

A1.1

ges: $D_{41}(\alpha, \beta, \gamma)$

$$D_{41} = D_{43(\alpha)}(\gamma) \cdot D_{32(\beta)}(\beta) \cdot D_{21(\gamma)}(\alpha)$$

$$D_{41} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} \cos \beta & \sin \beta & 0 \\ -\sin \beta & \cos \beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos \gamma & 0 & -\sin \gamma \\ 0 & 1 & 0 \\ \sin \gamma & 0 & \cos \gamma \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & -\sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} \cos \beta & \sin \beta & -\sin \gamma \cos \beta \\ -\sin \beta & \cos \beta & \sin \beta \sin \alpha \\ \sin \alpha & 0 & \cos \alpha \end{pmatrix}$$

$$= \begin{pmatrix} \cos \alpha \cos \beta & \sin \beta & -\sin \alpha \cos \beta \\ -\cos \alpha \sin \beta \cos \gamma + \sin \alpha \sin \gamma & \cos \beta \cos \gamma & \sin \alpha \sin \beta \cos \gamma + \cos \alpha \sin \gamma \\ \cos \alpha \sin \beta \sin \gamma + \sin \alpha \cos \gamma & -\cos \beta \sin \gamma & -\sin \alpha \sin \beta \sin \gamma + \cos \alpha \cos \gamma \end{pmatrix}$$

A1.2

$$U(x, y, z) = \frac{Q}{4\pi \epsilon_0 \sqrt{x^2 + y^2 + z^2}}$$

A1.3

$$\alpha = -\frac{\pi}{2}, \quad G = 2, \quad \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$d = \begin{pmatrix} \frac{i\sqrt{5}}{2} \\ \frac{j\sqrt{5}}{2} \\ 0 \end{pmatrix}$$

$$D\left(\begin{pmatrix} \frac{i\sqrt{5}}{2} \\ \frac{j\sqrt{5}}{2} \\ 0 \end{pmatrix}, -\frac{\pi}{2}\right) = \cos\left(-\frac{\pi}{2}\right) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \left(1 - \cos\left(-\frac{\pi}{2}\right)\right) \begin{pmatrix} \frac{i\sqrt{5}}{2} \\ \frac{j\sqrt{5}}{2} \\ 0 \end{pmatrix} \otimes \begin{pmatrix} \frac{i\sqrt{5}}{2} \\ \frac{j\sqrt{5}}{2} \\ 0 \end{pmatrix} - \sin\left(-\frac{\pi}{2}\right) \cdot \begin{pmatrix} 0 & 0 & \frac{i\sqrt{5}}{2} \\ 0 & 0 & -\frac{j\sqrt{5}}{2} \\ \frac{i\sqrt{5}}{2} & \frac{j\sqrt{5}}{2} & 0 \end{pmatrix}$$

$$= \cos\left(-\frac{\pi}{2}\right) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \left(1 - \cos\left(-\frac{\pi}{2}\right)\right) \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \sin\left(-\frac{\pi}{2}\right) \cdot \begin{pmatrix} 0 & 0 & \frac{i\sqrt{5}}{2} \\ 0 & 0 & -\frac{j\sqrt{5}}{2} \\ \frac{i\sqrt{5}}{2} & \frac{j\sqrt{5}}{2} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\left(-\frac{\pi}{2}\right) & 0 & 0 \\ 0 & \cos\left(-\frac{\pi}{2}\right) & 0 \\ 0 & 0 & \cos\left(-\frac{\pi}{2}\right) \end{pmatrix} + \begin{pmatrix} 1 & -\frac{1}{2} \cos\left(-\frac{\pi}{2}\right) & \frac{1}{2} \cos\left(-\frac{\pi}{2}\right) \\ -\frac{1}{2} \cos\left(-\frac{\pi}{2}\right) & 1 & -\frac{1}{2} \cos\left(-\frac{\pi}{2}\right) \\ \frac{1}{2} \cos\left(-\frac{\pi}{2}\right) & -\frac{1}{2} \cos\left(-\frac{\pi}{2}\right) & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & \frac{1}{2} \sin\left(\frac{\pi}{2}\right) \\ 0 & 0 & -\frac{1}{2} \sin\left(\frac{\pi}{2}\right) \\ \frac{1}{2} \sin\left(\frac{\pi}{2}\right) & \frac{1}{2} \sin\left(\frac{\pi}{2}\right) & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} + \frac{1}{2} \cos\left(-\frac{\pi}{2}\right) & \frac{1}{2} - \frac{1}{2} \cos\left(-\frac{\pi}{2}\right) & -\frac{\sqrt{5}}{2} \sin\left(-\frac{\pi}{2}\right) \\ \frac{1}{2} - \frac{1}{2} \cos\left(-\frac{\pi}{2}\right) & \frac{1}{2} + \frac{1}{2} \cos\left(-\frac{\pi}{2}\right) & +\frac{\sqrt{5}}{2} \sin\left(-\frac{\pi}{2}\right) \\ +\frac{\sqrt{5}}{2} \sin\left(-\frac{\pi}{2}\right) & -\frac{\sqrt{5}}{2} \sin\left(-\frac{\pi}{2}\right) & \cos\left(-\frac{\pi}{2}\right) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

Matrix aus
A 1.1 $\Rightarrow \beta = \frac{1}{6}\pi$

$$\Rightarrow \alpha = \gamma = 0,955$$

Mit: $D_{41} = D_{43(\alpha)}(\alpha) \cdot D_{32(\beta)}(\beta) \cdot D_{21(\gamma)}(\gamma)$

A 1.4 $\alpha = 30^\circ \quad \beta = 45^\circ$

$$D_{21} = D_{32(\beta)}(\beta) \cdot D_{21(\alpha)}(\alpha)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \beta & \sin \beta \\ 0 & -\sin \beta & \cos \beta \end{pmatrix} \cdot \begin{pmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha \cos \beta & \cos \alpha \cos \beta & \sin \beta \\ \sin \alpha \sin \beta & -\cos \alpha \cos \beta & \cos \beta \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{2}+1}{4} \\ \frac{-\sqrt{2}+1}{4} \end{pmatrix}$$

A 1.5

$$A = \frac{1}{3} \begin{pmatrix} -7 & -4 & 4 \\ 4 & 1 & -8 \\ 4 & -8 & 1 \end{pmatrix}$$

A1.6

$$F = (47,65^\circ N, 5,48^\circ O, 400m \text{ o. NN})$$

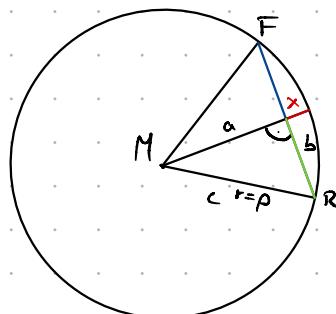
$$R = (47,57^\circ N, \approx 38^\circ O)$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \rho \cdot \cos \phi \cdot \sin \vartheta \\ \rho \cdot \sin \phi \cdot \sin \vartheta \\ \rho \cdot \cos \vartheta \end{pmatrix}$$

$$F = \begin{pmatrix} 6370,4 \cdot \cos(5,48^\circ) \cdot \sin(47,65^\circ) \\ 6370,4 \cdot \sin(5,48^\circ) \cdot \sin(47,65^\circ) \\ 6370,4 \cdot \cos(47,65^\circ) \end{pmatrix} = \begin{pmatrix} 4643,71 \\ 275,42 \\ 4281,47 \end{pmatrix}$$

$$R = \begin{pmatrix} 6370,4 \cdot \cos(5,38^\circ) \cdot \sin(47,57^\circ) \\ 6370,4 \cdot \sin(5,38^\circ) \cdot \sin(47,57^\circ) \\ 6370,4 \cdot \cos(47,57^\circ) \end{pmatrix} = \begin{pmatrix} 4639,14 \\ 766,34 \\ 4298,04 \end{pmatrix}$$

$$\vec{FR} = \begin{pmatrix} 4,57 \\ 5,08 \\ -6,57 \end{pmatrix} \quad |\vec{FR}| = \underline{12,1 \text{ km}}$$



$$b = 6,05 \text{ km}$$

$$r = 6370,4 \text{ km}$$

$$a = \sqrt{r^2 - b^2} = \sqrt{6370,4^2 - 6,05^2} = 6370,3971$$

$$x = r - a = 0,0029 \approx \underline{2,9 \text{ m}}$$

A2.1

$$A = \begin{pmatrix} 4 & 0 & -2 & 4 \\ 0 & 9 & 9 & 0 \\ -2 & 9 & 11 & -3 \\ 4 & 0 & -3 & 30 \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} = \begin{pmatrix} b_{11} & 0 & 0 & 0 \\ b_{21} & b_{22} & 0 & 0 \\ b_{31} & b_{32} & b_{33} & 0 \\ b_{41} & b_{42} & b_{43} & b_{44} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} & b_{31} & b_{41} \\ 0 & b_{21} & b_{22} & b_{42} \\ 0 & 0 & b_{33} & b_{43} \\ 0 & 0 & 0 & b_{44} \end{pmatrix}$$

$$b_{11} = \sqrt{a_{11}} \quad b_{22} = \sqrt{a_{22} - b_{11}^2} \quad b_{33} = \sqrt{a_{33} - b_{31}^2 - b_{32}^2} \quad b_{44} = \sqrt{a_{44} - b_{41}^2 - b_{42}^2 - b_{43}^2}$$

$$b_{21} = \frac{a_{21}}{b_{11}} \quad b_{32} = \frac{a_{32} - b_{31}b_{21}}{b_{22}} \quad b_{43} = \frac{a_{43} - b_{31}b_{41} - b_{42}b_{32}}{b_{33}}$$

$$b_{31} = \frac{a_{31}}{b_{11}} \quad b_{42} = \frac{a_{42} - b_{41}b_{21}}{b_{22}}$$

$$b_{41} = \frac{a_{41}}{b_{11}}$$

$$\Rightarrow b_{11} = \sqrt{4} = 2 \quad b_{22} = \sqrt{9 - 0} = 3 \quad b_{33} = \sqrt{11 - 1 - 9} = 1$$

$$b_{21} = \frac{0}{2} = 0 \quad b_{32} = \frac{9+0}{3} = 3 \quad b_{43} = \frac{-3+2}{1} = -1$$

$$b_{31} = -\frac{2}{2} = -1 \quad b_{42} = \frac{0-0}{3} = 0 \quad b_{44} = \sqrt{30 - 4 - 1} = 5$$

$$b_{41} = \frac{4}{2} = 2$$

$$\Rightarrow \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ -1 & 2 & 1 & 0 \\ 2 & 0 & -1 & 5 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 & -1 & 2 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 5 \end{pmatrix} \Rightarrow 30 \cdot 30 = 900$$

