{--}A_{++} - \mathrm{i}[A_{++}, A_{--}]$ is the ordinary field strength. Note that A_{++} acts on $\lambda_-, \overline{\lambda}_-$ by adjoint action. From the components, we see that the field strengths are covariant Fermi multiplets living in the gauge adjoint representation.

Decompose $(0,2)$ multiplets into $(0,1)$ multiplets

The decomposition of $(0,2)$ multiplets into $(0,1)$ multiplets depends on the choice of $(0,1)$ section of supersymmetry algebra. In this note, we take the section to be determined by (4.1) and then the θ parameters are related by (4.8). [12] provides a good analysis about these decompositions.

There are two strategies for doing the decomposition. First, for a $(0,2)$ multiplet $\mathscr{F}(x, \theta^+, \overline{\theta}^+)$, the compositing $(0,1)$ multiplets with respect to Q_{1+} (or Q_{2+}) can be obtained from taking θ^{2+} (or θ^{2+}) to zero. Second, try to write $(0,2)$ multiplets into "nested forms", meaning that, a form of $(0,1)$ multiplet, but each component is another $(0,1)$ multiplet with respect to the other supercharge. This is how we decompose a 4d $\mathcal{N}=2$ vector multiplet into $\mathcal{N}=1$ multiplets.

need to check the explicit susy transform

The second method can give the full decomposition, but does not apply to all cases, for example, for 4d $\mathcal{N}=2$ hypermultiplets. On the other hand, the first method might not give the full decomposition.

for $(0,2)$ chiral multiplet The $(0,2)$ chiral multiplet (4.22) has complex components, which can be expanded as

$$\varphi = \varphi_1 + \mathrm{i}\varphi_2 \ , \quad \psi_+ = \psi_{1+} + \mathrm{i}\psi_{2+} \ , \tag{4.55}$$

with real scalars φ_1, φ_2 and Majorana-Weyl fermions ψ_{1+}, ψ_{2+}. In terms of $(0,1)$ theta parameters, (4.22) then becomes (in y-coordinate)

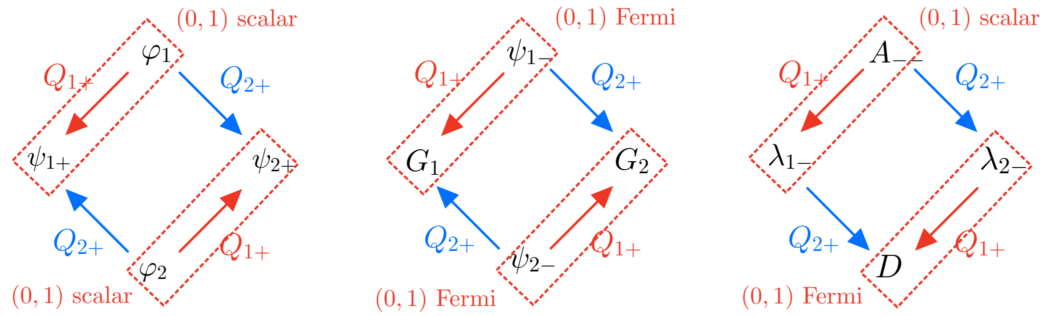
$$\Phi = \varphi_1 + \frac{\mathrm{i}}{2}\left(\theta^{1+}\psi_{1+} + \theta^{2+}\psi_{2+}\right) + \mathrm{i}\left\{\varphi_2 + \frac{\mathrm{i}}{2}\left(\theta^{1+}\psi_{2+} - \theta^{2+}\psi_{1+}\right)\right\} \ . \tag{4.56}$$

By taking θ^{2+} or θ^{1+} to zero, we obtain

$$\begin{aligned} \Phi &\overset{\theta^{2+}\to 0}{=} \left(\varphi_1 + \frac{\mathrm{i}}{2}\theta^{1+}\psi_{1+}\right) + \mathrm{i}\left(\varphi_2 + \frac{\mathrm{i}}{2}\theta^{1+}\psi_{2+}\right) \\ &\overset{\theta^{1+}\to 0}{=} \left(\varphi_1 + \frac{\mathrm{i}}{2}\theta^{2+}\psi_{2+}\right) + \mathrm{i}\left(\varphi_2 - \frac{\mathrm{i}}{2}\theta^{2+}\psi_{1+}\right) \ . \end{aligned} \tag{4.57}$$

Therefore, we see that if $\theta^{2+}=0$, φ_1 and ψ_{1+} form a $(0,1)$ scalar multiplet, so do φ_2 and ψ_{2+}. On the other hand, if we take $\theta^{1+}=0$, we will see that, φ_1 and ψ_{2+} form a $(0,1)$ scalar multiplet, so do φ_2 and ψ_{1+}.

In summary, a $(0,2)$ chiral multiplet (φ, ψ_+) can be decomposed into two $(0,1)$ scalar multiplets, see Figure 1.



$(0,2)$ chiral (φ, ψ_+) $(0,2)$ Fermi (ψ_-, G) $(0,2)$ vector $(A_{\pm\pm}, \lambda_-, D)$

Figure 1. Decomposing (0,2) multiplets into (0,1) multiplets (with respect to Q_{1+}).

for $(0,2)$ **Fermi multiplet** For simplicity, here assume $E = 0$, then the analysis is the same as for a chiral multiplet. An $(0,2)$ Fermi multiplet (ψ_-, G) can be decomposed into two $(0,1)$ Fermi multiplets, see Figure 1, where the complex components are decomposed into real components

$$\psi_- = \psi_{1+} + \mathrm{i}\psi_{2+} \ , \quad G = G_1 + \mathrm{i}G_2 \ . \tag{4.58}$$

for $(0,2)$ **vector multiplet** In Wess-Zumino gauge, it is evident that Ψ of (4.29) is simply the V_+ of the $(0,1)$ vector multiplet. For V_{--}, it can be written as the "nested form"

$$V_{--} = \left(A_{--} + \theta^{1+}\lambda_{1-}\right) - \theta^{2+}\left(\lambda_{2-} - 2\mathrm{i}\theta^{1+}D\right) \ , \tag{4.59}$$

with

$$\lambda_- = \lambda_{1-} + \mathrm{i}\lambda_{2-} \tag{4.60}$$

for Majorana-Weyl $\lambda_{1-}, \lambda_{2-}$. Therefore, a $(0,2)$ vector multiplet $(A_{\pm\pm}, \lambda_-, \overline{\lambda}_-, D)$ can be decomposed into a $(0,1)$ vector multiplet $(A_{\pm\pm}, \lambda_{1-})$ and a $(0,1)$ Fermi multiplet (λ_{2-}, D), see Figure 1.

4.2 $\mathcal{N} = (0,2)$ Lagrangians

There are two ways to write down (0,2)-SUSY invariant Lagrangian terms:

D-terms Integrate over the all super-directions:

$$\mathcal{L}_{(D)} = \int d\theta^+ d\overline{\theta}^+ \mathcal{K}_{(D)} \ , \tag{4.61}$$

for a real scalar (0,2) superfield $\mathcal{K}_{(D)}$ being a function of the compositing superfields in this theory.

F-terms Integrate over the half of the super-directions:

$$\mathcal{L}_{(F)} = \int d\theta^+ d\mathcal{W}_{(F)}|_{\overline{\theta}^+=0} + \text{c.c.} \ , \tag{4.62}$$

for a real scalar (0,2) chiral superfield $\mathcal{W}_{(D)}$ being a holomorphic function of the chiral superfields in this theory.

4.2.1 (0,2) gauge theory

Now let's construct the gauge invariant Lagrangian terms for the multiplets defined in the previous section. Assume there are n_Φ chiral multiplets Φ_i in gauge representation $\mathfrak{R}_i^\Phi$, n_Γ Fermi multiplets $\Gamma_{-,i}$ in $\mathfrak{R}_i^\Gamma$ together with a vector multiplet (Ψ, V_{--}) in the adjoint representation.

D-terms The simplest gauge invariant kinetic terms for chiral multiplets Φ is

$$\mathcal{L}_{\Phi,\text{kin}} = -2\mathrm{i} \int d\theta^+ d\overline{\theta}^+ \sum_{i=1}^{n_\Phi} \overline{\boldsymbol{\Phi}}_i \mathscr{D}_{--} \boldsymbol{\Phi}_i \ , \tag{4.63}$$

where the right-hand side is the expression in terms of covariant chiral multiplet $\boldsymbol{\Phi} \equiv e^\Psi \Phi$. The gauge invariance is a result of (4.34) and

$$\overline{\boldsymbol{\Phi}}_i \mathscr{D}_{--} \boldsymbol{\Phi}_i = \overline{\Phi}_i e^\Psi \mathscr{D}_{--} e^\Psi \Phi_i = \overline{\Phi}_i e^{2\Psi} e^{-\Psi} \mathscr{D}_{--} e^\Psi \Phi_i \ , \tag{4.64}$$

where the property (4.42) is used. In component form (much more easier to expand using covariant multiplets)

$$\begin{aligned} \mathcal{L}_{\Phi,\text{kin}} = \sum_{i=1}^{n_\Phi} \Big(& 4(\nabla_{++}\varphi_i)^\dagger \nabla_{--}\varphi_i + 2\mathrm{i}\overline{\psi}_{+,i}\nabla_{--}\psi_{+,i} \\ & + 2\overline{\psi}_{+,i}\overline{\lambda}_-\varphi_i + 2\overline{\varphi}_i\lambda_-\psi_{+,i} - 2\overline{\varphi}_i D\varphi_i \Big) \ . \end{aligned} \tag{4.65}$$

For Fermi multiplets, the kinetic terms can be constructed as

$$\begin{aligned} \mathcal{L}_{\Gamma,\text{kin}} = & - \int d\theta^+ d\overline{\theta}^+ \sum_{i=1}^{n_\Gamma} \overline{\Gamma}_{-,i} e^{2\Psi} \Gamma_{-,i} \equiv - \int d\theta^+ d\overline{\theta}^+ \sum_{i=1}^{n_\Gamma} \overline{\boldsymbol{\Gamma}}_{-,i} \boldsymbol{\Gamma}_{-,i} = \\ & \sum_{i=1}^{n_\Gamma} \left(2\mathrm{i}\overline{\psi}_{-,i}\nabla_{++}\psi_{-,i} + |G_i|^2 - |E_i|^2 - \overline{\psi}_{-,i}E_{+,i} - \overline{E}_{+,i}\psi_{-,i} \right) \ . \end{aligned} \tag{4.66}$$

Typically, the $\boldsymbol{E}_i$ multiplets are holomorphic functions of the covariant chiral multiplets in the theory

$$\boldsymbol{E}_i = E_i(\boldsymbol{\Phi}) = E_i(\varphi) + \theta^+ \sum_{j=1}^{n_\Phi} \frac{\partial E_i(\varphi)}{\partial \varphi_j} \psi_{+,j} - \mathrm{i}\theta^+ \overline{\theta}^+ \nabla_{++} E_i(\varphi) \ , \tag{4.67}$$

and thus the kinetic term becomes

$$\mathcal{L}_{\Gamma,\text{kin}} = \sum_{i=1}^{n_\Gamma} \left(2\mathrm{i}\overline{\psi}_{-,i}\nabla_{++}\psi_{-,i} + |G_i|^2 - |E_i(\varphi)|^2 - \sum_{j=1}^{n_\Phi} \left(\overline{\psi}_{-,i}\psi_{+,j} \frac{\partial E_i(\varphi)}{\partial \varphi_j} + \text{c.c.} \right) \right) \ . \tag{4.68}$$

