Abstract

The basic premise of Ramsey Theory states that in a sufficiently large system, complete disorder is impossible. One instance from the world of graph theory says that given two fixed graphs F and H, there exists a finitely large graph G such that any red/blue edge coloring of the edges of G will produce a red copy of F or a blue copy of H. Much research has been conducted in recent decades on quantifying exactly how large G must be if we consider different classes of graphs for F and H. In this thesis, we explore several Ramsey-type problems with a particular focus on paths and cycles. We first examine the bipartite size Ramsey number of a path on n vertices, $\hat{br}(P_n)$, and give an upper bound using a random graph construction motivated by prior upper bound improvements in similar problems. Next, we consider the size Ramsey number $\hat{R}(\mathcal{C}, P_n)$ and provide a significant improvement to the upper bound using a very structured graph, the cube of a path, as opposed to a random construction. We also prove a small improvement to the lower bound and show that the r-colored version of this problem is asymptotically linear in rn. Lastly, we give an upper bound for the online Ramsey number $\tilde{R}(\mathcal{C}, P_n)$.

MONTCLAIR STATE UNIVERSITY

Bipartite, size, and online Ramsey numbers of some cycles and paths

by

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1 Introduction

A *graph* $G = (V, E)$ is an ordered pair that consists of a set of vertices V and a set of edges, E, connecting pairs of vertices. The origin of graph theory is widely attributed to Leonard Euler's famous solution to the Königsberg bridge problem in his paper titled, *Solutio Problematis ad Geometriam Situs Pertinentis* [1]. The field of graph theory has since become a central part of combinatorics and has had a variety of applications in computer science, biology, and numerous other disciplines that impact our society as a whole. We begin this thesis with the definitions of several well-studied graphs that are pertinent to the work discussed herein. As a note, many of the definitions in this thesis follow those from *Introduction to Graph Theory* by Douglas West [2].

Perhaps the most prevalent graph in this paper is a *path* P_n on n vertices. P_n is a graph on vertex set $[n] = \{1, 2, \ldots, n\}$ whose vertices can be ordered such that there is an edge between two vertices if and only if they are ordered consecutively. We call the vertices only adjacent to one other vertex in the path the *endpoints*. A *cycle* on n vertices, denoted C_n, is a graph whose vertices may be laid along a circle such that two vertices are adjacent if and only if they are next to each other along the circle. We define a *forest* as a graph with no cycles and a *tree* as a connected forest (i.e., there is a path connecting any two vertices in the forest). Figure 1 shows the graphical representation of a path and a cycle; note that the path is a tree since it is connected and contains no cycle. Lastly, the *complete graph* K_n on n vertices is a graph on n vertices with all $\binom{n}{2}$ possible edges between pairs of vertices.

In 1930, English mathematician Frank Ramsey [3] proved two theorems that ultimately birthed a subfield of Graph Theory and Combinatorics known as Ramsey Theory. The basic premise of Ramsey Theory is that complete disorder is impossible. One primary instance of this idea in the context of Graph Theory is given fixed graphs F and H, there exists a finitely

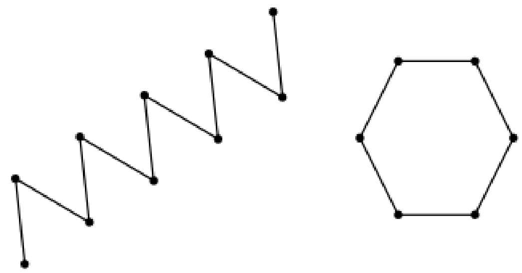

Figure 1: On the left is the path P_{10}. On the right is the cycle C_6.

large graph G such that any red/blue edge coloring of the edges of G will produce a red copy of F or a blue copy of H.

A classic introductory question to Ramsey Theory is the following: How many people are necessary to ensure that three people are mutual friends or three people are mutual strangers? (Here, we're assuming two people must either be friends or strangers.) To relate this problem to graphs, we represent people as vertices and their relationships as edges. If two people are friends, we color the edge between them blue; otherwise, we color the edge between them red. Therefore, this problem really challenges us to find the smallest integer n such that any red/blue edge coloring of K_n will produce a monochromatic copy of K_3.

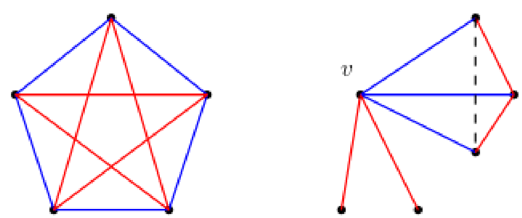

Figure 2: The graph on the left shows why we must have more than 5 people. The graph on the right shows 6 people is enough.

The solution to this problem is $n = 6$. The graph on the left in Figure 2 demonstrates a coloring of K_5 such that there is no monochromatic K_3, thus proving that $n > 5$. The graph on the right shows why $n \leq 6$. Consider an arbitrary vertex v and the 5 vertices adjacent to v (called the *neighbors* of v). By the pigeonhole principle, v must have 3 neighbors that are all red or all blue; without loss of generality, we assume v has 3 blue neighbors. Then none of those vertices can have a blue edge between them, since then we would have a blue K_3. Thus all of the edges between them must be red which produces red K_3. So any red/blue edge coloring of K_6 must contain a monochromatic K_3.

Let F and H be fixed graphs. We say $G \to (F, H)$ if every red/blue coloring of the edges of G yields a red copy of F or a blue copy of H. The (ordinary) *Ramsey number* $R(F, H)$ is defined as

$$R(F, H) = \min \{n : K_n \to (F, H)\}.$$

To prove $R(F, H) = n$ for some integer n, one must show that $R(F, H) > n - 1$ and $R(F, H) \leq n$. To prove $R(F, H) > n - 1$, we must give a red/blue coloring of K_{n-1} such that there is no red F and no blue H. To prove $R(F, H) \leq n$, we must show that every red/blue coloring of K_n produces a red F or a blue H. The above result therefore proves that $R(K_3, K_3) = 6$ since we showed $R(K_3, K_3) > 5$ and $R(K_3, K_3) \leq 6$ by the two graphs in Figure 2.

Some research has been done to find Ramsey numbers for special graphs. A *bipartite graph* B is a graph whose vertices can be partitioned into two disjoint sets X and Y such that there are no edges within X or Y. The *complete bipartite graph* $K_{n,n}$ is a bipartite graph with each part of order n and each vertex adjacent to all n vertices in the other part. Faudree and Schelp [4] initiated the study of the *bipartite Ramsey number* $br(F, H)$, which has a related definition to the ordinary Ramsey number:

$$br(F, H) = \min \{n : K_{n,n} \to (F, H)\}.$$

Rather than studying the *order* of a complete graph necessary to guarantee a monochromatic copy of two fixed graphs F and H, Erdős, Faudree, Rousseau, and Schelp [5] introduced the concept of the *size Ramsey number*, which studies the sufficient *size* required to produce two fixed graphs. Let F and H be fixed graphs. The size Ramsey number $\hat{R}(F, H)$ is defined as

$$\hat{R}(F, H) = \min \{|E(G)| : G \to (F, H)\}.$$

In the case where $F = H$, we write $\hat{R}(F, H)$ as $\hat{R}(F)$. If $\mathcal{F}$ is a family of graphs, we say $G \to (\mathcal{F}, H)$ if every red/blue coloring of the edges of G contains a monochromatic red copy of some graph from $\mathcal{F}$ or a monochromatic blue copy of H, and we define $\hat{R}(\mathcal{F}, H)$ accordingly. To prove $\hat{R}(F, H) \geq m$, we must show that $G \nrightarrow (F, H)$ for every graph G with $m - 1$ edges. To prove $\hat{R}(F, H) \leq m$, we must show the existence of a graph G with m edges such that $G \to (F, H)$. By definition of $R(F, H)$, one can see that $\hat{R}(F, H) \leq \binom{R(F,H)}{2}$. It was shown in [5] that this bound is tight when $F = K_n$ and $H = K_m$ for some $n, m \in \mathbb{N}$. Moreover, [5] also initiated the size Ramsey number of trees in proving that $\hat{R}(K_{1,m}, K_{1,n}) = m + n - 1$.

The concentration on trees continued and remains one of the most studied areas in size Ramsey numbers today. Friedman and Pippenger [6] showed that for every n, there exists a graph G with $O(n)$ edges such that after the removal of all but $\delta|E(G)|$ edges for some $\delta > 0$, G continues to contain every tree with n vertices and maximum degree at most d. Research continued for the size Ramsey number of trees in general [7, 8, 9], but there is particular interest in $\hat{R}(P_n)$. The pursuit in finding the best bounds for $\hat{R}(P_n)$ became widespread when Erdős [10] famously offered \$100 for a proof or disproof that

$$\hat{R}(P_n)/n \to \infty \quad \text{and} \quad \hat{R}(P_n)/n^2 \to 0.$$

This was first solved by Beck [11] who found that $\hat{R}(P_n) < 900n$ for sufficiently large n (we note that all size Ramsey numbers discussed in this thesis are for sufficiently large n). Following a series of incremental improvements on the upper [12, 13, 14, 15] and lower bounds [16, 15, 12, 17], the best current bounds are $(3.75 + o(1))n \leq \hat{R}(P_n) \leq 74n$ for sufficiently large

n. This motivated our work in the following section.

2 On the bipartite size Ramsey number $\hat{br}(P_n)$

2.1 Definitions and Proof Idea

A *bipartite graph* B is a graph whose vertices can be partitioned into two disjoint sets X and Y such that there are no edges within X or Y. Let F and H be fixed graphs. We define the *bipartite size Ramsey number* $\hat{br}(F, H)$ as

$$\hat{br}(F, H) = \min\{|E(B)| : B \to (F, H),\ B \text{ is bipartite}\},$$

Sun and Li showed [18] that for a fixed integer m, $m2^m n/e \leq \hat{br}(K_{m,n}) \leq 4m^2 2^m n$ and $n^2 2^n/15 \leq \hat{br}(K_{n,n}) \leq 3n^3 2^n$ for sufficiently large n. Our work is on the upper bound of the bipartite size Ramsey number $\hat{br}(P_n)$. Our motivation for this problem is from Dudek and Prałat's [15] bound of $\hat{R}(P_n) \leq 74n$. We were curious as to how the bipartite property would influence the linear coefficient of n.