A2.5

$$b_{11} = 1 \quad b_{22} = 2 \quad b_{33} = 3 \quad b_{44} = 1$$

$$b_{21} = -2 \quad b_{32} = -3 \quad b_{43} = -2$$

$$b_{31} = 0 \quad b_{42} = 0$$

$$b_{41} = 0$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & -2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -2 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot x = y$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & -2 & 1 \end{pmatrix} \cdot z = y$$

$$\begin{pmatrix} 1 & -2 & 0 & 0 \\ 0 & 2 & -3 & 0 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot x = z$$

$$\begin{aligned} z_1 &= y_1 \\ -2z_1 + 2z_2 &= y_2 \quad |+2I \\ -3z_2 + 3z_3 &= y_3 \quad |+1,5II \\ -2z_3 + z_4 &= y_4 \quad |+\frac{2}{3}III \end{aligned}$$

$$\begin{aligned} x_1 - 2x_2 &= z_1 \\ 2x_2 - 3x_3 &= z_2 \\ 3x_3 - 2x_4 &= z_3 \\ x_4 &= z_4 \end{aligned} \quad \left| \begin{array}{l} +II \\ +III \\ +2IV \end{array} \right.$$

$$\begin{aligned} 2z_2 - 3z_3 &= y_2 + 2y_1 \\ 3z_3 - 2z_4 &= y_3 + \frac{3}{2}y_2 + 3y_1 \\ z_4 &= y_4 + \frac{2}{3}y_3 + y_2 + 2y_1 \end{aligned}$$

$$\begin{aligned} x_1 &= z_1 + z_2 + z_3 + 2z_4 \\ 2x_2 &= z_2 + z_3 + 2z_4 \\ 3x_3 &= z_3 + 2z_4 \end{aligned}$$

$$\Rightarrow z = \begin{pmatrix} 4 \\ -5 \\ -1 \\ 2 \end{pmatrix}$$

$$\Rightarrow x = \begin{pmatrix} 2 \\ -1 \\ 1 \\ 2 \end{pmatrix}$$

A 3.1

$$x^2 + y^2 - 14x + 12y + 60 = 0$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$(1-2)^2 \Rightarrow \lambda_{1,2} = 1$$

$$\begin{aligned} (1-2)_x &= 0 \\ (1-2)_y &= 0 \end{aligned} \Rightarrow v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\xi^2 + \eta^2 + (-14, 12) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

$$\xi^2 + \eta^2 + -14\xi + 12\eta + 60 = 0$$

$$(\xi + \frac{-14}{2})^2 - \frac{196}{4} = (\xi - 7)^2 - 49$$

$$(\eta + 6)^2 - 36$$

$$\Rightarrow (\xi - 7)^2 - 49 + (\eta + 6)^2 - 36 + 60 = 0$$

$$v^2 + w^2 = 25$$

\Rightarrow Kreis \checkmark

$\Rightarrow M(7+6)$

A3.2

$$5x^2 + 6xy + 5y^2 - 2 = 0$$

$$(x \ y) \cdot \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} - 2 = 0$$

$$(5-2)^2 - 9 = 0$$

$$25 - 10k + 2^2 - 9 = 0$$

$$2^2 - 10k + 16 = 0$$

$$\rightarrow \lambda_1 = 2 \quad \Rightarrow \quad \lambda_2 = 8$$

$$\begin{array}{r} 2^2 - 10k + 16 : 2 \cdot 2 = 2 - 8 \\ - \underline{2^2 + 2k} \\ -8k + 16 \end{array}$$

$$\begin{pmatrix} 5-2 & 3 \\ 3 & 5-2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \frac{1}{\sqrt{2}}$$

$$\begin{pmatrix} -3 & 3 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}}$$

eigentlich unnötig weil hier nicht benötigt

$$(x \ y) \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 \\ 0 & 8 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} - 2 = 0$$

$$2\xi^2 + 8\eta^2 - 2 = 0$$

$$\xi^2 + 4\eta^2 = 1$$

$$\frac{\xi^2}{1^2} + \frac{\eta^2}{(\frac{1}{2})^2} = 1$$

große Halbachse kleine Halbachse

3.3

$$5x^2 + 5y^2 + 26xy - 34\sqrt{2}x - 2\sqrt{2}y - 75 = 0$$

$$\begin{pmatrix} 5 & 13 \\ 13 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}^2 - 13^2 = 25 - 10x + 2^2 - 16y$$

$$= 2^2 - 10x - 144 \Rightarrow \underline{x_1 = -8}$$

$$-(2^2 + 8x) \quad \underline{-18x - 144}$$

$$\begin{pmatrix} 13 & 13 \\ 13 & 13 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \underline{x = 18} \rightarrow \underline{x_2 = 18}$$

$$\begin{pmatrix} -13 & 13 \\ 13 & -13 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow V = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \quad \underline{\lambda = \begin{pmatrix} -8 & 0 \\ 0 & 18 \end{pmatrix}}$$

$$\frac{1}{\sqrt{2}} (-34\sqrt{2} - 2\sqrt{2}) \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} (-34\sqrt{2} - 2\sqrt{2}) \begin{pmatrix} \xi + \eta \\ -\xi + \eta \end{pmatrix}$$

$$= (-34 \quad) (\xi + \eta) + (-2 \quad) (-\xi + \eta)$$

$$= -32\xi - 36\eta$$

$$\Rightarrow -8\xi^2 + 18\eta^2 - 32\xi - 36\eta - 75 = 0$$

$$\Rightarrow -8\left(\xi + \frac{32}{16}\right)^2 + \frac{32^2}{32} = -8(\xi + 2)^2 + 32$$

$$\Rightarrow 18\left(\eta - \frac{36}{36}\right)^2 - \frac{36^2}{4 \cdot 18} = 18(\eta - 1)^2 - 18$$

$$\Rightarrow -8v^2 + 32 + 18w^2 - 18 - 75 = 0$$

$$-8v^2 + 18w^2 = 61$$

$$-\frac{v^2}{\frac{61}{8}} + \frac{w^2}{\frac{61}{18}} = 1 \Rightarrow \text{Ellipse}$$

Verschiebungsvektor $\begin{pmatrix} +2 \\ -1 \end{pmatrix}$

$$\text{D} \cdot \vec{x} = \begin{pmatrix} +2 \\ -1 \end{pmatrix}$$

$$\begin{aligned} \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y &= +2 \\ -\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y &= -1 \quad |+I \end{aligned}$$

$$\frac{2}{\sqrt{2}}y = 1$$

$$y = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}} = 2$$

$$x = \frac{3\sqrt{2}}{2}$$

$$\Rightarrow P\left(\frac{3\sqrt{2}}{2}, \frac{1}{\sqrt{2}}\right)$$

3.4

$$9x^2 + 4y^2 + 12xy - 4x + 5 = 0$$

$$\begin{pmatrix} 9 & 6 \\ 6 & 4 \end{pmatrix} \quad (5-2)(4-2) - 36 = 0$$

$$36 - 36 - 132 + 2^2 = 0$$

$$2^2 - 132 = 0 \quad \rightarrow x_1 = 0 \\ x_2 = 13$$

$$\begin{pmatrix} 9 & 6 \\ 6 & 4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow v_1 = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \frac{1}{\sqrt{13}}$$

$$\begin{pmatrix} -4 & 6 \\ 6 & -9 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow v_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \frac{1}{\sqrt{13}}$$

→ wgg. Lösung hier $\det(V) = -1$

$$\Rightarrow V = \begin{pmatrix} 3 & 2 \\ 2 & -3 \end{pmatrix} \frac{1}{\sqrt{13}} \quad \Delta = \begin{pmatrix} 13 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\frac{1}{\sqrt{13}} (-4 \ 0) \begin{pmatrix} 3 & 2 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} \varepsilon \\ \eta \end{pmatrix}$$

$$= \frac{1}{\sqrt{13}} (-4 \ 0) (3\varepsilon + 2\eta)$$

$$= \frac{1}{\sqrt{13}} (-12\varepsilon - 8\eta)$$

$$\rightarrow 13\varepsilon^2 - \frac{12}{\sqrt{13}}\varepsilon - \frac{8}{\sqrt{13}}\eta + 5 = 0$$

$$13\left(\varepsilon - \frac{6}{\sqrt{13}}\right)^2 - \frac{144}{13 \cdot 13} = 13\left(\varepsilon - \frac{6}{\sqrt{13}}\right)^2 - \frac{36}{169}$$

$$13\left(\varepsilon - \frac{6}{\sqrt{13}}\right)^2 - \frac{8}{\sqrt{13}}\eta + \frac{809}{169} = 0$$

$$13\left(\varepsilon - \frac{6}{\sqrt{13}}\right)^2 - \frac{8}{\sqrt{13}}\left(\eta - \frac{809}{169} \cdot \frac{\sqrt{13}}{8}\right) = 0$$

$$13v^2 - \frac{8}{\sqrt{13}}w = 0$$

⇒ Parabel

35

$$\frac{1}{2}x^2 + \frac{5}{12}(y^2 + z^2) - \frac{1}{6}yz - 1 = 0$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{5}{12} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{5}{12} \end{pmatrix} \left(\frac{1}{2} - \lambda \right) \left(\frac{5}{12} - \lambda \right)^2 - \left(\frac{1}{12} \cdot \left(\frac{1}{2} - \lambda \right) \right) = 0$$

$$\left(\frac{1}{2} - \lambda \right) \left(\frac{25}{144} - \frac{10}{12} \lambda + \lambda^2 \right) - \left(\frac{1}{12} \cdot \frac{1}{2} - \frac{\lambda}{12} \right) = 0$$

$$\frac{25}{144}\lambda^2 - \frac{10}{24}\lambda + \frac{\lambda}{2} - \frac{25}{144} + \frac{10}{12}\lambda^2 - \lambda^3 - \frac{1}{144}\lambda + \frac{\lambda}{12^2} = 0$$

