Branes and DAHA Representations

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Based on work with S.Nawata, Y.Zhang, S.Zhuang [arXiv:2412.19647]

Contents

INTRODUCTION TO BRANE QUANTIZATION

Quantization of a symplectic manifold

Classical system

symplectic manifold (M, ω)+Observable functions f over M

Quantization

- Construct the Hilbert space $\mathcal{H}(M, \omega)$
- Convert the functions f over M into Hermitian operators $\hat{f}$ acting on $\mathcal{H}(M, \omega)$, i.e. construct a quantization map

$$Q: C^{\infty}(M) \to \mathrm{Herm}(\mathcal{H}(M, \omega))$$

Dirac axioms of quantization

Dirac proposed $Q: C^\infty(M) \to \mathrm{Herm}(\mathcal{H}(M,\omega))$ should satisfy:

1. Q is linear
2. $[Q(f), Q(g)] = -i\hbar Q([f,g]_{\mathrm{P.B.}})$ for any $f, g \in C^\infty(M)$
3. $Q(1) = \mathrm{id}$

The natural guess is to take

$$Q(f) = -i\hbar X_f \equiv -i\hbar\left(\frac{\partial f}{\partial p_j}\frac{\partial}{\partial q^j} - \frac{\partial f}{\partial q^j}\frac{\partial}{\partial p_j}\right)$$

since $[X_f, X_g] = X_{[f,g]_{\mathrm{P.B.}}}$, but axiom 3 is violated.

Prequantization

Locally $\omega = d\eta$, and define the prequantization map

$$Q(f) = -i\hbar X_f + \left(-i_{X_f}\eta + f\right)\mathrm{id}$$

which satisfies all three axioms. But it has two issues:

1. η is defined locally
2. η has a redundancy $\eta \mapsto \eta + d\phi$

Solution: view η as defining a $U(1)$ connection over M on some complex line bundle $\mathcal{N}$ called the prequantum line bundle, then

$Q(f) = -i\hbar\nabla_{X_f} + f\,\mathrm{id}$ and ω is simply the curvature on $\mathcal{N}$.

Geometric quantization

However, the space $Q(f)$ acts is $\Gamma(\mathcal{N})$: sections over M,

while wavefunctions are local sections over the configuration space with half dimension as M.

Solution: Introduce a polarization on the space $\Gamma(\mathcal{N})$, and declare the polarized sections to be the wavefunctions, resulting in a Hilbert space $\mathcal{H}(M,\omega)$.

Deformation quantization

Deform a function $f \in C^\infty(M)$ to be a formal series $f \mapsto \sum_{k=0}^{\infty} f_k \hbar^k$

w.r.t. a formal parameter $\hbar$, and introduce a star product $f \star g$ s.t.

$$[f, g] \equiv f \star g - g \star f \xrightarrow{\hbar \to 0} -i\hbar [f, g]_{\mathrm{P.B.}} + O(\hbar)$$

The resulting algebra $(C^\infty(M)^q, \star)$ is called the deformation quantization of (M, ω) where $q = e^{2\pi i \hbar}$ and Kontsevich proved:

Every Poisson manifold has a deformation quantization.

Topological A-model

Consider the topological A-model with target a Kähler manifold $\mathfrak{X}$ with Kähler form $\omega_{\mathfrak{X}} \equiv \mathrm{Im}\Omega$ for some holomorphic 2-form Ω.

The ring $\mathcal{O}(\mathfrak{X})$ of holomorphic functions over $\mathfrak{X}$ has a deformation quantization with respect to Ω, denoted as $\mathcal{O}^q(\mathfrak{X})$ for some $q = e^{2\pi i\hbar}$

Topological A-model

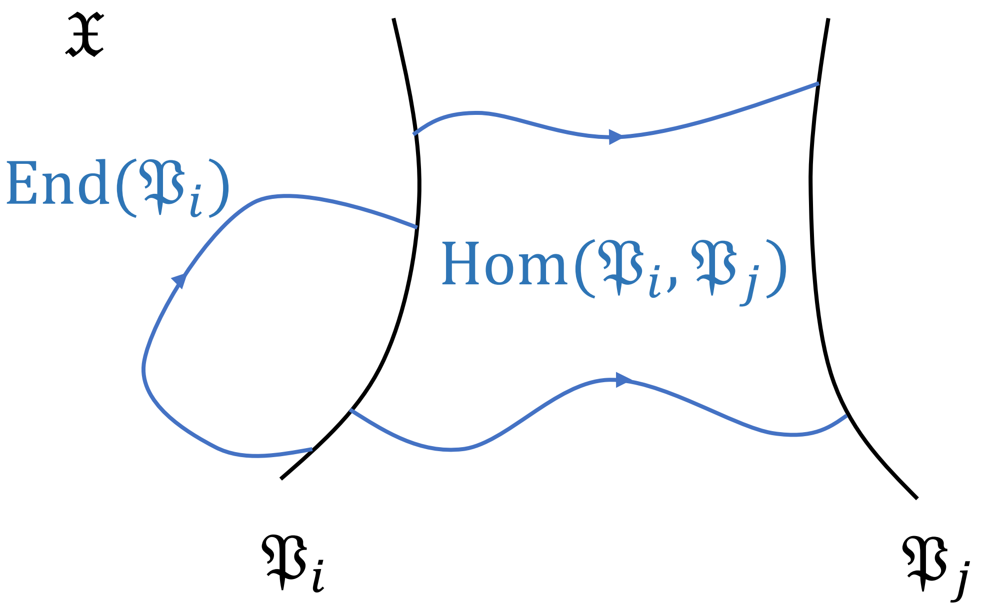
An (rank-1) A-brane is defined as a complex line bundle $\mathfrak{P}: \mathcal{L} \to Y$ for a submanifold $Y \subset \mathfrak{X}$ with a $U(1)$ connection that can provide an A-type boundary condition to the worldsheet theory.

