A 1.1

ges: $D_{41}(\alpha,\beta,\gamma)$

$$D_{41} = D_{43(x)}(\gamma) \cdot D_{32(z)}(\beta) \cdot D_{21(y)}(\alpha)$$

$$D_{41} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & \sin\gamma \\ 0 & -\sin\gamma & \cos\gamma \end{pmatrix} \cdot \begin{pmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos & \sin \\ 0 & -\sin & \cos \end{pmatrix} \cdot \begin{pmatrix} \cos\beta\cos\alpha & \sin\beta & -\sin\alpha\cos\beta \\ -\sin\beta\cos\alpha & \cos\beta & \sin\beta\sin\alpha \\ \sin\alpha & 0 & \cos\alpha \end{pmatrix}$$

$$= \begin{pmatrix} \cos\alpha\cos\beta & \sin\beta & -\sin\alpha\cos\beta \\ -\cos\alpha\sin\beta\cos\gamma+\sin\alpha\sin\gamma & \cos\beta\cos\gamma & \sin\alpha\sin\beta\cos\gamma+\cos\alpha\sin\gamma \\ \cos\alpha\sin\beta\sin\gamma+\sin\alpha\cos\gamma & -\cos\beta\sin\gamma & -\sin\alpha\sin\beta\sin\gamma+\cos\alpha\cos\gamma \end{pmatrix}$$

A1.2

$$U(x,y,z) = \frac{Q}{4\pi\varepsilon_0\sqrt{x^2+y^2+z^2}}$$

A1.3

$\alpha = -\frac{\pi}{2}$ $\quad G = \lambda \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$d = \begin{pmatrix} \sqrt{0{,}5} \\ \sqrt{0{,}5} \\ 0 \end{pmatrix}$

$$D\left(\begin{pmatrix} \sqrt{0{,}5} \\ \sqrt{0{,}5} \\ 0 \end{pmatrix}, -\frac{\pi}{2}\right) = \cos\left(-\frac{\pi}{2}\right) \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \left(1-\cos\left(-\frac{\pi}{2}\right)\right) \cdot \begin{pmatrix} \sqrt{0{,}5} \\ \sqrt{0{,}5} \\ 0 \end{pmatrix} \otimes \begin{pmatrix} \sqrt{0{,}5} \\ \sqrt{0{,}5} \\ 0 \end{pmatrix}' - \sin\left(-\frac{\pi}{2}\right) \cdot \begin{pmatrix} 0 & 0 & \sqrt{0{,}5} \\ 0 & 0 & -\sqrt{0{,}5} \\ -\sqrt{0{,}5} & \sqrt{0{,}5} & 0 \end{pmatrix}$$

$$= \cos\left(-\frac{\pi}{2}\right) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \left(1-\cos\left(-\frac{\pi}{2}\right)\right) \cdot \begin{pmatrix} 0{,}5 & 0{,}5 & 0 \\ 0{,}5 & 0{,}5 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \sin\left(-\frac{\pi}{2}\right) \cdot \begin{pmatrix} 0 & 0 & \sqrt{0{,}5} \\ 0 & 0 & -\sqrt{0{,}5} \\ -\sqrt{0{,}5} & \sqrt{0{,}5} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos(-\frac{\pi}{2}) & 0 & 0 \\ 0 & \cos(-\frac{\pi}{2}) & 0 \\ 0 & 0 & \cos(\frac{\pi}{2}) \end{pmatrix} + \begin{pmatrix} \frac{1}{2}-\frac{1}{2}\cos(-\frac{\pi}{2}) & \frac{1}{2}-\frac{1}{2}\cos(-\frac{\pi}{2}) & 0 \\ \frac{1}{2}-\frac{1}{2}\cos(-\frac{\pi}{2}) & \frac{1}{2}-\frac{1}{2}\cos(-\frac{\pi}{2}) & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & \sqrt{\frac{1}{2}}\sin(\frac{\pi}{2}) \\ 0 & 0 & -\sqrt{\frac{1}{2}}\sin(\frac{\pi}{2}) \\ -\sqrt{\frac{1}{2}}\sin(\frac{\pi}{2}) & \sqrt{\frac{1}{2}}\sin(\frac{\pi}{2}) & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}+\frac{1}{2}\cos(-\frac{\pi}{2}) & \frac{1}{2}-\frac{1}{2}\cos(-\frac{\pi}{2}) & -\sqrt{\frac{1}{2}}\sin(-\frac{\pi}{2}) \\ \frac{1}{2}-\frac{1}{2}\cos(-\frac{\pi}{2}) & \frac{1}{2}+\frac{1}{2}\cos(-\frac{\pi}{2}) & +\sqrt{\frac{1}{2}}\sin(-\frac{\pi}{2}) \\ +\sqrt{\frac{1}{2}}\sin(-\frac{\pi}{2}) & -\sqrt{\frac{1}{2}}\sin(-\frac{\pi}{2}) & \cos(-\frac{\pi}{2}) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

Matrix aus **A 1.1** $\Rightarrow \beta = \frac{1}{6}\pi$

$$\Rightarrow \alpha = \gamma = 0{,}955$$

Mit: $D_{41} = D_{43(x)}(\gamma) \cdot D_{32(z)}(\beta) \cdot D_{21(y)}(\alpha)$

A 1.4

$\alpha = 30°$ $\beta = 45°$

$$D_{31} = D_{32(x)}(\beta) \cdot D_{21(z)}(\alpha)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c\beta & s\beta \\ 0 & -s\beta & c\beta \end{pmatrix} \cdot \begin{pmatrix} c\alpha & s\alpha & 0 \\ -s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} c\alpha & s\alpha & 0 \\ -s\alpha c\beta & c\alpha c\beta & s\beta \\ s\alpha s\beta & -c\alpha s\beta & c\beta \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{\sqrt{2}}{4} & \frac{\sqrt{6}}{4} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{4} & -\frac{\sqrt{6}}{4} & \frac{\sqrt{2}}{2} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{6}+2\sqrt{2}}{4} \\ \frac{-\sqrt{6}+2\sqrt{2}}{4} \end{pmatrix}$$

A1.5

$$A = \frac{1}{9} \begin{pmatrix} -7 & -4 & -4 \\ 4 & 1 & -8 \\ 4 & -8 & 1 \end{pmatrix}$$

A1.6

$F = (47{,}65° N,\ 9{,}48° O,\ 400 m\ \ddot{u}.\ NN)$

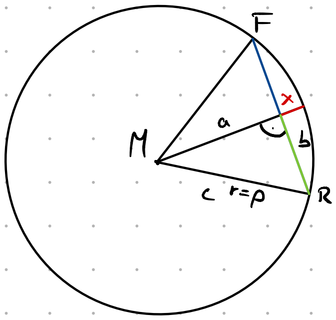
$R = (47{,}57° N,\ 9{,}38° O)$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \rho \cdot \cos\phi \cdot \sin\vartheta \\ \rho \cdot \sin\phi \cdot \sin\vartheta \\ \rho \cdot \cos\vartheta \end{pmatrix}$$

$$F = \begin{pmatrix} 6370{,}4 \cdot \cos(9{,}48°) \cdot \sin(47{,}65°) \\ 6370{,}4 \cdot \sin(9{,}48°) \cdot \sin(47{,}65°) \\ 6370{,}4 \cdot \cos(47{,}65°) \end{pmatrix} = \begin{pmatrix} 4643{,}71 \\ 775{,}42 \\ 4291{,}47 \end{pmatrix}$$

$$R = \begin{pmatrix} 6370{,}4 \cdot \cos(9{,}38°) \sin(47{,}57°) \\ 6370{,}4 \cdot \sin(9{,}38°) \sin(47{,}57°) \\ 6370{,}4 \cdot \cos(47{,}57°) \end{pmatrix} = \begin{pmatrix} 4639{,}14 \\ 766{,}34 \\ 4298{,}04 \end{pmatrix}$$

$$\vec{FR} = \begin{pmatrix} 4{,}57 \\ 9{,}08 \\ -6{,}57 \end{pmatrix} \qquad |\vec{FR}| = \underline{\underline{12{,}1\ km}}$$