$$-\lambda^3 + \frac{16}{12}\lambda^2 - \frac{84}{12}\lambda + \frac{24}{12^2} = 0$$

$$\underline{\lambda_1 = \frac{1}{2}}$$

$$\begin{aligned} -\lambda^3 + \frac{4}{3}\lambda^2 - \frac{7}{12}\lambda + \frac{1}{12} : 2 - \frac{1}{2} = -\lambda^2 + \frac{5}{6}\lambda - \frac{2}{12} \\ \underline{-(-\lambda^3 + \frac{4}{3}\lambda^2)} \end{aligned}$$

$$\begin{aligned} \frac{5}{6}\lambda^2 - \frac{7}{12}\lambda \\ -(\frac{5}{6}\lambda^2 - \frac{5}{6}\lambda) \\ -\frac{2}{12}\lambda + \frac{1}{12} \end{aligned}$$

$$\lambda_{2,3} = \frac{-\frac{5}{6} \pm \sqrt{\frac{25}{36} - \frac{2}{3}}}{-2} = \frac{-\frac{5}{6} \pm \sqrt{\frac{1}{36}}}{-2} = \frac{-\frac{5}{6} \pm \frac{1}{6}}{-2}$$

$$\Rightarrow \underline{\lambda_2 = \frac{1}{2}}$$

$$\underline{\lambda_3 = \frac{1}{3}}$$

$$\begin{pmatrix} \frac{1}{2} - \lambda & 0 & 0 \\ 0 & \frac{5}{12} - \lambda & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{5}{12} - \lambda \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} -\frac{1}{2}v_2 - \frac{1}{12}v_3 \\ -\frac{1}{12}v_2 - \frac{1}{12}v_3 \end{aligned} \Rightarrow \begin{matrix} \vec{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \\ \vec{v}_2 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \end{matrix}$$

$$\begin{aligned} \frac{1}{6}v_1 \\ \frac{1}{12}v_2 - \frac{1}{12}v_3 \\ -\frac{1}{12}v_2 \quad \frac{1}{12}v_3 \end{aligned} \Rightarrow \begin{matrix} \vec{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \end{matrix}$$

$$\begin{pmatrix} \frac{1}{12} & \frac{2}{12} & 0 \\ -\frac{1}{12} & \frac{1}{12} & 1 \\ \frac{1}{12} & \frac{1}{12} & \frac{1}{12} \end{pmatrix}$$

$$\Rightarrow V = \begin{pmatrix} \frac{1}{12} & \frac{2}{12} & 0 \\ -\frac{1}{12} & \frac{1}{12} & 1 \\ \frac{1}{12} & \frac{1}{12} & \frac{1}{12} \end{pmatrix} \text{ mit } \Delta = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$$

$$\frac{1}{2}\xi^2 + \frac{1}{2}\eta^2 + \frac{1}{3}\mu^2 = 1$$

\Rightarrow Ellipsoid mit $a = \sqrt{2}$, $b = \sqrt{2}$, $c = \sqrt{3}$

3.6

$$x^2 + 2y^2 - xy + 4y + 1 = 0$$

$$\begin{pmatrix} 1 & \frac{1}{2} \\ -\frac{1}{2} & 2 \end{pmatrix} \quad (1-2)(2-2) - \frac{1}{4} = 2^2 - 3 \cdot 2 + \frac{3}{4} \quad (1-\sqrt{2})(\sqrt{2}-1)$$

$$\Rightarrow \frac{3 \pm \sqrt{3-7}}{2} \quad \lambda_1 = \frac{3+\sqrt{2}}{2} \quad \lambda_2 = \frac{3-\sqrt{2}}{2}$$

$$2\sqrt{2} - 2 - 1$$

$$-1 \over 4-2\sqrt{2}$$

$$\begin{pmatrix} -1-\sqrt{2} \\ 2 \end{pmatrix} v_1 - \frac{1}{2} v_2 = 0 \quad \Rightarrow v_1 = \begin{pmatrix} 1-\sqrt{2} \\ 1 \end{pmatrix} = \frac{1}{\sqrt{4-2\sqrt{2}}} \begin{pmatrix} 1-\sqrt{2} \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -1+\sqrt{2} \\ 2 \end{pmatrix} v_1 - \frac{1}{2} v_2 = 0 \quad \Rightarrow v_2 = \begin{pmatrix} 1 \\ -1+\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{4-2\sqrt{2}}} \begin{pmatrix} 1 \\ -1+\sqrt{2} \end{pmatrix}$$

$$\Rightarrow V = \frac{1}{\sqrt{4-2\sqrt{2}}} \begin{pmatrix} 1 & 1-\sqrt{2} \\ \sqrt{2}-1 & 1 \end{pmatrix} \quad \Delta = \begin{pmatrix} \frac{3-\sqrt{2}}{2} & 0 \\ 0 & \frac{3+\sqrt{2}}{2} \end{pmatrix}$$

$$(0 \ 4) \cdot \begin{pmatrix} 1 & 1-\sqrt{2} \\ \sqrt{2}-1 & 1 \end{pmatrix} \cdot \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

$$(0 \ 4) \cdot \begin{pmatrix} 1\xi + (1-\sqrt{2})\eta \\ (\sqrt{2}-1)\xi + \eta \end{pmatrix}$$

$$(4\sqrt{2}-4)\xi + 4\eta$$

$$\Rightarrow \frac{3-\sqrt{2}}{2} \xi^2 + \frac{3+\sqrt{2}}{2} \eta^2 + (4\sqrt{2}-4)\xi + 4\eta + 1 = 0$$

$$\frac{3-\sqrt{2}}{2} \left(\xi + \frac{4(\sqrt{2}-1)}{3-\sqrt{2}} \right)^2 - \frac{16(\sqrt{2}-1)^2}{2(3-\sqrt{2})} \quad \frac{8(2-2\sqrt{2}+1)}{3-\sqrt{2}} \quad \frac{8(3-2\sqrt{2})}{3-\sqrt{2}} \cdot \frac{-3-\sqrt{2}}{(3-\sqrt{2})}$$

$$\frac{3-\sqrt{2}}{2} \left(\xi + \frac{4\sqrt{2}-1}{3-\sqrt{2}} \right)^2 + \frac{-40+24\sqrt{2}}{7}$$

$$\frac{3+\sqrt{2}}{2} \left(\eta + \frac{4}{3+\sqrt{2}}\right)^2 - \frac{16}{2(3+\sqrt{2})}$$

$$\frac{3+\sqrt{2}}{2} \left(\eta + \frac{4}{3+\sqrt{2}}\right)^2 + \frac{-24+8\sqrt{2}}{2}$$

$$\Rightarrow \frac{3-\sqrt{2}}{2} v^2 + \frac{3+\sqrt{2}}{2} \omega^2 = \frac{21-32\sqrt{2}}{7}$$

$$\frac{21-3\sqrt{2}}{142-64\sqrt{2}} v^2 + \frac{21+3\sqrt{2}}{142-64\sqrt{2}} \omega^2 = 1$$

2,15 1,29

4.1

$$a) y'' - 4y = 0$$

$$\lambda^2 - 4 = 0$$

$$\lambda_1, \lambda_2 = \pm 2$$

$$y = c_1 e^{2t} + c_2 e^{-2t}$$

$$b) y'' - 5y' + 6y = 0$$

$$\lambda^2 - 5\lambda + 6 = 0$$

$$\lambda_1 = 2 \quad \lambda_2 = 3$$

$$\Rightarrow y = c_1 e^{2t} + c_2 e^{3t}$$

4.2

$$a) y'' - 4y = 0$$

\downarrow \downarrow
 λ_1 λ_2

$$\begin{pmatrix} 2 & 1 \\ 4 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\lambda^2 - 4 = 0$$

$$\lambda_1, \lambda_2 = \pm 2$$

$$\begin{pmatrix} 2 & 1 \\ 4 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\Rightarrow e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad e^{2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} \Rightarrow y = c_1 e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$b) \quad y'' - 5y' + 6y = 0 \quad \Leftrightarrow \quad y'' = -\frac{5}{2}y' + \frac{6}{2}y$$

$$\begin{pmatrix} z_0 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix} \cdot \begin{pmatrix} z_0 \\ z_1 \end{pmatrix}$$

$$\begin{aligned} 0 &= -2(5-2) + 6 \\ &= 2^2 - 5 \cdot 2 + 6 \\ &= (2-2) \cdot (2-3) \quad \Rightarrow \lambda_1 = 2, \lambda_2 = 3 \end{aligned}$$

$$\begin{pmatrix} -2 & 1 \\ -6 & 3 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} -3 & 1 \\ -6 & 2 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\Rightarrow e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad e^{3t} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad \Rightarrow y = c_1 e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

4.3

$$\dot{x} = 2x + 4y$$

$$\dot{y} = -x - 2y$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{aligned} 0 &= (2-2)(-2-2) + 4 \\ &= 2^2 \end{aligned}$$

$$\Rightarrow \lambda_1 = 0$$

$$\begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$\Rightarrow e^{0t} \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad e^{0t} \begin{pmatrix} -1+2t \\ 1-t \end{pmatrix}$$

$$\begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\Leftrightarrow c_1 e^{0t} \begin{pmatrix} 2 \\ -1 \end{pmatrix} + c_2 e^{0t} \begin{pmatrix} -1+2t \\ 1-t \end{pmatrix}$$