Since the superfield strength is a covariant Fermi multiplet, so the gauge kinetic term can be naturally defined as

$$\mathcal{L}_{\Upsilon,\text{kin}} = -\frac{4}{g^2} \int d\theta^+ d\overline{\theta}^+ \,\mathrm{tr}\, \overline{\boldsymbol{\Upsilon}}_- \boldsymbol{\Upsilon}_- = \frac{4}{g^2} \,\mathrm{tr} \left(2\mathrm{i}\overline{\lambda}_-\nabla_{++}\lambda_- + F^2 + D^2 \right) \ . \tag{4.69}$$

F-terms Apart from D-terms, there can be superpotential term (or called J-term):

$$\mathcal{L}_J = \int d\theta^+ \sum_{i=1}^{n_\Gamma} \mathbf{\Gamma}_{-,i} J^i(\mathbf{\Phi})|_{\overline{\theta}=0} + \text{c.c.} \tag{4.70}$$

where $J^i(\mathbf{\Phi})$ is a holomorphic function in dual representation of $\mathbf{\Gamma}_{-,i}$. In components

$$\begin{aligned}\sum_{i=1}^{n_\Gamma} \mathbf{\Gamma}_{-,i} J^i(\mathbf{\Phi}) = \sum_{i=1}^{n_\Gamma} \psi_{-,i} J^i(\varphi) + \theta^+ \sum_{i=1}^{n_\Gamma} \left(G_i J^i(\varphi) + \sum_{j=1}^{n_\Phi} \psi_{+,j}\psi_{-,i} \frac{\partial J^i(\varphi)}{\partial \varphi_j} \right) - \overline{\theta}^+ \sum_{i=1}^{n_\Gamma} E_i(\varphi) J^i(\varphi) \\ + \theta^+\overline{\theta}^+ \sum_{i=1}^{n_\Gamma} \left(-\mathrm{i}\nabla_{++}(\psi_{-,i}J^i(\varphi)) + \sum_{j=1}^{n_\Phi} \psi_{+,j} \frac{\partial}{\partial\varphi_j} \left(E_i(\varphi)\, J^i(\varphi)\right) \right) \ ,\end{aligned} \tag{4.71}$$

In order to maintain the supersymmetry, it ought to be a $(0,2)$ multiplet. Here it can only be a chiral multiplet. And we can easily see that, it is a chiral multiplet only if the $E \cdot J$ *constraint* is satisfied:

$$\sum_{i=1}^{n_\Gamma} E_i(\varphi) J^i(\varphi) = 0 \ . \tag{4.72}$$

Then we have

$$\mathcal{L}_J = \sum_{i=1}^{n_\Gamma} \left(G_i J^i(\varphi) + \sum_{j=1}^{n_\Phi} \psi_{+,j}\psi_{-,i} \frac{\partial J^i(\varphi)}{\partial\varphi_j} \right) + \text{c.c.} \ , \tag{4.73}$$

FI-terms Finally, for abelian theory, there can also be an *FI-term*

$$\mathcal{L}_{FI} = -\frac{t}{2}\int d\theta^+ \mathbf{\Upsilon}_-|_{\overline{\theta}^+=0} + \text{c.c.} = \frac{t}{2}(F - \mathrm{i}D) + \text{c.c.} = \frac{\vartheta}{2\pi}F + rD \ , \tag{4.74}$$

with FI parameter $t = \frac{\vartheta}{2\pi} + \mathrm{i}r$.

Full Lagrangian The full Lagrangian can then be written as

$$\mathcal{L} = \mathcal{L}_{\Phi,\text{kin}} + \mathcal{L}_{\Gamma,\text{kin}} + \mathcal{L}_{\Upsilon,\text{kin}} + \mathcal{L}_J + \mathcal{L}_{FI} \ , \tag{4.75}$$

After eliminating the auxiliary fields, the Lagrangian becomes

$$\begin{aligned}\mathcal{L} = \sum_{i=1}^{n_\Phi} \Big(4(\nabla_{++}\varphi_i)^\dagger \nabla_{--}\varphi_i + 2\mathrm{i}\overline{\psi}_{+,i}\nabla_{--}\psi_{+,i} + 2\overline{\psi}_{+,i}\overline{\lambda}_-\varphi_i + 2\overline{\varphi}_i\lambda_-\psi_{+,i} \Big) + \frac{4}{g^2}\,\mathrm{tr}\left(2\mathrm{i}\overline{\lambda}_-\nabla_{++}\lambda_- + F^2\right) \\ + \sum_{i=1}^{n_\Gamma} \left(2\mathrm{i}\overline{\psi}_{-,i}\nabla_{++}\psi_{-,i} - |J^i(\varphi)|^2 - |E_i(\varphi)|^2\right) + \left[\sum_{i=1}^{n_\Gamma}\sum_{j=1}^{n_\Phi} \left(\psi_{+,j}\psi_{-,i}\frac{\partial J^i(\varphi)}{\partial\varphi_j} - \overline{\psi}_{-,i}\psi_{+,j}\frac{\partial E_i(\varphi)}{\partial\varphi_j} \right) + \text{c.c.} \right] \\ + \frac{\xi}{2\pi}F - \frac{g^2}{4C}\sum_a \left(\sum_{i=1}^{n_\Phi} \overline{\varphi}_i T^a(\mathfrak{R}_i^\Phi)\varphi_i - \frac{r}{2} \right)^2 \ ,\end{aligned} \tag{4.76}$$

and the scalar potential is

$$V(\varphi) = \frac{g^2}{4C}\sum_a \left(\sum_{i=1}^{n_\Phi} \overline{\varphi}_i T^a(\mathfrak{R}_i^\Phi)\varphi_i - \frac{r}{2} \right)^2 + \sum_{i=1}^{n_\Gamma} \left(|J^i(\varphi)|^2 + |E^i(\varphi)|^2 \right) \ , \tag{4.77}$$

where $\{T^a(\mathfrak{R}_i^\Phi)\}$ are the generators of the gauge representation $\mathfrak{R}_i^\Phi$, and $C = \sum_{i=1}^{n_\Phi} C(\mathfrak{R}_i^\Phi)$ is the total Dynkin index.

Anomalies and central charges Since (0,2) theories are chiral, one must pay attention to anomalies. Consider a (0,2) theory endowed with a global symmetry F, characterized by a simple Lie algebra. The associated 't Hooft anomaly coefficient, k_F, is determined via:

$$\mathrm{Tr}\left(\Gamma^3 f^a f^b\right) = k_F \delta^{ab}, \tag{4.78}$$

where f^a denotes the generators of symmetry F, and the trace is computed over Weyl fermions within the theory. Therefore, the evaluations of anomalies involve calculating the discrepancy between contributions from sets of (0,2) chiral and Fermi multiplets:

$$k_F = \sum_{\Phi \in (0,2) \text{ chiral}} C(\mathfrak{R}_F^\Phi) - \sum_{\Gamma \in (0,2) \text{ Fermi}} C(\mathfrak{R}_F^\Gamma), \tag{4.79}$$

with $C(\mathfrak{R}_F)$ representing the Dynkin index of the representation $\mathfrak{R}_F$ under F. Notably, the gauge-invariant field strength within a (0,2) supermultiplet behaves analogously to a Fermi multiplet, and thus gauginos contribute to anomalies when charged under the symmetry F. For example, for SU(N) group, the Dynkin indices are explicitly $C(\square) = \frac{1}{2}$ and $C(\mathbf{adj}) = N$.

Moreover, if the theory includes two distinct U(1) symmetries—$\mathrm{U}(1)_1$ and $\mathrm{U}(1)_2$, with respective charges f_1 and f_2—a mixed 't Hooft anomaly can manifest:

$$k_{12} = \mathrm{Tr}\left(\Gamma^3 f_1 f_2\right). \tag{4.80}$$

Importantly, gauge anomalies must vanish for the theory to be well-defined.

Specifically, the anomaly associated with the $\mathrm{U}(1)_R$-symmetry connects directly to the right-moving central charge c_R via:

$$c_R = 3\,\mathrm{Tr}\left(\Gamma_3 R^2\right). \tag{4.81}$$

To ascertain the charges under the $\mathrm{U}(1)_{R'}$ symmetry, one typically employs c-extremization [13, 14], provided the theory satisfies two critical criteria:

1. The theory must exhibit boundedness, ensuring the energy spectrum is bounded from below.

2. The vacuum moduli space must be compact, yielding normalizable ground-state wavefunctions.

If chiral multiplets parameterize compact directions within the theory, applying c-extremization to these fields is appropriate. On the other hand, for chiral fields describing noncompact directions without constraints from superpotential, the R-charge must be zero. This stems from potential non-holomorphic currents associated with flavor symmetries of noncompact directions. Such currents cannot mix with the R-symmetry current, thus rendering naive applications of c-extremization invalid [14, 15].

Indeed, for the majority of $\mathcal{N} = (0,2)$ gauge theories, vacuum moduli spaces are intrinsically noncompact, with chiral multiplets parameterizing exclusively noncompact directions. Consequently, c-extremization becomes inapplicable in such situations. In these situations, the right-moving central charge is equal to three times the complex dimension of the moduli space, identified as the target space of the infrared non-linear sigma model [16, 17].

4.2.2 (0,2) non-linear sigma models

4.3 (0,2) triality

[18]

4.3.1 (0,2) topological twisted theories

4.4 More topics

(0,2) superconformal theories.

The Landau-Ginzberg model.

5 $\mathcal{N} = (2,2)$

$\mathcal{N} = (2,2)$ supersymmetry requires two copies of right-moving Majorana-Weyl supercharges Q_{A+} and two copies of left-moving Majorana-Weyl supercharges Q_{A-}, $A = 1,2$. Just like the $\mathcal{N} = (0,2)$ cases, they can be assembled into two complex Weyl spinors moving in opposite directions.

$$\begin{aligned} \mathcal{Q}_+ &\equiv Q_{1+} + \mathrm{i}Q_{2+}, \quad \overline{\mathcal{Q}}_+ \equiv Q_{1+} - \mathrm{i}Q_{2+}, \\ \mathcal{Q}_- &\equiv Q_{1-} + \mathrm{i}Q_{2-}, \quad \overline{\mathcal{Q}}_- \equiv Q_{1-} - \mathrm{i}Q_{2-}, \end{aligned} \tag{5.1}$$

subject to the relation

$$\{\mathcal{Q}_\pm, \overline{\mathcal{Q}}_\pm\} = -2P_{\pm\pm} = H \mp P. \tag{5.2}$$

The Lorentz transforms are

$$[M, \mathcal{Q}_\pm] = \mp\frac{\mathrm{i}}{2}\mathcal{Q}_\pm, \quad [M, \overline{\mathcal{Q}}_\pm] = \mp\frac{\mathrm{i}}{2}\overline{\mathcal{Q}}_\pm. \tag{5.3}$$

There are possible non-trivial anti-commutators between supercharges with opposite directions

$$\begin{aligned} \{\overline{\mathcal{Q}}_+, \overline{\mathcal{Q}}_-\} &= \mathcal{Z}, \quad \{\mathcal{Q}_+, \mathcal{Q}_-\} = \mathcal{Z}^*, \\ \{\mathcal{Q}_-, \overline{\mathcal{Q}}_+\} &= \widetilde{\mathcal{Z}}, \quad \{\overline{\mathcal{Q}}_-, \mathcal{Q}_+\} = \widetilde{\mathcal{Z}}^*, \end{aligned} \tag{5.4}$$

for some central charges $\mathcal{Z}, \widetilde{\mathcal{Z}}$.