It is proved that

- Y must be a co-isotropic submanifold of $\mathfrak{X}$
- the curvature should satisfy certain A-brane conditions.

Topological A-model

The A-brane category $\text{A-Brane}(\mathfrak{X}, \omega_{\mathfrak{X}})$.

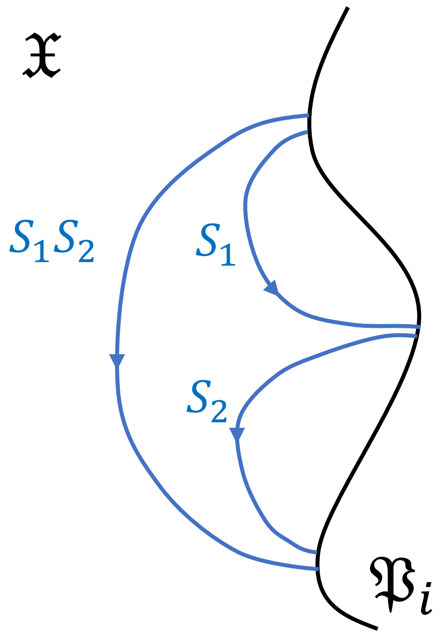


$\mathrm{Hom}(\mathfrak{P}_i, \mathfrak{P}_j)$:
The space of the topological strings between $\mathfrak{P}_i$ and $\mathfrak{P}_j$.

$\mathrm{End}(\mathfrak{P}_i)$:
The space of the topological strings from $\mathfrak{P}_i$ to itself.

Canonical co-isotropic A-brane

$\mathrm{End}(\mathfrak{P}_i)$ has an algebra structure by joining strings.



There is a unique A-brane $\mathfrak{P}_{\mathrm{cc}}: \mathcal{L} \to \mathfrak{X}$, called the canonical co-isotropic A-brane, parametrized by a parameter $\hbar$ that:

$$\mathrm{End}(\mathfrak{P}_{\mathrm{cc}}) \cong \underline{\mathcal{O}^q(\mathfrak{X})}$$

deformation quantization

for $q = e^{2\pi i \hbar}$ satisfying the A-brane condition

$$F + B = \mathrm{Re}\Omega$$

Lagrangian A-branes

Every Lagrangian submanifold $M \subset \mathfrak{X}$, i.e. $\omega_{\mathfrak{X}}|_M = 0$ has a symplectic structure by the restriction $\mathrm{Re}\Omega|_M$.

It can support an A-brane $\mathfrak{P}_M\colon \tilde{\mathcal{L}} \otimes K_M^{-1/2} \to M$ with the A-brane condition becoming a flatness condition

$$\tilde{F} + B\Big|_M = 0$$

Lagrangian A-branes

Then it can be shown

$$\mathrm{Hom}(\mathfrak{P}_M, \mathfrak{P}_{\mathrm{cc}}) \cong \underline{\mathcal{H}\left(M, \mathrm{Re}\Omega\big|_M\right)}$$

Hilbert space after geometric quantization

w.r.t the prequantum line bundle $\mathcal{N} = \mathfrak{P}_M \otimes \mathfrak{P}_{\mathrm{cc}}{}^{-1}$.

When $\dim_{\mathbb{C}} \mathfrak{X} = 2$ we can use the B-model perspective to calculate the dimension of the Hilbert space to be