4.4

$$\dot{x} = -3x - y$$

$$\dot{y} = x - y$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -3 & -1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

$$0 = (-3-\lambda)(-1-\lambda) + 1 \\ = \lambda^2 + 4\lambda + 4 \\ = (\lambda+2)^2$$

$$\Rightarrow \lambda_{1/2} = -2$$

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$\Rightarrow c_1 e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$x = c_1 e^{-2t} - c_2 e^{-2t} + c_2 t e^{-2t}$$

$$y = -c_1 e^{-2t} - c_2 t e^{-2t}$$

$$\stackrel{t=0}{\rightarrow} x(0) = 1$$

$$y(0) = 1$$

$$1 = c_1 - c_2 \Rightarrow c_2 = -2$$

$$1 = -c_1 \Rightarrow c_1 = -1$$

$$x = (-1 - 2t)e^{-2t}$$

$$y = (1 + 2t)e^{-2t}$$

4.5

$$\dot{x} = y$$

$$\dot{y} = x + e^+ + e^-$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow x^2 - 1 \Rightarrow x_{1,2} = \pm 1$$

$$\lambda = 1: \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow c_1 \cdot e^{\lambda t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-\lambda t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\lambda = -1: \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\dot{x} = y \Rightarrow \dot{x} = y$$

$$\Rightarrow \ddot{x} = x + e^+ + e^-$$

$$\ddot{x} - x = e^+ + e^-$$

2

4.6

$$\begin{aligned}\ddot{x} &= -\left(\frac{k_1+k_2}{m_1}\right)x + \frac{k_2}{m_1}y \Rightarrow \begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} -\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} \\ \frac{k_2}{m_2} & -\frac{k_2+k_3}{m_2} \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} \\ \ddot{y} &= \frac{k_2}{m_2}x - \left(\frac{k_2+k_3}{m_2}\right)y\end{aligned}$$

$$\begin{pmatrix} -70 & 20 \\ 40 & -140 \end{pmatrix} \Rightarrow \begin{pmatrix} -70-x \\ 40+140 \end{pmatrix} = \begin{pmatrix} -70-x \\ 20+210x \end{pmatrix} - 800$$

$$\begin{array}{r} 65 \cdot 65 \\ \hline 390 \\ 1325 \\ \hline 4225 \end{array} \quad \begin{array}{r} 210 \cdot 65 \\ \hline 1260 \\ 1050 \\ \hline 13650 \end{array}$$

5.1

$$f(x, y) = x^2 + 3xy + y^2$$

$$\text{Gradient: } \nabla f = (f_x, f_y) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right)$$

$$f_x = 2x + 3y \quad f_y = 3x + 2y$$

$$f_{xx} = 2 \quad f_{yy} = 2$$

$$f_{xy} = 3 \quad f_{yx} = 3$$

$$\nabla f = (f_x, f_y) = (2x + 3y, 3x + 2y) \quad \text{Gradient}$$

$$df = \begin{pmatrix} 2x + 3y \\ 3x + 2y \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix} \quad \text{totales Differential}$$

5.2

$$z = x^2 + 3xy + 5y^2 \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

$$\text{ges: } \frac{dz}{dt}$$

$$\frac{dz}{dx} = 2x + 3y \quad \frac{dx}{dt} = \cos t$$

$$\frac{dz}{dy} = 3x + 10y \quad \frac{dy}{dt} = -\sin t$$

$$\begin{aligned} \frac{dz}{dt} &= (2x + 3y) \frac{dx}{dt} + (3x + 10y) \frac{dy}{dt} \\ &= (2\sin t + 3\cos t) \cos t + (3\sin t + 10\cos t) (-\sin t) \\ &= 2\sin t \cos t - 10\cos t \sin t + 3\cos^2 t - 3\sin^2 t \\ &= 3\cos^2 t - 3\sin^2 t - 8\sin t \cos t \end{aligned}$$

5.3

$$f(x,y) = 3x - 3y - 2x^3 - xy^2 + 2x^2y + y^3$$

$$\Rightarrow \nabla f(x,y) = 0$$

$$f_x = 3 - 6x^2 - y^2 + 4xy = -6x^2 + 4xy - y^2 + 3$$

$$f_y = -3 - 2xy + 2x^2 + 3y^2 = 3y^2 - 2xy + 2x^2 - 3$$

$$\Rightarrow \begin{aligned} 0 &= -6x^2 + 4xy - y^2 + 3 \\ 0 &= 2x^2 - 2xy + 3y^2 - 3 \end{aligned} \quad |+2\text{II}$$

$$0 = -16x^2 + 10xy + 6$$

$$y = \frac{16x^2 - 6}{10x}$$

$$0 = -2x^2 + 5y^2 - 3$$

$$\begin{aligned} 0 &= -2x^2 + 5 \cdot \left(\frac{16x^2 - 6}{10x} \right)^2 - 3 \\ &= -2x^2 + 5 \cdot \left(\frac{256x^4 - 192x^2 + 36}{100x^2} \right) - 3 \quad | \cdot 100x^2 \text{ for } x \neq 0 \\ &= -200x^4 + 1280x^4 - 960x^2 + 180 - 300x^2 \end{aligned}$$

$$= 1080x^4 - 1260x^2 + 180$$

$$= 108x^4 - 126x^2 + 18$$

$$= 54x^4 - 63x^2 + 9$$

$$= 18x^4 - 21x^2 + 3$$

$$= 6u^4 - 7u^2 + 1$$

$$= 6u^2 - 7u + 1$$

$$u_{1,2} = \frac{7 \pm \sqrt{49 - 24}}{12} = \frac{7 \pm \sqrt{25}}{12} = \frac{7 \pm 5}{12}$$

$$u_1 = 1 \quad u_2 = \frac{1}{6}$$

$$\Rightarrow x_{1,2} = \pm 1 \quad x_{3,4} = \pm \frac{1}{\sqrt{6}}$$

$$\Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ \frac{1}{\sqrt{6}} \end{pmatrix}, \begin{pmatrix} -1 \\ -\frac{\sqrt{6}}{3} \end{pmatrix}, \begin{pmatrix} -\frac{1}{\sqrt{6}} \\ \frac{1}{3} \end{pmatrix}$$

$$y\left(\frac{1}{\sqrt{6}}\right) = \frac{\frac{16}{6} - 6}{\frac{10}{\sqrt{6}}} = -\frac{20 \cdot \sqrt{6}}{10 \cdot 6} = -\frac{\sqrt{6}}{3}$$

$$y\left(-\frac{1}{\sqrt{6}}\right) = \frac{\frac{16}{6} - 6}{-\frac{10}{\sqrt{6}}} = \frac{\sqrt{6}}{3}$$

$$f_{xx} = -12x + 4y$$

$$f_{yy} = -2x + 6y$$

$$f_{xy} = -2y + 4x$$

$$f_{yx} = -2y + 4x$$

$$H(x_0, y_0) = \begin{pmatrix} -12x+4y & -2y+4x \\ -2y+4x & -2x+6y \end{pmatrix}$$

$$\begin{aligned} D &= (-12x+4y)(-2x+6y) - (4x-2y)^2 \\ &= 24x^2 + 24y^2 - 72xy - 8xy - 16x^2 + 32xy - 4y^2 \\ &= 8x^2 + 20y^2 - 48xy \end{aligned}$$

$$\begin{aligned} D(1,1) &= 8+20-48 = -20 && \swarrow \\ D(-1,-1) &= 8+20-48 = -20 && \searrow \end{aligned}$$

$$D\left(\frac{1}{\sqrt{6}}, -\frac{\sqrt{6}}{2}\right) = \frac{8}{6} + \frac{20 \cdot 6}{9} + 48 \cdot \frac{1}{3} = \frac{4}{3} + \frac{40}{3} + \frac{48}{3} = \frac{91}{3} \quad \checkmark$$

$$D\left(-\frac{1}{\sqrt{6}}, \frac{\sqrt{6}}{3}\right) = \frac{8}{6} + \frac{20 \cdot 6}{9} + 48 \cdot \frac{1}{3} = \frac{91}{3} \quad \checkmark$$

$$f_{xx}\left(\frac{1}{\sqrt{6}}, -\frac{\sqrt{6}}{3}\right) = -12 \cdot \frac{1}{\sqrt{6}} - 4 \cdot \frac{\sqrt{6}}{3} < 0 \Rightarrow \text{Maximum} \quad \left(\frac{1}{\sqrt{6}}, -\frac{\sqrt{6}}{3}\right)$$

$$f_{xx}\left(-\frac{1}{\sqrt{6}}, \frac{\sqrt{6}}{3}\right) = 12 \cdot \frac{1}{\sqrt{6}} + 4 \cdot \frac{\sqrt{6}}{3} > 0 \Rightarrow \text{Minimum} \quad \left(-\frac{1}{\sqrt{6}}, \frac{\sqrt{6}}{3}\right)$$

5.3

$$f(x,y) = 3x - 3y - 2x^3 - xy^2 + 2x^2y + y^3$$

$$f_x = 3 - 6x^2 - y^2 + 4xy$$

$$f_y = -3 - 2xy + 2x^2 + 3y^2$$

$$\begin{aligned} 0 &= -6x^2 + 4xy - y^2 + 3 \quad |+3\text{II} \\ 0 &= 2x^2 - 2xy + 3y^2 - 3 \quad |+2\text{II} \end{aligned}$$

$$0 = -2xy + 8y^2 - 6$$

$$x = \frac{8y^2 - 6}{2y}$$

$$0 = -2x^2 + 5y^2 - 3$$

$$0 = -2\left(\frac{8y^2 - 6}{2y}\right) + 5y^2 - 3$$

$$= -\frac{64y^4 - 96y^2 + 36}{2y^2} + 5y^2 - 3 \quad | \cdot y^2$$

$$= -32y^4 + 48y^2 - 18 + 5y^4 - 3y^2$$

$$= -27y^4 + 45y^2 - 18$$

$$= -9y^4 + 15y^2 - 6$$

$$= -3y^4 + 5y^2 - 2 \quad | y^2 = u$$

$$u_{1,2} = \frac{-5 \pm \sqrt{25 - 24}}{-6} = \frac{-5 \pm 1}{-6}$$

$$u_1 = \frac{2}{3} \quad u_2 = 1$$

$$y_{1,2} = \pm \frac{\sqrt{2}}{\sqrt{3}} \quad y_{3,4} = \pm 1$$

$$= \pm \frac{\sqrt{6}}{3}$$

$$x_1 = \frac{\frac{6}{3} - 6}{2 \frac{\sqrt{6}}{3}} = \frac{3}{\sqrt{6}} \cdot \left(\frac{24}{3} - 3\right) = \frac{8}{\sqrt{6}} - \frac{9}{\sqrt{6}} = -\frac{1}{\sqrt{6}} \quad | \frac{\sqrt{6}}{3}$$

$$x_2 = \frac{\frac{6}{3} - 6}{-2 \frac{\sqrt{6}}{3}} = \frac{1}{\sqrt{6}} \quad | -\frac{\sqrt{6}}{3}$$