$b = 6{,}05\ km$

$r = 6370{,}4\ km$

$a = \sqrt{r^2 - b^2} = \sqrt{6370{,}4^2 - 6{,}05^2} = 6370{,}3971$

$x = r - a = 0{,}0029 \cong \underline{\underline{2{,}9 m}}$

A2.1

$$A = \begin{pmatrix} 4 & 0 & -2 & 4 \\ 0 & 9 & 9 & 0 \\ -2 & 9 & 11 & -3 \\ 4 & 0 & -3 & 30 \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} = \begin{pmatrix} b_{11} & 0 & 0 & 0 \\ b_{21} & b_{22} & 0 & 0 \\ b_{31} & b_{32} & b_{33} & 0 \\ b_{41} & b_{42} & b_{43} & b_{44} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{21} & b_{31} & b_{41} \\ 0 & b_{22} & b_{32} & b_{42} \\ 0 & 0 & b_{33} & b_{43} \\ 0 & 0 & 0 & b_{44} \end{pmatrix}$$

$b_{11} = \sqrt{a_{11}}$ $\quad$ $b_{22} = \sqrt{a_{22} - b_{21}^2}$ $\quad$ $b_{33} = \sqrt{a_{33} - b_{31}^2 - b_{32}^2}$ $\quad$ $b_{44} = \sqrt{a_{44} - b_{41}^2 - b_{42}^2 - b_{43}^2}$

$b_{21} = \frac{a_{21}}{b_{11}}$ $\quad$ $b_{32} = \frac{a_{32} - b_{31} b_{21}}{b_{22}}$ $\quad$ $b_{43} = \frac{a_{43} - b_{31} b_{41} - b_{42} b_{32}}{b_{33}}$

$b_{31} = \frac{a_{31}}{b_{11}}$ $\quad$ $b_{42} = \frac{a_{42} - b_{41} b_{21}}{b_{22}}$

$b_{41} = \frac{a_{41}}{b_{11}}$

$\Rightarrow b_{11} = \sqrt{4} = 2$ $\quad$ $b_{22} = \sqrt{9 - 0} = 3$ $\quad$ $b_{33} = \sqrt{11 - 1 - 9} = 1$

$b_{21} = \frac{0}{2} = 0$ $\quad$ $b_{32} = \frac{9+0}{3} = 3$ $\quad$ $b_{43} = \frac{-3+2}{1} = -1$

$b_{31} = -\frac{2}{2} = -1$ $\quad$ $b_{42} = \frac{0-0}{3} = 0$ $\quad$ $b_{44} = \sqrt{30 - 4 - 1} = 5$

$b_{41} = \frac{4}{2} = 2$

$$\Rightarrow \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ -1 & 3 & 1 & 0 \\ 2 & 0 & -1 & 5 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 & -1 & 2 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 5 \end{pmatrix} \Rightarrow 30 \cdot 30 = 900$$

A2.5

$b_{11} = 1 \quad b_{22} = 2 \quad b_{33} = 3 \quad b_{44} = 1$

$b_{21} = -2 \quad b_{32} = -3 \quad b_{43} = -2$

$b_{31} = 0 \quad b_{42} = 0$

$b_{41} = 0$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & -2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -2 & 0 & 0 \\ 0 & 2 & -3 & 0 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot x = y$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & -2 & 1 \end{pmatrix} \cdot z = y \qquad \begin{pmatrix} 1 & -2 & 0 & 0 \\ 0 & 2 & -3 & 0 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot x = z$$

$$\begin{array}{rcll} z_1 & = & y_1 & \\ -2z_1 + 2z_2 & = & y_2 & \mid +2\,\mathrm{I} \\ -3z_2 + 3z_3 & = & y_3 & \mid +1{,}5\,\mathrm{II}' \\ -2z_3 + z_4 & = & y_4 & \mid +\frac{2}{3}\,\mathrm{III}' \end{array}$$

$$\begin{array}{rcll} x_1 - 2x_2 & = & z_1 & \mid +\mathrm{II}' \\ 2x_2 - 3x_3 & = & z_2 & \mid +\mathrm{III}' \\ 3x_3 - 2x_4 & = & z_3 & \mid +2\,\mathrm{IV} \\ x_4 & = & z_4 & \end{array}$$

$$\begin{array}{rcl} 2z_2 & = & y_2 + 2y_1 \\ 3z_3 & = & y_3 + \frac{3}{2}y_2 + 3y_1 \\ z_4 & = & y_4 + \frac{2}{3}y_3 + y_2 + 2y_1 \end{array}$$

$$\begin{array}{rcl} x_1 & = & z_1 + z_2 + z_3 + 2z_4 \\ 2x_2 & = & z_2 + z_3 + 2z_4 \\ 3x_3 & = & z_3 + 2z_4 \end{array}$$

$$\Rightarrow z = \begin{pmatrix} 4 \\ -5 \\ -1 \\ 2 \end{pmatrix} \qquad \Rightarrow x = \underline{\underline{\begin{pmatrix} 2 \\ -1 \\ 1 \\ 2 \end{pmatrix}}}$$

A3.1

$$x^2+y^2-14x+12y+60=0$$

$$\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$(1-\lambda)^2 \Rightarrow \lambda_{1/2}=1$$

$$\begin{aligned}(1-\lambda)x&=0\\(1-\lambda)y&=0\end{aligned} \Rightarrow v_1=\begin{pmatrix}1\\0\end{pmatrix} \quad v_2=\begin{pmatrix}0\\1\end{pmatrix}$$

$$\xi^2+\eta^2+(-14\ \ 12)\cdot\begin{pmatrix}1&0\\0&1\end{pmatrix}\cdot\begin{pmatrix}\xi\\\eta\end{pmatrix}$$

$$\xi^2+\eta^2+-14\xi+12\eta+60=0$$

$$\left(\xi+\frac{-14}{2}\right)^2-\frac{196}{4}=(\xi-7)^2-49$$

$$(\eta+6)^2-36$$

$$\Rightarrow (\xi-7)^2-49+(\eta+6)^2-36+60=0$$

$$v^2+w^2=25$$

$\Rightarrow$ Kreis ✓

$\Rightarrow M(7|-6)$

A3.2

$5x^2 + 6xy + 5y^2 - 2 = 0$

$$(x\ y) \cdot \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} - 2 = 0$$

$(5-\lambda)^2 - 9 = 0$

$25 - 10\lambda + \lambda^2 - 9 = 0$

$\lambda^2 - 10\lambda + 16 = 0$

$\Rightarrow \lambda_1 = 2 \qquad \Rightarrow \lambda_2 = 8$

$$\begin{array}{l} \lambda^2 - 10\lambda + 16 : \lambda - 2 = \lambda - 8 \\ \underline{-\ \lambda^2 + 2\lambda} \\ \quad -8\lambda + 16 \end{array}$$

$$\begin{pmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \vec{\lambda}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \frac{1}{\sqrt{2}}$$

$$\begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \vec{\lambda}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}}$$

eigentlich unnötig weil hier nicht benötigt

$$(x\ y) \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 \\ 0 & 8 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} - 2 = 0$$