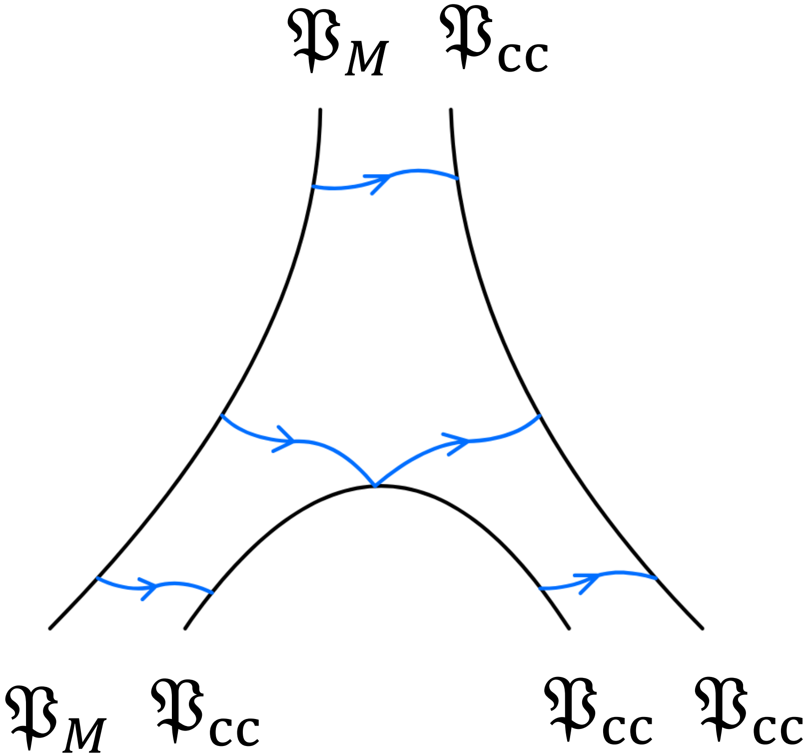
$$\dim \mathcal{H}\left(M, \mathrm{Re}\Omega\big|_M\right) = \frac{\mathrm{vol}(M)}{\hbar} = \boxed{\int_M \frac{\mathrm{Re}\Omega}{2\pi} \in \mathbb{Z}}$$

quantization condition

which provides a criterion of the existence of a Lagrangian A-brane on M.

Relate Geometry to Algebra

$\mathrm{End}(\mathfrak{P}_{cc})$ acts naturally on $\mathrm{Hom}(\mathfrak{P}_M, \mathfrak{P}_{cc})$ by right composition:



This provides a correspondence:

$$\mathrm{A{-}Brane}(\mathfrak{X}, \omega_{\mathfrak{X}}) \leftrightarrow \mathrm{Rep}(\mathcal{O}^q(\mathfrak{X}))$$

In particular:

compact A-branes $\leftrightarrow$ finite-dim reps

non-compact A-branes $\leftrightarrow$ infinite-dim reps

In our project

$\mathfrak{X}$ is the Coulomb branch of vacua of the 4d $\mathcal{N} = 2\, SU(2)$ SQCD with 4 hypermultiplets, compactified on S^1, which is a hyper-Kähler manifold with:

complex structures $I\,, J, K$, Kähler forms $\omega_I, \omega_J, \omega_K$

holomorphic symplectic forms

$$\Omega_I = \omega_J + i\omega_K, \qquad \Omega_J = \omega_K + i\omega_I, \qquad \Omega_K = \omega_I + i\omega_J$$

In our project

Set $\Omega = \Omega_J / i\hbar$, the deformation quantization $\mathcal{O}^q(\mathfrak{X}) \cong S\ddot{H}$ is the spherical Double Affine Hecke Algebra(DAHA) of type $C^\vee C_1$.

Assume $\hbar = |\hbar| e^{i\vartheta}$ then

$$\mathrm{Re}\Omega = \frac{1}{|\hbar|}(\omega_I \cos\vartheta - \omega_K \sin\vartheta)$$

$$\omega_{\mathfrak{X}} = \mathrm{Im}\Omega = -\frac{1}{|\hbar|}(\omega_I \sin\vartheta + \omega_K \cos\vartheta)$$

If $\hbar$ is real $\mathrm{Re}\Omega = \omega_I / |\hbar|$, $\omega_{\mathfrak{X}} = -\omega_K / |\hbar|$

If $\hbar$ is pure imaginary $\mathrm{Re}\Omega = -\omega_K / |\hbar|$, $\omega_{\mathfrak{X}} = \omega_I / |\hbar|$

In our project

1. **Geometric side:**
   Identify the Lagrangian submanifolds M i.e. $\omega_{\mathfrak{X}}|_M = 0$ in $\mathfrak{X}$, and find their quantization conditions $\mathrm{vol}(M)/\hbar = \int_M \mathrm{Re}\Omega/2\pi \in \mathbb{Z}$
2. **Algebraic side:**
   Construct the representations of $S\ddot{H}$, find the conditions of appearance of these representations: shortening conditions.
3. **Matching A-branes and representations:**
   By explicitly matching the quantization conditions and the shortening conditions.