We first introduce several useful definitions. A *directed graph* ("digraph") D is a triple $D = (V, E, f)$, where V is the set of vertices, E is the set of edges, and f is a function assigning each edge an ordered pair of vertices. The first vertex in the ordered pair is called the "tail" of the edge and the second vertex is called the "head." We say an edge in a digraph goes from its tail vertex to the head vertex. The $out-degree$ of a vertex v in a digraph is the number of edges for which v is the tail. A *random r-out graph* $\mathscr{G}_{n,r}$ is a graph on n vertices where each vertex has out-degree r and the nr edge heads are independently and uniformly distributed over the vertices. A *random r-out bipartite graph* $\mathscr{B}_{n,n,r}$ is a bipartite graph where each vertex has out-degree r and the nr edge heads from one part are independently and uniformly distributed over the vertices in the other part. Lastly, for two subsets X, Y of vertices, we use $e(X, Y)$ to represent the number of edges with one endpoint in X and one in Y. Using these definitions, we state the main result of this section.

Theorem 2.1. *Let $r = 18$ and $c = 3.614$. If $B = \mathscr{B}_{cn,cn,r}$ is a random r-out bipartite graph on $2cn$ vertices, then asymptotically almost surely $B \to P_n$. Thus, $\hat{br}(P_n) \leq 130.104n$ for sufficiently large n.*

To show an upper bound on a bipartite size-Ramsey number $\hat{br}(P_n) \leq k$, we must show there exists a red/blue coloring of a bipartite graph B on k edges such that there is always a monochromatic copy of P_n. Using the Lemma 2.2, we can show an upper bound by finding a graph B with no sufficiently large bipartite "holes." By bipartite holes, we mean two sets of vertices, one set in each part of the graph, in which there are no edges between the sets of vertices. We construct a graph with no sufficiently large bipartite holes by considering a random r-out bipartite graph on $2cn$ vertices. First, we mimic a lemma in [15] and utilize the depth first search ("DFS") algorithm to show the following:

Lemma 2.2. *Let B be a bipartite graph with parts X, Y each of order cn for some $c > 2$. Assume that for every two disjoint sets of vertices S' and T' such that $|S'| = |T'| = n(c-1.5)/4$ we have $e(S', T') \neq 0$. Then, $B \to P_n$.*

Proof. We prove the lemma by contrapositive; namely, we assume $B \nrightarrow P_n$ and show there exists two disjoint sets of vertices S' and T' such that $|S'| = |T'| = n(c-1.5)/4$. We find such sets by first conducting the DFS algorithm on the red edges in B as follows: Let v_1 be an arbitrary vertex in X, and let $P = (v_1), U = V(B) \setminus \{v_1\}, W = \emptyset$. We look for any red edge incident to v_1; if such an edge exists between v_1 and another vertex, say v_2, then we extend the red path to $P = (v_1, v_2)$ and we remove v_2 from U. We now continue the search for a red edge from v_2, repeating the same process we did to find v_2. Since $B \nrightarrow P_n$, we will reach a vertex v_k for some $k < n$ such that there are no red edges extending from v_k. When this occurs we put the vertex v_k in W and remove it from P. We now continue the search from vertex v_{k-1} until we find another vertex v_i that has no other red edges other than the one incident to it and v_{i-1}. If P is ever reduced to a single point, we choose another arbitrary vertex in X and begin the search again.

We make several important observations. Firstly, there is never a red edge between U and W and $|U| - |W| > 0$ at the beginning of the algorithm. Since B is a bipartite graph and by assumption there is no red P_n, the red path P can have at most $n/2$ vertices in each part; hence $|U \cup W| \geq n(2c-1)$ and $|X \backslash (P \cap X)| = |Y \backslash (P \cap Y)| \geq n(2c-1)/2$. At each step of the algorithm, U either decreases in order by one or W increases in order by one, so $|U| - |W|$ decreases by exactly one at every iteration of the algorithm. Thus at some point there must be a step in the algorithm in which $|U| - |W| = 0$ with $|U| = |W| = n(2c-1)/2$, which is when we stop the algorithm on the red edges.

Let $U' = U \cap Y, U'' = U \cap X, W' = W \cap X$, and $W'' = W \cap Y$. Then one can see that $|U| = |W| = |U' \cup U''| = |W' \cup W''| = |U' \cup W''| = |W' \cup U''| = n(2c-1)/2$, so it follows that either $|U'| = |W'| \geq n(2c-1)/4$ or $|U''| = |W''| \geq n(2c-1)/4$. Without loss of generality, assume $|U'| = |W'| \geq n(2c-1)/4$ and consider the blue subgraph B' induced on $U' \cup W'$ (note this gives us a bipartite graph B' with parts U' and W'). We now run the DFS algorithm on the blue edges of B' starting with $P' = (u_1), S = V(B') \backslash \{v_1\}, T = \emptyset$ for an arbitrary vertex u_1 in B'. Similar to the algorithm on the red edges, there are no blue edges between S and T and the value $|S| - |T|$ is positive at the beginning of the algorithm. Also by supposition there is no blue P_n, so at some point in the algorithm we must have $|S| = |T| = n(c-1.5)/2$. We now let $S' = S \cap W', S'' = S \cap U', T' = T \cap U'$, and $T'' = T \cap W'$ and similarly note that $|S| = |T| = |S' \cup S''| = |T' \cup T''| = |S' \cup T''| = |S' \cup T''| = n(2c-3)/4$. Then either $|S'| = |T'| \geq n(c-1.5)/4$ or $|S''| = |T''| \geq n(c-1.5)/4$, so without loss of generality we let $|S'| = |T'| \geq n(c-1.5)/4$. Now we have sets S' and T' such that $|S'| = |T'| = n(c-1.5)/4$

(we may discard any "leftover" vertices) and $e(S', T') = 0$ since there are no red or blue edges between S' and T', so the proof is finished. $\square$

2.2 Proof of Theorem 2.1

Before we begin the proof of the main theorem for this section, we first mention some useful well-known lemmas.

Lemma 2.3 (Stirling's Formula)**.** *Let n be an integer. Then as $n \to \infty$,*

$$n! = (1 + o(1)) \left(\frac{n}{e}\right)^n \sqrt{2\pi n}.$$

Lemma 2.4 (Markov's Inequality)**.** *Let X be a random variable with $\mathbb{E}[X] < \infty$. Then,*

$$\mathbb{P}[X \geq n] \leq \frac{\mathbb{E}[X]}{n}.$$

Proof of Theorem 2.1. Let $p = \frac{c-1.5}{4c} < 1$ and let K be the random variable counting the number of bipartite holes of order pcn in B. Then the expectation of K can be expressed as

$$\mathbb{E}[K] = \binom{cn}{pcn}^2 \left(1 - \frac{pcn}{cn}\right)^{2pcnr} = \binom{cn}{pcn}^2 (1-p)^{2pcnr}.$$

We derive the above expression by using the fact that there are $\binom{cn}{pcn}$ ways to choose a hole in each part of the graph, so there are $\binom{cn}{pcn}^2$ ways to select a pair of holes from X and Y. Further, each hole is of order pcn and each vertex has out-degree r so there are $2pcnr$ possible edges between the two holes. Since the choice of each edge's head is uniformly distributed among the other part's vertices, an arbitrary edge has probability $1 - \frac{pcn}{cn}$ of not landing in a given hole. Thus the probability that there are no edges between two given holes is $\left(1 - \frac{pcn}{cn}\right)^{2pcnr}$. We note that by our definition of r-out random graphs, if $u \in X$ and $v \in Y$, each vertex is independently assigned a head vertex, so it is possible that there are two edges between u and v (one going from u to v, the other from v to u).

Using Stirling's formula, we get that

$$\binom{cn}{pcn} = \frac{(cn)!}{(pcn)!(cn(1-p))!} = \frac{\left(\frac{cn}{e}\right)^{cn}}{\left(\frac{pcn}{e}\right)^{pcn}\left(\frac{cn(1-p)}{e}\right)^{cn(1-p)}} \cdot \Theta(1/\sqrt{n}) = p^{-pcn}(1-p)^{-cn(1-p)} \cdot \Theta(1/\sqrt{n})$$

Now we can express the expectation of K as an exponential function,

$$\mathbb{E}[K] = \exp\{-2pcn\log p - 2cn(1-p)\log(1-p) + 2pcnr\log(1-p) + O(\log n)\}$$

$$= \exp\left\{2cn\left(p\log\left(\frac{1}{p}\right) + (1-p)\log\left(\frac{1}{1-p}\right) - pr\log\left(\frac{1}{1-p}\right) + o(1)\right)\right\}.$$

We define the function

$$f(p,r) = p\log\left(\frac{1}{p}\right) + (1-p)\log\left(\frac{1}{1-p}\right) - pr\log\left(\frac{1}{1-p}\right)$$

and note that $\lim_{n\to\infty}\mathbb{E}[K] = 0$ if $f(p,r) < 0$. To find an optimal upper bound, we wish to minimize the function $2cr$ subject to $f(p,r) < 0$. Computer assisted numerical analysis shows $f(p,r) < -0.000039 < 0$ when $r = 18$ and $c = 3.614$ (recall that p is a function of c). By our choice of r and c, we get $\lim_{n\to\infty}\mathbb{E}[K] = 0$ and by Markov's inequality we see $\mathbb{P}[K \geq 1] \to 0$. So by Lemma 2.2, $B \to P_n$. The values of r and c also give that $|E(B)| = 130.104n$, which completes the proof. $\square$

We observe an interesting result for anti-directed paths on n vertices $\overleftrightarrow{P_n}$ that follows from Theorem 2.1. An *anti-directed path* $\overleftrightarrow{P_n}$ is a directed graph in which every non-endpoint vertex on the path serves as two heads or two tails. To prove the upper bound $\hat{R}(\overleftrightarrow{P_n}) \leq m$, we must show there exists a directed graph G with m edges such that $G \to \overleftrightarrow{P_n}$. This is accomplished by first finding a graph G such that $G \to P_n$, and then giving an orientation to G such that $G \to \overleftrightarrow{P_n}$.