$2\xi^2 + 8\eta^2 - 2 = 0$

$\xi^2 + 4\eta^2 = 1$

$$\frac{\xi^2}{1^2} + \frac{\eta^2}{\left(\frac{1}{2}\right)^2} = 1$$

große Halbachse kleine Halbachse

3.3

$$5x^2+5y^2+26xy-34\sqrt{2}\cdot x-2\sqrt{2}\,y-75=0$$

$$\begin{pmatrix}5&13\\13&5\end{pmatrix}\quad (5-\lambda)^2-13^2 = 25-10\lambda+\lambda^2-169$$

$$= \lambda^2-10\lambda-144 \qquad \Rightarrow \underline{\underline{\lambda_1=-8}}$$

$$\frac{-(\lambda^2+8\lambda)}{-18\lambda-144}$$

$$\lambda-18 \quad \to \underline{\underline{\lambda_2=18}}$$

$$\begin{pmatrix}13&13\\13&13\end{pmatrix}\cdot\begin{pmatrix}v_1\\v_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \Rightarrow \underline{\underline{\vec{v}_1=\begin{pmatrix}1\\-1\end{pmatrix}}}$$

$$\begin{pmatrix}-13&13\\13&-13\end{pmatrix}\cdot\begin{pmatrix}v_1\\v_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \Rightarrow \vec{v}_2=\begin{pmatrix}1\\1\end{pmatrix}$$

$$\Rightarrow V=\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\-1&1\end{pmatrix} \qquad D=\begin{pmatrix}-8&0\\0&18\end{pmatrix}$$

$$\frac{1}{\sqrt{2}}(-34\sqrt{2}\;\; -2\sqrt{2})\begin{pmatrix}1&1\\-1&1\end{pmatrix}\cdot\begin{pmatrix}\xi\\\eta\end{pmatrix}$$

$$=\frac{1}{\sqrt{2}}(-34\sqrt{2}\;\; -2\sqrt{2})\begin{pmatrix}\xi+\eta\\-\xi+\eta\end{pmatrix}$$

$$=(-34\quad)(\xi+\eta)+(-2\quad)(-\xi+\eta)$$

$$=-32\xi-36\eta$$

$$\Rightarrow -8\xi^2+18\eta^2-32\xi-36\eta-75=0$$

$$\to -8\left(\xi+\frac{32}{16}\right)^2+\frac{32^2}{32}=-8(\xi+2)^2+32$$

$$\to 18\left(\eta-\frac{36}{36}\right)^2-\frac{36^2}{4\cdot 18}=18(\eta-1)^2-18$$

$$\Rightarrow -8v^2+32+18w^2-18-75=0$$

$$-8v^2+18w^2=61$$

$$-\frac{v^2}{\sqrt{\frac{61}{8}}^2}+\frac{w^2}{\sqrt{\frac{61}{18}}^2}=1 \quad \Rightarrow \text{Ellipse}$$

Verschiebungsvektor: $\begin{pmatrix}+2\\-1\end{pmatrix}$

$$D\cdot\vec{x}=\begin{pmatrix}+2\\-1\end{pmatrix}$$

$$\frac{1}{\sqrt{2}}x+\frac{1}{\sqrt{2}}y=+2$$

$$-\frac{1}{\sqrt{2}}x+\frac{1}{\sqrt{2}}y=-1 \quad |+I$$

$$\frac{2}{\sqrt{2}}y=1$$

$$y=\frac{\sqrt{2}}{2}=\frac{1}{\sqrt{2}}$$

$$\Rightarrow \frac{1}{\sqrt{2}}x+\frac{1}{2}=2$$

$$x=\frac{3\sqrt{2}}{2}$$

$$\Rightarrow P\left(\frac{3\sqrt{2}}{2}\;\middle|\;\frac{1}{\sqrt{2}}\right)$$

3.4

$9x^2+4y^2+12xy-4x+5=0$

$\begin{pmatrix}9&6\\6&4\end{pmatrix}$ $\quad (9-\lambda)(4-\lambda)-36=0$

$36-36-13\lambda+\lambda^2=0$

$\lambda^2-13\lambda=0 \quad \to \lambda_1=0$

$\lambda_2=13$

$\begin{pmatrix}9&6\\6&4\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \to v_1=\begin{pmatrix}2\\-3\end{pmatrix}\frac{1}{\sqrt{13}}$

$\begin{pmatrix}-4&6\\6&-9\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \to v_2=\begin{pmatrix}3\\2\end{pmatrix}\frac{1}{\sqrt{13}}$

$\to$ weg. Lösung hier $\det(V)=-1$

$\Rightarrow V=\begin{pmatrix}3&2\\2&-3\end{pmatrix}\frac{1}{\sqrt{13}} \quad \Delta=\begin{pmatrix}13&0\\0&0\end{pmatrix}$

$\frac{1}{\sqrt{13}}(-4 \quad 0)\begin{pmatrix}3&2\\2&-3\end{pmatrix}\begin{pmatrix}\xi\\\eta\end{pmatrix}$

$=\frac{1}{\sqrt{13}}(-4 \quad 0)\begin{pmatrix}3\xi+2\eta\\2\xi-3\eta\end{pmatrix}$

$=\frac{1}{\sqrt{13}}(-12\xi-8\eta)$

$\Rightarrow 13\xi^2-\frac{12}{\sqrt{13}}\xi-\frac{8}{\sqrt{13}}\eta+5=0$

$13\left(\xi-\frac{6}{\sqrt{13}^3}\right)^2-\frac{144}{13\cdot4\cdot13}=13\left(\xi-\frac{6}{\sqrt{13}^3}\right)^2-\frac{36}{169}$

$13\left(\xi-\frac{6}{\sqrt{13}^3}\right)^2-\frac{8}{\sqrt{13}}\eta+\frac{809}{169}=0$

$13\left(\xi-\frac{6}{\sqrt{13}^3}\right)^2-\frac{8}{\sqrt{13}}\left(\eta-\frac{809}{169}\frac{\sqrt{13}}{8}\right)=0$

$13v^2-\frac{8}{\sqrt{13}}w=0$

$\Rightarrow$ Parabel

3.5

$$\frac{1}{2}x^2 + \frac{5}{12}(y^2+z^2) - \frac{1}{6}yz - 1 = 0$$

$$\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{5}{12} & -\frac{1}{12} \\ 0 & -\frac{1}{12} & \frac{5}{12} \end{pmatrix}$$

$$\left(\frac{1}{2}-\lambda\right)\left(\frac{5}{12}-\lambda\right)^2 - \left(\frac{1}{12^2}\cdot\left(\frac{1}{2}-\lambda\right)\right) = 0$$

$$\left(\frac{1}{2}-\lambda\right)\left(\frac{25}{12^2} - \frac{10}{12}\lambda + \lambda^2\right) - \left(\frac{1}{12^2\cdot 2} - \frac{\lambda}{12^2}\right) = 0$$

$$\cancel{\frac{25}{12^2\cdot 2}} - \frac{10}{24}\lambda + \frac{\lambda^2}{2} - \frac{25}{12^2}\lambda + \frac{10}{12}\lambda^2 - \lambda^3 - \cancel{\frac{1}{12^2\cdot 2}} + \frac{\lambda}{12^2} = 0$$

$$-\lambda^3 + \frac{16}{12}\lambda^2 - \frac{84}{12^2}\lambda + \frac{24}{12^2\cdot 2} = 0$$

$$\lambda_1 = \frac{1}{2}$$

$$-\lambda^3 + \frac{4}{3}\lambda^2 - \frac{7}{12}\lambda + \frac{1}{12} : \lambda - \frac{1}{2} = -\lambda^2 + \frac{5}{6}\lambda - \frac{2}{12}$$

$$-\left(-\lambda^3 + \frac{1}{2}\lambda^2\right)$$

$$\frac{5}{6}\lambda^2 - \frac{7}{12}\lambda$$

$$-\left(\frac{5}{6}\lambda^2 - \frac{5}{12}\lambda\right)$$

$$-\frac{2}{12}\lambda + \frac{1}{12}$$

$$\lambda_{2/3} = \frac{-\frac{5}{6} \pm \sqrt{\frac{25}{36} - \frac{2}{3}}}{-2} = \frac{-\frac{5}{6} \pm \sqrt{\frac{1}{36}}}{-2} = \frac{-\frac{5}{6} \pm \frac{1}{6}}{-2}$$

$$\Rightarrow \lambda_2 = \frac{1}{2}$$

$$\lambda_3 = \frac{1}{3}$$

$$\begin{pmatrix} \frac{1}{2}-\lambda & 0 & 0 \\ 0 & \frac{5}{12}-\lambda & -\frac{1}{12} \\ 0 & -\frac{1}{12} & \frac{5}{12}-\lambda \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-\frac{1}{12}v_2 \quad -\frac{1}{12}v_3$$

$$-\frac{1}{12}v_2 \quad -\frac{1}{12}v_3$$

$$\Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

$$\frac{1}{6}v_1$$

$$\frac{1}{12}v_2 \quad -\frac{1}{12}v_3$$

$$-\frac{1}{12}v_2 \quad \frac{1}{12}v_3$$

$$\Rightarrow \vec{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{matrix} \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{matrix}$$

$$\Rightarrow V = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix} \text{ mit } \Lambda = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$$

$$\frac{1}{2}\xi^2 + \frac{1}{2}\eta^2 + \frac{1}{3}\mu^2 = 1$$

$\Rightarrow$ Ellipsoid mit $a = \sqrt{2}$ $b = \sqrt{2}$ $c = \sqrt{3}$

3.6

$x^2+2y^2-xy+4y+1=0$

$\begin{pmatrix}1 & -\frac{1}{2}\\ -\frac{1}{2} & 2\end{pmatrix}$ $(1-\lambda)(2-\lambda)-\frac{1}{4}=\lambda^2-3\lambda+\frac{7}{4}$