GEOMETRIC SIDE: A-BRANES

Geometry of Coulomb branch

Views from distinct complex structures provide multiple tools to analyze the geometry of $\mathfrak{X}$:

In complex structure I $\mathfrak{X}$ is the total space of a Hitchin system

In complex structure J $\mathfrak{X}$ is an affine cubic surface

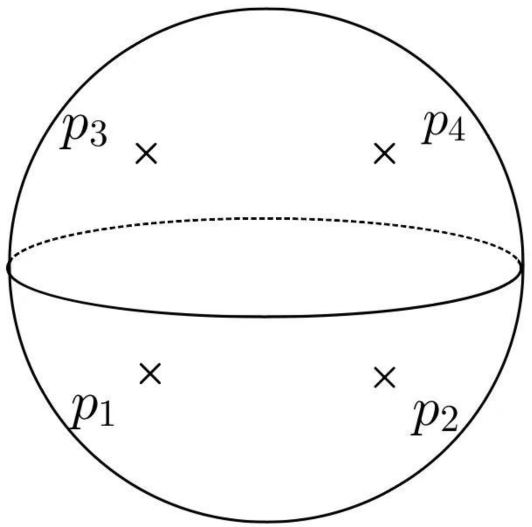
In complex structure K $\mathfrak{X}$ is the conjugation as in complex structure I

In complex structure I: Hitchin system

$\mathfrak{X}$ is the moduli space $\mathcal{M}_H\left(C_{0,4}, SU(2)\right)$ of Higgs bundles (E, A, φ) over $C_{0,4}$, with ramification parameters $\alpha_j, \beta_j, \gamma_j$ around puncture p_j ($j = 1,2,3,4$):

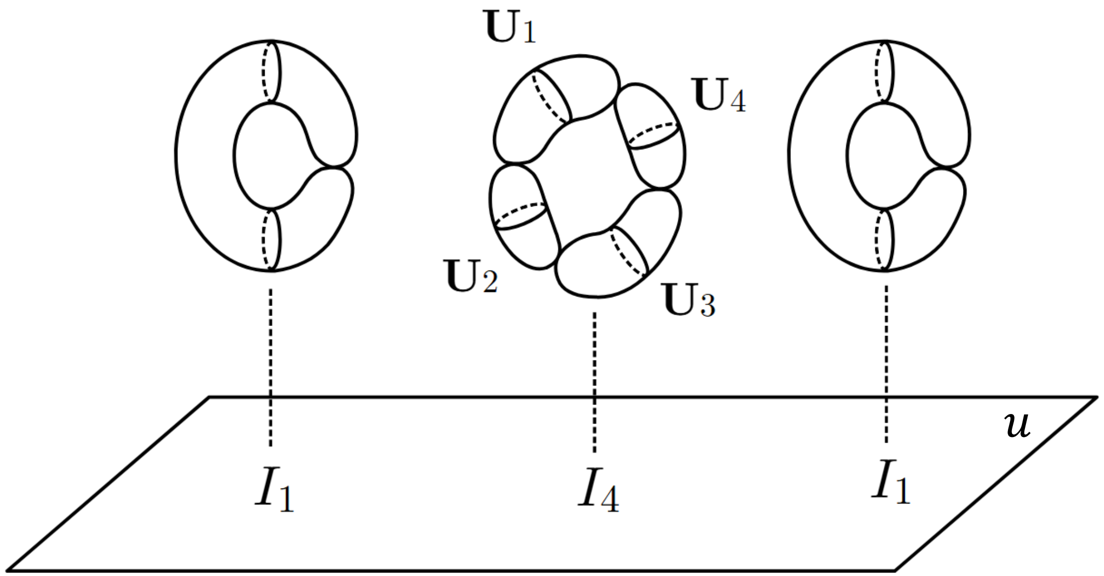
$$A = \alpha_j d\vartheta + \cdots$$
$$\varphi = (\beta_j + \gamma_j)\frac{dz}{2z} + \cdots$$

where $z = re^{i\vartheta}$ is the local coordinates.



In complex structure I: Hitchin system

$\mathfrak{X}$ admits a fibration structure, called the Hitchin fibration, projecting to the Hitchin base (u-plane).



Generic fibers

compact tori

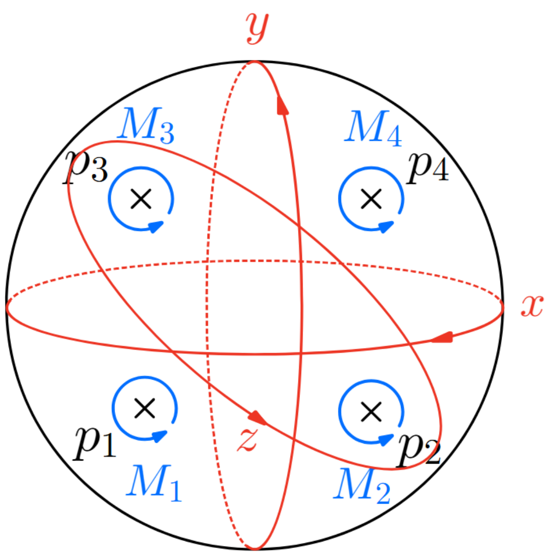
Singular fibers

by Kodaira classification

In complex structure J: cubic surface

$\mathfrak{X}$ is the moduli space $\mathcal{M}_{\text{flat}}(C_{0,4}, SL(2,\mathbb{C}))$ of flat $SL(2,\mathbb{C})$-connections over $C_{0,4}$, described by the cubic equation