$$x_3 = \frac{8-6}{2} = 1 \quad | 1$$

$$x_4 = \frac{8-6}{2} = -1 \quad | -1$$

$$f_{xx} = -12x + 4y$$

$$f_{xy} = -2y + 4x$$

$$f_{yy} = -2x + 6y$$

$$\begin{aligned} D(x,y) &= (-12x+4y)(-2x+6y) - (-2y+4x)^2 \\ &= \cancel{24x^2} + \cancel{24y^2} - \cancel{72xy} - \cancel{8xy} - \cancel{16x^2} + \cancel{16xy} - \cancel{4y^2} \\ &= 8x^2 + 20y^2 - 64xy \end{aligned}$$

$$D\left(-\frac{1}{\sqrt{6}}, \frac{\sqrt{6}}{3}\right) = 8 \frac{1}{6} + 20 \frac{6}{9} - 64 \left(-\frac{1}{3}\right) > 0 \quad \checkmark$$

$$D\left(\frac{1}{\sqrt{6}}, -\frac{\sqrt{6}}{3}\right) = 8 \frac{1}{6} + 20 \frac{6}{9} - 64 \left(-\frac{1}{3}\right) > 0 \quad \checkmark$$

$$D(1,1) = 8 + 20 - 64 < 0 \quad \times$$

$$D(-1,-1) = 8 + 20 - 64 < 0 \quad \times$$

$$f_{xx}\left(-\frac{1}{\sqrt{6}}, \frac{\sqrt{6}}{3}\right) = -12\left(-\frac{1}{\sqrt{6}}\right) + 4\frac{\sqrt{6}}{3} > 0 \rightarrow \text{Minimum } \left(-\frac{1}{\sqrt{6}}, \frac{\sqrt{6}}{3}\right)$$

$$f_{xx}\left(\frac{1}{\sqrt{6}}, -\frac{\sqrt{6}}{3}\right) = -12\frac{1}{\sqrt{6}} + 4\left(-\frac{\sqrt{6}}{3}\right) < 0 \rightarrow \text{Maximum } \left(\frac{1}{\sqrt{6}}, -\frac{\sqrt{6}}{3}\right)$$

5.4

$$T(x,y) = 1 + xy \quad x^2 + y^2 = 1$$

$$f_x + \lambda \cdot g_x = y + \lambda \cdot 2x = 0$$

$$f_y + \lambda \cdot g_y = x + \lambda \cdot 2y = 0$$

$$0 = \lambda \cdot 2(-2 \cdot 2y) + y$$

$$0 = -4\lambda^2 y + y$$

$$0 = y(-4\lambda^2 + 1)$$

$$0 = -4\lambda^2 + 1 \quad \Rightarrow \lambda = \pm \frac{1}{2}$$

$$\Rightarrow 0 = y + x \quad 0 = y - x$$

$$y = -x \quad y = x$$

$$\Rightarrow y = \pm x$$

$$x^2 + y^2 = 1$$

$$x^2 + x^2 = 1$$

$$2x^2 = 1$$

$$x^2 = \frac{1}{2}$$

$$x = \pm \frac{1}{\sqrt{2}} = y$$

$$T\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 1 + \frac{1}{2} = 1,5 = T\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \Rightarrow P_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \quad P_2 = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

5.4

$$T(x,y) = 1+xy \quad x^2 + y^2 = 1$$

$$g(x,y) = x^2 + y^2 - 1$$

$$\begin{matrix} T_x = y \\ T_y = x \end{matrix} \quad \begin{matrix} g_x = 2x \\ g_y = 2y \end{matrix}$$

$$\begin{matrix} y + 2x = 0 \\ x + 2y = 0 \end{matrix}$$

$$y + 2 \cdot 2(-2x) = 0$$

$$y - 4x^2 = 0$$

$$y(1-4x^2) = 0 \Rightarrow y=0$$

$$1 = 4x^2$$

$$\frac{1}{4} = x^2 \Rightarrow x = \pm \frac{1}{2}$$

$$\Rightarrow y+x=0 \quad y-x=0 \quad x-y=0 \quad x+y=0$$

$$y = -x \quad y = x \quad y = x \quad y = -x$$

$$\Rightarrow y = \pm x$$

$$x^2 + y^2 = 1$$

$$x^2 + x^2 = 1$$

$$2x^2 = 1$$

$$x = \pm \sqrt{\frac{1}{2}} \quad y = \pm \sqrt{\frac{1}{2}}$$

$$\Rightarrow ++, -- \Rightarrow \underbrace{\begin{pmatrix} \sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{2}} \end{pmatrix} \quad \begin{pmatrix} -\sqrt{\frac{1}{2}} \\ -\sqrt{\frac{1}{2}} \end{pmatrix}}$$

5.5

$$H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

$$\begin{aligned} 0 &= (f_{xx} - \lambda)(f_{yy} - \lambda) - (f_{xy})^2 \\ &= \lambda^2 - 2f_{xx} - 2f_{yy} + f_{xx}f_{yy} - (f_{xy})^2 \\ &= \lambda^2 - 2(f_{xx} + f_{yy}) + f_{xx}f_{yy} - f_{xy}^2 \end{aligned}$$

$$\begin{aligned} \lambda_{1,2} &= \frac{(f_{xx} + f_{yy}) \pm \sqrt{(f_{xx} + f_{yy})^2 - 4(f_{xx}f_{yy} - f_{xy}^2)}}{2} \\ &= \frac{f_{xx} + f_{yy}}{2} \pm \sqrt{\frac{(f_{xx} + f_{yy})^2}{4} - (f_{xx}f_{yy} - f_{xy}^2)} \end{aligned}$$

mit $\lambda > 0 \Rightarrow f_{xx} + f_{yy} > 0$

$$\begin{aligned} \text{für } (f_{xx}f_{yy} - f_{xy}^2) = 0 &\rightarrow \frac{f_{xx} + f_{yy}}{2} \pm \frac{f_{xx} + f_{yy}}{2} \Leftrightarrow 0 \\ &\Rightarrow (f_{xx}f_{yy} - f_{xy}^2) > 0 \quad (\rightarrow \text{Wurzel kleiner als 3 nach vorne}) \end{aligned}$$

5.6

$$f(x,y) = x - y \quad \tan x = 3 \tan y \quad 0 \leq x \leq \frac{\pi}{2} \quad 0 \leq y \leq \frac{\pi}{2}$$

$$f_x + \lambda g_x = 1 + \lambda \cdot \left(\frac{\cos x}{\cos y} - \sin x \cos^2 y (-\sin x) \right) = 1 + \lambda \cdot \frac{1}{\cos^2 y} \Rightarrow \lambda = -\cos^2 x$$

$$f_y + \lambda g_y = -1 - \lambda \cdot 3 \cdot \frac{1}{\cos^2 y} = -1 - 3\lambda \frac{1}{\cos^2 y}$$

$$\text{mit } \lambda = -\cos^2 x \Rightarrow 0 = -1 + 3 \cos^2 x \frac{1}{\cos^2 y} \quad \cos^2 x = \frac{\cos^2 y}{3}$$

$$\cos x = \pm \frac{\cos y}{\sqrt{3}}$$

$$\Rightarrow \tan x = 3 \tan y$$

$$\sqrt{3} \frac{\sin x}{\cos y} = 3 \frac{\sin y}{\cos y}$$

$$\sin x = \sqrt{3} \sin y$$

$$\text{mit: } \sin^2 x + \cos^2 x = 1 = 3 \sin^2 y + \frac{1}{3} \cos^2 y$$

$$\text{und } \sin^2 y = 1 - \cos^2 y$$

$$\Rightarrow 1 = 3 - 3 \cos^2 y + \frac{1}{3} \cos^2 y$$

$$1 = \frac{4}{3} \cos^2 y$$

$$\cos^2 y = \frac{3}{4}$$

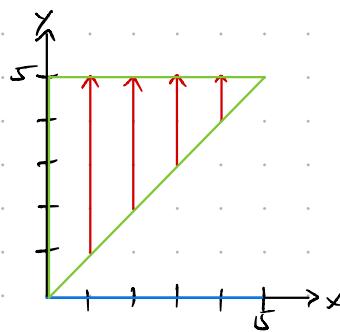
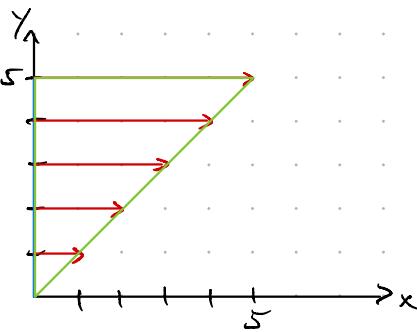
$$\cos y = \frac{\sqrt{3}}{2} \quad \Rightarrow \quad y = \frac{\pi}{6}$$

$$\Rightarrow \cos x = \frac{1}{2} \quad \Rightarrow \quad y = \frac{\pi}{3}$$

6.1

$$\iint_D f(x,y) dx dy$$

$$= \int_0^5 \int_x^5 f(x,y) dy dx$$

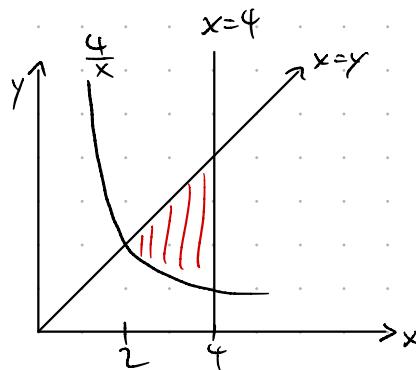


6.2

$$x \cdot y = 4$$

$$x = y$$

$$x = 4$$



$$\frac{4}{x} = x - 1 \cdot x \text{ für } x \neq 0 \Rightarrow \iint_D dy dx = \int_2^4 x - \frac{4}{x} dx$$

$$= \left[\frac{1}{2}x^2 - 4 \ln(x) \right]_2^4$$

$$\approx \underline{\underline{3,227}} \text{ FE}$$

6.3

$$A = 2\pi \int_0^R -\frac{h}{R} x + h dx$$

$$= 2\pi \left[-\frac{h}{2R} x^2 + hx \right]_0^R$$

$$= 2\pi \left(-\frac{hR}{2} + hR \right)$$

$$= 2\pi \frac{hR}{2} = \pi \cdot h \cdot R$$

$$\int_0^h 2\pi \left(R - \frac{R}{h} x \right) dx$$

$$2\pi R x - \frac{R}{2h} x^2$$

$$R h - \frac{Rh}{2}$$

$$R \cdot h \cdot \pi$$

Übungsaufgabe 6.3: Gegeben sei ein Kreiskegel, für den gilt: Radius R = Höhe h . Berechnen Sie seine Mantelfläche.