$(1-\sqrt{2})(\sqrt{2}-1)$

$2\sqrt{2}-2-1$

$\Rightarrow \frac{3\pm\sqrt{9-7}}{2}$ $\lambda_1=\frac{3+\sqrt{2}}{2}$

$\lambda_2=\frac{3-\sqrt{2}}{2}$

$-\frac{1}{4-2\sqrt{2}}$

$\left(\frac{-1-\sqrt{2}}{2}\right)v_1-\frac{1}{2}v_2=0$

$-\frac{1}{2}v_1+\left(\frac{1-\sqrt{2}}{2}\right)v_2=0$ $\Rightarrow v_1=\begin{pmatrix}1-\sqrt{2}\\1\end{pmatrix}=\frac{1}{\sqrt{4-2\sqrt{2}}}\begin{pmatrix}1-\sqrt{2}\\1\end{pmatrix}$

$\frac{-1+\sqrt{2}}{2}v_1-\frac{1}{2}v_2=0$

$-\frac{1}{2}v_1\quad\frac{1+\sqrt{2}}{2}v_2=0$ $\Rightarrow \vec{v}_2=\begin{pmatrix}1\\-1+\sqrt{2}\end{pmatrix}=\frac{1}{\sqrt{4-2\sqrt{2}}}\begin{pmatrix}1\\-1+\sqrt{2}\end{pmatrix}$

$\Rightarrow V=\frac{1}{\sqrt{4-2\sqrt{2}}}\begin{pmatrix}1 & 1-\sqrt{2}\\ \sqrt{2}-1 & 1\end{pmatrix}$ $\Lambda=\begin{pmatrix}\frac{3-\sqrt{2}}{2} & 0\\ 0 & \frac{3+\sqrt{2}}{2}\end{pmatrix}$

$(0\ 4)\cdot\begin{pmatrix}1 & 1-\sqrt{2}\\ \sqrt{2}-1 & 1\end{pmatrix}\cdot\begin{pmatrix}\xi\\ \eta\end{pmatrix}$

$(0\ 4)\cdot\begin{pmatrix}1\xi+(1-\sqrt{2})\eta\\ (\sqrt{2}-1)\xi+\eta\end{pmatrix}$

$(4\sqrt{2}-4)\xi+4\eta$

$\Rightarrow \frac{3-\sqrt{2}}{2}\xi^2+\frac{3+\sqrt{2}}{2}\eta^2+(4\sqrt{2}-4)\xi+4\eta+1=0$

$\frac{3-\sqrt{2}}{2}\left(\xi+\frac{4(\sqrt{2}-1)}{3-\sqrt{2}}\right)^2-\frac{16(\sqrt{2}-1)^2}{2(3-\sqrt{2})}$ $\quad\frac{8(2-2\sqrt{2}+1)}{3-\sqrt{2}}$ $\quad\frac{8(3-2\sqrt{2})}{3-\sqrt{2}}\cdot\frac{-3-\sqrt{2}}{(-3-\sqrt{2})}$

$\frac{3-\sqrt{2}}{2}\left(\xi+\frac{4\sqrt{2}-1}{3-\sqrt{2}}\right)^2+\frac{-40+24\sqrt{2}}{7}$

$$\frac{3+\sqrt{2}}{2}\left(\eta+\frac{4}{3+\sqrt{2}}\right)^2-\frac{16}{2(3+\sqrt{2})}$$

$$\frac{3+\sqrt{2}}{2}\left(\eta+\frac{4}{3+\sqrt{2}}\right)^2+\frac{-24+8\sqrt{2}}{7}$$

$$\Rightarrow \frac{3-\sqrt{2}}{2}v^2+\frac{3+\sqrt{2}}{2}w^2=\frac{71-32\sqrt{2}}{7}$$

$$\frac{21-7\sqrt{2}}{142-64\sqrt{2}}v^2+\frac{21+7\sqrt{2}}{142-64\sqrt{2}}w^2=1$$

2,15 1,29

4.1

a) $y'' - 4y = 0$

$\lambda^2 - 4 = 0$

$\lambda_{1/2} = \pm 2$

$y = c_1 e^{2t} + c_2 e^{-2t}$

b) $y'' - 5y' + 6y = 0$

$\lambda^2 - 5\lambda + 6 = 0$

$\lambda_1 = 2 \quad \lambda_2 = 3$

$\Rightarrow y = c_1 e^{2t} + c_2 e^{3t}$

4.2

a) $\underset{\downarrow \atop z_1'}{y''} - 4\underset{\downarrow \atop z_0}{y} = 0$

$\begin{pmatrix} z_0' \\ z_1' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix} \cdot \begin{pmatrix} z_0 \\ z_1 \end{pmatrix}$

$\lambda^2 - 4 = 0$

$\lambda_{1/2} = \pm 2$

$\begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\Rightarrow e^{2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad e^{-2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad \Rightarrow y = c_1 \cdot e^{2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \cdot e^{-2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

b) $y'' - 5y' + 6y = 0 \quad \Leftrightarrow \quad \underset{z_1'}{y''} = -6\underset{z_0}{y} + 5\underset{\substack{z_1 \\ z_0'}}{y'}$

$$\begin{pmatrix} z_0' \\ z_1' \end{pmatrix} = \begin{array}{c} \begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix} \\ \scriptstyle z_0 \quad z_1 \end{array} \cdot \begin{pmatrix} z_0 \\ z_1 \end{pmatrix}$$

$$\begin{aligned} 0 &= -\lambda(5-\lambda) + 6 \\ &= \lambda^2 - 5\lambda + 6 \\ &= (\lambda - 2)\cdot(\lambda - 3) \quad \Rightarrow \lambda_1 = 2, \ \lambda_2 = 3 \end{aligned}$$

$$\begin{pmatrix} -2 & 1 \\ -6 & 3 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} -3 & 1 \\ -6 & 2 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\Rightarrow e^{2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad e^{3t}\begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad \Rightarrow y = c_1 \cdot e^{2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{3t}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

4.3

$$\dot{x} = 2x + 4y$$

$$\dot{y} = -x - 2y$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{aligned} 0 &= (2-\lambda)(-2-\lambda) + 4 \\ &= \lambda^2 \end{aligned}$$

$$\Rightarrow \lambda_{1/2} = 0$$

$$\begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\Rightarrow e^{0t}\begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad e^{0t}\begin{pmatrix} -1+2t \\ 1-t \end{pmatrix}$$

$$\hookrightarrow c_1 e^{0t}\begin{pmatrix} 2 \\ -1 \end{pmatrix} + c_2 e^{0t}\begin{pmatrix} -1+2t \\ 1-t \end{pmatrix}$$

4.4

$\dot{x} = -3x - y$

$\dot{y} = x - y$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -3 & -1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{aligned} 0 &= (-3-\lambda)(-1-\lambda) + 1 \\ &= \lambda^2 + 4\lambda + 4 \\ &= (\lambda+2)^2 \end{aligned}$$

$$\Rightarrow \lambda_{1/2} = -2$$

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$\Rightarrow c_1 e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} -1+t \\ -t \end{pmatrix}$$

$t=0$

$x(0) = 1$

$y(0) = 1$

$$x = c_1 e^{-2t} - c_2 e^{-2t} + c_2 t e^{-2t}$$

$$y = -c_1 e^{-2t} - c_2 t e^{-2t}$$

$$1 = c_1 - c_2 \quad \Rightarrow c_2 = -2$$

$$1 = -c_1 \quad \Rightarrow c_1 = -1$$

$$x = (1 - 2t) e^{-2t}$$

$$y = (1 + 2t) e^{-2t}$$

4.5

$\dot{x} = y$

$\dot{y} = x + e^{t} + e^{-t}$

$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \lambda^2 - 1 \Rightarrow \lambda_{1/2} = \pm 1$

$\lambda = 1: \quad \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\Rightarrow c_1 \cdot e^{t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda = -1: \quad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\dot{x} = y \Rightarrow \ddot{x} = \dot{y}$

$\Rightarrow \ddot{x} = x + e^{t} + e^{-t}$

$\ddot{x} - x = e^{t} + e^{-t}$

?