$$-xyz + x^2 + y^2 + z^2 + \theta_1 x + \theta_2 y + \theta_3 z + \theta_4 = 0$$



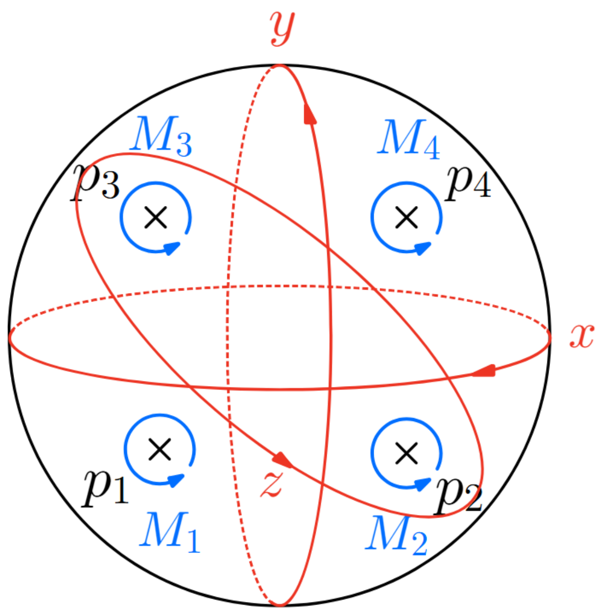
$$\theta_1 = \bar{t}_1\bar{t}_2 + \bar{t}_3\bar{t}_4$$

$$\theta_2 = \bar{t}_1\bar{t}_3 + \bar{t}_2\bar{t}_4$$

$$\theta_3 = \bar{t}_1\bar{t}_4 + \bar{t}_2\bar{t}_3$$

for four parameters t_j .

In complex structure J: cubic surface



parameters t_j can be interpreted as monodromy parameters around p_j that

$$t_j + {t_j}^{-1} = \mathrm{Tr}(M_j)$$

related to $\alpha_j, \beta_j, \gamma_j$ by

$$t_j = \exp(-2\pi(\gamma_j + i\alpha_j))$$

Identifying Lagrangian submanifolds

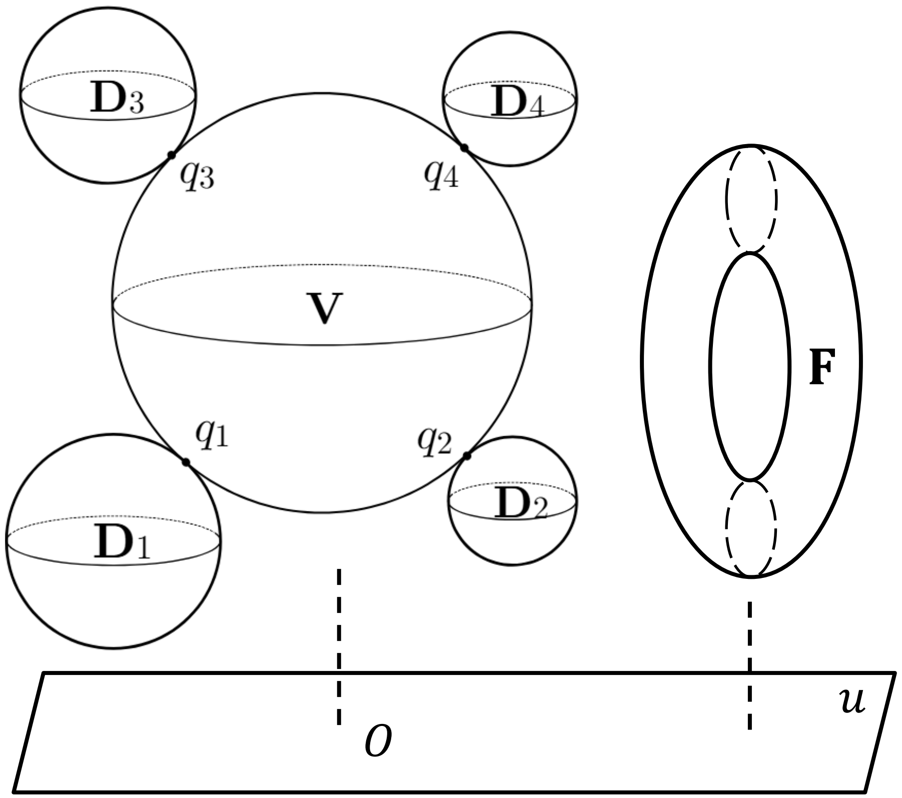
Set $\beta_j = \gamma_j = 0$ there is a single singular fiber of type I_0^* at $u = 0$.