The maximal R-symmetry is $SO(2)_- \times SO(2)_+ \cong U(1)_- \times U(1)_+$, rotating the supercharges

$$\begin{aligned} U(1)_+ : \quad & \mathcal{Q}_+ \mapsto e^{-\mathrm{i}\alpha}\mathcal{Q}_+, \quad \overline{\mathcal{Q}}_+ \mapsto e^{\mathrm{i}\alpha}\overline{\mathcal{Q}}_+ \ , \\ U(1)_- : \quad & \mathcal{Q}_- \mapsto e^{-\mathrm{i}\alpha}\mathcal{Q}_-, \quad \overline{\mathcal{Q}}_- \mapsto e^{\mathrm{i}\alpha}\overline{\mathcal{Q}}_- \ . \end{aligned} \tag{5.5}$$

However, it is more conventional to define another basis $U(1)_- \times U(1)_+ \cong U(1)_V \times U(1)_A$

$$\begin{aligned} U(1)_V : \quad & \mathcal{Q}_\pm \mapsto e^{-\mathrm{i}\alpha}\mathcal{Q}_\pm, \quad \overline{\mathcal{Q}}_\pm \mapsto e^{\mathrm{i}\alpha}\overline{\mathcal{Q}}_\pm \ , \\ U(1)_A : \quad & \mathcal{Q}_\pm \mapsto e^{\mp\mathrm{i}\alpha}\mathcal{Q}_\pm, \quad \overline{\mathcal{Q}}_\pm \mapsto e^{\pm\mathrm{i}\alpha}\overline{\mathcal{Q}}_\pm \ , \end{aligned} \tag{5.6}$$

i.e. $U(1)_V, U(1)_A$ are the diagonal and anti-diagonal subgroups, and they are called the *vector R-symmetry* and *axial R-symmetry* respectively. Denote their generators as R_V and R_A, then

$$\begin{aligned} [R_V, \mathcal{Q}_\pm] &= -\mathcal{Q}_\pm, \qquad [R_V, \overline{\mathcal{Q}}_\pm] = \overline{\mathcal{Q}}_\pm \ , \\ [R_A, \mathcal{Q}_\pm] &= \mp\mathcal{Q}_\pm, \qquad [R_A, \overline{\mathcal{Q}}_\pm] = \pm\overline{\mathcal{Q}}_\pm \ . \end{aligned} \tag{5.7}$$

We see that $\mathcal{Z}, \mathcal{Z}^*$ carry R_V charges and $\widetilde{\mathcal{Z}}, \widetilde{\mathcal{Z}}^*$ carry R_A charges. As a consequence, $\mathcal{Z}, \mathcal{Z}^*$ must vanish if R_V is conserved, and $\widetilde{\mathcal{Z}}, \widetilde{\mathcal{Z}}^*$ must vanish if R_A is conserved.

5.1 $\mathcal{N}=(2,2)$ superfields

The $\mathcal{N}=(2,2)$ superspace is parametrized by $x^{\pm\pm}, \theta^{\pm}, \overline{\theta}^{\pm}$, with natural derivatives $\partial_{\pm\pm}, \partial_{\pm}, \overline{\partial}_{\pm}$. The multiplication of translations can be easily write down from the $(0,2)$ case (4.12)

$$G(y^{\pm\pm},\xi^{\pm},\overline{\xi}^{\pm})G(x^{\pm\pm},\theta^{\pm},\overline{\theta}^{\pm}) = G\left(x^{\pm\pm}+y^{\pm\pm}+\mathrm{i}\xi^{\pm}\overline{\theta}^{\pm}+\mathrm{i}\overline{\xi}^{\pm}\theta^{\pm},\theta^{\pm}+\xi^{\pm},\overline{\theta}^{\pm}+\overline{\xi}^{\pm}\right), \tag{5.8}$$

from which the supercharge generators are

$$\mathcal{Q}_{\pm}=\mathrm{i}\partial_{\pm}-\overline{\theta}^{\pm}\partial_{\pm\pm}, \quad \overline{\mathcal{Q}}_{\pm}=-\mathrm{i}\overline{\partial}_{\pm}+\theta^{\pm}\partial_{\pm\pm}\ , \tag{5.9}$$

and the super-derivatives are

$$\mathcal{D}_{\pm}=\partial_{\pm}-\mathrm{i}\overline{\theta}^{\pm}\partial_{\pm\pm}, \quad \overline{\mathcal{D}}_{\pm}=-\overline{\partial}_{\pm}+\mathrm{i}\theta^{\pm}\partial_{\pm\pm}\ . \tag{5.10}$$

These generators satisfies the $(2,2)$ supersymmetry algebra

$$\left\{\mathcal{Q}_{\pm},\overline{\mathcal{Q}}_{\pm}\right\}=\left\{\mathcal{D}_{\pm},\overline{\mathcal{D}}_{\pm}\right\}=2\mathrm{i}\partial_{\pm\pm}\ , \tag{5.11}$$

and all the other anti-commutators among $\mathcal{Q}$ or $\mathcal{D}$ vanish.

Generically, an $(2,2)$ superfield can be expanded as

$$\begin{aligned}\mathscr{F}(x^{\pm\pm},\theta^{\pm},\overline{\theta}^{\pm}) &= f+\theta^{+}g_{+}+\theta^{-}g_{-}-\overline{\theta}^{+}\overline{g}_{+}-\overline{\theta}^{-}\overline{g}_{-}+\theta^{+}\overline{\theta}^{+}h_{++}+\theta^{-}\overline{\theta}^{-}h_{--}+\theta^{+}\overline{\theta}^{-}h_{+-}+\theta^{-}\overline{\theta}^{+}h_{-+}\\ &+\theta^{+}\theta^{-}F-\overline{\theta}^{+}\overline{\theta}^{-}\overline{F}+\overline{\theta}^{+}\overline{\theta}^{-}\theta^{+}l_{+}+\overline{\theta}^{+}\overline{\theta}^{-}\theta^{-}l_{-}-\theta^{+}\theta^{-}\overline{\theta}^{+}\overline{l}_{+}-\theta^{+}\theta^{-}\overline{\theta}^{-}\overline{l}_{-}+\theta^{+}\theta^{-}\overline{\theta}^{+}\overline{\theta}^{-}D\ .\end{aligned} \tag{5.12}$$

Sometimes, it is convenient to combine $\theta^{\pm}$ into a Dirac spinor θ^{α} and similarly $\overline{\theta}^{\dot{\alpha}}$ for $\alpha,\dot{\alpha}=+,-$. Then the expansion can be rewritten as

$$\begin{aligned}\mathscr{F}(x^{\pm\pm},\theta^{\pm},\overline{\theta}^{\pm}) &= f+\theta^{\alpha}g_{\alpha}+\overline{\theta}_{\dot{\alpha}}\overline{g}^{\dot{\alpha}}-\frac{1}{2}\theta\theta F-\frac{1}{2}\overline{\theta\theta F}+\theta^{\alpha}\sigma^{I}_{\alpha\dot{\alpha}}\overline{\theta}^{\dot{\alpha}}H_{I}\\ &+\frac{1}{2}\overline{\theta\theta}\theta^{\alpha}l_{\alpha}+\frac{1}{2}\theta\theta\overline{\theta}_{\dot{\alpha}}\overline{l}^{\dot{\alpha}}+\frac{1}{4}\theta\theta\overline{\theta\theta}D\ ,\end{aligned} \tag{5.13}$$

where we assume the $\alpha,\dot{\alpha}$ indices are raised and lowered by the antisymmetric rank 2 matrix $\epsilon^{\alpha\beta}$ as in 4d, and abbreviate $\theta^{\alpha}\theta_{\alpha}\equiv\theta\theta$, $\overline{\theta}_{\dot{\alpha}}\overline{\theta}^{\dot{\alpha}}\equiv\overline{\theta\theta}$. H_I fields for $I=0,1,2,3$ are related to $h_{\alpha\beta}$ by

$$\begin{aligned}H_0&=\frac{1}{2}(h_{++}+h_{--}), \quad H_3=\frac{1}{2}(h_{++}-h_{--}),\\ H_1&=\frac{1}{2}(h_{+-}+h_{-+}), \quad H_2=\frac{i}{2}(h_{+-}-h_{-+})\ .\end{aligned} \tag{5.14}$$

We see the similarity of the superfields with 4d $\mathcal{N}=1$ case, see [5]. Indeed, these two theories, though with distinct dimensions, both have 4 supercharges, and thus have the same supersymmetry multiplet structures. In fact, many 2d $(2,2)$ theories can be obtained from 4d $\mathcal{N}=1$ theories compactified on a two torus. We will discuss the relations between them in later sections.

The supersymmetric transformations of the component fields under

$$\delta=-\mathrm{i}\xi^{+}\mathcal{Q}_{+}-\mathrm{i}\xi^{-}\mathcal{Q}_{-}+\mathrm{i}\overline{\xi}^{+}\overline{\mathcal{Q}}_{+}+\mathrm{i}\overline{\xi}^{-}\overline{\mathcal{Q}}_{-} \tag{5.15}$$

can be directly obtained by applying the differential operators.

Among all the possible superfields, we are primarily interested in the following three cases:

$\mathcal{N}=(2,2)$ **chiral multiplet** It is defined by imposing the chirality condition

$$\overline{\mathcal{D}}_\pm \Phi = 0 \ . \tag{5.16}$$

Define the y coordinates by

$$y^{\pm\pm} \equiv x^{\pm\pm} - \mathrm{i}\theta^\pm \overline{\theta}^\pm \ , \tag{5.17}$$

satisfying

$$\overline{\mathcal{D}}_\pm y^{\pm\pm} = 0 \ . \tag{5.18}$$

Then the chiral multiplet can be expanded as

$$\begin{aligned}\Phi(x^{\pm\pm},\theta^\pm,\overline{\theta}^\pm) = \Phi(y^{\pm\pm},\theta^\pm) &= \varphi(y) + \theta^+\psi_+(y) + \theta^-\psi_-(y) + \theta^+\theta^- F(y)\\ &= \varphi + \theta^+\psi_+ + \theta^-\psi_- - \mathrm{i}\theta^+\overline{\theta}^+\partial_{++}\varphi - \mathrm{i}\theta^-\overline{\theta}^-\partial_{--}\varphi + \theta^+\theta^- F\\ &\quad - \mathrm{i}\theta^+\theta^-\overline{\theta}^-\partial_{--}\psi_+ + \mathrm{i}\theta^+\theta^-\overline{\theta}^+\partial_{++}\psi_- + \theta^+\theta^-\overline{\theta}^+\overline{\theta}^-\partial_{++}\partial_{--}\varphi \ .\end{aligned} \tag{5.19}$$

Anti-chiral multiplets can be defined similarly.