4.6

$$\ddot{x} = -\left(\frac{k_1+k_2}{m_1}\right)x + \frac{k_2}{m_1}y$$
$$\ddot{y} = \frac{k_2}{m_2}x - \left(\frac{k_2+k_3}{m_2}\right)y$$
$$\Rightarrow \begin{pmatrix}\ddot{x}\\ \ddot{y}\end{pmatrix} = \begin{pmatrix}-\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1}\\ \frac{k_2}{m_2} & -\frac{k_2+k_3}{m_2}\end{pmatrix} \cdot \begin{pmatrix}x\\ y\end{pmatrix}$$

$$\begin{pmatrix}-70 & 20\\ 40 & -140\end{pmatrix} \Rightarrow (-70-\lambda)(-140-\lambda) - 800$$
$$0 = \lambda^4 + 210\lambda^2 + 9000$$

$$\begin{array}{r} 65 \cdot 65 \\ \hline 390 \\ 325 \\ \hline 4225 \end{array} \qquad \begin{array}{r} 210 \cdot 65 \\ \hline 1260 \\ 1050 \\ \hline 13650 \end{array}$$

5.1

$f(x,y) = x^2 + 3xy + y^2$

Gradient: $\nabla f = (f_x, f_y) = \left(\frac{\delta}{\delta x}, \frac{\delta}{\delta y}\right)$

$f_x = 2x + 3y \qquad f_y = 3x + 2y$

$f_{xx} = 2 \qquad f_{yy} = 2$

$f_{xy} = 3 \qquad f_{yx} = 3$

$\nabla f = (f_x, f_y) = (2x+3y, 3x+2y)$ Gradient

$df = \begin{pmatrix} 2x+3y \\ 3x+2y \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}$ totales Differential

5.2

$z = x^2 + 3xy + 5y^2 \qquad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$

ges: $\frac{dz}{dt}$

$\frac{dz}{dx} = 2x + 3y \qquad \frac{dx}{dt} = \cos t$

$\frac{dz}{dy} = 3x + 10y \qquad \frac{dy}{dt} = -\sin t$

$$\frac{dz}{dt} = (2x+3y)\frac{dx}{dt} + (3x+10y)\frac{dy}{dt}$$
$$= (2\sin t + 3\cos t)\cos t + (3\sin t + 10\cos t)(-\sin t)$$
$$= 2\sin t\cos t - 10\cos t\sin t + 3\cos^2 t - 3\sin^2 t$$
$$= 3\cos^2 t - 3\sin^2 t - 8\sin t\cos t$$

5.3

$$f(x,y) = 3x - 3y - 2x^3 - xy^2 + 2x^2y + y^3$$

$$\Rightarrow \quad \nabla f(x,y) = 0$$

$$f_x = 3 - 6x^2 - y^2 + 4xy = -6x^2 + 4xy - y^2 + 3$$

$$f_y = -3 - 2xy + 2x^2 + 3y^2 = 3y^2 - 2xy + 2x^2 - 3$$

$$\Rightarrow \quad \begin{aligned} 0 &= -6x^2 + 4xy - y^2 + 3 \\ 0 &= 2x^2 - 2xy + 3y^2 - 3 \quad | + 3\mathrm{I} \end{aligned} \qquad \mathrm{I} + 2\mathrm{II}$$

$$0 = -16x^2 + 10xy + 6$$

$$y = \frac{16x^2 - 6}{10x}$$

$$0 = -2x^2 + 5y^2 - 3$$

$$0 = -2x^2 + 5 \cdot \left(\frac{16x^2 - 6}{10x}\right)^2 - 3$$

$$= -2x^2 + 5 \cdot \left(\frac{256x^4 - 192x^2 + 36}{100x^2}\right) - 3 \qquad | \cdot 100x^2 \text{ für } x \neq 0$$

$$= -200x^4 + 1280x^4 - 960x^2 + 180 - 300x^2$$

$$= 1080x^4 - 1260x^2 + 180$$

$$= 108x^4 - 126x^2 + 18$$

$$= 54x^4 - 63x^2 + 9$$

$$= 18x^4 - 21x^2 + 3$$

$$= 6x^4 - 7x^2 + 1$$

$$= 6u^2 - 7u + 1$$

$$u_{1/2} = \frac{7 \pm \sqrt{49 - 24}}{12} = \frac{7 \pm \sqrt{25}}{12} = \frac{7 \pm 5}{12}$$

$$u_1 = 1 \qquad u_2 = \frac{1}{6}$$

$$\Rightarrow x_{1/2} = \pm 1 \qquad x_{3/4} = \pm \frac{1}{\sqrt{6}}$$

$$y\left(\tfrac{1}{\sqrt{6}}\right) = \frac{\frac{16}{6} - 6}{\frac{10}{\sqrt{6}}} = -\frac{20 \cdot \sqrt{6}}{10 \cdot 6} = -\frac{\sqrt{6}}{3}$$

$$y\left(-\tfrac{1}{\sqrt{6}}\right) = \frac{\frac{16}{6} - 6}{-\frac{10}{\sqrt{6}}} = \frac{\sqrt{6}}{3}$$

$$\Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \begin{pmatrix} -1 \\ -1 \end{pmatrix} \quad \begin{pmatrix} \frac{1}{\sqrt{6}} \\ -\frac{\sqrt{6}}{3} \end{pmatrix} \quad \begin{pmatrix} -\frac{1}{\sqrt{6}} \\ \frac{\sqrt{6}}{3} \end{pmatrix}$$

$f_{xx} = -12x + 4y$ $\qquad$ $f_{yy} = -2x + 6y$

$f_{xy} = -2y + 4x$ $\qquad$ $f_{yx} = -2y + 4x$

$$H(x_0, y_0) = \begin{pmatrix} -12x+4y & -2y+4x \\ -2y+4x & -2x+6y \end{pmatrix}$$

$$\begin{aligned} D &= (-12x+4y)(-2x+6y) - (4x-2y)^2 \\ &= \cancel{24x^2} + \cancel{24y^2} - \cancel{72xy} - \cancel{8xy} - \cancel{16x^2} + \cancel{32xy} - \cancel{4y^2} \\ &= 8x^2 + 20y^2 - 48xy \end{aligned}$$

$$\left.\begin{aligned} D(1,1) &= 8+20-48 = -20 \\ D(-1,-1) &= 8+20-48 = -20 \end{aligned}\right\rangle ↯$$

$D\left(\frac{1}{\sqrt{6}}, -\frac{\sqrt{6}}{3}\right) = \frac{8}{6} + \frac{20 \cdot 6}{9} + 48 \cdot \frac{1}{3} = \frac{4}{3} + \frac{40}{3} + \frac{48}{3} = \frac{91}{3}$ ✓

$D\left(-\frac{1}{\sqrt{6}}, +\frac{\sqrt{6}}{3}\right) = \frac{8}{6} + \frac{20 \cdot 6}{9} + 48\,\frac{1}{3} = \frac{91}{3}$ ✓

$f_{xx}\left(\frac{1}{\sqrt{6}}, -\frac{\sqrt{6}}{3}\right) = -12\,\frac{1}{\sqrt{6}} - 4\,\frac{\sqrt{6}}{3} < 0 \Rightarrow$ Maximum $\begin{pmatrix} \frac{1}{\sqrt{6}} \\ -\frac{\sqrt{6}}{3} \end{pmatrix}$

$f_{xx}\left(-\frac{1}{\sqrt{6}}, \frac{\sqrt{6}}{3}\right) = 12\,\frac{1}{\sqrt{6}} + 4\,\frac{\sqrt{6}}{3} > 0 \Rightarrow$ Minimum $\begin{pmatrix} -\frac{1}{\sqrt{6}} \\ \frac{\sqrt{6}}{3} \end{pmatrix}$