Set $\hbar$ real the compact Lagrangian submanifolds are the components of fibers:

Generic fiber $\mathbf{F}$ with unit volume

Components $\mathbf{D}_j, \mathbf{V}$ of the I_0^* singular fiber satisfying

$$2\mathrm{vol}(\mathbf{V}) + \sum_{j=1}^{4} \mathrm{vol}(\mathbf{D}_j) = 1$$



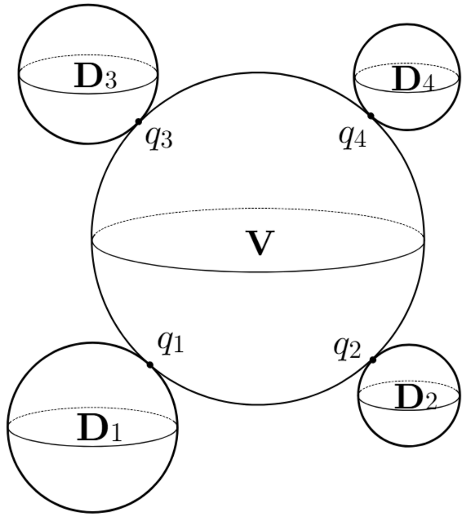
Volumes and wall-crossing

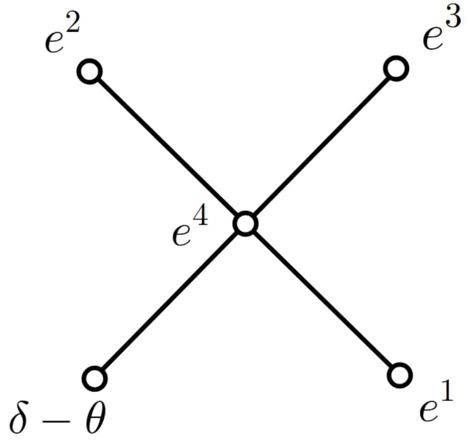
The volumes are parametrized by the ramification parameter α_j as

$$\mathrm{vol}(\mathbf{D}_1) = 1 - \theta \cdot \alpha, \qquad \mathrm{vol}(\mathbf{D}_2) = e^1 \cdot \alpha,$$
$$\mathrm{vol}(\mathbf{D}_3) = e^2 \cdot \alpha, \qquad \mathrm{vol}(\mathbf{D}_4) = e^3 \cdot \alpha,$$
$$\mathrm{vol}(\mathbf{V}) \quad = e^4 \cdot \alpha$$

with e^a the simple D_4 roots, θ the highest D_4 root and

$$\theta \cdot \alpha = \sum_{j=1}^{4} \theta_j \alpha_j\,, \qquad e^a \cdot \alpha \equiv \sum_{j=1}^{4} (e^a)_j \alpha_j$$





Volumes and wall-crossing

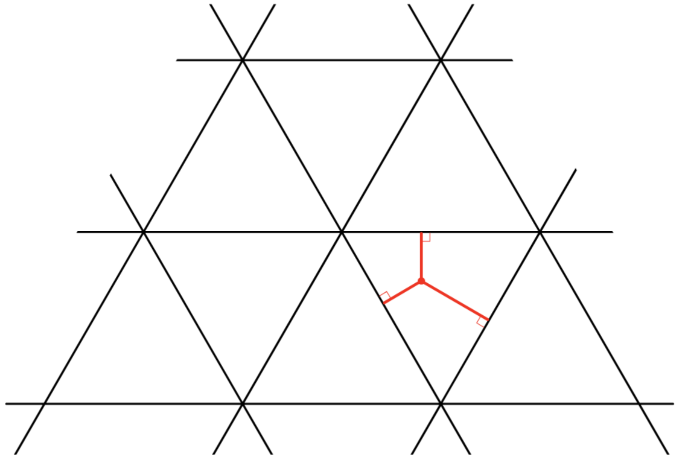
The volumes exhibit complicated wall-crossing behavior with walls determined by

$$t^r = t_1^{r_1} t_2^{r_2} t_3^{r_3} t_4^{r_4} = 1, \qquad \forall\, r = (r_1, r_2, r_3, r_4) \in \mathrm{R}(D_4)$$

When crossing the wall determined by a root r^a, the volumes transform by Weyl reflecting the root vectors

$$r^b \longmapsto s_{r^a}(r^b)$$

The chambers then correspond to the affine Weyl alcove of type D_4.



ALGEBRAIC SIDE: REPRESENTATIONS OF DAHA

Spherical DAHA of type $C^{\vee}C_1$

$S\ddot{H}$ is parametrized by 4 parameters t_j and a deformation parameter q, and can be presented as

$$[x,y]_q = (q^{-1}-q)z + \left(q^{-\frac{1}{2}} - q^{\frac{1}{2}}\right)\theta_3,$$

$$[y,z]_q = (q^{-1}-q)x + \left(q^{-\frac{1}{2}} - q^{\frac{1}{2}}\right)\theta_1,$$

$$[z,x]_q = (q^{-1}-q)y + \left(q^{-\frac{1}{2}} - q^{\frac{1}{2}}\right)\theta_2,$$

$$-q^{-\frac{1}{2}}xyz + q^{-1}x^2 + qy^2 + q^{-1}z^2 + q^{-\frac{1}{2}}\theta_1 x + q^{\frac{1}{2}}\theta_2 y + q^{-\frac{1}{2}}\theta_3 z + \theta_4(q) = 0,$$