Theorem 2.5. *Let $B = \mathscr{B}_{cn,cn,r}$, where $r = 18$ and $c = 3.614$. Then $B \to \overleftrightarrow{P_n}$, so $\hat{R}(\overleftrightarrow{P_n}) < 130.092n$.*

Proof. Consider the graph B we constructed for Theorem 2.1. If we orient each edge on B from the partition X to the partition Y, then B forms a graph with only anti-directed paths. We know $B \to P_n$ by Theorem 2.1 and we oriented the edges of B such that the edges of the path P_n form an anti-directed path $\overleftrightarrow{P_n}$. Thus, $\hat{R}(\overleftrightarrow{P_n}) < 130.092n$. $\square$

We note that this result provides an interesting difference from the size-Ramsey number of normal directed paths $\hat{R}(\overrightarrow{P_n})$, which was found to be $\hat{R}(\overrightarrow{P_n}) = \Omega(n^2 \log n)$ by Bucic, Letzter, and Sudakov [19]. Our result shows that the size-Ramsey number of anti-directed paths is asymptotically linear, whereas the size-Ramsey number of normal directed paths is super-quadratic.

3 On the r-colored size Ramsey number $\hat{R}_r(\mathcal{C}, \ldots, \mathcal{C}, P_n)$

We say $G \to (\mathcal{C}, \ldots, \mathcal{C}, P_n)_r$ if every r-coloring of G either contains a monochromatic cycle in one of the first $r-1$ colors or contains a P_n in the r-th color. For the purposes of this thesis, we will assume the r-th color is blue. The size Ramsey number of C_n, the cycle of length n, was first proven to be linear in n by Haxell, Kohayakawa, and Łuczak [20] with use of the sparse regularity lemma. A proof of this avoiding the use of regularity and providing explicit constants was given by Javadi, Khoeini, Omidi and Pokrovskiy [21], who proved that $\hat{R}(C_n) \leq 10^6 cn$ where $c = 843$ if n is even and $c = 113482$ if n is odd. The proofs of these upper bounds as well as the best known upper bounds for $\hat{R}(P_n)$ use random (regular) graphs as their construction.

For any $c \in \mathbb{R}_+$, let $\mathcal{C}_{\leq cn}$ be the family of all cycles of length at most cn and let $\mathcal{C}$ be the family of all cycles. In [22], Dudek, Khoeini and Prałat initiated the study of $\hat{R}(\mathcal{C}_{\leq cn}, P_n)$ and $\hat{R}(\mathcal{C}, P_n)$. We remark that the parameter $\hat{R}(\mathcal{C}, P_n)$ is perhaps a natural one to study. If $G \to (\mathcal{C}, P_n)$, then G contains a path of order n after the removal of the edges of any spanning forest.

Concerning lower bounds, first note that for any $c \in \mathbb{R}_+$, $\hat{R}(\mathcal{C}_{\leq cn}, P_n) \geq \hat{R}(\mathcal{C}, P_n) \geq 2(n-1)$. The first inequality follows from the fact that any coloring of a graph which avoids all cycles in red, clearly avoids all cycles of length at most cn in red. For the second inequality, take any (connected) graph on $2(n-1)-1$ edges (and at least n vertices), color any spanning tree red, and note that there are not enough edges remaining to form a blue P_n. It is not immediately clear how one can move away from this trivial lower bound, but in [22], the authors managed to prove that for sufficiently large n and any $c \in \mathbb{R}_+$, $\hat{R}(\mathcal{C}_{\leq cn}, P_n) \geq \hat{R}(\mathcal{C}, P_n) \geq 2.00365n$.

For the upper bound, the authors of [22] use a random graph construction and techniques similar to those in [13, 15, 14] to prove that

$$\hat{R}(\mathcal{C}_{\leq cn}, P_n) \leq \begin{cases} \frac{80\log(e/c)}{c} n & \text{for } c < 1 \\ 31n & \text{for } c \geq 1 \end{cases} \tag{1}$$

Note that as $c \to 0$, this upper bound tends to infinity. It is mentioned in [22] that due to monotonicity ($m_1 \geq m_2 \implies \hat{R}(\mathcal{C}_{\leq m_1}, P_n) \leq \hat{R}(\mathcal{C}_{\leq m_2}, P_n)$), it is perhaps plausible that there is some decreasing function $\beta(c)$, such that for each fixed $c > 0$, $\hat{R}(\mathcal{C}_{\leq cn}, P_n) \sim \beta(c)n$. They mention that the "limiting case" $c \to \infty$ corresponds to $\hat{R}(\mathcal{C}, P_n)$ but they are only able to prove the upper bound $\hat{R}(\mathcal{C}, P_n) \leq \hat{R}(\mathcal{C}_{\leq cn}, P_n) \leq 31n$.

The main focus of our work in this problem is when $r = 2$, which produces the most significant result. Specifically, we show that a significant improvement in the upper bound for $\hat{R}(\mathcal{C}, P_n)$ can be attained, not by considering the limit as c grows large, but rather by considering very *small* values of c. In fact, for our improvement, it is enough to only consider red cycles of length $3, 4$ or 5. This fact may seem surprising given the behavior of the upper bound provided in (1) as $c \to 0$, but in light of the construction we provide, the surprise diminishes. Recall that for a graph G, the k-th power, G^k is a graph on vertex set V in which two vertices are adjacent if they are of distance at most k in graph G. In our main theorem, we abandon random constructions altogether and show that a very structured graph, the third power of a path, suffices.

Theorem 3.1. *Let $n \geq 2$ and let $N \geq \frac{7}{4}n + 10$. Then $P_N^3 \to (\mathcal{C}_{\leq 5}, P_n)$.*

By monotonicity, this result improves the entire range of results stated in (1).

Corollary 3.2. *For any $c \in \mathbb{R}^+$,*

$$\hat{R}(\mathcal{C}, P_n) \leq \hat{R}(\mathcal{C}_{\leq cn}, P_n) \leq \hat{R}(\mathcal{C}_{\leq 5}, P_n) \leq \frac{21}{4}n + 27.$$

Proof. The first two inequalities follow from monotonicity. Let $N = \left\lceil \frac{7}{4}n + 10 \right\rceil$. Then

$$|E(P_N^3)| = 3(N-3) + 2 + 1 = 3N - 6 \leq \frac{21}{4}n + 27.$$

$\square$

Making use of a lemma proved with a computer check (described in Section 3.4), we have the following improvement.

Theorem 3.3. *Let $n \geq 2$ and let $N \geq \frac{25}{19}n + 43$. Then $P_N^3 \to (\mathcal{C}_{\leq 8}, P_n)$. Thus we have the bound*

$$\hat{R}(\mathcal{C}, P_n) \leq \hat{R}(\mathcal{C}_{\leq 8}, P_n) \leq \frac{75}{19}n + O(1) < 3.947n + O(1).$$

We remark that one interesting fact about P_N^3 is that it is a maximal planar graph and is in fact an *Apollonian network*. That is, it can be drawn by starting with a triangle in the plane and then repeatedly adding a new vertex inside of a current face and connecting it to each vertex of the containing face. Such a planar drawing is shown in Figure 3.

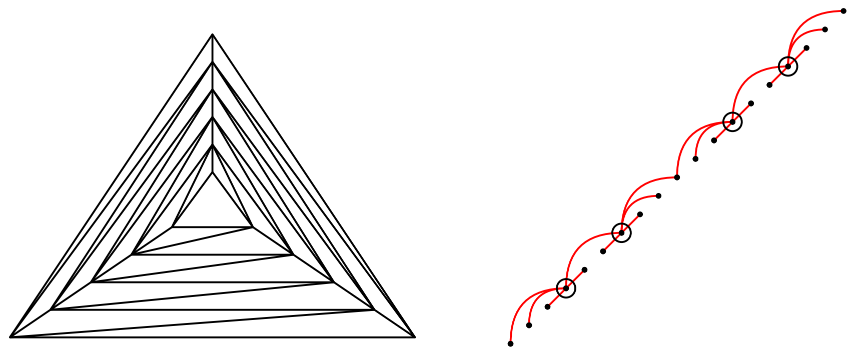

Figure 3: On the left is a planar drawing of P_N^3. On the right is a spanning tree of P_N^3 whose removal leaves behind a path of density $\sim 7/9$.

In this paper we also consider the lower bound. By improving upon the ideas in [22], we prove the following theorem.

Theorem 3.4. *Suppose n is sufficiently large and G is a graph with at most $\left(2+\frac{43}{651}\right)n-O(1)$ edges. Then there exists a red/blue coloring of $E(G)$ such that the red graph is acyclic and the blue graph contains no path of order n. Thus*

$$2.066n < \left(2+\frac{43}{651}\right)n - O(1) \leq \hat{R}(\mathcal{C}, P_n)$$

3.1 Proof Idea and Notation

3.1.1 Upper Bound

Given an integer vertex set $[N] = \{1, 2, \ldots, N\}$, we call the path with $i \sim (i+1)$ for all $i = 1, \ldots, N-1$ a *base path*. Let $N \geq \frac{7}{4}n + 10$ and let $P := P_N$ be the base path on vertex set $[N]$. Define $G := P_N^3$. We will prove that every red/blue coloring of $E(G)$ with no red C_3, C_4 or C_5 contains a blue path of order at least n.

Suppose Q is a base path on vertex set $\{0, 1, \ldots, \ell\}$ and $H = Q^3$. The *density* of a path

P in H with endpoint 0 is defined as

$$r(P) := \frac{|V(P) \cap \{1, 2, \ldots, \ell\}|}{\ell}.$$

The following observation shows that one can "stitch together" paths while maintaining the density of the longer path.

Observation 3.5. *Suppose Q is a base path on vertex set $\{0, 1, \ldots, k, k+1, \ldots, k+\ell\}$ and $H = Q^3$. Suppose that P_1 is a path in $H[\{0, 1, \ldots, k\}]$ with endpoints 0 and k and $r(P_1) = d_1$, and that P_2 is a path in $H[\{k, k+1, \ldots, k+\ell\}]$ with endpoints k and $k+\ell$ and $r(P_2) = d_2$. Then $P_1 \cup P_2$ is a path in H with endpoints 0 and $k+\ell$ and $r(P_1 \cup P_2) \geq \min\{d_1, d_2\}$.*

Proof. The fact that $P_1 \cup P_2$ forms a path in H is obvious. For the density, suppose $\hat{d} = \min\{d_1, d_2\}$. Then we have

$$\begin{aligned} r(P_1 \cup P_2) &= \frac{|V(P_1 \cup P_2) \cap \{1, 2, \ldots, k+\ell\}|}{k+\ell} \\ &= \frac{|V(P_1) \cap \{1, \ldots, k\}| + |V(P_2) \cap \{k+1, \ldots, k+\ell\}|}{k+\ell} \\ &= \frac{d_1 k + d_2 \ell}{k+\ell} \geq \hat{d}. \end{aligned}$$

□

Throughout this section, we will make use of the underlying order of the vertex set of $G = P_N^3$. Each vertex of G in $\{4, 5, \ldots, N-3\}$ has exactly 6 neighbors: $v \pm i$ where $i \in [3]$. For each vertex $v \in [N-3]$, we refer to the neighbors $v+i$, $i \in [3]$ as the *up-neighbors* of v. Given a red/blue (or R/B for short) coloring of $E(G)$, for each vertex $v \in [N-3]$, we may associate an element of $\{R, B\}^3$ (i.e. a string of length 3 with entries from $\{R, B\}$) representing the colors assigned to the edges between v and its up-neighbors. We use the notation $\mathrm{up}(v) = c_1 c_2 c_3$ to mean that the edges $\{v, v+1\}, \{v, v+2\}, \{v, v+3\}$ are colored with c_1, c_2, c_3 respectively.