5.3

$$f(x,y) = 3x - 3y - 2x^3 - xy^2 + 2x^2y + y^3$$

$$f_x = 3 - 6x^2 - y^2 + 4xy$$

$$f_y = -3 - 2xy + 2x^2 + 3y^2$$

$$0 = -6x^2 + 4xy - y^2 + 3 \qquad | \mathrm{I} + 3\mathrm{II} \qquad | \mathrm{I} + 2\mathrm{II}$$
$$0 = 2x^2 - 2xy + 3y^2 - 3$$

$$0 = -2xy + 8y^2 - 6$$

$$x = \frac{8y^2 - 6}{2y}$$

$$0 = -2x^2 + 5y^2 - 3$$

$$0 = -2\left(\frac{8y^2-6}{2y}\right) + 5y^2 - 3$$

$$= -\frac{64y^4 - 96y^2 + 36}{2y^2} + 5y^2 - 3 \qquad | \cdot y^2$$

$$= -\cancel{32}y^4 + \cancel{48}y^2 - 18 + \cancel{5y^4} - \cancel{3y^2}$$

$$= -27y^4 + 45y^2 - 18$$

$$= -9y^4 + 15y^2 - 6$$

$$= -3y^4 + 5y^2 - 2 \qquad | y^2 := u$$

$$u_{1/2} = \frac{-5 \pm \sqrt{25 - 24}}{-6} = \frac{-5 \pm 1}{-6}$$

$$u_1 = \frac{2}{3} \qquad u_2 = 1$$

$$y_{1/2} = \pm\frac{\sqrt{2}}{\sqrt{3}} \qquad y_{3/4} = \pm 1$$

$$= \pm\frac{\sqrt{6}}{3}$$

$$x_1 = \frac{8 \cdot \frac{6}{9} - 6}{2\frac{\sqrt{6}}{3}} = \frac{3}{\sqrt{6}} \cdot \left(\frac{24}{9} - 3\right) = \frac{8}{\sqrt{6}} - \frac{9}{\sqrt{6}} = -\frac{1}{\sqrt{6}} \qquad | \frac{\sqrt{6}}{3}$$

$$x_2 = \frac{8\frac{6}{9} - 6}{-2\frac{\sqrt{6}}{3}} = \frac{1}{\sqrt{6}} \qquad | -\frac{\sqrt{6}}{3}$$

$$x_3 = \frac{8-6}{2} = 1 \qquad | 1$$

$$x_4 = \frac{8-6}{-2} = -1 \qquad | -1$$

$f_{xx} = -12x + 4y$

$f_{xy} = -2y + 4x$

$f_{yy} = -2x + 6y$

$$D(x,y) = (-12x+4y)(-2x+6y) - (-2y+4x)^2$$

$$= \cancel{24x^2} + \cancel{24y^2} - \cancel{72xy} - \cancel{8xy} - \cancel{16x^2} + \cancel{16xy} - \cancel{4y^2}$$

$$= 8x^2 + 20y^2 - 64xy$$

$D(-\frac{1}{\sqrt{6}}, \frac{\sqrt{6}}{3}) = 8\,\frac{1}{6} + 20\,\frac{6}{9} - 64(-\frac{1}{3}) \quad > 0 \quad \checkmark$

$D(\frac{1}{\sqrt{6}}, -\frac{\sqrt{6}}{3}) = 8\,\frac{1}{6} + 20\,\frac{6}{9} - 64(-\frac{1}{3}) \quad > 0 \quad \checkmark$

$D(1,1) = 8 + 20 - 64 \quad < 0 \quad X$

$D(-1,-1) = 8 + 20 - 64 \quad < 0 \quad X$

$f_{xx}(-\frac{1}{\sqrt{6}}, \frac{\sqrt{6}}{3}) = -12(-\frac{1}{\sqrt{6}}) + 4\frac{\sqrt{6}}{3} \quad > 0 \rightarrow$ Minimum $(-\frac{1}{\sqrt{6}}, \frac{\sqrt{6}}{3})$

$f_{xx}(\frac{1}{\sqrt{6}}, -\frac{\sqrt{6}}{3}) = -12\,\frac{1}{\sqrt{6}} + 4(-\frac{\sqrt{6}}{3}) \quad < 0 \rightarrow$ Maximum $(\frac{1}{\sqrt{6}}, -\frac{\sqrt{6}}{3})$

S.4

$T(x,y) = 1 + xy \qquad x^2+y^2=1$

$f_x + \lambda \cdot g_x = y + \lambda \cdot 2x = 0$

$f_y + \lambda \cdot g_y = x + \lambda \cdot 2y = 0$

$0 = \lambda \cdot 2(-\lambda \cdot 2y) + y$

$0 = -4\lambda^2 y + y$

$0 = y(-4\lambda^2 + 1)$

$0 = -4\lambda^2 + 1 \qquad \Rightarrow \lambda = \pm \frac{1}{2}$

$\Rightarrow 0 = y + x \qquad 0 = y - x$

$y = -x \qquad y = x$

$\Rightarrow y = \pm x$

$x^2 + y^2 = 1$

$x^2 + x^2 = 1$

$2x^2 = 1$

$x^2 = \frac{1}{2}$

$x = \pm \frac{1}{\sqrt{2}} = y$

$T(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = 1 + \frac{1}{2} = 1{,}5 = T(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$

$T(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) = 1 - \frac{1}{2} = 0{,}5 = T(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$

$\Rightarrow P_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \quad P_2 = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$

5.4

$T(x,y) = 1 + xy \qquad x^2 + y^2 = 1$

$g(x,y) = x^2 + y^2 - 1$

$T_x = y \qquad g_x = 2x$
$T_y = x \qquad g_y = 2y$

$y + \lambda\, 2x = 0$
$x + \lambda\, 2y = 0$

$y + \lambda \cdot 2(-\lambda 2y) = 0$

$y - 4\lambda^2 y = 0$

$y\,(1 - 4\lambda^2) = 0 \qquad \Rightarrow y = 0$

$1 = 4\lambda^2$

$\frac{1}{4} = \lambda^2 \Rightarrow \lambda = \pm \frac{1}{2}$

$\Rightarrow y + x = 0 \qquad y - x = 0 \qquad x - y = 0 \qquad x + y = 0$

$y = -x \qquad y = x \qquad y = x \qquad y = -x$

$\Rightarrow y = \pm x$

$x^2 + y^2 = 1$

$x^2 + x^2 = 1$

$2x^2 = 1$

$x = \pm\sqrt{\frac{1}{2}} \qquad y = \pm\sqrt{\frac{1}{2}}$

$\Rightarrow ++, -- \Rightarrow \begin{pmatrix} \sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} -\sqrt{\frac{1}{2}} \\ -\sqrt{\frac{1}{2}} \end{pmatrix}$

5.5

$$H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}$$

$$0 = (f_{xx} - \lambda)(f_{yy} - \lambda) - (f_{xy})^2$$
$$= \lambda^2 - \lambda f_{xx} - \lambda f_{yy} + f_{xx} f_{yy} - (f_{xy})^2$$
$$= \lambda^2 - \lambda (f_{xx} + f_{yy}) + f_{xx} f_{yy} - f_{xy}^2$$

$$\lambda_{1/2} = \frac{(f_{xx}+f_{yy}) \pm \sqrt{(f_{xx}+f_{yy})^2 - 4(f_{xx}f_{yy} - f_{xy}^2)}}{2}$$
$$= \frac{f_{xx}+f_{yy}}{2} \pm \sqrt{\frac{(f_{xx}+f_{yy})^2}{4} - (f_{xx}f_{yy} - f_{xy}^2)}$$

mit $\lambda > 0 \Rightarrow \boxed{f_{xx} + f_{yy} > 0}$

für $(f_{xx}f_{yy} - f_{xy}^2) = 0 \rightarrow \frac{f_{xx}+f_{yy}}{2} \pm \frac{f_{xx}+f_{yy}}{2}$ ↯ $= 0$

$\Rightarrow \boxed{(f_{xx}f_{yy} - f_{xy}^2) > 0}$ (-> Wurzel kleiner als Bruch davor)