where

$$[x,y]_q \equiv q^{-\frac{1}{2}}xy - q^{\frac{1}{2}}yx$$

Spherical DAHA of type $C^\vee C_1$

$$\theta_1 = \bar{t}_1\bar{t}_2 + \bar{t}_3\bar{t}_4 = \chi_{\mathrm{V}}$$

$$\theta_2 = \bar{t}_1\bar{t}_3 + \bar{t}_2\bar{t}_4 = \chi_{\mathrm{S}}$$

$$\theta_3 = \bar{t}_1\bar{t}_4 + \bar{t}_2\bar{t}_3 = \chi_{\mathrm{C}}$$

$$\theta_4(q) = \bar{t}_1{}^2 + \bar{t}_2{}^2 + \bar{t}_3{}^2 + \bar{t}_4{}^2 + \bar{t}_1\bar{t}_2\bar{t}_3\bar{t}_4 - \bar{q} - 2 = \chi_{\mathrm{adj}} - \bar{q} + 2$$

with $\bar{t} \equiv t + t^{-1}$and $\chi_{\mathfrak{R}}$ the character of the representation $\mathfrak{R}$ of D_4 Lie algebra. $\Leftarrow$ $SO(8)$ flavor symmetry

Interpretation from physics

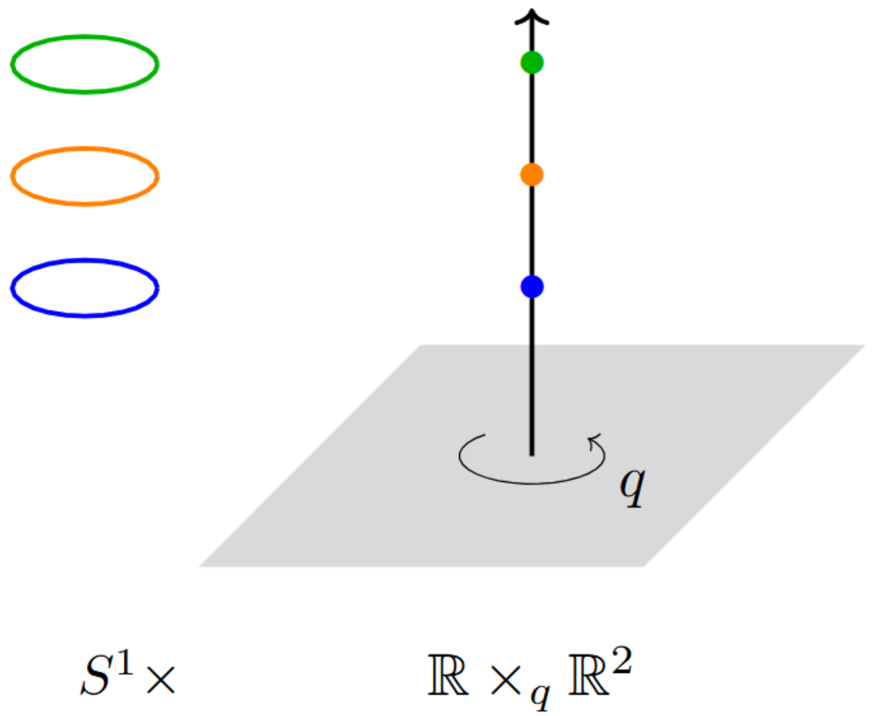
The coordinate ring $\mathcal{O}(\mathfrak{X})$ is generated by x, y, z. Physically

$$x \leftrightarrow \text{Wilson loop}$$

$$\text{y} \leftrightarrow \text{' t Hooft loop}$$

$$z \leftrightarrow \text{dyonic loop}$$

The deformation quantization $\mathcal{O}^q(\mathfrak{X}) \cong S\ddot{H}$ can be interpreted as putting the 4d theory on an Ω-background with parameter q.



Symmetries of $S\ddot{H}$

Weyl group $W(D_4)$ Since parameters are in D_4 characters.

Braid group B_3

$$\tau_+ : (x, y, z, \theta_1, \theta_2, \theta_3, \theta_4) \longmapsto (x, q^{-\frac{1}{2}}(xy - q^{-\frac{1}{2}}z - \theta_3), y, \theta_1, \theta_3, \theta_2, \theta_4)$$

$$\tau_- : (x, y, z, \theta_1, \theta_2, \theta_3, \theta_4) \longmapsto (q^{-\frac{1}{2}}(xy - q^{-\frac{1}{2}}z - \theta_3), y, x, \theta_3, \theta_2, \theta_1, \theta_4)$$

$$\sigma : (x, y, z, \theta_1, \theta_2, \theta_3, \theta_4) \longmapsto (y, x, q^{-\frac{1}{2}}(xy - q^{-\frac{1}{2}}z - \theta_3), \theta_2, \theta_1, \theta_3, \theta_4)$$

Sign-flip $\mathbb{Z}_2 \times \mathbb{Z}_2$

$$\xi_1 : (x, y, z, \theta_1, \theta_2, \theta_3, \theta_4) \longmapsto (-x, y, -z, -\theta_1, \theta_2, -\theta_3, \theta_4)$$

$$\xi_2 : (x, y, z, \theta_1, \theta_2, \theta_3, \theta_4) \longmapsto (x, -y, -z, \theta_1, -\theta_2, -\theta_3, \theta_4)$$

$$\xi_3 : (x, y, z, \theta_1, \theta_2, \theta_3, \theta_4) \longmapsto (-x, -y, z, -\theta_1, -\theta_2, \theta_3, \theta_4)$$