6.4

$$z_s = \frac{1}{V} \cdot \underset{V}{\iint \iint} z \, dx \, dy \, dz \quad \Rightarrow \quad \frac{1}{V} \cdot \int_0^h z \cdot \pi R(z)^2 \, dz$$

mit $R(z) = z \cdot \frac{R}{h}$

$$\begin{aligned} z_s &= \frac{1}{V} \cdot \pi \cdot \int_0^h z^3 \cdot \frac{R^2}{h^2} \, dz \\ &= \frac{1}{V} \cdot \pi \left[\frac{1}{4} z^4 \cdot \frac{R^2}{h^2} \right]_0^h \\ &= \frac{1}{V} \cdot \frac{\pi}{4} R^2 h^2 \end{aligned}$$

$$V = \underset{V}{\iiint} dxdydz$$

$$= \pi \int_0^h R(z)^2 \, dz$$

$$= \pi \int_0^h z^2 \cdot \frac{R^2}{h^2} \, dz$$

$$= \pi \left[\frac{1}{3} z^3 \cdot \frac{R^2}{h^2} \right]_0^h$$

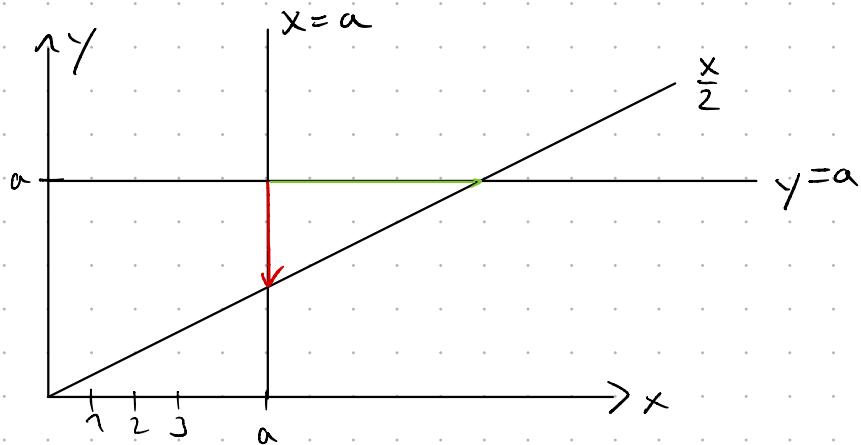
$$= \frac{1}{3} \pi R^2 h$$

$$\Rightarrow \frac{1}{\frac{1}{3} \pi R^2 h} \cdot \frac{\pi}{4} R^2 h^2$$

$$= \underline{\underline{\frac{3}{4} h}}$$

6.5

$$y = \frac{x}{2} \quad x=a \quad y=a$$



$$\begin{aligned} I_x &= \iint_A y^2 \, dy \, dx \\ &= \int_a^{2a} \int_{\frac{x}{2}}^a y^2 \, dy \, dx \\ &= \int_a^{2a} \left[\frac{1}{3} y^3 \right]_{\frac{x}{2}}^a \, dx \\ &= \int_a^{2a} \frac{1}{3} a^3 - \frac{1}{3} \frac{x^3}{8} \, dx \\ &= \frac{1}{3} \int_a^{2a} a^3 - \frac{x^3}{8} \, dx \\ &= \frac{1}{3} \left[a^3 x - \frac{1}{32} x^4 \right]_a^{2a} \\ &= \frac{1}{3} \left(2a^4 - \frac{1}{2} a^4 - \cancel{a^4} + \frac{1}{32} a^4 \right) \\ &= \frac{1}{3} \left(\frac{1}{2} a^4 + \frac{1}{32} a^4 \right) \\ &= \underline{\underline{\frac{17}{36} a^4}} \end{aligned}$$

6.6

Kreisverschiebung Radius

$$(x-b)^2 + z^2 \leq a^2 \Leftrightarrow \text{Kreisfläche}$$

$$\Rightarrow A = \pi a^2 \quad d = 2\pi b$$

$$\Rightarrow A \cdot d = V = \underline{\underline{2\pi^2 a^2 b}}$$

7.2

von $(0,0,0) \Rightarrow (1,1,1)$ $\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} \nu \\ \nu \\ \nu \end{pmatrix}}_{r(\nu)} ; \nu \in [0;1]$

$$\int_{\nu_1}^{\nu_2} F(r(\nu)) \cdot r'(\nu) \cdot d\nu$$

$$\int_0^1 \begin{pmatrix} \nu + \nu^3 \\ \nu + \nu^3 \\ \nu + \nu^3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot d\nu = \int_0^1 3(\nu + \nu^3) d\nu = 3 \left[\frac{1}{2}\nu^2 + \frac{1}{4}\nu^4 \right]_0^1 = 3 \cdot \left(\frac{1}{2} + \frac{1}{4} \right) = \underline{\underline{\frac{9}{4}}} = 2,25$$

7.1

$$\textcircled{1} \quad \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} \Rightarrow x = x \quad \checkmark$$

$$= -1 \cdot \left(\frac{1}{3} \right) = \underline{\underline{-\frac{1}{3}}}$$

$$\textcircled{2} \quad \frac{\partial F_1}{\partial y} = 0 \quad \frac{\partial F_2}{\partial x} = 0$$

$$= -1 \cdot \left[\nu - \nu^2 + \frac{1}{3}\nu^3 \right]_0^1$$

$$\frac{\partial F_1}{\partial z} = 3 \quad \frac{\partial F_3}{\partial x} = 0 \quad \checkmark$$

$$= - \int_0^1 1 - 2\nu + \nu^2 d\nu$$

$$\textcircled{3} \quad \frac{\partial F_1}{\partial y} = 4 \quad \frac{\partial F_2}{\partial x} = 4$$

$$\int_0^1 F(x,y) \cdot r'(\nu) d\nu = \int_0^1 -(1-\nu)^2 d\nu$$

$$\frac{\partial F_1}{\partial z} = -3 \quad \frac{\partial F_3}{\partial x} = -3$$

$$F = \begin{pmatrix} x^2 + xy \\ y^2 + \frac{1}{2}x^2 \end{pmatrix} \quad r(\nu) = \begin{pmatrix} 1-\nu \\ 0 \end{pmatrix} \quad \nu \in [0,1]$$

$$\frac{\partial F_2}{\partial z} = -4 \quad \frac{\partial F_3}{\partial y} = -4 \quad \checkmark$$

$$F = \begin{pmatrix} (1-\nu)^2 \\ \frac{1}{2}(1-\nu)^2 \end{pmatrix} \quad r'(\nu) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} (1-\nu)^2 \\ \frac{1}{2}(1-\nu)^2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \end{pmatrix} = -(1-\nu)^2$$

7.3

$$\oint_C F_{1x}(x,y) \cdot \begin{pmatrix} dx \\ dy \end{pmatrix} \quad r(t) = \begin{pmatrix} \cos \frac{t}{2} \\ \frac{t^2}{8+3+t^2+4t} \sin t \end{pmatrix} + e[0, \pi]$$

$$r(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad r(\pi) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

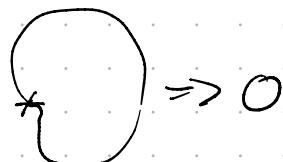
$$\Rightarrow r(\nu) = \begin{pmatrix} 1-\nu \\ 0 \end{pmatrix} \quad \nu \in [0, 1]$$

$$r'(0) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$\oint_C \left(\frac{(1-\nu)^2}{\frac{1}{2}(1-\nu)^2} \cdot \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right) d\nu = \int_0^1 (1-\nu)^2 d\nu = \int_0^1 -1 + 2\nu - \nu^2 d\nu = \left[-\frac{1}{3}\nu^3 + \nu^2 - \nu \right]_0^1 = -\frac{1}{3} + 1 - 1 = -\frac{1}{3}$$

7.4

$$W = \oint_C \frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$$



$$(1) \quad r(t) = \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix} + e[0, 2\pi]$$

$$F_1 = \frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}} \quad F_2 = \frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}} \quad F_3 = \frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

$$\frac{\partial F_1}{\partial y} = x \cdot \left(-\frac{3}{2} (x^2+y^2+z^2)^{-\frac{5}{2}} \cdot 2y \right) \quad \checkmark$$

$$\frac{\partial F_2}{\partial x} = y \cdot \left(-\frac{3}{2} (x^2+y^2+z^2)^{-\frac{5}{2}} \cdot 2x \right)$$

$$\frac{\partial F_1}{\partial z} = \frac{\partial F_2}{\partial x} \quad \checkmark \quad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y} \quad \checkmark \quad \Rightarrow r(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = r(2\pi) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow = \underline{0}$$

$$\textcircled{2} \quad r(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = r(2\pi) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \underline{\underline{O}}$$

$$\textcircled{3} \quad r_{\text{Anfang}} = r_{\text{Ende}} \Rightarrow \underline{\underline{O}}$$

7.5

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \cdot \cos t \\ b \cdot \sin t \end{pmatrix} \quad \frac{dx}{dt} = -a \cdot \sin t \quad \frac{dy}{dt} = b \cdot \cos t$$

$$\frac{1}{2} \oint_C (x dy - y dx)$$

$$\frac{1}{2} \oint_C (a \cdot \cos t \cdot b \cdot \cos t - b \cdot \sin t \cdot a \cdot \sin t) dt$$

$$\frac{1}{2} \oint_C (a \cdot b \cos^2 t + a \cdot b \sin^2 t) dt$$

$$\frac{1}{2} \oint_C a \cdot b (\underbrace{\cos^2 t + \sin^2 t}_1) dt$$

$$\frac{1}{2} a \cdot b \int_0^{2\pi} 1 dt = \frac{1}{2} ab [+]_0^{2\pi} = \underline{\underline{ab \cdot \pi}}$$