$\mathcal{N}=(2,2)$ **twisted chiral multiplet** It is defined by imposing the twisted chiral condition

$$\overline{\mathcal{D}}_+ U = \mathcal{D}_- U = 0 \ . \tag{5.20}$$

Define the z coordinates by

$$z^{\pm\pm} \equiv x^{\pm\pm} + \mathrm{i}\theta^\pm\overline{\theta}^\pm \ , \tag{5.21}$$

satisfying $\mathcal{D}_\pm z^{\pm\pm} = 0$. Then the twisted chiral multiplet can be expanded as

$$\begin{aligned} U(x^{\pm\pm},\theta^\pm,\overline{\theta}^\pm) &= U(y^{++},z^{--},\theta^+,\overline{\theta}^-)\\ &= u(y^{++},z^{--}) + \theta^+\overline{\chi}_+(y^{++},z^{--}) + \overline{\theta}^-\chi_-(y^{++},z^{--}) + \theta^+\overline{\theta}^- h(y^{++},z^{--})\\ &= u + \theta^+\overline{\chi}_+ + \overline{\theta}^-\chi_- + \theta^+\overline{\theta}^- h - \mathrm{i}\theta^+\overline{\theta}^+\partial_{++}u + \mathrm{i}\theta^-\overline{\theta}^-\partial_{--}u\\ &\quad + \mathrm{i}\theta^+\theta^-\overline{\theta}^-\partial_{--}\overline{\chi}_+ - \mathrm{i}\theta^+\overline{\theta}^+\overline{\theta}^-\partial_{++}\chi_- - \theta^+\theta^-\overline{\theta}^+\overline{\theta}^-\partial_{++}\partial_{--}u \ .\end{aligned} \tag{5.22}$$

Twisted anti-chiral multiplet is defined by

$$\mathcal{D}_+\overline{U} = \overline{\mathcal{D}}_-\overline{U} = 0 \ . \tag{5.23}$$

$\mathcal{N}=(2,2)$ **vector multiplet** For $(2,2)$ theories, we have enough supersymmetry to place the two components of the gauge vector field $A_{\pm\pm}$ in a single multiplet. As is mentioned, the $\mathcal{N}=(2,2)$ theories are very similar to the 4d $\mathcal{N}=1$ theories, and the vector multiplet can be defined in similar way by imposing the reality condition on the generic superfield (5.12)

$$\begin{aligned} V(x^{\pm\pm},\theta^\pm,\overline{\theta}^\pm) &= f + \theta^+\chi_+ + \theta^-\chi_- - \overline{\theta}^+\overline{\chi}_+ - \overline{\theta}^-\overline{\chi}_- + \theta^+\theta^-\rho - \overline{\theta}^+\overline{\theta}^-\overline{\rho} - \theta^+\overline{\theta}^+A_{++} - \theta^-\overline{\theta}^-A_{--}\\ &\quad - \theta^-\overline{\theta}^+\sigma - \theta^+\overline{\theta}^-\overline{\sigma} + \overline{\theta}^+\overline{\theta}^-\theta^+\lambda_+ + \overline{\theta}^+\overline{\theta}^-\theta^-\lambda_- + \theta^+\theta^-\overline{\theta}^+\overline{\lambda}_+ + \theta^+\theta^-\overline{\theta}^-\overline{\lambda}_- + \theta^+\theta^-\overline{\theta}^+\overline{\theta}^- D \ ,\end{aligned} \tag{5.24}$$

for real scalars $f, A_{\pm\pm}, D$, complex scalars ρ,σ and complex Weyl fermions $\chi_\pm,\lambda_\pm$.

To motivate the Wess-Zumino gauge, we first define the gauge transform of a chiral multiplet Φ as

$$\Phi \mapsto e^{-\mathrm{i}\Omega}\cdot\Phi \ , \tag{5.25}$$

in some representation for a complex chiral multiplet

$$\Omega = \alpha(y) + \theta^+\beta_+(y) + \theta^-\beta_-(y) + \theta^+\theta^-\gamma(y) \ , \tag{5.26}$$

valued in the complexified gauge Lie algebra $\mathfrak{g}_{\mathbb{C}}$.

Just like in 4d, we want to construct gauge invariant kinetic terms for a chiral multiplet Φ by $\overline{\Phi}e^{2V}\Phi$, which implies that the vector multiplet should transform by

$$e^{2V} \mapsto e^{-\mathrm{i}\overline{\Omega}}e^{2V}e^{\mathrm{i}\Omega} = \exp\big[2V + \mathrm{i}(\Omega - \overline{\Omega}) + \cdots\big] \equiv V' \ . \tag{5.27}$$

It can be proved that we can always choose the multiplet Ω such that $V'|_1, V'|_{\theta^\pm}, V'|_{\theta^+\theta^-}$ and their conjugates vanish, see for example [19, 20]. Therefore, we can restrict the component expansion to be in the *Wess-Zumino gauge*

$$\begin{aligned} V(x^{\pm\pm},\theta^\pm,\overline{\theta}^\pm) &= -\theta^+\overline{\theta}^+A_{++} - \theta^-\overline{\theta}^-A_{--} - \theta^-\overline{\theta}^+\sigma - \theta^+\overline{\theta}^-\overline{\sigma} \\ &\quad + \overline{\theta}^+\overline{\theta}^-\theta^+\lambda_+ + \overline{\theta}^+\overline{\theta}^-\theta^-\lambda_- + \theta^+\theta^-\overline{\theta}^+\overline{\lambda}_+ + \theta^+\theta^-\overline{\theta}^-\overline{\lambda}_- + \theta^+\theta^-\overline{\theta}^+\overline{\theta}^-D \ , \end{aligned} \tag{5.28}$$

Then we have

$$V^2 = \theta^+\theta^-\overline{\theta}^+\overline{\theta}^- \left(-A_{++}A_{--} - A_{--}A_{++} + \sigma\overline{\sigma} + \overline{\sigma}\sigma\right) \ , \quad V^3 = 0 \ . \tag{5.29}$$

We have the immediate properties

$$e^V = 1 + V + \frac{1}{2}V^2 \ , \quad e^{-V} = 1 - V + \frac{1}{2}V^2 \ , \tag{5.30}$$

and

$$e^Ve^V = 1 + 2V + 2V^2 = e^{2V} \ . \tag{5.31}$$

The residual gauge transforms are to set $\beta_\pm = \gamma = 0$ and α to be real[20]:

$$\Omega(x^{\pm\pm},\theta^\pm,\overline{\theta}^\pm) = \alpha(x) - \mathrm{i}\theta^+\overline{\theta}^+\partial_{++}\alpha(x) - \mathrm{i}\theta^-\overline{\theta}^-\partial_{--}\alpha(x) + \theta^+\theta^-\overline{\theta}^+\overline{\theta}^-\alpha(x) \ . \tag{5.32}$$

Then we can expand the gauge transformation (5.27)

$$e^{2V} \mapsto e^{-\mathrm{i}\overline{\Omega}}e^{2V}e^{\mathrm{i}\Omega} = e^{-\mathrm{i}\overline{\Omega}}e^{\mathrm{i}\Omega} + 2e^{-\mathrm{i}\overline{\Omega}}Ve^{\mathrm{i}\Omega} + 2e^{-\mathrm{i}\overline{\Omega}}V^2e^{\mathrm{i}\Omega}, \tag{5.33}$$

and make use of the following expansion

$$\begin{aligned} e^{\mathrm{i}\Omega} &= \exp\left(\mathrm{i}\alpha + \theta^+\overline{\theta}^+\partial_{++}\alpha + \theta^-\overline{\theta}^-\partial_{--}\alpha + \mathrm{i}\theta^+\theta^-\overline{\theta}^+\overline{\theta}^-\alpha\right) \\ &= \left(1 - \mathrm{i}\theta^+\overline{\theta}^+\partial_{++} - \mathrm{i}\theta^-\overline{\theta}^-\partial_{--} + \theta^+\theta^-\overline{\theta}^+\overline{\theta}^-(1 - \partial_{++}\partial_{--})\right)e^{\mathrm{i}\alpha}. \end{aligned} \tag{5.34}$$

Then we obtain the gauge transforms of the components of the vector multiplet as

$$A_{\pm\pm} \mapsto e^{-\mathrm{i}\alpha}A_{\pm\pm}e^{\mathrm{i}\alpha} + \mathrm{i}e^{-\mathrm{i}\alpha}\partial_{\pm\pm}e^{\mathrm{i}\alpha} \ ,$$

and all the other fields $\sigma, \overline{\sigma}, \lambda_\pm, \overline{\lambda}_\pm, D$ live in the gauge adjoint representation.

Similar to (4.30), we can define the covariant super-derivatives as [21]

$$\mathscr{D}_\pm = e^{-V}\mathcal{D}_\pm e^V \ , \quad \overline{\mathscr{D}}_\pm = e^V\mathcal{D}_\pm e^{-V} \ , \tag{5.35}$$

and thus

$$\begin{aligned}\mathscr{D}_\pm &= \left(1+V+\frac{1}{2}V^2\right)\mathcal{D}_\pm\left(1-V+\frac{1}{2}V^2\right)\\ &= \mathcal{D}_\pm + \mathcal{D}_\pm V - \frac{1}{2}\left[V,\mathcal{D}_\pm V\right] - \frac{1}{2}V(\mathcal{D}_\pm V)V + \frac{1}{4}V^2(\mathcal{D}_\pm V)V \ .\end{aligned} \tag{5.36}$$

Since V^2 is already full of θ and every term of $\mathcal{D}_\pm V$ has at least one θ

$$V^2(\mathcal{D}_\pm V) = (\mathcal{D}_\pm V)V^2 = 0 \ , \tag{5.37}$$

and since $V^3=0$

$$0 = \mathcal{D}_\pm V^3 = (\mathcal{D}_\pm V)V^2 + V(\mathcal{D}_\pm V)V + V^2(\mathcal{D}_\pm V) = V(\mathcal{D}_\pm V)V \ . \tag{5.38}$$

Therefore, the last two terms of (5.36) vanish and similar for $\overline{\mathscr{D}}_\pm$. So we conclude

$$\begin{aligned}\mathscr{D}_\pm &= e^{-V}\mathcal{D}_\pm e^V = \mathcal{D}_\pm + \mathcal{D}_\pm V - \frac{1}{2}\left[V,\mathcal{D}_\pm V\right],\\ \overline{\mathscr{D}}_\pm &= e^{V}\overline{\mathcal{D}}_\pm e^{-V} = \overline{\mathcal{D}}_\pm - \overline{\mathcal{D}}_\pm V - \frac{1}{2}\left[V,\overline{\mathcal{D}}_\pm V\right] \ .\end{aligned} \tag{5.39}$$