As an illustration of this notation, we highlight one fact which we will use repeatedly without mention. If G contains no red cycles, and $\text{up}(v) = RRR$, then vertices $v+1, v+2, v+3$ form a blue triangle (else there would be a red C_3). See Figure 4.

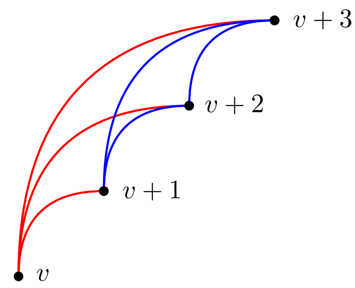

Figure 4: Blue C_3 when $\text{up}(v) = RRR$.

The main idea of the proof is to suppose that G has been R/B colored such that there is no red cycle of length at most 5 and to show that in this case, there must be a blue path of order at least n. We will find the long blue path by showing that starting at any vertex v with $\text{up}(v) \neq RRR$, one can find a blue path of density at least $4/7$ in the next 10 consecutive vertices with endpoints v and w where $\text{up}(w) \neq RRR$. These short high density blue paths can then be stitched together as in Observation 3.5 to form the long blue path. The following lemma which is the main ingredient in our proof of Theorem 3.1, says that the short high density paths can always be found.

Lemma 3.6. *Let $Q = P_{11}$ on vertex set $\{0, 1, \ldots 10\}$ and let $H = Q^3$. Suppose that H has been 2-colored with no red cycles from $\mathcal{C}_{\leq 5}$. Further suppose that in H, up(0) contains at least one B. Then there is a $k \in \{1, \ldots 9\}$ such that $H[\{0, \ldots, k\}]$ contains a blue path P_B with endpoints 0 and k such that up(k) contains at least one B and*

$$r(P_B) := \frac{|V(P_B) \cap \{1, \ldots, k\}|}{k} \geq \frac{4}{7}.$$

With this lemma in hand (proved in Section 3.3), we can prove the main theorem.

Proof of Theorem 3.1. Let $N \geq \frac{7}{4}n + 10$, let $G = P_N^3$ and suppose that $E(G)$ has been 2-colored with red and blue such that there is no red cycle from $\mathcal{C}_{\leq 5}$. It cannot be the case that vertices 1 and 2 both have 3 red up-neighbors. Hence we may apply Lemma 3.6 starting at one of these vertices. We then repeatedly apply Lemma 3.6 to find an extension of the current blue path to another with density at least 4/7 (by Observation 3.5). We continue extending the blue path until we have found one, P_B, whose endpoint lies in $\{N-9, \ldots, N\}$ (if the last blue endpoint is smaller than $N-9$, then Lemma 3.6 can be applied again). Then since $r(P_B) \geq 4/7$, we have

$$|V(P_B)| \geq \frac{4}{7} \cdot (N-11) + 1 \geq n$$

where we have used $N-11$ since P_B may start at vertex 2 and the additional 1 accounts for the very first vertex of P_B. □

The largest blue path density one could hope for in P_N^3 is 7/9 since we may color the edges red in a repeating pattern as indicated by Figure 3 . At most 2 of the circled vertices may be used in a blue path (as endpoints) since they would have blue degree 1. Thus we have the following.

Observation 3.7. *The best upper bound that one could ever prove using the cube of a path is* $\hat{R}(\mathcal{C}, P_n) \leq \frac{9}{7}n \cdot 3 + O(1) \approx 3.857n + O(1)$.

3.1.2 Lower Bound

In order to improve the lower bound, we must show that every graph G with at most $(2+\alpha)n$ edges contains a forest whose removal destroys all the paths of order n. One approach to accomplish this is to find a forest which contains many vertices of full degree (that is, vertices with the same degree in the forest as in the graph G). Such full degree vertices cannot be used in a blue path. This is the approach taken in [22]. One snag is that it is not so simple to find such forests in graphs with unbounded degree. The proof of Theorem 3.4 shows how to deal with high degree vertices and also gives an improved approach for bounded degree graphs than the one in [22].

3.2 Notation and outline

We use $N(v)$ to refer to the open neighborhood of vertex v. In Section 3, we deal with a graph on vertex set $\{0, 1, \ldots 10\}$ and since we do not refer to vertex 10 in the proof, we choose to omit commas when naming paths and cycles. For example the path $(0, 1, 3, 4)$ will be denoted by 0134 and the cycle on those same vertices will be denoted (0134).

In Section 3.3 we prove Lemma 3.6. In Section 3.4 we briefly describe the computer assisted improvement to Lemma 3.6 which implies Theorem 3.3. In Section 3.5 we prove Theorem 3.4.

3.3 Proof of Main Lemma

Proof of Lemma 3.6. We split into 7 cases depending on up(0). Note by assumption, we do not consider the case up$(0) = RRR$. Cases 3 and 6 are much more involved than the other cases so the reader may wish to read those last. We provide `python` code at the url `http://msuweb.montclair.edu/~bald/research.html` which can help with the verification of this proof.

- **Case 1** (up$(0) = BRR$)

  If up$(1) = RRR$, then edges $02, 03, 12$ and 13 are all red and so (0213) would form a red C_4, a contradiction. Thus up(1) must contain at least one B and we can take $P_B = 01$ which satisfies $r(P_B) = 1$.

- **Case 2** (up$(0) = RBR$)

  Suppose edge 12 is red. Then 023 is a blue path since edge 23 must be blue (else (0123) is a red cycle). If up(3) $= RRR$, then edge 14 must be blue (else (0143) is a red cycle), and so we can take $P_B = 02314$ since 4 has blue up-neighbors 5 and 6 and $r(P_B) = 1$. Otherwise up(3) contains a B and we can take $P_B = 023$ which satisfies $r(P_B) = 2/3$.

Now, suppose edge 12 is blue. In this case, 0213 is a blue path (edge 13 must be blue otherwise (013) is a red cycle). If up(3) contains a B, then we may take $P_B = 0213$. Otherwise up(3) $= RRR$. In this case, edges 14, 45 and 56 are all blue. Thus we can take $P_B = 02145$ where $r(P_B) = 4/5$.

- **Case 3** (up(0) $= RRB$)

Suppose edge 23 is red. Then edge 13 must be blue (else (0132) is a red cycle) and so 031 is a blue path.

If edge 14 were red, then edge 24 must be blue (else (0142) is a red cycle) and so 03124 is a blue path. If up(4) contains a B, then we may take $P_B = 03124$. If up(4) $= RRR$, then 320145 is a red path, and so any other edge among these vertices must be blue. Thus we may take $P_B = 03425$ since vertex 6 is a blue up-neighbor of vertex 5 and $r(P_B) = 4/5$.

If edge 14 were blue, then 0314 is a blue path. If up(4) contains a B, then we may take $P_B = 0314$ which has $r(P_B) = 3/4$. Otherwise, suppose up(4) $= RRR$ (which recall implies that vertices $5, 6$ and 7 form a blue triangle).

If edge 24 were red, then edges 34 and 25 must be blue (else we have red cycles (234) or (245) respectively). Thus we may take $P_B = 034125$ since vertex 6 is a blue up-neighbor of vertex 5.

So we assume edge 24 is blue. If edge 35 is red then edge 25 must be blue (else (235) is a red cycle). Thus we may again take $P_B = 031425$. So assume that edge 35 is blue. In this case, we have 03567 is a blue path. Now if up(7) contains a B, when we may take $P_B = 03567$ which has $r(P_B) = 4/7$ (this specific case is illustrated in Figure 5 just as an example). Otherwise if $\text{up}(7) = RRR$, then edge 68 is blue (else (4687) is a red C_4). In this case we may take $P_B = 0357689$ which has $r(P_B) = 2/3$.

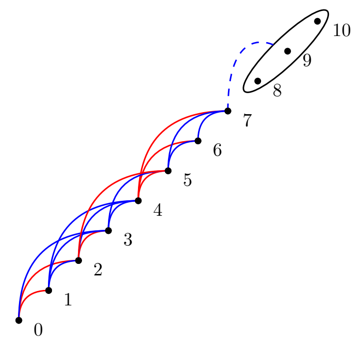

Figure 5: An illustration of the situation when the proof has led us to the assumptions $\text{up}(0) = RRB$, edge 23 is red, 14 is blue, $\text{up}(4) = RRR$, 24 is blue, 35 is blue and up(7) contains a B. In this case, we take $P_B = 03567$ which has $r(P_B) = 4/7$.

Now we assume edge 23 is blue. Then 0321 forms a blue path.

Suppose edge 14 is red. Then edge 24 is blue (else (0142) is a red cycle) and so 0324 is a blue path. If up(4) contains a B, then we may take $P_B = 0324$. So suppose that $\text{up}(4) = RRR$. Then (567) is a blue triangle and edge 25 must be blue (else (01452) is a red cycle) and so we may take $P_B = 03256$ which has $r(P_B) = 2/3$.

Now suppose edge 14 is blue. If up(4) contains a B, then we may take $P_B = 03214$. Else suppose $\text{up}(4) = RRR$ so that (567) forms a blue triangle. If edge 25 is blue, then we may take $P_B = 03256$ which has $r(P_B) = 2/3$. So suppose edge 25 is red.

If edge 35 is blue, then 03567 is blue path. If up(7) contains a B, then we may take $P_B = 03567$ with $r(P_B) = 4/7$. Otherwise suppose $\text{up}(7) = RRR$. Then we may take $P_B = 0357689$ which has $r(P_B) = 2/3$.