7.6 $-x^2 - 2y^2$

$$\oint_C (x^2 + 2y^2) dx + (4xy + 5x) dy \quad r = \phi^2, \quad -\pi \leq \phi \leq \pi$$

$$\frac{F_1}{\partial x} + \frac{F_2}{\partial y} = 4y + 5 - 4y = \underline{\underline{5}}$$

$$\begin{aligned} \Rightarrow 5 \iint_A dA &= 5 \iint_A dx dy \quad r \cdot dr d\phi \\ &= 5 \iint_{-\pi}^{\pi} \int_0^{\phi^2} r dr d\phi = \frac{5}{2} \int_{-\pi}^{\pi} \phi^4 d\phi = \frac{5}{2} \left[\frac{1}{5} \phi^5 \right]_{-\pi}^{\pi} = \frac{5}{2} \left(\frac{1}{5} \pi^5 + \frac{1}{5} (-\pi)^5 \right) \\ &= \frac{5}{2} \cdot \frac{2}{5} \pi^5 \\ &= \underline{\underline{\pi^5}} \end{aligned}$$

8.1

$$\int_1^2 \frac{1}{x} dx \quad h = 1, \frac{1}{2}, \frac{1}{4}$$

$$h=1: \frac{1}{2} (f(1) + f(2)) \cdot (2-1) = \frac{1}{2} (1 + \frac{1}{2}) = \underline{\underline{\frac{3}{4}}} = \frac{18}{24} = \frac{1260}{1680}$$

$$\begin{aligned} h = \frac{1}{2}: & \frac{1}{2} \cdot (f(1) + 2 \cdot (f(1 + \frac{1}{2}(2-0)) + f(2)) \cdot (2-1) \\ &= \frac{1}{4} \cdot (1 + 2 \cdot \frac{2}{3} + \frac{1}{2}) \cdot 1 \\ &= \frac{1}{4} \cdot \frac{17}{6} = \underline{\underline{\frac{17}{24}}} = \frac{1150}{1680} \end{aligned}$$

$$\begin{aligned} h = \frac{1}{4}: & \frac{1}{2 \cdot 4} \cdot (f(1) + 2 \cdot (f(\frac{5}{4}) + f(\frac{3}{2}) + f(\frac{7}{4})) + f(2)) \\ &= \frac{1}{8} \cdot (1 + 2 \cdot (\frac{4}{5} + \frac{2}{3} + \frac{4}{7}) + \frac{1}{2}) \\ &= \frac{1}{8} \cdot \left(\frac{3}{2} + 2 \cdot \left(\frac{84}{105} + \frac{70}{105} + \frac{60}{105} \right) \right) \\ &= \frac{1}{8} \cdot \left(\frac{315}{210} + \frac{856}{210} \right) \\ &= \frac{1}{8} \cdot \frac{1171}{210} \\ &= \frac{1171}{1680} \end{aligned}$$

$$\begin{array}{ccc} h: & 1 & \frac{1}{2} & \frac{1}{4} \\ \hline \int: & \frac{1260}{1680} & \frac{1150}{1680} & \frac{1171}{1680} \\ \approx: & 0,75 & 0,70833 & 0,68702 \end{array}$$

8.2

$$\int_0^1 \sqrt{1+x^4} dx \quad n=2$$

$$\begin{aligned} A &\approx \frac{1}{6 \cdot 2} \cdot [f(0) + 4 \cdot \sum_{i=0}^1 \{f(\frac{1}{2} \cdot (a_i + a_{i+1}))\} + 2 \cdot \sum_{i=1}^1 f(a_i) + f(1)] \cdot 1 \\ &= \frac{1}{12} \cdot [1 + 4 \cdot (f(\frac{1}{2} \cdot \frac{1}{2}) + f(\frac{1}{2} \cdot \frac{3}{2})) + 2 \cdot (f(\frac{1}{2})) + f(1)] \\ &= \frac{1}{12} \cdot [1 + 4 \cdot \left(\frac{\sqrt{253}}{16} + \frac{\sqrt{337}}{16}\right) + 2 \cdot \frac{\sqrt{17}}{4} + \sqrt{2}] \\ &= \frac{1}{12} \cdot [1 + \frac{\sqrt{253} + \sqrt{337}}{4} + \frac{\sqrt{17}}{2} + \sqrt{2}] \end{aligned}$$

$$TR = 1,08541$$

$$a_0 = 0 + \frac{0}{2} \cdot 1 = 0$$

$$a_1 = 0 + \frac{1}{2} \cdot 1 = \frac{1}{2}$$

$$a_2 = 0 + \frac{2}{2} \cdot 1 = 1$$

$$\begin{aligned} f(\frac{1}{4}) &= \sqrt{1 + (\frac{1}{4})^4} \\ &= \sqrt{1 + \frac{1}{256}} \end{aligned}$$

$$f(\frac{3}{4}) = \sqrt{1 + \frac{81}{256}}$$

$$f(\frac{1}{2}) = \sqrt{1 + \frac{1}{16}}$$

8.3

$$y'(x) = \sqrt{1 - y^2(x)} \quad y(0) = 0$$

ges: $y(1)$ mit $\Delta x = 0,1$

$$y_{0,1} = y_0 + h \cdot f(x_0, y_0) \quad y_{0,2} = y_{0,1} + h \cdot f(x_{0,1}, y_{0,1})$$

$$y_{0,1} = 0 + \frac{1}{10} \cdot 1 \quad = 0,1 + \frac{1}{10} \cdot \sqrt{0,99}$$

$$y_{0,1} = 0,1 \quad \approx 0,199499$$

$$y_{0,3} = y_{0,2} + \frac{1}{10} \cdot f(x_{0,2}, y_{0,2})$$

$$y_{0,3} = 0,199499 + \frac{1}{10} \cdot \sqrt{1 - 0,199499^2}$$

$$\approx 0,297489$$

$$y_{0,4} = y_{0,3} + \frac{1}{10} \cdot f(x_{0,3}, y_{0,3})$$

$$= 0,297489 + \frac{1}{10} \cdot \sqrt{1 - 0,297489^2}$$

$$\approx 0,392962$$

Übungsaufgabe 8.3: Gegeben sei das Anfangswertproblem $y'(x) = \sqrt{1 - y^2(x)}$ mit $y(0) = 0$. Bestimmen Sie den Wert $y(1)$ mittels der Näherungslösung mit Hilfe des Eulerschen Polygonzugverfahrens (Schrittweite $\Delta x = 0,1$) und vergleichen Sie diesen mit der exakten Lösung.

2.4

$$y'(x) = 2x \quad y(0) = 0$$

$$\int 2x$$

$$\begin{aligned} y_1 &= y_0 + h \cdot y'(x_0) \\ &= 0 + \frac{x}{n} \cdot 0 = 0 \end{aligned}$$

$$\begin{aligned} y_2 &= 0 + \frac{x}{n} \cdot \frac{2x}{n} \\ &= 0 + \frac{2x^2}{n^2} \end{aligned}$$

$$\begin{aligned} y_3 &= 0 + \frac{2x^2}{n^2} + \frac{1}{n} \cdot \frac{4x^2}{n} \\ &= 0 + \frac{2x^2}{n^2} + \frac{4x^2}{n^2} + \frac{6x^2}{n^2} \end{aligned}$$

$$y_n = \frac{\sum_{i=0}^{n-1} 2i}{n^2} x^2$$

$$\begin{aligned} \text{mit } n=1: \quad 0 \cdot x^2 &= 0 \\ n=2: \quad \frac{2}{4} x^2 &= \frac{1}{2} x^2 \\ n=4: \quad \frac{12}{16} x^2 &= \frac{3}{4} x^2 \end{aligned}$$

$$\begin{array}{c} x^2 \\ \frac{1}{2} x^2 \\ \frac{3}{4} x^2 \end{array}$$

Mittelpunktsregel:

$$y_{\frac{1}{2}} = y_0 + \frac{x}{2n} \cdot y'(0) = 0$$

$$y_1 = y_0 + \frac{x}{n} \cdot y'\left(y_{\frac{1}{2}}, x_0 + \frac{h}{2}\right)$$

$$= 0 + \frac{x}{n} \cdot 2 \cdot \frac{x}{2n}$$

$$= \frac{x^2}{n^2}$$

$$\text{mit } n=1: \quad y_1 = x^2 \quad 0$$

Übungsaufgabe 8.4: Gegeben sei das Anfangswertproblem $y'(x) = 2x$ mit $y(0) = 0$. Bestimmen Sie den Wert $y(x)$ numerisch mittels

1. Euler-Verfahren (1, 2 und 4 Schritte),
2. Mittelpunktsregel (1 Schritt) und
3. Runge-Kutta-Verfahren 4. Ordnung (1 Schritt).

Geben Sie das Ergebnis der Näherungsverfahren jeweils als analytischen Ausdruck an und berechnen Sie die Differenz zwischen Näherungslösung und exakter Lösung.

Hinweis: Hier gilt offensichtlich $f(x, y) = f(x)$.

Runge-Kutta: 4. Ordnung

$$h = \frac{x}{n}$$

$$K_1 = \frac{x}{n} \cdot 2 \cdot 0 = 0$$

$$K_2 = \frac{x}{n} \cdot 2 \cdot \left(0 + \frac{x}{2n}\right) = \frac{x^2}{n^2}$$

$$K_3 = \frac{x}{n} \cdot 2 \cdot \left(\frac{x}{2n}\right) = \frac{x^2}{n^2}$$

$$K_4 = \frac{x}{n} \cdot 2 \cdot \left(\frac{x}{n}\right) = \frac{2x^2}{n^2}$$

$$y_1 = 0 + \frac{1}{6} \cdot \left(0 + \frac{2x^2}{n^2} + \frac{2x^2}{n^2} + \frac{2x^2}{n^2}\right)$$

$$= \frac{1}{6} \cdot \frac{6x^2}{n^2} = \frac{x^2}{n^2}$$

$$\Rightarrow \text{mit } n=1: \quad y_1 = x^2 \quad 0$$

8.5

$$y'(x) = e^x = y(x)$$

$$h = \frac{1}{n}$$

$$K_1 = \frac{1}{n} \cdot f(0) = \frac{1}{n}$$

$$K_2 = \frac{1}{n} \cdot \left(y_0 + \frac{K_1}{2}\right) = \frac{1}{n} \cdot \left(1 + \frac{1}{2n}\right) = \frac{1}{n} + \frac{1}{2n^2}$$

$$K_3 = \frac{1}{n} \cdot \left(y_0 + \frac{K_2}{2}\right) = \frac{1}{n} \cdot \left(1 + \frac{1}{2n} + \frac{1}{4n^2}\right) = \frac{1}{n} + \frac{1}{2n^2} + \frac{1}{4n^3}$$