Then we can expand in components:

$$\begin{aligned}\mathscr{D}_+ &= \partial_+ - \mathrm{i}\overline{\theta}^+\nabla_{++} - \overline{\theta}^-\overline{\sigma} + \overline{\theta}^+\overline{\theta}^-\lambda_+ + \theta^-\overline{\theta}^+\overline{\lambda}_+ + \theta^-\overline{\theta}^-\overline{\lambda}_- - \mathrm{i}\theta^+\overline{\theta}^+\overline{\theta}^-\nabla_{++}\overline{\sigma}\\ &\qquad - \mathrm{i}\theta^-\overline{\theta}^+\overline{\theta}^-\left(\partial_{++}A_{--} + \mathrm{i}D - \frac{\mathrm{i}}{2}[A_{++},A_{--}] + \frac{\mathrm{i}}{2}[\overline{\sigma},\sigma]\right) - \mathrm{i}\theta^+\theta^-\overline{\theta}^+\overline{\theta}^-\left(\nabla_{++}\overline{\lambda}_- - \mathrm{i}[\overline{\lambda}_+,\overline{\sigma}]\right) \ ,\\ \mathscr{D}_- &= \partial_- - \mathrm{i}\overline{\theta}^-\nabla_{--} - \overline{\theta}^+\sigma + \overline{\theta}^+\overline{\theta}^-\lambda_- - \theta^+\overline{\theta}^+\overline{\lambda}_+ - \theta^+\overline{\theta}^-\overline{\lambda}_- + \mathrm{i}\theta^-\overline{\theta}^+\overline{\theta}^-\nabla_{--}\sigma\\ &\qquad + \mathrm{i}\theta^+\overline{\theta}^+\overline{\theta}^-\left(\partial_{--}A_{++} + \mathrm{i}D + \frac{\mathrm{i}}{2}[A_{++},A_{--}] - \frac{\mathrm{i}}{2}[\overline{\sigma},\sigma]\right) + \mathrm{i}\theta^+\theta^-\overline{\theta}^+\overline{\theta}^-\left(\nabla_{--}\overline{\lambda}_+ - \mathrm{i}[\overline{\lambda}_-,\sigma]\right) \ ,\\ \overline{\mathscr{D}}_+ &= -\overline{\partial}_+ + \mathrm{i}\theta^+\nabla_{++} + \theta^-\sigma - \theta^+\overline{\theta}^-\lambda_+ - \theta^-\overline{\theta}^-\lambda_- + \theta^+\theta^-\overline{\lambda}_+ + \mathrm{i}\theta^+\theta^-\overline{\theta}^+\nabla_{++}\sigma\\ &\qquad + \mathrm{i}\theta^+\theta^-\overline{\theta}^-\left(\partial_{++}A_{--} - \mathrm{i}D - \frac{\mathrm{i}}{2}[A_{++},A_{--}] - \frac{\mathrm{i}}{2}[\overline{\sigma},\sigma]\right) - \mathrm{i}\theta^+\theta^-\overline{\theta}^+\overline{\theta}^-\left(\nabla_{++}\lambda_- - \mathrm{i}[\lambda_+,\sigma]\right) \ ,\\ \overline{\mathscr{D}}_- &= -\overline{\partial}_- + \mathrm{i}\theta^-\nabla_{--} + \theta^+\overline{\sigma} + \theta^+\overline{\theta}^+\lambda_+ + \theta^-\overline{\theta}^+\lambda_- + \theta^+\theta^-\overline{\lambda}_- - \mathrm{i}\theta^+\theta^-\overline{\theta}^-\nabla_{--}\overline{\sigma}\\ &\qquad - \mathrm{i}\theta^+\theta^-\overline{\theta}^+\left(\partial_{--}A_{++} - \mathrm{i}D + \frac{\mathrm{i}}{2}[A_{++},A_{--}] + \frac{\mathrm{i}}{2}[\overline{\sigma},\sigma]\right) + \mathrm{i}\theta^+\theta^-\overline{\theta}^+\overline{\theta}^-\left(\nabla_{--}\lambda_+ - \mathrm{i}[\lambda_-,\overline{\sigma}]\right)\end{aligned} \tag{5.40}$$

where $\nabla_{\pm\pm}$ are the usual covariant derivatives. It may be checked that the covariant super-derivatives satisfy the supersymmetry algebra (5.11)

$$\left\{\mathscr{D}_\pm,\overline{\mathscr{D}}_\pm\right\} = 2\mathrm{i}\nabla_{\pm\pm}, \tag{5.41}$$

and all the other anti-commutators vanish.

By (5.27), we know that the e^V conjugations of covariant super-derivatives are in the gauge adjoint representation

$$e^{-V}\mathscr{D}_\pm e^V \mapsto e^{-\mathrm{i}\Omega}(e^{-V}\mathscr{D}_\pm e^V)e^{\mathrm{i}\Omega} \ , \quad e^V\overline{\mathscr{D}}_\pm e^{-V} \mapsto e^{-\mathrm{i}\overline{\Omega}}(e^V\overline{\mathscr{D}}_\pm e^{-V})e^{\mathrm{i}\overline{\Omega}} \ . \tag{5.42}$$

$\mathcal{N} = (2,2)$ **covariant multiplets** Define the covariant (twisted) chiral multiplet by imposing the covariant (twisted) chiral condition instead

$$\overline{\mathscr{D}}_\pm \mathbf{\Phi} = 0 \quad \text{or} \ \overline{\mathscr{D}}_+ \mathbf{\Phi} = \mathscr{D}_- \mathbf{\Phi} = 0 \ . \tag{5.43}$$

A covariant chiral multiplet can be obtained by $\mathbf{\Phi} = e^V \Phi$ for any usual chiral multiplet Φ, while there is no such simple ansatz for a covariant twisted chiral multiplet. In components, the covariant chiral multiplet $e^V \Phi$ can be expanded as

$$\mathbf{\Phi} \qquad =, \tag{5.44}$$

while a covariant twisted chiral can be expanded as

$\mathcal{N} = (2,2)$ **superfield strength** One can then define the field-strength superfield

$$\Sigma = \frac{1}{2}\left\{\overline{\mathscr{D}}_+, \mathscr{D}_-\right\}, \quad \overline{\Sigma} = \frac{1}{2}\left\{\overline{\mathscr{D}}_-, \mathscr{D}_+\right\} \ , \tag{5.45}$$

which can be expanded using (5.40) to be

$$\begin{aligned}
\Sigma = {} & \sigma - \overline{\theta}^- \lambda_- - \theta^+ \overline{\lambda}_+ - \mathrm{i}\theta^+ \overline{\theta}^- (F - \mathrm{i}D) + \mathrm{i}\theta^- \overline{\theta}^- \nabla_{--}\sigma - \mathrm{i}\theta^+ \overline{\theta}^+ \nabla_{++}\sigma \\
& - \mathrm{i}\theta^+\theta^-\overline{\theta}^- \nabla_{--}\overline{\lambda}_+ + \mathrm{i}\theta^+\overline{\theta}^+\overline{\theta}^- \nabla_{++}\lambda_- + \theta^+\overline{\theta}^+\theta^- [\lambda_+, \sigma] - \theta^+\theta^-\overline{\theta}^- \left[\overline{\lambda}_-, \sigma\right] \\
& + \frac{1}{2}\theta^+\theta^-\overline{\theta}^+\overline{\theta}^- \Big(- \{\nabla_{++}, \nabla_{--}\}\sigma + \mathrm{i}\Big[\partial_{++}A_{--} + \partial_{--}A_{++} - \mathrm{i}\left[\overline{\sigma}, \sigma\right], \sigma\Big]\Big) \ , \\
\overline{\Sigma} = {} & \overline{\sigma} + \overline{\theta}^+ \lambda_+ + \theta^- \overline{\lambda}_- - \mathrm{i}\theta^- \overline{\theta}^+ (-F - \mathrm{i}D) + \mathrm{i}\theta^+ \overline{\theta}^+ \nabla_{++}\overline{\sigma} - \mathrm{i}\theta^- \overline{\theta}^- \nabla_{--}\overline{\sigma} \\
& - \mathrm{i}\theta^+\theta^-\overline{\theta}^+ \nabla_{++}\overline{\lambda}_- + \mathrm{i}\theta^-\overline{\theta}^+\overline{\theta}^- \nabla_{--}\lambda_+ + \theta^-\overline{\theta}^+\theta^- [\lambda_-, \overline{\sigma}] - \theta^+\theta^-\overline{\theta}^+ \left[\overline{\lambda}_+, \overline{\sigma}\right] \\
& + \frac{1}{2}\theta^+\theta^-\overline{\theta}^+\overline{\theta}^- \Big(- \{\nabla_{++}, \nabla_{--}\}\overline{\sigma} + \mathrm{i}\Big[\partial_{++}A_{--} + \partial_{--}A_{++} + \mathrm{i}\left[\overline{\sigma}, \sigma\right], \overline{\sigma}\Big]\Big) \ ,
\end{aligned} \tag{5.46}$$

where $F = \partial_{++}A_{--} - \partial_{--}A_{++} - \mathrm{i}\,[A_{++}, A_{--}]$ is the ordinary field strength associated with $A_{\pm\pm}$.

Σ is a covariant twisted chiral multiplet, which can be seen either by compairing with the component expansion (**??**) or by the algebra (5.41).

Decompose $(2,2)$ **multiplets into** $(0,2)$ **multiplets**

A (2,2) chiral multiplet can be rewritten into a nested form (in y coordinates):

$$\begin{aligned}
\Phi &= \left(\varphi + \theta^+\psi_+\right) + \theta^- \left(\psi_- - \theta^+ F\right) \\
&= \left(\varphi + \theta^-\psi_-\right) + \theta^+ \left(\psi_+ + \theta^- F\right) \ .
\end{aligned} \tag{5.47}$$

Therefore, a (2,2) chiral multiplet can be decomposed into a (0,2) chiral multiplet and a (0,2) Fermi multiplet.

A (2,2) twisted chiral multiplet can be rewritten as:

$$\begin{aligned}
U &= \left(u + \theta^+\overline{\chi}_+\right) + \overline{\theta}^- \left(\chi_- - \theta^+ h\right) \\
&= \left(u + \overline{\theta}^- \chi_-\right) + \theta^+ \left(\overline{\chi}_+ + \overline{\theta}^- h\right) \ .
\end{aligned} \tag{5.48}$$

Therefore, a (2,2) twisted chiral multiplet can be decomposed into a (0,2) chiral multiplet and a (0,2) Fermi multiplet as well, but they are assembled in a “twisted” way.

A (2,2) vector multiplet (5.28) can be rewritten into a nested form

$$V = \theta^+\overline{\theta}^+ A_{++} + \left\{\overline{\theta}^- \left(\sigma + \overline{\theta}^+ \lambda_+\right) + \text{c.c.}\right\} + \theta^-\overline{\theta}^- \left(A_{--} + \theta^+\overline{\lambda}_- - \overline{\theta}^+\lambda_- - \theta^+\overline{\theta}^+ D\right) \ , \tag{5.49}$$

from which we see that a (2,2) vector multiplet can be decomposed into a (0,2) vector multiplet $(A_{\pm\pm}, \lambda_-, \overline{\lambda}_-, D)$ and a (0,2) chiral multiplet (σ, λ_+).

Decompose $(2,2)$ multiplets into $(1,1)$ multiplets

We can also decompose (2,2) multiplets into (1,1) multiplets.

5.2 $\mathcal{N} = (2,2)$ Lagrangians

[10]

Due to the existence of twisted chiral multiplets, there are three ways to write (2,2)-SUSY invariant Lagrangian terms:

D-terms Integrate over the all super-directions:

$$\mathcal{L}_{(D)} = \int d\theta^\pm d\overline{\theta}^\pm \mathcal{K}_{(D)} \ , \tag{5.50}$$

for a real scalar (2,2) superfield $\mathcal{K}_{(D)}$ being a functional of the compositing superfields in this theory.

F-terms Integrate over the half of the super-directions:

$$\mathcal{L}_{(F)} = \int d\theta^\pm \mathcal{W}_{(F)}|_{\overline{\theta}^\pm=0} \ , \tag{5.51}$$

for a real scalar (2,2) chiral superfield $\mathcal{W}_{(D)}$ being a functional of the chiral superfields in this theory.

twisted F-terms Integrating over the twisted half of the $(2,2)$ super-directions:

$$\mathcal{L}_{(\widetilde{F})} = \int d\theta^+ d\overline{\theta}^- \widetilde{\mathcal{W}}_{(\widetilde{F})}|_{\theta^-=\overline{\theta}^+=0} \ , \tag{5.52}$$

for a real scalar (2,2) twisted chiral superfield $\widetilde{\mathcal{W}}_{(\widetilde{F})}$ being a functional of the twisted chiral superfields in this theory.