So suppose edge 35 is red. Then edge 36 is blue (else (3546) is a red C_4). So 03657 is a blue path. If up(7) contains a B, then we take $P_B = 03657$ with

$r(P_B) = 4/7$. Otherwise suppose up(7) $= RRR$ and so edge 58 is blue (else (4578) is a red C_4). So we may take $P_B = 0367589$ which has $r(P_B) = 2/3$.

- **Case 4** (up(0) $= BBR$)

  If up(1) contains a B, then we may take $P_B = 01$. Otherwise suppose up(1) $= RRR$. In this case, (234) is a blue C_3 and so we may take $P_B = 023$ which has $r(P_B) = 2/3$.

- **Case 5** (up(0) $= BRB$)

  If up(1) contains a B, then we may take $P_B = 01$. Otherwise suppose up(1) $= RRR$ so that (234) is a blue C_3. Then 0324 is a blue path. If up(4) contains a B, then we may take $P_B = 0324$ which has $r(P_B) = 3/4$. Otherwise suppose up(4) $= RRR$ so that (567) is a blue C_3 and so that edge 25 is blue (else (1254) is a red C_4). Then we may take $P_B = 034256$ which has $r(P_B) = 5/6$.

- **Case 6** (up(0) $= RBB$)

  Suppose edge 23 is blue. If up(3) contains a B, then we may take $P_B = 023$ which has $r(P_B) = 2/3$. Otherwise suppose up(3) $= RRR$. Then edges 24 and 25 cannot both be red (else (2435) is a red C_4). If edge 24 blue, then we may take $P_B = 03245$ which has $r(P_B) = 4/5$. If edge 25 is blue, then we may take $P_B = 0325$ which has $r(P_B) = 3/5$.

  So suppose edge 23 is red. Then edges 12 and 13 cannot both be red (else (123) is a red C_3).

  First suppose both edges 12 and 13 are both blue. Then 0213 is a blue path. If up(3) contains a B, then we may take $P_B = 0213$. Otherwise up(3) $= RRR$ in which case edge 24 is blue (else (234) is a red C_3) and so we may take $P_B = 031245$.

  Now suppose that exactly one of 12 or 13 is blue and the other is red. Denote the blue edge as 1α and the red edge as 1β, where $\alpha, \beta \in \{2, 3\}, \alpha \neq \beta$. Notice that the edge $\alpha\beta$ is red since this is the edge 23.

Suppose edge 14 is blue. If up(4) contains a B, then we may take $P_B = 0\alpha 14$. Otherwise suppose up(4) $= RRR$.

If edge $\beta 4$ is red, then the red graph on vertices $\{0, \dots, 7\}$ forms a tree, and so any other edge on these vertices must be blue. In particular, edges $25, 35$ and 36 are blue and so we may take $P_B = 02536$ which has $r(P_B) = 2/3$.

So suppose edge $\beta 4$ is blue. If edge $\beta 5$ is blue, then we may take $P_B = 0\alpha 14\beta 56$. If edge $\beta 5$ is red, then again, the red graph on vertices $\{0, \dots, 7\}$ forms a tree, and so any other edge is blue. In particular, edge $\alpha 5$ is blue and so we may take $P_B = 0\beta 41\alpha 56$.

Suppose edge 14 is red. Then edges 24 and 34 are both blue since the red graph on vertices $\{0, \dots, 4\}$ forms a tree.

Suppose edge 25 is red. Then the red graph on vertices $\{0, \dots, 5\}$ forms a tree and so any other edge on these vertices must be blue. In particular edges 35 and 45 are blue. So 02435 forms a blue path. If up(5) contains a B, then we may take $P_B = 02435$. Otherwise suppose up(5) $= RRR$, in which case we may take $P_B = 0245367$.

Suppose edge 25 is blue. If up(5) contains a B, then we may take $P_B = 02435$. Otherwise suppose that up(5) $= RRR$. If edge 46 is red, then the red graph on $\{0, \dots, 8\}$ forms a tree and so we may take $P_B = 0245367$. If edge 46 is blue, then we may take $P_B = 03467$ which has $r(P_B) = 4/7$.

- **Case 7** (up(0) $= BBB$)

If up(1) contains a B, then we may take $P_B = 01$. Otherwise suppose up(1) $= RRR$ in which case we may take $P_B = 023$.

$\square$

3.4 Computer assisted improvement

With the use of a computer program (written in `python`, making use of the `networkx` package, and made available at the url[1] `http://msuweb.montclair.edu/~bald/research.html`) we have a proof of the following lemma which finds a higher density path than Lemma 3.6.

Lemma 3.8. *Let $Q = P_{43}$ on vertex set $\{0, 1, \ldots 42\}$ and let $H = Q^3$. Suppose that H has been 2-colored with no red cycles from $\mathcal{C}_{\leq 8}$. Further suppose that in H, up$(0) \notin \{RRR, RRB\}$. Then there is a $k \in \{1, \ldots 39\}$ such that $H[\{0, \ldots, k\}]$ contains a blue path P_B with endpoints 0 and k such that up$(k) \notin \{RRR, RRB\}$ and*

$$r(P_B) := \frac{|V(P_B) \cap \{1, \ldots, k\}|}{k} \geq \frac{19}{25}.$$

Using this improved density of $19/25 = .76$, Theorem 3.3 follows just as Theorem 3.1 followed from Lemma 3.6. The algorithm proceeds much as our proof of Lemma 3.6 proceeds. Suppose $\mathrm{up}(0), \ldots, \mathrm{up}(k-1)$ have been assigned and one finds neither a red cycle nor a blue path of the desired ratio ending at $k-1$. Then we iterate through all 8 possibilities for $\mathrm{up}(k)$, again searching for a red cycle or a high density blue path (ending at k) and deepening the recursion when neither is found. In order to cut down on cases checked, we forced the program to avoid the most work intensive "Case 3", hence the requirement $\mathrm{up}(0), \mathrm{up}(k) \notin \{RRR, RRB\}$. Note that any coloring of P_N^3 with no red cycles must satisfy $\{\mathrm{up}(0), \mathrm{up}(1)\} \not\subseteq \{RRR, RRB\}$ and so this is an okay assumption. As a demonstration of the growth of complexity, we mention that the output of the program which verifies a density of $4/7$ (i.e. equivalent to the proof of Lemma 3.6) is a `.txt` file of size 85 KB. The file which verifies the density of $3/4$ has size 1.7 MB and the file which verifies the density of $19/25$ has size 34 MB. As discussed in Section 3.1.1, the best density one could hope for in P_N^3 is $7/9 \approx 0.7777$. Due to our proof method

[1] A `.txt` file containing the output of the program is also available.

(stitching together segments), it is unlikely that our program (as currently written) will be able to prove the exact bound of 7/9; one can color the portion near vertex 0 'badly' in a way that lowers the overall density of the segment.

3.5 Lower Bound

In this section we prove Theorem 3.4 by improving on the ideas which appear in [22]. The following reduction lemma essentially appears as a lemma in [17]. In that paper, the lemma concerns avoidance monochromatic paths in both colors rather than cycles in red and a path in blue. However, the proof is almost identical, so we have decided to omit it. This lemma allows us to concentrate on graphs with minimum degree at least 3.

Lemma 3.9. *Let n be a positive integer with $n \geq 6$. If every connected graph with at most m edges and minimum degree at least 3 has a 2-coloring such that the red graph is acyclic, and every blue path has order less than $n-2$, then every graph with at most m edges has a 2-coloring such that the red graph is acyclic and every blue path has order less than n.*

We also make use of the following lemma which shows how to find a forest in a bounded degree graph whose removal creates many vertices of degree 0 or 1 (thus unsuitable for paths in the remaining graph).

Lemma 3.10. *Suppose G is connected and has n vertices and maximum degree d. Then G contains a forest F and disjoint subsets $A_0, A_1 \subseteq V(G)$ such that*

1. *$A_0 \cup A_1$ is an independent set*
2. *$d_F(v) = d_G(v)$ for all $v \in A_0$*
3. *$d_F(v) \geq d_G(v) - 1$ for all $v \in A_1$*

4. $|A_0| + \frac{1}{2}|A_1| \geq \gamma_\Delta n$ *where*

$$\gamma_\Delta = \left(\frac{1}{\Delta^2+\Delta+2} + \frac{3}{2(\Delta^2+2\Delta+3)}\right)$$

Proof. We greedily build the forest F and maintain disjoint sets A_0, A_1, X, Y. Throughout, $X = N(A_0 \cup A_1)$ and $Y = V(G) \setminus (A_0 \cup A_1 \cup X)$, and so there are no edges between Y and $A_0 \cup A_1$. Initialize $A_0, A_1, X, F = \emptyset$ and $Y = V(G)$.

We start with *Phase 1.* Begin by adding an arbitrary vertex to A_0, removing it from Y and updating X. At each subsequent step of Phase 1, we look for a vertex $v \in Y$ such that $|N(v) \cap X| \leq 1$. If such a vertex v exists, we add v to A_0, add all of v's incident edges to F, and include all of v's neighbors in X. When no such vertex v exists, then Phase 1 ends. At the end of Phase 1, every vertex in Y has at least 2 neighbors in X and every vertex in X has at most $(\Delta - 1)$ neighbors in Y (since each vertex in X has a neighbor in A_0), so $2|Y| \leq e(X,Y) \leq (\Delta-1)|X|$ and also $|X| \leq \Delta|A_0|$. So at the end of Phase 1

$$n = |A_0| + |X| + |Y| \leq |A_0| + \Delta|A_0| + \frac{\Delta-1}{2}\Delta|A_0| = \left(\frac{\Delta^2+\Delta+2}{2}\right)|A_0|$$

so $|A_0| \geq \frac{2}{\Delta^2+\Delta+2}n$.

In *Phase 2* we add vertices to A_1 which have $|N(v) \cap X| \leq 2$. If there is a vertex with $|N(v) \cap X| \leq 1$, we handle it as above. If no such v exists, then we next look for a vertex $v \in Y$ such that $|N(v) \cap X| = 2$. In this case, we move v to A_1, we add to F, any edges incident to v and not X. Of the two edges incident to both v and X, we arbitrarily choose one to add to F. If no such v exists, we terminate the process. At the end of Phase 2, every vertex in Y has at least 3 neighbors in X.