Polynomial representations

The Askey-Wilson polynomials are defined by

$$P_n(X;q,t) = \frac{(qab,qac,qad;q)_n}{q^{\frac{n}{2}}a^n(abcdq^{n+1};q)_n}\,{}_4\phi_3\left(\begin{matrix} q^{-n}, q^{n+1}abcd, q^{\frac{1}{2}}aX, q^{\frac{1}{2}}aX^{-1} \\ qab, qac, qad \end{matrix}; q, q\right)$$

$$a = -t_1t_2, \qquad b = -\frac{t_1}{t_2}, \qquad c = -t_3t_4, \qquad d = -\frac{t_3}{t_4}$$

which is acted by $S\ddot{H}$ by

$$x \cdot P_n(X) = P_{n+1}(X) + (\cdots)P_n(X) + (\cdots)P_{n-1}(X)$$

$$y \cdot P_n(X) = (\cdots)P_n(X)$$

$$z \cdot P_n(X) = (\cdots)P_{n+1}(X) + (\cdots)P_n(X) + (\cdots)P_{n-1}(X)$$

Polynomial representations

Construct raising and lowering operators

$$R_n = -q^{1-n}x - qt_1t_3z + A_n, \qquad L_n = -q^{n+2}x - \frac{q}{t_1t_3}z + B_n$$

for some constants A_n, B_n. They act by

$$R_n \cdot P_n(X) = r_nP_{n+1}(X), \qquad L_n \cdot P_n(X) = l_nP_{n-1}(X)$$

for some constant coefficients r_n and l_n depending on q and t_j.

By imposing the Weyl group actions and the braid group actions on $P_n(X)$, we can obtain 24 distinct polynomial representations.

Shortening conditions

Under certain conditions (shortening conditions), l_n can vanish for some n, then $P_n, P_{n+1} \cdots$ generate an infinite-dimensional submodule. And thus, we can construct a finite-dimensional representation by taking the quotient with the submodule.

$$0 \xleftarrow[L_0]{} P_0 \underset{L_1}{\overset{R_0}{\rightleftarrows}} P_1 \underset{L_2}{\overset{R_1}{\rightleftarrows}} \cdots \underset{L_{n-1}}{\overset{R_{n-2}}{\rightleftarrows}} P_{n-1} \underset{L_n}{\overset{R_{n-1}}{\rightleftarrows}} P_n \rightleftarrows \cdots$$

The shortening conditions of all the 24 polynomial representations can be expressed as

$$q^n \equiv t_1^{-r_1} t_2^{-r_2} t_3^{-r_3} t_4^{-r_4} = t^{-r}, \forall\, r = (r_1, r_2, r_3, r_4) \in \mathrm{R}(D_4) \cup \{0,0,0,0\}$$

MATCHING GEOMETY & ALGEBRA

Explicit matching

A-branes	A-brane volumes	A-brane conditions	Shortening conditions
$\mathfrak{P}_{\mathbf{F}}$	1	$\frac{1}{\hbar} = m \in \mathbb{Z}$	$q^m = 1$
$\mathfrak{P}_{\mathbf{D}_1}$	$1 - \theta \cdot \alpha$	$\frac{1 - \theta \cdot \alpha}{\hbar} = l_1 \in \mathbb{Z}$	$q^{l_1} = t^{\theta}$
$\mathfrak{P}_{\mathbf{D}_2}$	$e^1 \cdot \alpha$	$\frac{e^1 \cdot \alpha}{\hbar} = l_2 \in \mathbb{Z}$	$q^{l_2} = t^{-e^1}$
$\mathfrak{P}_{\mathbf{D}_3}$	$e^2 \cdot \alpha$	$\frac{e^2 \cdot \alpha}{\hbar} = l_3 \in \mathbb{Z}$	$q^{l_3} = t^{-e^2}$
$\mathfrak{P}_{\mathbf{D}_4}$	$e^3 \cdot \alpha$	$\frac{e^3 \cdot \alpha}{\hbar} = l_4 \in \mathbb{Z}$	$q^{l_4} = t^{-e^3}$
$\mathfrak{P}_{\mathbf{V}}$	$e^4 \cdot \alpha$	$\frac{e^4 \cdot \alpha}{\hbar} = k \in \mathbb{Z}$	$q^k = t^{-e^4}$

$$q = e^{2\pi i \hbar}$$

$$t_j = e^{-2\pi i \alpha_j}$$

Summary

1. Analysis of the Lagrangian submanifolds of $\mathfrak{X}$, calculation of their volumes and obtain their quantization conditions.
2. Classification of the finite dimensional representations of the $S\ddot{H}$ algebra by shortening conditions of the polynomial representations
3. One-to-one correspondence between quantization conditions and shortening conditions.