5.2.1 (2,2) sigma model

Consider the sigma model of n chiral multiplets $\Phi^1, \cdots \Phi^n$ with Lagrangian given by

$$\mathcal{L}_\sigma = \int d^2\theta d^2\overline{\theta} K(\Phi, \overline{\Phi}) + \left(\int d^2\theta W(\Phi) + \text{c.c.}\right) \ , \tag{5.53}$$

for some real function $K(\Phi, \overline{\Phi})$ called the *Kähler potential* and holomorphic function $W(\Phi)$ called the *superpotential*. In components, the Lagrangian becomes

$$\begin{aligned}
\mathcal{L}_\sigma = &-\frac{\partial^2 K(\varphi,\overline{\varphi})}{\partial\varphi^i\partial\overline{\varphi}^{\bar\imath}} \left(2\partial_{++}\overline{\varphi}^{\bar\imath}\partial_{--}\varphi^i + 2\partial_{--}\overline{\varphi}^{\bar\imath}\partial_{++}\varphi^i + 2\mathrm{i}\overline{\psi}^{\bar\imath}_+\partial_{--}\psi^i_+ + 2\mathrm{i}\overline{\psi}^{\bar\imath}_-\partial_{++}\psi^i_- - \overline{F}^{\bar\imath}F^i\right) \\
&- 2\frac{\partial^3 K(\varphi,\overline{\varphi})}{\partial\varphi^i\partial\varphi^j\partial\overline{\varphi}^{\bar\imath}} \left(\mathrm{i}\overline{\psi}^{\bar\imath}_+\psi^i_+\partial_{--}\varphi^j + \mathrm{i}\overline{\psi}^{\bar\imath}_-\psi^i_-\partial_{++}\varphi^j\right) - \frac{\partial^3 K(\varphi,\overline{\varphi})}{\partial\varphi^i\partial\varphi^j\partial\overline{\varphi}^{\bar\imath}}\psi^i_+\psi^j_-\overline{F}^{\bar\imath} + \frac{\partial^3 K(\varphi,\overline{\varphi})}{\partial\varphi^i\partial\overline{\varphi}^{\bar\imath}\partial\overline{\varphi}^{\bar\jmath}}\overline{\psi}^{\bar\imath}_+\overline{\psi}^{\bar\jmath}_-F^i \\
&+ \frac{\partial^4 K(\varphi,\overline{\varphi})}{\partial\varphi^i\partial\varphi^j\partial\overline{\varphi}^{\bar\imath}\partial\overline{\varphi}^{\bar\jmath}}\overline{\psi}^{\bar\imath}_+\overline{\psi}^{\bar\jmath}_-\psi^i_+\psi^j_- + \left(\frac{\partial W}{\partial\varphi^i}F^i - \frac{\partial^2 W}{\partial\varphi^i\partial\varphi^j}\psi^i_+\psi^j_- + \text{c.c}\right) \ .
\end{aligned} \tag{5.54}$$

After solving for the auxiliary fields $F^i, \overline{F}^{\bar{\imath}}$, the Lagrangian becomes

$$\begin{aligned}\mathcal{L}_\sigma = &-2g_{i\bar{\imath}}\left(\partial_{++}\overline{\varphi}^{\bar{\imath}}\partial_{--}\varphi^i + \partial_{--}\overline{\varphi}^{\bar{\imath}}\partial_{++}\varphi^i + \mathrm{i}\overline{\psi}^{\bar{\imath}}_+\mathfrak{D}_{--}\psi^i_+ + \mathrm{i}\overline{\psi}^{\bar{\imath}}_-\mathfrak{D}_{++}\psi^i_-\right)\\ &+\mathfrak{R}_{i\bar{\imath}j\bar{\jmath}}\overline{\psi}^{\bar{\imath}}_+\overline{\psi}^{\bar{\jmath}}_-\psi^i_+\psi^j_- + g^{\bar{\imath}i}\frac{\partial\overline{W}}{\partial\overline{\varphi}^{\bar{\imath}}}\frac{\partial W}{\partial\varphi^i} - \left(\mathfrak{D}_i\frac{\partial W}{\partial\varphi^j}\right)\psi^i_+\psi^j_- + \left(\mathfrak{D}_{\bar{\imath}}\frac{\partial\overline{W}}{\partial\overline{\varphi}^{\bar{\jmath}}}\right)\overline{\psi}^{\bar{\imath}}_+\overline{\psi}^{\bar{\jmath}}_- ,\end{aligned} \tag{5.55}$$

where

$$\begin{aligned} g_{i\bar{\jmath}} = \frac{\partial^2 K}{\partial\varphi^i\partial\overline{\varphi}^{\bar{\jmath}}}\ , \quad \mathfrak{R}_{i\bar{\imath}j\bar{\jmath}} = \frac{\partial}{\partial\varphi^i}\left(g_{j\bar{k}}\Gamma^{\bar{k}}_{\bar{\imath}\bar{\jmath}}\right) - g_{k\bar{k}}\Gamma^k_{ij}\Gamma^{\bar{k}}_{\bar{\imath}\bar{\jmath}},\\ \mathfrak{D}_{\pm\pm}\psi^i_\mp = \partial_{\pm\pm}\psi^i_\mp + \partial_{\pm\pm}\varphi^j\Gamma^i_{jk}\psi^k_\mp, \quad \left(\mathfrak{D}_i\frac{\partial W}{\partial\varphi^j}\right) = \frac{\partial^2 W}{\partial\varphi^i\partial\varphi^j} - \Gamma^k_{ij}\frac{\partial W}{\partial\varphi^k}\end{aligned} \tag{5.56}$$

with

$$\Gamma^i_{jk} = g^{\bar{\jmath}i}\partial_j g_{k\bar{\jmath}}, \quad \Gamma^{\bar{\imath}}_{\bar{\jmath}\bar{k}} = g^{\bar{\imath}j}\partial_{\bar{\jmath}}g_{j\bar{k}}\ . \tag{5.57}$$

In order for the kinetic terms to be non-degenerate and have correct signs, $g_{i\bar{\jmath}}$ should be positive definite. Therefore, the target space must be a Kähler manifold X parameterized by $\varphi^i, \overline{\varphi}^{\bar{\imath}}$ with Kähler potential K. And the superpotential W is a holomorphic function over X.

From the Lagrangian, the vacua are determined by the critical points of the superpotential

$$\frac{\partial W}{\partial\varphi^i} = 0\ . \tag{5.58}$$

5.2.2 (2,2) gauge theory

ST: if we use the mirror book convention...

The Lagrangian consisting of the vector multiplets is usually called the gauged linear sigma model (GLSM). And the modified Lagrangian (covariant kinetic terms) introduced before is expanded as

$$\begin{aligned} L_{\mathrm{kin}} =& \int d^4\theta\ \overline{\Phi}e^V\Phi\\ =&|D_0\phi|^2 - |D_1\phi|^2 + i\overline{\psi}_-\left(D_0+D_1\right)\psi_- + i\overline{\psi}_+\left(D_0-D_1\right)\psi_+\\ &+D|\phi|^2 + |F|^2 - |\sigma|^2|\phi|^2 - \overline{\psi}_-\sigma\psi_+ - \overline{\psi}_+\overline{\sigma}\psi_- - i\overline{\phi}\lambda_-\psi_+\\ &+i\overline{\phi}\lambda_+\psi_- + i\overline{\psi}_+\overline{\lambda}_-\phi - i\overline{\psi}_-\overline{\lambda}_+\phi.\end{aligned} \tag{5.59}$$

The kinetic term of the vector superfield, or the super Yang-Mills term, is written in terms of the super field-strength Σ,

$$\begin{aligned} L_{\mathrm{SYM}} &= -\frac{1}{2e^2}\int d^4\theta\ \overline{\Sigma}\Sigma\\ &= \frac{1}{2e^2}\left(|\partial_0\sigma|^2 - |\partial_1\sigma|^2 + i\overline{\lambda}_-\left(\partial_0+\partial_1\right)\lambda_- + i\overline{\lambda}_+\left(\partial_0-\partial_1\right)\lambda_+ + v_{01}^2 + D^2\right).\end{aligned} \tag{5.60}$$

One can also add a twisted superpotential (FI term) $\widetilde{W}_{FI,\theta} = -t\Sigma$, with a dimensionless complex parameter $t = r - i\theta$,

$$L_{FI,\theta} = \frac{1}{2}\left(-t\int d^2\widetilde{\theta}\ \Sigma + c.c.\right) = -rD + \theta v_{01} \tag{5.61}$$

here, r is indeed the Fayet–Iliopoulos parameter, and θ is the theta angle.

5.3 $\mathcal{N} = (2,2)$ Chiral rings

A-type chiral ring

B-type chiral ring

5.4 $\mathcal{N} = (2,2)$ superconformal field theories

$\mathcal{N} = (2,2)$ superconformal algebra

The (2,2) superconformal field theories can be classified by study the solitonic solutions of their massive deformation[22]. In particular, the minimal models have ADE classification, with the Cartan matrices given by the solitons between different vacua.

5.5 $\mathcal{N} = (2,2)$ topologically twisted theories

The usual procedure of constructing a topological field theory, or more exactly cohomological field theory[23], from a supersymmetric field theory is to mix the holonomy group of spacetime with the R-symmetry group, which is called *topological twist*. As a result, the R-symmetry is required to be an exact quantum symmetry of the system.

Assume that we have a 2d $(2,2)$ field theory, which has classical R-symmetries R_V and R_A. R_V is non-chiral thus anomaly-free, while R_A can be anomalous. The anomaly of R_A is simply the instanton class, which is proportional to first Chern class $c_1(\mathcal{M})$ of the target space $\mathcal{M}$. Therefore, in order to preserve R_A, we can impose $c_1(M) = 0$. Therefore it is natural to assume that the target space $\mathcal{M}$ is a Calabi-Yau manifold.