By construction, one can observe that $A_0 \cup A_1$ remains independent since we only add vertices from Y. Also by construction, F remains a forest and the degree conditions in (ii) and

(iii) are met. It remains to show that at the end of the process, $|A_0| + \frac{1}{2}|A_1| \geq \gamma_\Delta n$.

Note that $|X| \leq \Delta|A_0 \cup A_1|$ and that at the end of Phase 2, we have $3|Y| \leq e(X,Y) \leq (\Delta - 1)|X|$. Thus at termination, we have $|Y| \leq \frac{\Delta-1}{3}|X|$ and so

$$\begin{aligned} n = |A_0 \cup A_1| + |X| + |Y| &\leq |A_0 \cup A_1| + \frac{\Delta+2}{3}|X| \\ &\leq |A_0 \cup A_1| + \frac{\Delta+2}{3} \cdot \Delta|A_0 \cup A_1| \\ &= \frac{\Delta^2 + 2\Delta + 3}{3}|A_0 \cup A_1| \end{aligned}$$

and so $|A_0| + |A_1| \geq \frac{3}{\Delta^2+2\Delta+3}n$. To finish, we observe

$$\begin{aligned} |A_0| + \frac{1}{2}|A_1| &= \frac{1}{2}|A_0| + \frac{1}{2}(|A_0| + |A_1|) \\ &\geq \left(\frac{1}{2}\frac{2}{\Delta^2 + \Delta + 2} + \frac{1}{2}\frac{3}{\Delta^2 + 2\Delta + 3}\right) n = \gamma_\Delta n \end{aligned}$$

□

Proof of Theorem 3.4. Suppose $G = (V, E)$ is connected, has $e = (2 + \alpha)n$ edges and $G \to (\mathcal{C}, P_n)$. In light of Lemma 3.9, we also assume that $\delta(G) \geq 3$. We note that technically, by using Lemma 3.9, we should now change our goal to finding a coloring such that the red graph is acyclic and every blue path is of order less than $n - 2$. For readability, we continue to forbid paths of order n and mention that the $O(1)$ in the statement of Theorem 3.4 takes care of the issue. We may assume that G has $N = (1 + \beta)n$ vertices where $\beta \leq \alpha$ (else we may take any spanning tree, color it red and note that there are too few remaining edges to have a blue path of order n).

Let X be the set of vertices of degree at least 4. Then $|X| \geq 2n - N$. To see this, note that by the assumption $G \to (\mathcal{C}, P_n)$, must have a path of order n and we may color its edges red (which is acyclic in red). Then the uncolored edges must have a path of order

n (otherwise we could color them all blue). Thus we have two edge-disjoint paths P_1, P_2 on vertex sets A_1, A_2, each of size n, and any vertex in $A_1 \cap A_2$ has degree at least 4. Thus we have $|X| \geq |A_1 \cap A_2| = |A_1| + |A_2| - |A_1 \cup A_2| \geq 2n - N$.

Let B be the set of vertices of degree at least $d+1$ (we will end up taking $d = 5$). Then

$$\begin{aligned}(4+2\alpha)n = 2e = \sum_v d(v) &\geq (d+1)|B| + 4(|X| - |B|) + 3(N - |X|) \\ &= (d-3)|B| + |X| + 3N \\ &\geq (d-3)|B| + 2n + 2N\end{aligned}$$

and so rearranging, we have

$$|B| \leq \frac{1}{d-3}\left((2+2\alpha)n - 2N\right) = \frac{2}{d-3}\left((1+\alpha)n - N\right) = \frac{2}{d-3}(\alpha - \beta)n.$$

Let

$$\gamma_d := \left(\frac{1}{d^2+d+2} + \frac{3}{2(d^2+2d+3)}\right).$$

Note that $G[V \setminus B]$ has maximum degree d and so we may apply Lemma 3.10 to each component of G in order to find a forest F and sets A_0, A_1 with

$$|A_0| + \frac{1}{2}|A_1| \geq \gamma_d \cdot (N - |B|)\,.$$

We color all edges in F with red, complete this forest to a red tree in G and then color the remaining edges in G with blue. Let $R = V \setminus (A_0 \cup A_1 \cup B)$. So every vertex in A_0 has only red edges to R and every vertex in A_1 has at most one blue edge to R. Suppose $P = (v_1, v_2, \ldots, v_k)$ is a blue path. Note that if $v_i \in A_0$ for some $1 < i < k$, then v_{i-1} and v_{i+1} must both be in B. Also, if $v_i \in A_0$ for some $1 < i < k$, then at least one of v_{i-1} and v_{i+1} is in B. For $X \in \{A_0, A_1, B, R\}$, let $X' = V(P) \cap X$. So if we let $e_P(A_0 \cup A_1, B)$ count the number of edges in P with one end in $A_0 \cup A_1$ and the other end in B, we have $2|A_0'| + |A_1'| - 2 \leq e_P(A_0 \cup A_1, B) \leq 2|B'|$, and so $|A_0'| + |A_1'| \leq |B'| + \frac{1}{2}|A_1'| + 1$. We then have

$$\begin{aligned}
|V(P)| &= |R'| + |A_0'| + |A_1'| + |B'| \\
&\leq |R| + \frac{1}{2}|A_1'| + 2|B'| + 1 \\
&\leq N - |A_0| - |A_1| - |B| + \frac{1}{2}|A_1'| + 2|B'| + 1 \\
&\leq N - |A_0| - \frac{1}{2}|A_1| + |B| + 1.
\end{aligned}$$

We see that if $N - (|A_0| + \frac{1}{2}|A_1|) + |B| < n - 1$, then there is no blue path of order n.

$$\begin{aligned}
\frac{1}{n-1}\left(N - (|A_0| + \frac{1}{2}|A_1|) + |B|\right) &\leq \frac{1}{n-1}\left(N - \gamma_d\left(N - |B|\right) + |B|\right) \\
&= \frac{1}{n-1}\left((1-\gamma_d)N + (1+\gamma_d)|B|\right) \\
&\leq (1-\gamma_d)(1+\beta) + (1+\gamma_d)\cdot\frac{2}{d-3}(\alpha - \beta) + O(1/n) \\
&= (1-\gamma_d) + \beta\left(1 - \gamma_d - \frac{2}{d-3}(1+\gamma_d)\right) \\
&\qquad + \alpha\left(\frac{2}{d-3}\right)(1+\gamma_d) + O(1/n)
\end{aligned}$$

We set

$$f(\alpha, \beta, d) := (1-\gamma_d) + \beta\left(1 - \gamma_d - \frac{2}{d-3}(1+\gamma_d)\right) + \alpha\left(\frac{2}{d-3}\right)(1+\gamma_d).$$

This function is decreasing in β for $d = 4, 5$ and increasing in β for $d \geq 6$. When $d = 5$, we may maximize this function by setting $\beta = 0$, and in this case we get

$$f(\alpha, 0, 5) = \frac{651}{608}\alpha + \frac{565}{608}.$$

So we have that $N - (|A_0| + \frac{1}{2}|A_1|) + |B| < n - 1$ whenever $\alpha < \frac{43}{651} - \Omega(1/n)$. One can check that using $d = 4, 6$ yields the bounds $\alpha < 5/109$ and $\alpha < 39/709$ (recalling that for $d = 6$, one must set $\beta = \alpha$ when maximizing) both of which are worse than $43/651$. For all $d \geq 7$, the bound is also worse. $\square$

3.6 Asymptotic bounds for $\hat{R}_r(\mathcal{C}, \ldots, \mathcal{C}, P_n)$

Theorem 3.11. *Let $\mathcal{C}$ denote the family of all cycles and P_n be a path on n vertices. Then for every integer $r \geq 2$ and n sufficiently large, $\hat{R}_r(\mathcal{C}, \ldots, \mathcal{C}, P_n) = \Theta(r)n$.*

To show $\hat{R}_r(\mathcal{C}, \ldots, \mathcal{C}, P_n) = \Theta(r)n$, we must prove

$$c_1 rn < \hat{R}_r(\mathcal{C}, \ldots, \mathcal{C}, P_n) < c_2 rn$$

for some $c_1, c_2 \in \mathbb{R}$. To prove the lower bound, we provide an r-coloring of an arbitrary graph H with $c_1 rn$ edges such that $H \nrightarrow (\mathcal{C}, \ldots, \mathcal{C}, P_n)_r$. We do so with the following lemma.

Lemma 3.12. *Let $c_1 = (n-1)/n$ and $r \geq 2$. If $\mathcal{H}$ is the family of graphs with $c_1 rn = (n-1)r$ edges, then $H \nrightarrow (\mathcal{C}, \ldots, \mathcal{C}, P_n)_r$ for every $H \in \mathcal{H}$. Thus, $\hat{R}_r(\mathcal{C}, \ldots, \mathcal{C}, P_n) > c_1 rn$.*

Proof. We prove the above lemma by induction on r with base case $r = 2$. Let $H_r = (V, E)$ be a graph with $r(n-1)$ edges and consider H_2. We may assume that $|V(H_2)| \geq n$ since otherwise we can color all of H_2's edges blue without creating a P_n. We can also assume that H_2 is connected, else we can apply the lemma to each component of H_2. To color H_2 in a way that guarantees no blue P_n or red cycle, color a spanning forest F of H_2 red. Note that $|E(F)| \geq n - 1$, so $|E(H_2 \backslash F)| \leq n - 1$. Thus we can color all edges in $H_2 \backslash F$ blue without creating a blue P_n. So $H_2 \nrightarrow (\mathcal{C}, P_n)$.

We now assume that $H_k \nrightarrow (\mathcal{C}, \ldots, \mathcal{C}, P_n)_k$ where $k \leq r$, and consider the graph H_{k+1} with $(n-1)(k+1)$ edges. Similar to the base case, we also assume that H_{k+1} is connected and $|V(H_{k+1})| \geq n$. Again we find a spanning forest F of H_{k+1} and color it with the first color. Note that $|E(F)| \geq n-1$ so $|E(H_{k+1} \backslash F)| \leq (n-1)k$. Then by the inductive hypothesis, $H_{k+1} \backslash F$ can be k-colored such that $H_{k+1} \backslash F \nrightarrow (\mathcal{C}, \ldots, \mathcal{C}, P_n)_k$. Thus the k-coloring of $H_{k+1} \backslash F$ and the coloring of F provide a $k+1$-coloring of H_{k+1} such that there is no blue P_n or non-blue cycle. So $H_{k+1} \nrightarrow (\mathcal{C}, \ldots, \mathcal{C}, P_n)_{k+1}$, and by induction the proof is finished. $\square$

We now approach the upper bound by adapting the same lemma in [15] we used to prove Lemma 2.2.