$$K_4 = \frac{1}{n} \cdot \left(y_0 + K_3\right) = \frac{1}{n} \cdot \left(1 + \frac{1}{n} + \frac{1}{2n^2} + \frac{1}{4n^3}\right) = \frac{1}{n} + \frac{1}{n^2} + \frac{1}{2n^2} + \frac{1}{4n^3}$$

$$\text{für } n=1: \quad y_1 = 1 + \frac{1}{6} \left(1 + 2 \cdot 1,5 + 2 \cdot \frac{7}{4} + 2,25\right)$$

$$= 1 + \frac{1}{6} \cdot (10,25)$$

$$= \frac{65}{24}$$

$$h = I/n = (x_n - x_0)/n$$

$$x_{k-1} = x_0 + h \cdot (k-1)$$

$$= x_{k-2} + h$$

$$K_1 = h \cdot f(y_{k-1}, x_{k-1})$$

$$K_2 = h \cdot f(y_{k-1} + \frac{K_1}{2}, x_{k-1} + \frac{h}{2})$$

$$K_3 = h \cdot f(y_{k-1} + \frac{K_2}{2}, x_{k-1} + \frac{h}{2})$$

$$K_4 = h \cdot f(y_{k-1} + K_3, x_{k-1} + h)$$

$$y_k = y_{k-1} + \frac{1}{6} \cdot (K_1 + 2 \cdot K_2 + 2 \cdot K_3 + K_4)$$

$$= y_{k-1} + \Delta y_{k-1}$$

die Eulersche Zahl e unter Verwendung von Euler-Verfahrens vierter Ordnung. Wieviele Schritte des Euler-Verfahrens wären jeweils mindestens nötig, um eine mindestens gleichgute Näherung zu berechnen?

$$y_1 = 1 + \frac{1}{n} + \frac{2}{12n^2} + \frac{1}{8n^3}$$

$$K_1 = \frac{1}{n} \cdot \left(1 + \frac{1}{n} + \frac{2}{12n^2} + \frac{1}{8n^3}\right)$$

$$= \frac{1}{n} + \frac{1}{n^2} + \frac{2}{12n^3} + \frac{1}{8n^4}$$

$$K_2 = \frac{1}{n} \cdot \left(1 + \cancel{\frac{1}{n}} + \cancel{\frac{2}{12n^2}} + \cancel{\frac{1}{8n^3}} + \frac{1}{2n} + \cancel{\frac{1}{2n^2}} + \cancel{\frac{2}{24n^3}} + \frac{1}{16n^4}\right)$$

$$= \frac{1}{n} + \frac{3}{2n^2} + \frac{13}{12n^3} + \frac{5}{12n^4} + \frac{1}{16n^5}$$

$$K_3 = \frac{1}{n} \cdot \left(1 + \cancel{\frac{1}{n}} + \cancel{\frac{2}{12n^2}} + \cancel{\frac{1}{8n^3}} + \frac{1}{2n} + \cancel{\frac{3}{2n^2}} + \cancel{\frac{13}{24n^3}} + \frac{5}{24n^4} + \frac{1}{32n^5}\right)$$

$$= \frac{1}{n} + \frac{3}{2n^2} + \frac{16}{24n^3} + \frac{12}{24n^4} + \frac{5}{24n^5} + \frac{1}{32n^6}$$

$$K_4 = \frac{1}{n} \cdot \left(1 + \cancel{\frac{1}{n}} + \cancel{\frac{2}{12n^2}} + \cancel{\frac{1}{8n^3}} + \frac{1}{n} + \cancel{\frac{1}{2n^2}} + \cancel{\frac{30}{24n^3}} + \frac{2}{3n^4} + \frac{5}{24n^5} + \frac{1}{32n^6}\right)$$

$$= \frac{1}{n} + \frac{2}{n^2} + \frac{25}{12n^3} + \frac{35}{24n^4} + \frac{2}{3n^5} + \frac{5}{24n^6} + \frac{1}{32n^7}$$

$$y_2 = 1 + \frac{1}{n} + \frac{2}{12n^2} + \frac{1}{8n^3} + \frac{1}{6} \left(\cancel{\frac{1}{n}} + \cancel{\frac{1}{n^2}} + \cancel{\frac{2}{12n^2}} + \cancel{\frac{1}{8n^3}} + \cancel{\frac{1}{n}} + \cancel{\frac{3}{2n^2}} + \cancel{\frac{16}{24n^3}} + \cancel{\frac{10}{24n^4}} + \cancel{\frac{4}{24n^5}} + \cancel{\frac{1}{32n^6}} + \cancel{\frac{3}{24n^7}} + \cancel{\frac{16}{24n^8}} + \cancel{\frac{10}{24n^9}} + \frac{1}{32n^{10}} \right)$$

$$= 1 + \frac{1}{n} + \frac{2}{12n^2} + \frac{1}{8n^3} + \frac{1}{6} \left(\frac{6}{n} + \frac{9}{n^2} + \frac{90}{12n^3} + \frac{80}{24n^4} + \frac{20}{24n^5} + \frac{12}{48n^6} + \frac{1}{32n^7} \right)$$

$$= \cancel{1} + \cancel{\frac{1}{n}} + \cancel{\frac{2}{12n^2}} + \cancel{\frac{1}{8n^3}} + \cancel{\frac{1}{n}} + \cancel{\frac{16}{24n^4}} + \cancel{\frac{15}{24n^5}} + \cancel{\frac{2}{24n^6}} + \cancel{\frac{13}{288n^7}} + \cancel{\frac{1}{152n^8}}$$

$$= 1 + \frac{2}{n} + \frac{25}{12n^2} + \frac{11}{8n^3} + \frac{15}{24n^4} + \frac{29}{144n^5} + \frac{13}{288n^6} + \frac{1}{192n^7}$$

for $n=2$

$$y_2 = 2,729$$

$$\text{Euler: } \left(1 + \frac{1}{n}\right)^n$$

10.4

$$\frac{d^2y}{dx^2} \approx \frac{y_{k+1} - 2y_k + y_{k-1}}{\Delta x^2} = y_{k+1} - 2y_k + y_{k-1}$$

mit $\Delta x = 1$

$$f(y) = y^2$$

$$f'(y) = 2y$$

$$f''(y) = 2$$

$$y_{k+1} - 2y_k + y_{k-1} = 2 \quad \text{mit } k+1=k$$

$$y_k = 2 + 2y_{k-1} - y_{k-2}$$

Einsetzen: $y_{k-2} = 0 = 0^2$

$$y_{k-1} = 1 = 1^2$$

$$y_k = 2 + 2 - 0 = 4 = 2^2$$

$$y_{k+1} = 2 + 2 \cdot 4 - 1 = 9 = 3^2$$

$$y_{k+2} = 2 + 2 \cdot 9 - 4 = 16 = 4^2$$

10.1

$$y_n = 3y_{n-1} - 4$$

$$y_6 = -1406$$

$$y_{n-1} = \frac{y_n + 4}{3}$$

$$y_5 = -\frac{1452}{3} = -484$$

$$y_4 = -\frac{480}{3} = -160$$

$$y_3 = -\frac{156}{3} = -52$$

$$y_2 = -\frac{48}{3} = -16$$

$$y_1 = -\frac{12}{3} = -4$$

$$y_0 = -\frac{0}{3} = 0$$

$$L_{a+/-1}(k) = (-a)^k \cdot C + \frac{b}{1+a}$$

$$\text{mit } a = -3, b = -4$$

$$L(k) = 3^k \cdot C + 2$$

$$L(1) = 3C + 2$$

$$-4 = 3C + 2$$

$$-6 = 3C$$

$$C = -2$$

$$\Rightarrow L(k) = 3^k(-2) + 2$$

10.2

$S_0 = \text{Kredit zu } t=0$
 $p \text{ Zinsatz}$

$$S_n = S_{n-1} (1+p) - A$$

\Rightarrow Allg Lösung

$$L(k) = (-1+p)^k \cdot C + \frac{A}{1+p}$$

$$p = 0,1 \quad A = 0,1 \cdot S_0$$

$$L(k) = (-1,1)^k \cdot C + \frac{0,1 \cdot S_0}{1,1}$$

$$S_1 = S_0 (1,1) - S_0 \cdot 0,1$$

$$S_1 = S_0 (1,1 - 0,1) = S_0$$

$$\Rightarrow S_{10} = S_0$$

10.3

$$x_n = 3x_{n-1} - 2x_{n-2} + y_{n-1} \quad y_{n-1} = x_n - 3x_{n-1} + 2x_{n-2}$$

$$y_n = y_{n-2} - x_{n-1} \quad y_n = -3x_{n-2} + 2x_{n-3}$$

$$x_{n-1} = y_{n-2} - y_n$$

$$x_n = y_{n-1} - y_{n+1}$$

$$y_{n-1} - y_{n+1} = 3(y_{n-2} - y_n) - 2(y_{n-3} - y_{n-1}) + y_{n-1}$$

$$y_{n-1} - y_{n+1} = 3y_{n-2} - 3y_n - 2y_{n-3} + 2y_{n-1} + y_{n-1}$$

$$-y_{n+1} = -2y_{n-3} + 3y_{n-2} + 2y_{n-1} - 3y_n$$

$$y_n = 2y_{n-4} - 3y_{n-3} - 2y_{n-2} + 3y_{n-1}$$

\Rightarrow 4 Anfangswerte

10.4

$$f_y = y^2$$

$$y' = 2y$$

$$y'' = 2$$

$$y'' - 2 = 0$$

$$\frac{y_n - 2y_{n-1} + y_{n-2}}{(\Delta x)^2} - 2 = 0$$

$$y_n - 2y_{n-1} + y_{n-2} = 2$$

$$y_n = 2 + 2y_{n-1} - y_{n-2}$$

$$y_1 = 1 \quad y_2 = 4$$

$$y_n = e^{2t}$$

10.6

$$S_n = S_{n-1} \cdot (1+p) - A \quad S_n - S_{n-1} \cdot (1+p) = -A$$

$$y' = \frac{y_n - y_{n-1}}{\Delta x} = (1+p)y_n - (1+p)y_{n-1}$$

$$y = y_n$$

$$(1+p)y' - y = -A \quad \text{mit } C = p$$