Recall that a cohomological field theory requires a scalar Grassmannian nilpotent symmetry operator $\mathcal{Q}$ that the energy-momentum tensor is $\mathcal{Q}$-exact. In particular, the momentum P and Hamiltonian H should be $\mathcal{Q}$-exact. Therefore, supercharges $\mathcal{Q}_\pm, \overline{\mathcal{Q}}_\pm$ cannot play the role. It can be check that, there are only two independent ways to mix supercharges(up to automorphisms of the whole supersymmetry algebra) to obtain such a $\mathcal{Q}$:

$$\mathcal{Q}_A := \overline{\mathcal{Q}}_+ + \mathcal{Q}_-, \quad \mathcal{Q}_B := \overline{\mathcal{Q}}_+ + \overline{\mathcal{Q}}_- \ , \tag{5.62}$$

which can be verified to be nilpotent and H, P are indeed exact:

$$\begin{aligned} H &= \{\mathcal{Q}_A, \frac{1}{2}\left(\mathcal{Q}_+ + \overline{\mathcal{Q}}_-\right)\} = \{\mathcal{Q}_B, \frac{1}{2}\left(\mathcal{Q}_+ + \mathcal{Q}_-\right)\} \ , \\ P &= \{\mathcal{Q}_A, \frac{1}{2}\left(\overline{\mathcal{Q}}_- - \mathcal{Q}_+\right)\} = \{\mathcal{Q}_B, \frac{1}{2}\left(\mathcal{Q}_- - \mathcal{Q}_+\right)\} \ . \end{aligned} \tag{5.63}$$

The Lorentz and R transformations of $\mathcal{Q}_A, \mathcal{Q}_B$ are

$$\begin{aligned} [M, \mathcal{Q}_A] &= -\frac{\mathrm{i}}{2}\left(\overline{\mathcal{Q}}_+ + \mathcal{Q}_-\right) \ , & [M, \mathcal{Q}_B] &= -\frac{\mathrm{i}}{2}\left(\overline{\mathcal{Q}}_+ + \overline{\mathcal{Q}}_-\right) \ , \\ [R_V, \mathcal{Q}_A] &= \overline{\mathcal{Q}}_+ - \mathcal{Q}_- \ , & [R_V, \mathcal{Q}_B] &= \overline{\mathcal{Q}}_+ + \overline{\mathcal{Q}}_- \ , \\ [R_A, \mathcal{Q}_A] &= \overline{\mathcal{Q}}_+ + \mathcal{Q}_- \ , & [R_A, \mathcal{Q}_B] &= \overline{\mathcal{Q}}_+ - \overline{\mathcal{Q}}_- \ , \end{aligned} \tag{5.64}$$

from which we see that if we define the twisted Lorentz operator

$$M_A := M + \frac{\mathrm{i}}{2} R_A \ , \quad M_B := M + \frac{\mathrm{i}}{2} R_V \ , \tag{5.65}$$

then

$$[M_A, \mathcal{Q}_A] = [M_B, \mathcal{Q}_B] = 0 \ . \tag{5.66}$$

Therefore, by doing topological twists associated with M_A, M_B, the charges $\mathcal{Q}_A, \mathcal{Q}_B$ are candidates of topological symmetries. We have to verify that the energy-momentum tensor is $\mathcal{Q}$-exact for a specific theory to be topological. These two types of topological twists are called *A-twist* and *B-twist*, and we conclude that B-twist can be performed only for Calabi-Yau target, while A-twist does not have this constraint.

topological A-model

[24]

topological B-model

5.6 Mirror symmetry

6 $\mathcal{N} = (0, 4)$

[25] Theories possessing $\mathcal{N} = (0, 4)$ supersymmetry feature four real, right-moving supercharges. Correspondingly, they exhibit an R-symmetry group

$$SO(4)_R \cong SU(2)^-_R \times SU(2)^+_R.$$

These supercharges transform as a $(\mathbf{2}, \mathbf{2})_+$ representation, where the subscript indicates chirality under the Lorentz group $SO(1, 1)$ and should not be confused with the $\pm$ superscripts that label the distinct R-symmetry factors.

Our objective in this section is to elucidate the multiplet structures and formulate Lagrangians for gauge theories endowed with $\mathcal{N} = (0, 4)$ supersymmetry. The method involves constructing these theories from $\mathcal{N} = (0, 2)$ supermultiplets that inherently manifest an enhanced $SO(4)_R$ symmetry, thus guaranteeing the existence of extended supersymmetry. Specifically, as outlined below, there are four fundamental multiplets associated with $\mathcal{N} = (0, 4)$ supersymmetry: the vector multiplet, hypermultiplet, twisted hypermultiplet, and Fermi multiplet.

$\mathcal{N} = (0, 4)$ **vector multiplet** The $\mathcal{N} = (0, 4)$ vector multiplet consists of an $\mathcal{N} = (0, 2)$ vector multiplet U paired with an adjoint-valued $\mathcal{N} = (0, 2)$ Fermi multiplet Θ. Alongside the gauge field, this multiplet includes a pair of left-moving complex fermions, denoted by ζ^a_- with $a = 1, 2$, which transform as $(\mathbf{2}, \mathbf{2})_-$ under the R-symmetry group. Additionally, there is a triplet of auxiliary fields that transform as $(\mathbf{3}, \mathbf{1})$.

The Fermi multiplet satisfies the condition

$$\bar{\mathcal{D}}_+ \Theta = E_\Theta, \tag{6.1}$$

where the function E_Θ depends explicitly on the matter content and will be elaborated upon subsequently. The Lagrangian describing the $\mathcal{N} = (0, 4)$ vector multiplet is obtained by combining

$\mathcal{N} = (0, 4)$ **twisted hypermultiplet** (0,4)twisted hyper=(0,2)chiral+(0,2)chiral

$\mathcal{N} = (0, 4)$ **hypermultiplet** (0,4)hyper=(0,2)chiral+(0,2)chiral

$\mathcal{N} = (0, 4)$ **fermi multiplet** (0,4)fermi=(0,2)fermi+(0,2)fermi

7 $\mathcal{N} = (4, 4)$

2d $\mathcal{N} = (4, 4)$ theory is similar to the 4d $\mathcal{N} = 2$ theory(without gravity) in some sense.

[26]

$\mathcal{N} = (4,4)$ **vector multiplet** (4,4)vector=(0,4)vec+(0,4)twisted hyper
(4,4)vector=(2,2)vec+(2,2)adjoint chiral

$\mathcal{N} = (4,4)$ **hyper multiplet** (4,4)hyper=(0,4)hyper+(0,4)fermi
(4,4)hyper=(2,2)chiral+(2,2)(conj.)chiral

8 $\mathcal{N} = (0,8)$

$\mathcal{N} = (0,8)$ **vector multiplet** (0,8)vector=(0,4)vector+(0,4)fermi??

$\mathcal{N} = (0,8)$ **hyper multiplet** (0,8)hyper=(0,4)hyper+(0,4)twisted hyper??

9 $\mathcal{N} = (8,8)$

2d $\mathcal{N} = (8,8)$ theory is similar to the 4d $\mathcal{N} = 4$ theory(without gravity) in some sense.

$\mathcal{N} = (8,8)$ **vector multiplet** (8,8)vector=(0,8)vec+(0,8)hyper??
(8,8)vector=(4,4)vec+(4,4)adjoint hyper??

10 2d theories from higher dimensions

10.1 From 3d

10.2 From 4d

Reducing 4d supersymmetric field theories to 2d is first discussed by [27], which we summarize here.

Reduce 4d Yang-Mills theory to 2d

Consider a 4d Yang-Mills theory on the four manifold which is the product of two Riemann surfaces $\Sigma \times C$ (assumed with no punctures), in Euclidean signature, with standard action

$$S_{4d} = \int_{\Sigma\times C} d^4x\sqrt{g}\,\mathrm{tr}\,F^{\mu\nu}F_{\mu\nu} = \int_{\Sigma\times C} d^4x\sqrt{g}\,\mathrm{tr}\,g^{\mu\nu}g^{\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\ . \tag{10.1}$$

The effective theory after compactification along C can be obtained by deforming the metric into $g_\varepsilon = g_\Sigma \oplus \varepsilon g_C$ and adjusting ε from 1 to 0. Denote the Σ indices by $\alpha, \beta, \cdots$, the C indices by m, n, p, q, then we can decompose

$$\begin{aligned}(g_\varepsilon)^{\mu\nu}(g_\varepsilon)^{\rho\sigma}F_{\mu\nu}F_{\rho\sigma} &= (g_\varepsilon)^{mn}(g_\varepsilon)^{pq}F_{mn}F_{pq} + 2(g_\varepsilon)^{\alpha\beta}(g_\varepsilon)^{mn}F_{\alpha m}F_{\beta n} + (g_\varepsilon)^{\alpha\beta}(g_\varepsilon)^{\gamma\lambda}F_{\alpha\beta}F_{\gamma\lambda}\\ &= \varepsilon^{-2}g^{mp}g^{nq}F_{mn}F_{pq} + 2\varepsilon^{-1}g^{\alpha\beta}g^{mn}F_{\alpha m}F_{\beta n} + g^{\alpha\gamma}g^{\beta\lambda}F_{\alpha\beta}F_{\gamma\lambda}\ ,\end{aligned} \tag{10.2}$$

and on the other hand, the volume factor scales as

$$\sqrt{g_\varepsilon} = \sqrt{\varepsilon^2 g} = \varepsilon\sqrt{g}\ , \tag{10.3}$$

where $g \equiv g_1 = g_\Sigma \oplus g_C$. Therefore, the 4d action becomes

$$S_{4d,\varepsilon} = \int_{\Sigma\times C} d^4x\sqrt{g}\left(\varepsilon^{-1}\,\mathrm{tr}\,F^{mn}F_{mn} + 2\,\mathrm{tr}\,F^{\alpha m}F_{\alpha m} + \varepsilon\,\mathrm{tr}\,F^{\alpha\beta}F_{\alpha\beta}\right)\ , \tag{10.4}$$

where the contraction of indices is with respect to $g = g_\Sigma \oplus g_C$.

As $\varepsilon \to 0$, the first term in (10.4) diverges, thus restricting the configurations (in the path-integral) to those that $F_{mn} = 0$. In another word, A_m should be a flat connection along C. More precisely, the result is a family of flat connections A_m parametrized by Σ, which is equivalent to a map $X : \Sigma \to \mathcal{M}_{\text{flat}}(C)$.

The third term of (10.4) vanishes unless $\operatorname{tr} F^{\alpha\beta}F_{\alpha\beta}$ diverges, which corresponds to the case where the connection is reducible and there are residual gauge redundancy in the 2d. Temporarily, we assume that the third term can be safely ignored in the limit.

The second term of (10.4) turns out to be the standard kinetic term of the σ-model $X : \Sigma \to \mathcal{M}_{\text{flat}}(C)$, as shown in the following. Firstly we have

$$\begin{aligned} F^{\alpha m}F_{\alpha m} &= \Big(\partial^\alpha A^m - \partial^m A^\alpha - \mathrm{i}[A^\alpha, A^m]\Big)\Big(\partial_\alpha A_m - \partial_m A_\alpha - \mathrm{i}[A_\alpha, A_m]\Big) \\ &= \Big(\partial^\alpha A^m - \nabla^m A^\alpha\Big)\Big(\partial_\alpha A_m - \nabla_m A_\alpha\Big) , \end{aligned} \tag{10.5}$$

which has no dynamical term for the field A_α, and therefore we can directly integrate out A_Σ, whose equation of motion is

$$\nabla^m(\partial_\alpha A_m - \nabla_m A_\alpha) = 0 . \tag{10.6}$$

Since A_m is flat, ∇_m is nilpotent, and the tangent space to the moduli space $\mathcal{M}_{\text{flat}}(C)$ can be identified with the first ∇_m-cohomology $H^1_{\nabla_m}(C)$. Therefore, (10.6) simply means that $\partial_\alpha A_m - \nabla_m A_\alpha$ is ∇_m-closed. Then we can choose a basis $\{\omega_I\}_I$ of $H^1_{\nabla_m}(C)$ such that

$$\partial_\alpha A_m - \nabla_m A_\alpha = \omega_{Im}\partial_\alpha X^I , \tag{10.7}$$

where $\partial_\alpha X^I$ is to pull ω_I back to the worldsheet Σ and therefore the effective action can be written as

$$S_{2d} = \int_\Sigma d^2x\sqrt{g_\Sigma}2\operatorname{tr}\left(\omega_I^m\partial^\alpha X^I\omega_{Jm}\partial_\alpha X^J\right) = \int_\Sigma d^2x\sqrt{g_\Sigma}G_{IJ}\partial^\alpha X^I\partial_\alpha X^J , \tag{10.8}$$

with $G_{IJ} \equiv 2\operatorname{tr}(\omega_I^m\omega_{Jm})$.