Lemma 3.13. *Let G be a graph with $2n$ vertices and assume that for every two disjoint sets of vertices S and T of G such that $|S| = |T| = n/2$ we have $e(S,T) > (r-1)(n-1)$. Then, $G \rightarrow (\mathcal{C}, \ldots, \mathcal{C}, P_n)_r$.*

Proof. We replicate our proof of Lemma 2.2 and prove the lemma by contrapositive: if $G \nrightarrow (\mathcal{C}, \ldots, \mathcal{C}, P_n)_r$, then there exists two disjoint sets of vertices S and T in G such that $|S| = |T| = n/2$ and $e(S,T) \leq (r-1)(n-1)$. We first conduct the DFS algorithm on the blue edges in G. Similar to the observations made in Lemma 2.2, since $G \nrightarrow (\mathcal{C}, \ldots, \mathcal{C}, P_n)_r$, we must have $e(S,T) \leq (r-1)(n-1)$ since otherwise we must have a blue edge between S and T or a monochromatic non-blue cycle. Furthermore, in each step of the algorithm, S either decreases in order by one or T increases in order by one and since there is no blue P_n, it again follows that $|P| < n$; so at some point in the algorithm we must have that $|S| = |T| = n/2$ as needed. $\square$

To prove $G \rightarrow (\mathcal{C}, \ldots, \mathcal{C}, P_n)_r$, we must show that any r-coloring of G either contains a monochromatic, non-blue cycle or a blue P_n. In doing so, we will make use of binomial random graphs, denoted $\mathcal{G}(n,p)$, as well as the Chernoff bound:

Lemma 3.14 (Chernoff Bound)**.** *Let X be a binomial random variable with n trials each with probability p and let $\mu = \mathbb{E}[X] = np$. Then,*

$$\mathbb{P}[X \leq (1-\delta)\mu] \leq e^{-\mu\delta^2/2} \quad \text{and} \quad \mathbb{P}[X \geq (1+\delta)\mu] \leq e^{\frac{-\mu\delta^2}{2+\delta}}.$$

Lemma 3.15. *Let $G = \mathcal{G}(2n, \frac{37r}{2n})$. Then for sufficiently large n, asymptotically almost surely (a.a.s.) $G \to (\mathcal{C}, \ldots, \mathcal{C}, P_n)_r$. Thus, $\hat{R}_r(\mathcal{C}, \ldots, \mathcal{C}, P_n) \leq 38rn$.*

Proof. Suppose G is r-colored; we will show that a.a.s., G satisfies Lemma 3.13. Let X be the number of disjoint sets $S, T \subset V(G)$ such that $|S| = |T| = n/2$ and $e(S,T) \leq (r-1)(n-1)$. Then,

$$\mathbb{E}[X] \leq \binom{2n}{n/2}^2 \mathbb{P}[e(S,T) \leq (r-1)(n-1)].$$

We also note that $e(S,T) \sim \text{Bin}(n^2/4, \frac{37r}{2n})$, so

$$\mathbb{E}[e(S,T)] = \frac{n^2}{4} \cdot \frac{37r}{2n} = \frac{37rn}{8},$$

and Chernoff bound says

$$\mathbb{P}[e(S,T) \leq (r-1)(n-1)] \leq \mathbb{P}[e(S,T) \leq rn] = \mathbb{P}[e(S,T) \leq \frac{8}{37}\mathbb{E}[e(S,T)]] \leq e^{-\frac{841}{592}rn},$$

where $\mu = \frac{37}{8}rn$ and $\delta = 29/37$. Because there are 2^{2n} possible subsets of a set of size $2n$, it follows that

$$\binom{2n}{n/2}^2 \leq \left(2^{2n}\right)^2 = 16^n,$$

which gives us to the following expression:

$$\mathbb{E}[X] \leq \exp\left\{\log 16n - \frac{841}{592}rn\right\}.$$

We observe that $\mathbb{E}[X] \to 0$ if $\log 16n - \frac{841}{592}rn < 0$. Since $r \geq 2$, we get

$$\log 16n - \frac{841}{592}rn \leq \log 16n - \frac{1682}{592}n < 0$$

as needed. Thus for sufficiently large n, we have no sets S, T such that $|S| = |T| = n/2$ and $e(S,T) \leq (r-1)(n-1)$. So G a.a.s. satisfies Lemma 3.13, hence $G \to (\mathcal{C}, \ldots, \mathcal{C}, P_n)_r$.

Lastly, we note that $e(G) \sim \mathrm{Bin}(\binom{2n}{2}, \frac{37r}{2n})$, so

$$\mathbb{E}[e(G)] = \binom{2n}{2} \cdot \frac{37r}{2n} = 37rn - O(1).$$

Then Chernoff bound gives us

$$\mathbb{P}[e(G) \geq 38rn] = \mathbb{P}[e(G) \geq (1 + \frac{1}{37})\mathbb{E}[e(G)]] \leq e^{\frac{-rn}{75}},$$

where $\mu = 37rn$ and $\delta = 1/37$. The probability tends to 0 as $n \to \infty$, so a.a.s. $e(G) \leq 38rn$.

$\square$

4 On the online Ramsey number $\tilde{R}(\mathcal{C}, P_n)$

4.1 Introduction and Proof Idea

In this section, we consider a variant of the size Ramsey number that was introduced independently by Beck [24] and Kurek and Ruciński [25]. The game is played on the edge set of an infinitely large complete graph by two players, Builder and Painter. Each round the Builder serves an edge and Painter colors the edge either red or blue. Painter loses by either painting a red copy of a fixed graph F or a blue copy of a fixed graph H. By nature of Ramsey-type problems, our interest is in finding the minimum number of rounds it takes Builder to win, assuming Painter uses an optimal strategy. Accordingly, the *online Ramsey number* $\tilde{R}(F,H)$ is the minimum number of rounds it takes for Painter to lose, regardless of Painter's strategy. By definition, one can see that $\tilde{R}(F,H) \leq \hat{R}(F,H)$. To prove an upper bound for the online Ramsey number $\tilde{R}(F,H) \leq m$, we must employ a Builder strategy that compels Painter to color a red copy of F or a blue copy of H within m rounds. To prove the lower bound for $\tilde{R}(F,H) > m$, we must show a Painter that avoids a red F or blue H in the first m rounds.

Online Ramsey numbers have been studied in a variety of contexts, including planar graphs [26]. A conjecture in the study of online Ramsey numbers, attributed by Kurek and Ruciński [25] to Rödl is if

$$\lim_{n\to\infty} \frac{\tilde{R}(K_n)}{\hat{R}(K_n)} = 0.$$

Conlon [27] approached a solution, proving there is a subsequence $\{t_1, t_2, \ldots\}$ of the integers such that

$$\lim_{i\to\infty} \frac{\tilde{R}(K_{t_i})}{\hat{R}(K_{t_i})} = 0.$$

Similar to other Ramsey numbers, there is a particular interest in researching the online Ramsey number of trees. Grytczuk, Kierstead, and Prałat [28] found $\tilde{R}(P_n) = 2n-3$ for $n = 2,3,4,5$ and $\tilde{R}(P_6) = 10$. They also give the upper bound $\tilde{R}(P_n) \leq 4n-7$.

Here, we expand upon our findings in Section 3 and give the upper bound $\tilde{R}(\mathcal{C}, P_n) \leq \left\lfloor \frac{5}{2}n \right\rfloor$. We accomplish this by showing that if Painter successfully avoids coloring "short" red cycles in the first $\left\lfloor \frac{5}{2}n \right\rfloor$ rounds, then Painter will have colored a blue P_n in the process. Builder constructs a long blue path by considering an existing blue path P_k with an edge between the endpoints, and extending the path by ℓ vertices in at most $\left\lfloor \frac{5}{2}\ell \right\rfloor$ rounds, where $1 \leq \ell \leq n-k$. Although we are not considering the cube of a path, we can still apply Observation 3.5 to see that Builder's strategy strategy gives the bound $\tilde{R}(\mathcal{C}, P_n) \leq \left\lfloor \frac{5}{2}n \right\rfloor + O(1)$. With this in mind, we state our main result of this section.

Theorem 4.1. *Let $n \geq 3$. Then there exists a Builder strategy such that after at most $\lfloor \frac{5}{2}n \rfloor + 5$ rounds, Painter will either color a red C_3 or C_4 or a blue P_n.*

Corollary 4.2. *For any $c \in \mathbb{R}^+$,*

$$\tilde{R}(\mathcal{C}, P_n) \leq \tilde{R}(\mathcal{C}_{\leq cn}, P_n) \leq \tilde{R}(\mathcal{C}_{\leq 4}, P_n) \leq \left\lfloor \frac{5}{2}n \right\rfloor + 5.$$

Proof. The first two inequalities follow by monotonicity. By Theorem 4.1, there exists a Builder strategy such that after at most $\lfloor \frac{5}{2}n \rfloor + 5$ rounds, Painter will lose by coloring a short red cycle or a blue P_n. So by definition $\tilde{R}(\mathcal{C}_{\leq 4}, P_n) \leq \lfloor \frac{5}{2}n \rfloor + 5$. □

4.2 Proof of Theorem 4.1

Proof of Theorem 4.1. We prove the theorem by induction on n. To show the base case when $n = 3$ (as illustrated in Figure 6), we must employ a Builder strategy that compels Painter to color a blue P_3 or a red cycle within at most $\lfloor \frac{5}{2} \cdot 3 \rfloor = 7$ rounds. Builder begins by serving edge v_1v_2. If v_1v_2 is red, serve edges v_1v_3 and v_2v_3 (the order in which the edges are served is arbitrary). Both of these edges cannot be red, else Painter would color the red cycle $(v_1v_2v_3)$. If both edges are blue we have the blue path $P_3 = v_1v_3v_2$ with edge v_1v_2 present, so assume only one edge is blue. Without loss of generality, suppose v_1v_3 is blue. Now serve edges v_1v_4 and v_3v_4, again in an arbitrary order. Both edges cannot be red, else Painter would color the red cycle $(v_1v_2v_3v_4)$. So suppose at least one edge is blue. If Painter colors edge v_1v_4 blue, then we have a blue $P_3 = v_3v_1v_4$ with edge v_3v_4 present. If Painter colors edge v_3v_4 blue, then we have a blue $P_3 = v_1v_3v_4$ with edge v_1v_4 present.