5.6

$f(x,y) = x - y$ $\quad\quad$ $\tan x = 3 \tan y$ $\quad\quad$ $0 \le x \le \pi/2$ $\quad$ $0 \le y \le \pi/2$

$$f_x + \lambda g_x = 1 + \lambda \cdot \left(\frac{\cos x}{\cos x} - \sin x \cos^{-2} x (-\sin x)\right) = 1 + \lambda \cdot \frac{1}{\cos^2 x} \Rightarrow \lambda = -\cos^2 x$$

$$f_y + \lambda g_y = -1 - \lambda \cdot 3 \cdot \frac{1}{\cos^2 y} \quad\quad = -1 - 3\lambda \frac{1}{\cos^2 y}$$

mit $\lambda = -\cos^2 x \rightarrow 0 = -1 + 3\cos^2 x \frac{1}{\cos^2 y}$ $\quad\quad$ $\cos^2 x = \frac{\cos^2 y}{3}$

$$\boxed{\cos x = \pm \frac{\cos y}{\sqrt{3}}}$$

$$\Rightarrow \quad \tan x = 3 \tan y$$

$$\sqrt{3}\,\frac{\sin x}{\cos y} = 3\,\frac{\sin y}{\cos y}$$

$$\sin x = \sqrt{3}\,\sin y$$

mit: $\sin^2 x + \cos^2 x = 1 = 3\sin^2 y + \frac{1}{3}\cos^2 y$

und $\sin^2 y = 1 - \cos^2 y$

$$\Rightarrow \quad 1 = 3 - 3\cos^2 y + \frac{1}{3}\cos^2 y$$

$$1 = \frac{4}{3}\cos^2 y$$

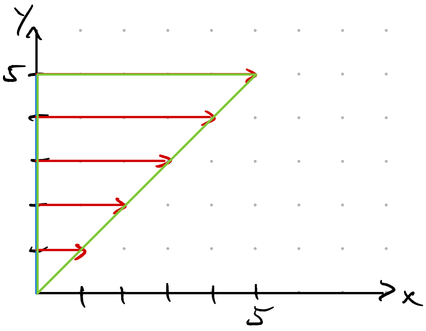
$$\cos^2 y = \frac{3}{4}$$

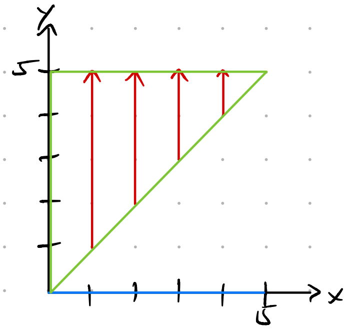
$$\cos y = \frac{\sqrt{3}}{2} \qquad \Rightarrow \quad y = \frac{\pi}{6}$$

$$\Rightarrow \quad \cos x = \frac{1}{2} \qquad \Rightarrow \quad y = \frac{\pi}{3}$$

6.1

$$\int_0^5\int_0^y f(x,y)\,dx\,dy = \int_0^5\int_x^5 f(x,y)\,dy\,dx$$



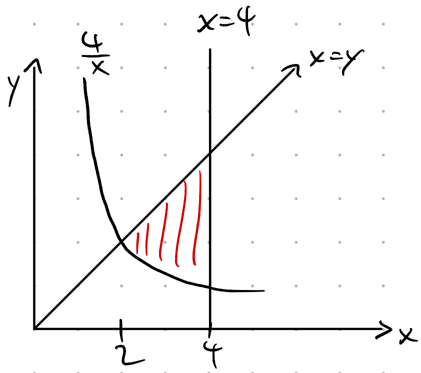


6.2

$x \cdot y = 4$

$x = y$

$x = 4$



$\frac{4}{x} = x \quad | \cdot x \quad$ für $x \neq 0$

$4 = x^2$

$x = \pm 2$

$$\Rightarrow \int_2^4\int_{\frac{4}{x}}^{x} dy\,dx = \int_2^4 x - \frac{4}{x}\,dx$$

$$= \left[\frac{1}{2}x^2 - 4\ln(x)\right]_2^4$$

$$\approx \underline{\underline{3{,}227}}\ \text{FE}$$

6.3

$$A = 2\pi \int_0^R -\frac{h}{R}x + h\,dx$$

$$= 2\pi\left[-\frac{h}{2R}x^2 + hx\right]_0^R$$

$$= 2\pi\left(-\frac{hR}{2} + hR\right)$$

$$= 2\pi\frac{hR}{2} = \pi \cdot h \cdot R$$

$$\int_0^h 2\pi\left(R - \frac{R}{h}\cdot x\right) dx$$

$$2\pi \quad Rx - \frac{R}{2h}x^2$$

$$Rh - \frac{Rh}{2}$$

$$R \cdot h \cdot \pi$$

Übungsaufgabe 6.3: Gegeben sei ein Kreiskegel, für den gilt: Radius R = Höhe h. Berechnen Sie seine Mantelfläche.

6.4

$$z_s = \frac{1}{V} \cdot \iiint_V \overbrace{z \, dx\,dy}^{\text{Fläche}}\,dz \quad \Rightarrow \frac{1}{V} \cdot \int_0^h z \cdot \pi \, R(z)^2 \, dz$$

mit $R(z) = z \cdot \frac{R}{h}$

$$z_s = \frac{1}{V} \cdot \pi \cdot \int_0^h z^3 \cdot \frac{R^2}{h^2} \, dz$$

$$= \frac{1}{V} \cdot \pi \left[\frac{1}{4} z^4 \frac{R^2}{h^2}\right]_0^h$$

$$= \frac{1}{V} \cdot \frac{\pi}{4} R^2 h^2$$

$$V = \iiint_V \overbrace{dx\,dy}^{\text{Fläche } \pi r^2}\,dz$$

$$= \pi \int_0^h R(z)^2 \, dz$$

$$= \pi \int_0^h z^2 \frac{R^2}{h^2} \, dz$$

$$= \pi \left[\frac{1}{3} z^3 \frac{R^2}{h^2}\right]_0^h$$

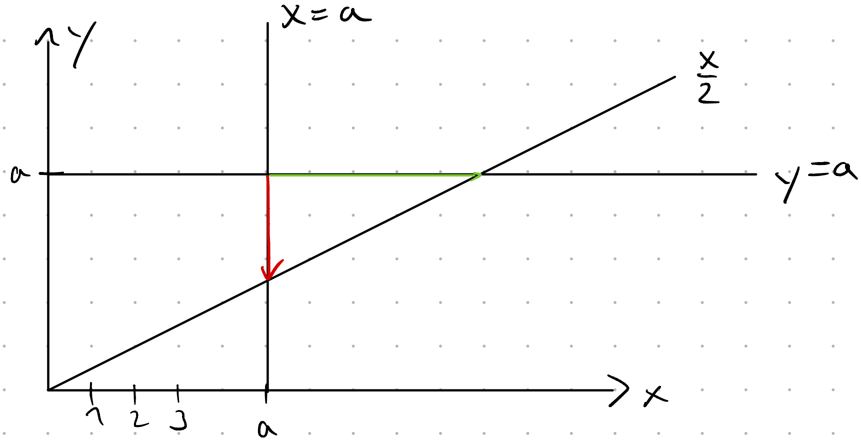
$$= \frac{1}{3} \pi R^2 \cdot h$$

$$\Rightarrow \frac{1}{\frac{1}{3} \cancel{\pi} \cancel{R^2} \cdot \cancel{h}} \cdot \frac{\cancel{\pi}}{4} \cancel{R^2} h^{\cancel{2}}$$

$$= \frac{3}{4} h$$

6.5

$y = \frac{x}{2} \qquad x = a \qquad y = a$



$$I_x = \iint_A y^2 \, dy\,dx$$

$$= \int_a^{2a} \int_{\frac{x}{2}}^{a} y^2 \, dy\,dx$$

$$= \int_a^{2a} \left[\frac{1}{3} y^3\right]_{\frac{x}{2}}^{a} dx$$

$$= \int_a^{2a} \frac{1}{3} a^3 - \frac{1}{3} \frac{x^3}{8} \, dx$$

$$= \frac{1}{3} \int_a^{2a} a^3 - \frac{x^3}{8} \, dx$$

$$= \frac{1}{3} \left[a^3 x - \frac{1}{32} x^4\right]_a^{2a}$$

$$= \frac{1}{3}\left(\cancel{2}a^4 - \frac{1}{2} a^4 - \cancel{a^4} + \frac{1}{32} a^4\right)$$

$$= \frac{1}{3}\left(\frac{1}{2} a^4 + \frac{1}{32} a^4\right)$$

$$= \underline{\underline{\frac{17}{96} a^4}}$$

6.6

$$(x-\overbrace{b}^{\text{Kreisverschiebung}})^2 + z^2 \leq \overbrace{a}^{\text{Radius}}{}^2$$ $\Leftarrow$ Kreisfläche