The moduli space $\mathcal{M}_{\text{flat}}(C)$ naturally has a Kähler structure, and the G defined above corresponds to the Kähler metric, so we can distinguish the holomorphic and anti-homolorphic indices $i, \bar{\jmath}$, and rewrite the action into

$$S_{2d} = \int_\Sigma d^2x\sqrt{g_\Sigma}G_{i\bar{\jmath}}\partial^\alpha X^i\partial_\alpha X^{\bar{\jmath}} , \tag{10.9}$$

which is the standard action of a σ-model with Kähler target $\mathcal{M}_{\text{flat}}(C)$.

We can also include the theta term to the 4d theory, and it turns out that, the 4d complexified coupling $\tau \equiv \frac{4\pi\mathrm{i}}{e^2} + \frac{\theta}{2\pi}$ is identified with the Kähler moduli of the effective 2d theory.

There are several generalizations we can consider to the analysis of this section.

Reduce 4d SYM theory to 2d

Now add the $\mathcal{N} = 1$

Reduce 4d quiver gauge theory to 2d

Adding punctures to C

2d (0,2) from 4d $\mathcal{N} = 1$ theories compactified on a 2-torus with magnetic flux[28, 29].

2d (2,2) from 4d $\mathcal{N} = 1$ theories compactified on a 2-torus

2d (0,2) from 4d $\mathcal{N} = 1$ theories twisted compactified on a 2-sphere[30].

2d (0,2) (0,4) from 4d $\mathcal{N} = 2$ theories compactified on 2-sphere [16]

10.3 From 6d

(0,1) from 6d (1,0) theories twisted compactified on four-manifolds[31]. When the four-manifold is Kähler, the 2d effective theory is enhanced to have (0,2) susy, while if the four-manifold is hyper-Kähler, it is enhanced to have (0,4) susy.

(0,2) from 6d (2,0) theories twisted compactified on four-manifolds[32]

10.4 From string theories and M-theories

The (2,2) SCFTs arise as the effective theories of type IIA string theory compactified on Calabi-Yau 4-folds[33].

2d (0,2) from compactification of heterotic strings.

(0,2) quiver gauge theories and brane constructions [11, 34, 35]

A Notations and Conventions

The imaginary unit is denoted as $\mathrm{i} \equiv \sqrt{-1}$ in order to be distinguished with the index i.

The hermitian conjugates of a superfield or component field is represented by an overline instead of a dagger symbol. For hermitian conjugate of a long expression, we use the dagger symbol.

To avoid heavy indices, we abbreviate by the following contraction rules

$$
\begin{aligned}
AB &\equiv A^{\mu}B_{\mu} = A^{++}B_{++} + A^{--}B_{--} \ , \\
\psi\lambda &\equiv \psi^{\alpha}\lambda_{\alpha} = \psi^{+}\lambda_{+} + \psi^{-}\lambda_{-} \ , \\
\overline{\psi\lambda} &\equiv \overline{\psi}_{\dot{\alpha}}\overline{\lambda}^{\dot{\alpha}} = \overline{\psi}_{+}\overline{\lambda}^{+} + \overline{\psi}_{-}\overline{\lambda}^{-} \ ,
\end{aligned}
\tag{A.1}
$$

for 2d vectors A, B , chiral spinors $\psi_{\pm}, \lambda_{\pm}$ and anti-chiral spinors $\overline{\psi}_{\pm}, \overline{\lambda}_{\pm}$. We use the $\pm$ indices by the convention that:

$$
\text{Every expression } f_{\pm}g_{\mp} \text{ always means } f_{+}g_{-} \text{ and } f_{-}g_{+} \tag{A.2}
$$

and in particular

$$
f_{\pm}g_{\mp} = h_{\pm} \ \text{ means } f_{+}g_{-} = h_{+} \text{ and } f_{-}g_{+} = h_{-}. \tag{A.3}
$$

B Complex Geometry

C Cohomological Field Theory

There are mainly two types of Topological Quantum Field Theories (TQFT). First are those theories which do not depend on the metric at all, called the Schwarz type. For instance, the Chern-Simons theories and more generally the BF theories are of this type.

The other type of TQFT are those constructed with a choice of metric, but turn out to be indepdendent of the metric, called the Witten type[23, 36], or also called *cohomological field theories*. Each of these theories is such a field theory that possesses a scalar Grassmannian symmetry operator Q that:

1. It is nilpotent: $Q^2 = 0$.

2. Physical operators are Q-closed: $\{Q, \mathcal{O}\} = 0$ for any physical operator $\mathcal{O}$[6].

[6] In this appendix, $\{\cdot, \cdot\}$ is the anti-commutator if both arguments are Grassmannian, otherwise the commutator.

3. Q is not spontaneously broken: $Q|0\rangle = 0$.

4. The energy-momentum tensor is Q-exact: $T_{\mu\nu} = \{Q, G_{\mu\nu}\}$ for some Grassmannian operator $G_{\mu\nu}$.

The first and second properties imply that Q behaves like a BRST operator. The second and third properties imply

$$\langle \mathcal{O}_1 \cdots \{Q, \Lambda\} \cdots \mathcal{O}_k \rangle = 0, \tag{C.1}$$

for any physical operators $\mathcal{O}_1, \cdots \mathcal{O}_k$ and any operator Λ. This means that, it is enough to consider the physical operators up to Q-exact operators. Equivalently, we can view physical operators as in the Q-cohomology of operators. This is the reason why these theories are called cohomological field theories.

The fourth property implies that the theory is independent of the metric of spacetime since

$$\begin{aligned} \frac{\delta}{\delta g^{\mu\nu}} \langle \mathcal{O}_1 \cdots \mathcal{O}_k \rangle &= \frac{\delta}{\delta g^{\mu\nu}} \left(\int \mathcal{D}\varphi \mathcal{O}_1 \cdots \mathcal{O}_k e^{\mathrm{i}S[\varphi]} \right) \\ &= \mathrm{i} \int \mathcal{D}\varphi \mathcal{O}_1 \cdots \mathcal{O}_k T_{\mu\nu} e^{\mathrm{i}S[\varphi]} \\ &= \mathrm{i} \langle \mathcal{O}_1 \cdots \mathcal{O}_k \{Q, G_{\mu\nu}\} \rangle = 0 \ , \end{aligned} \tag{C.2}$$

for any physical operators $\mathcal{O}_1, \cdots \mathcal{O}_k$ (independent of the metric). This is the reason why cohomological field theories are topological field theories.

By integrating $T_{\mu\nu}$ we obtain

$$P_\mu = \int_\Sigma d^{d-1}x T_{0\mu} = \{Q, G_\mu\} \ , \tag{C.3}$$

for Grassmannian operator $G_\mu \equiv \int_\Sigma d^{d-1}x G_{0\mu}$ over a spatial slice Σ, and then it is convenient to define the operator-valued 1-form $G \equiv G_\mu dx^\mu$. For any scalar physical local operator $\mathcal{O}^{(0)}$, we have

$$d\mathcal{O}^{(0)} = (\partial_\mu \mathcal{O}^{(0)}) dx^\mu = \mathrm{i}[P_\mu, \mathcal{O}^{(0)}] dx^\mu = \mathrm{i}[\{Q, G\}, \mathcal{O}^{(0)}] = \{Q, \mathrm{i}\{G, \mathcal{O}^{(0)}\}\} \ . \tag{C.4}$$

Next let's define the operator-valued 1-form $\mathcal{O}^{(1)} \equiv \mathrm{i}\{G, \mathcal{O}^{(0)}\} \equiv \mathcal{O}^{(1)}_\mu dx^\mu$, and then we can do the same thing on $\mathcal{O}^{(1)}$

$$\begin{aligned} d\mathcal{O}^{(1)} &= \partial_\nu \mathcal{O}^{(1)}_\mu dx^\nu \wedge dx^\mu = \mathrm{i}[P_\nu, \mathcal{O}^{(1)}_\mu] dx^\nu \wedge dx^\mu = -\mathrm{i}[\mathcal{O}^{(1)}_\mu, \{Q, G_\nu\}] dx^\nu \wedge dx^\mu \\ &= \pm \mathrm{i} \{\{Q, \mathcal{O}^{(1)}_\mu\}, G_\nu\} dx^\nu \wedge dx^\mu + 2\{Q, \mathcal{O}^{(2)}\} = \mp \{\{P_\mu, \mathcal{O}^{(0)}\}, G_\nu\} dx^\nu \wedge dx^\mu + 2\{Q, \mathcal{O}^{(0)}\} \\ &= [P_\mu, \{G_\nu, \mathcal{O}^{(0)}\}] dx^\nu \wedge dx^\mu + 2\{Q, \mathcal{O}^{(0)}\} = -d\mathcal{O}^{(1)} + 2\{Q, \mathcal{O}^{(2)}\} \ , \end{aligned} \tag{C.5}$$

therefore

$$d\mathcal{O}^{(1)} = \{Q, \mathcal{O}^{(2)}\} \ , \tag{C.6}$$

with the operator-valued 2-form

$$\mathcal{O}^{(2)} = \frac{1}{2} \{G_\mu, \{G_\nu, \mathcal{O}^{(0)}\}\} dx^\mu \wedge dx^\nu \ . \tag{C.7}$$

This procedure can be continued until the introduction of the operator-valued top form $\mathcal{O}^{(d)}$. Therefore, we see that every scalar physical operator $\mathcal{O}^{(0)}$ induces a tower of operator forms called the *topological descendents* of $\mathcal{O}^{(0)}$, satisfying the *descent equations*

$$d\mathcal{O}^k = \{Q, \mathcal{O}^{(k+1)}\} \ , \tag{C.8}$$

and trivially $d\mathcal{O}^{(d)} = 0$.

However, $\mathcal{O}^{(k)}$ is not a physical operator unless $k = 0$. In order to construct physical ones, we define the integral $\int_\Sigma \mathcal{O}^{(k)}$ of it along a closed k-submanifold Σ. Since

$$\left\{Q, \int_\Sigma \mathcal{O}^{(k)}\right\} = \int_\Sigma \left\{Q, \mathcal{O}^{(k)}\right\} = \int_\Sigma d\mathcal{O}^{(k-1)} = 0 \ , \tag{C.9}$$

for any physical local operator $\mathcal{O}^{(0)}$, we have a sequence of physical defect operators $\int_\Sigma \mathcal{O}^{(k)}$ of dimensions $k = 1, 2, \cdots, d$.

[37–40] provide nice reviews about cohomological field theories.

C.1 Topological twists

[23] put forward a brilliant method to construct a cohomological field theory from an ordinary supersymmetric field theory, called the *topological twist*.

Given a supersymmetric field theory on a flat spacetime M_0 with holonomy group K. If there is a nonanomalous R-symmetry group R, then a topological twist is to mix the spacetime holonomy with the R-symmetry. Each of these twist is characterized by a group homomorphism f from K to R. Mathematically, the topological twists are purely redefining the generators of global symmetries, without any modification to the physics on the flat spacetime M_0.

The crucial step is to generalize the theory to curved spacetime M. For generic curved spacetime, there is no covariantly constant spinor, and in particular, the supersymmetry will typically be broken by in this generalization. However, we can sometimes properly choose f such that some supercharges live as scalars under the twisted holonomy group. Since scalars can live on any curved spacetime, these supercharges will survive on the curved spacetime.

C.2 Topological twists in 4d

C.3 Holomorphic twists

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