Now let v_1v_2 be blue. Then Builder serves edges v_1v_3 and v_2v_3 in an arbitrary order. Then both edges must be red, else Painter would color a blue P_3 with an edge between the endpoints. Next, Builder serves edges v_1v_4 and v_2v_4. Both of these edges cannot be red since Painter would color the red cycle $(v_1v_2v_3v_4)$, so assume one of the edges is blue. Without loss of generality we let v_1v_4 be blue. Then Painter colored a blue $P_3 = v_2v_1v_4$ with edge v_2v_4 present.

Hence there exists a Builder strategy that compels Painter to either color a red cycle or a blue P_3 in $5 < 7$ rounds, so Theorem 4.1 is satisfied for $n = 3$. With our base case established, we now formulate an inductive hypothesis in the following lemma.

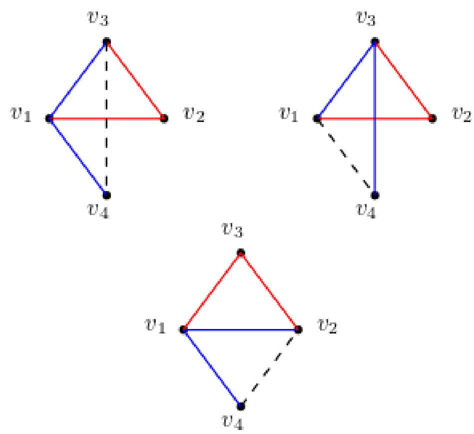

Figure 6: Builder's strategy to satisfy Theorem 4.1 when $n = 3$. The top two graphs depict Builder's strategy when v_1v_2 is red and the bottom graph shows Builder's strategy when v_1v_2 is blue.

Lemma 4.3. *Let $k \geq 3$ and P_k be the path $v_1v_2 \ldots v_k$ with edge v_1v_k present. Then there is a Builder strategy such that either:*

1. *Painter colors a red C_3 or C_4; or*

2. *There is an integer ℓ, $1 \leq \ell \leq n-k$, such that Painter colors a $P_{k+\ell}$ with an edge between the endpoints after at most $\left\lfloor \frac{5}{2}\ell \right\rfloor$ moves.*

Proof. Similar to the base case, there are naturally two cases to consider in our proof: when edge v_1v_k is red and when edge v_1v_k is blue.

- **Case 1** (v_1v_k is red)

  Let v_1v_k be red. Then Builder serves edges v_1v_{k+1} and v_kv_{k+1} (the order in which this occurs is irrelevant). Both v_1v_{k+1} and v_kv_{k+1} cannot be red, since Painter would be coloring the red cycle $(v_1v_kv_{k+1})$. So one edge must be blue; without loss of generality we let v_kv_{k+1} be blue. Then Painter colored a blue $P_{k+1} = v_1v_2\ldots v_kv_{k+1}$ with edge v_1v_{k+1} present in $\lfloor \frac{5}{2}\cdot 1 \rfloor = 2$ moves, so $\ell = 1$ and we are done.

- **Case 2** (v_1v_k is blue)

  Now let v_1v_k be blue. Builder again starts by serving edges v_1v_{k+1} and v_kv_{k+1} in an arbitrary order. If at least one of the new edges, say v_kv_{k+1}, is blue then Painter colored a blue $P_{k+1} = v_1v_2\ldots v_kv_{k+1}$ with edge v_1v_{k+1} present in two moves, so Lemma 4.3 is satisfied with $\ell = 1$. So assume both v_1v_{k+1} and v_kv_{k+1} are red. Next, Builder serves edge $v_{k+1}v_{k+2}$.

  Suppose $v_{k+1}v_{k+2}$ is colored blue. Then Builder serves edges v_1v_{k+2} and v_kv_{k+2} in an arbitrary order. If both edges are red, then Painter colors the red cycle $(v_kv_{k+2}v_1v_{k+1})$, so let at least one edge be blue. If edge v_kv_{k+2} is blue, then Painter colors a $P_{k+2} = v_1v_2\ldots v_kv_{k+2}v_{k+1}$ with edge v_1v_{k+1} present in five moves, so we are done. If edge v_1v_{k+2} is blue, then Painter colors a $P_{k+2} = v_{k+1}v_{k+2}v_1v_2\ldots v_k$ with edge v_kv_{k+1} present in five moves, so we are done. Thus if $v_{k+1}v_{k+2}$ is colored blue, then Lemma 4.3 is satisfied with $\ell = 2$.

  Assume $v_{k+1}v_{k+2}$ is red. Then Builder keeps serving edge $v_{k+1}v_{k+j}$ for $3 \leq j \leq n-k$ (note that we already accounted for when $j = 1, 2$) until Painter colors edge $v_{k+1}v_{k+j}$ blue if $j \leq n-k$ or colors edge $v_{k+1}v_n$.

  Suppose Painter colors the edge $v_{k+1}v_{k+j}$ blue for some $3 \leq j \leq n-k$. Then Builder serves the edge v_kv_{k+2} which Painter must color blue, else Painter would color the red cycle $(v_kv_{k+1}v_{k+2})$. Similarly, Builder then serves edge $v_{k+j-1}v_{k+j-2}$ which Painter must color blue, else Painter would color the red cycle $(v_{k+1}v_{k+j-1}v_{k+j-2})$. Builder then serves edges v_1v_{k+j} and $v_{k+j}v_{k+j-1}$ in an arbitrary order. If both edges are colored red then Painter colors the red cycle $(v_1 v_{k+1} v_{k+j-1} v_{k+j})$, so one of the edges must be blue.

If $v_1 v_{k+j}$ is blue, then Painter colors a blue $P_{k+j} = v_{k+j} v_1 v_2 \dots v_k v_{k+2} v_{k+3} \dots v_{k+j-1}$ with edge $v_{k+j} v_{k+j-1}$ present. This process took $2j+1$ rounds and extended the path by j vertices, so Lemma 4.3 is satisfied with $\ell = j$.

If $v_{k+j-1} v_{k+j}$ is blue, then Painter colors a blue $P_{k+j} = v_1 v_2 \dots v_k v_{k+2} v_{k+3} \dots v_{k+j} v_{k+1}$ with edge $v_1 v_{k+1}$ present. Similarly, this process took $2j+1$ rounds and extended the path by j vertices, so Lemma 4.3 is satisfied with $\ell = j$.

Suppose Painter colors the edge $v_{k+1} v_{k+j}$ red for every $3 \leq j \leq n-k$. Builder then serves edge $v_k v_{k+2}$ which Painter must color blue, otherwise Painter would color the red cycle $(v_k v_{k+1} v_{k+2})$. Builder similarly serves edge $v_{k+j-1} v_{k+j}$ for every $3 \leq j \leq n-k$. All of these edges must be colored blue since otherwise Painter would be coloring the red cycle $(v_{k+1} v_{k+j-1} v_{k+j})$ for some j. Lastly, Builder serves edge $v_1 v_n$. This must also be blue, else Painter would color the red cycle $v_1 v_{k+1} v_n$. Thus Painter has colored a blue $P_n = v_1 v_2 \dots v_n$ with edge $v_1 v_n$ present, and we are done.

So if Lemma 4.3 is not satisfied when $\ell = 1$, then we let $S = \{j : v_{k+1} v_{k+j} \text{ is blue}\}$ and $\ell = \min\{n-k, S\}$. Then with Builder's strategy, Painter will extend an existing blue P_k to a blue $P_{k+\ell}$ in at most $\left\lfloor \frac{5}{2}\ell \right\rfloor$ moves as needed, so we are done.

$\square$

$\square$

5 Concluding Remarks

In this thesis we have considered different variations of the ordinary Ramsey number for paths and cycles, with a particular focus on upper bounds. We first studied how the bipartite property of a graph affects the linear coefficient $\hat{R}(P_n) \leq 74n$ by using a construction related to the one given in [15]. With this approach we obtained the upper bound $\hat{br}(P_n) \leq 130.104n$ for sufficiently large n. Next, we considered the size Ramsey number for the family of cycles versus a path of order n, which provided the most significant result of this thesis. In contrast to many recent results on size Ramsey numbers of paths and cycles, we use a non-random construction to show $\hat{R}(\mathcal{C}, P_n) < 3.947n + O(1)$. This, however, is due to the fact that the question considered included forbidden short cycles. We note in passing that by considering the third power of a cycle C_N^3 with $N = \frac{25}{19}n + O(1)$, our proof easily implies that

$$\hat{R}(\mathcal{C}_{\leq 8}, \mathcal{C}_{\geq n}) \leq 3.947n$$

where $\mathcal{C}_{\geq n}$ is the family of all cycles of length at least n. The r-colored version of this problem was also explored, and we give the bounds $(n-1)r < \hat{R}_r(\mathcal{C}, \ldots, \mathcal{C}, P_n) \leq 38rn$ to show $\hat{R}_r(\mathcal{C}, \ldots, \mathcal{C}, P_n) = \Theta(r)n$. We lastly expanded on the study of Ramsey numbers for the family of cycles versus a path of order n and examine the online Ramsey number $\tilde{R}(\mathcal{C}, P_n)$. Specifically, we show $\tilde{R}(\mathcal{C}, P_n) \leq \left\lfloor \frac{5}{2}n \right\rfloor + 5$ by employing a Builder strategy that compels Painter to color short red cycles or extend an existing blue path. By slightly modifying Builder's strategy we were also able to attain the upper bound $\tilde{R}(\mathcal{C}, P_n) \leq \left\lfloor \frac{11}{5}n \right\rfloor + 5$, but the proof is omitted since it is quite detailed and there is no matching lower bound.

The most obvious open problem is to close the gap between the lower bound of $2.066n$ and the upper bound of $3.947n$ for $\hat{R}(\mathcal{C}, P_n)$. It is possible that there is a nice proof that every two coloring of P_N^3 contains a blue path of density $7/9$, but we were unable to find one.

Another question for future research is improving the bounds of $\hat{R}_r(\mathcal{C}, \ldots, \mathcal{C}, P_n)$. In particular, our construction in showing $\hat{R}(\mathcal{C}, P_n) < 3.947n + O(1)$ suggests there may be a non-random construction to give an upper bound better than $38rn$.

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