$\Rightarrow A = \pi a^2 \qquad d = 2\pi b$

$\Rightarrow A \cdot d = V = \underline{\underline{2\pi^2 a^2 b}}$

7.2

von $(0,0,0) \to (1,1,1)$ $\Rightarrow \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\text{Weg}} = \underbrace{\begin{pmatrix} \mu \\ \mu \\ \mu \end{pmatrix}}_{r(\mu)}$; $\mu \in [0;1]$

$$\oint_{\mu_1}^{\mu_2} F(r(\mu)) \cdot r'(\mu) \cdot d\mu$$

$$\oint_0^1 \begin{pmatrix} \mu+\mu^3 \\ \mu+\mu^3 \\ \mu+\mu^3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot d\mu = \oint_0^1 3(\mu+\mu^3)\, d\mu = 3\left[\tfrac{1}{2}\mu^2 + \tfrac{1}{4}\mu^4\right]_0^1 = 3 \cdot \left(\tfrac{1}{2} + \tfrac{1}{4}\right) = \underline{\underline{\tfrac{9}{4}}} = \underline{\underline{2{,}25}}$$

7.1

① $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} \Rightarrow x = x$ ✓

② $\frac{\partial F_1}{\partial y} = 0 \quad \frac{\partial F_2}{\partial x} = 0$

$\frac{\partial F_1}{\partial z} = 3 \quad \frac{\partial F_3}{\partial x} = 0$ ↯

③ $\frac{\partial F_1}{\partial y} = 4 \quad \frac{\partial F_2}{\partial x} = 4$

$\frac{\partial F_1}{\partial z} = -3 \quad \frac{\partial F_3}{\partial x} = -3$

$\frac{\partial F_2}{\partial z} = -4 \quad \frac{\partial F_3}{\partial y} = -4$ ✓

$F = \begin{pmatrix} x^2 + xy \\ y^3 + \frac{1}{2}x^2 \end{pmatrix} \qquad r(\mu) = \begin{pmatrix} 1-\mu \\ 0 \end{pmatrix} \quad \mu \in [0,1]$

$F = \begin{pmatrix} (1-\mu)^2 \\ \frac{1}{2}(1-\mu)^2 \end{pmatrix} \qquad r'(\mu) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} (1-\mu)^2 \\ \frac{1}{2}(1-\mu)^2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \end{pmatrix} = -(1-\mu)^2$$

$$\int_0^1 F_{(x,y)} \cdot r'(\mu)\, d\mu = \int_0^1 -(1-\mu)^2\, d\mu$$

$$= -\int_0^1 1 - 2\mu + \mu^2\, d\mu$$

$$= -1 \cdot \left[\mu - \mu^2 + \tfrac{1}{3}\mu^3\right]_0^1$$

$$= -1 \cdot \left(\tfrac{1}{3}\right) = \underline{\underline{-\tfrac{1}{3}}}$$

7.3

$$\oint_C F_{1a}(x,y)\cdot\begin{pmatrix}dx\\ dy\end{pmatrix} \qquad r(t)=\begin{pmatrix}\cos\frac{t}{2}\\ \frac{t^2}{8t^3+7t^2+4t2}\cdot\sin t\end{pmatrix} \qquad t\in[0,\pi]$$

$$r(0)=\begin{pmatrix}1\\0\end{pmatrix} \qquad r(\pi)=\begin{pmatrix}0\\0\end{pmatrix}$$

$$\Rightarrow r(p)=\begin{pmatrix}1-p\\0\end{pmatrix} \qquad p\in[0,1]$$

$$r'(p)=\begin{pmatrix}-1\\0\end{pmatrix}$$

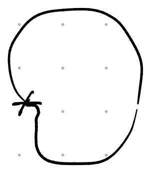
$$\oint_C \begin{pmatrix}(1-p)^2\\ \frac{1}{2}(1-p)^2\end{pmatrix}\cdot\begin{pmatrix}-1\\0\end{pmatrix}dp = \oint_C -(1-p)^2\, dp = \int_0^1 -1+2p-p^2\, dp = \left[-\frac{1}{3}p^3+p^2-p\right]_0^1$$

$$= -\frac{1}{3}+1-1$$

$$= \underline{\underline{-\frac{1}{3}}}$$

7.4

$$W=\oint_C \frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}dx\\dy\\dz\end{pmatrix}$$

 $\Rightarrow 0$

1. $r(t)=\begin{pmatrix}\cos^2 t\\ \sin t\\ 0\end{pmatrix} \qquad t\in[0,2\pi]$

$$F_1=\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}} \qquad F_2=\frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}} \qquad F_3=\frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

$$\frac{\partial F_1}{\partial y}=x\cdot\left(-\frac{3}{2}(x^2+y^2+z^2)^{-\frac{5}{2}}\cdot 2y\right)$$

✓

$$\frac{\partial F_2}{\partial x}=y\cdot\left(-\frac{3}{2}(x^2+y^2+z^2)^{-\frac{5}{2}}\cdot 2x\right)$$

$\frac{\partial F_1}{\partial z}=\frac{\partial F_3}{\partial x}$ ✓ $\qquad \frac{\partial F_2}{\partial z}=\frac{\partial F_3}{\partial y}$ ✓ $\qquad \Rightarrow r(0)=\begin{pmatrix}1\\0\\0\end{pmatrix}=r(2\pi)=\begin{pmatrix}1\\0\\0\end{pmatrix}$

$$\Rightarrow \underline{\underline{=0}}$$

② $r(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = r(2\pi) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \underline{\underline{0}}$

③ $r_{Anfang} = r_{Ende} \Rightarrow \underline{\underline{0}}$

7.5

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \cdot \cos t \\ b \cdot \sin t \end{pmatrix} \qquad \begin{matrix} dx & -a \cdot \sin t \, dt \\ dy & b \cdot \cos t \, dt \end{matrix}$$

$$\frac{1}{2} \oint_C (x \, dy - y \, dx)$$

$$\frac{1}{2} \oint_C (a \cdot \cos t \cdot b \cdot \cos t - b \cdot \sin t \cdot (-a) \cdot \sin t) \, dt$$

$$\frac{1}{2} \oint_C (a \cdot b \cos^2 t + a \cdot b \sin^2 t) \, dt$$

$$\frac{1}{2} \oint_C a \cdot b \underbrace{(\cos^2 t + \sin^2 t)}_{1} \, dt$$

$$\frac{1}{2} a \cdot b \int_0^{2\pi} 1 \, dt = \frac{1}{2} ab \, [t]_0^{2\pi} = \underline{\underline{a \cdot b \cdot \pi}}$$

7.6

$$-x^2 - 2y^2$$

$$\oint_\gamma (x^2 + 2y^2) \, dx + (4xy + 5x) \, dy \qquad r = \phi^2 \, , \quad -\pi \leq \phi \leq \pi$$

$$\frac{F_1}{\partial x} + \frac{F_2}{\partial y} = 4y + 5 - 4y = \underline{\underline{5}}$$

$$\Rightarrow 5 \iint_A dA = 5 \iint_A dx \, dy \qquad r \cdot dr \, d\phi$$

$$= 5 \int_{-\pi}^{\pi} \int_0^{\phi^2} r \, dr \, d\phi = \frac{5}{2} \int_{-\pi}^{\pi} \phi^4 \, d\phi = \frac{5}{2} \left[\frac{1}{5} \phi^5 \right]_{-\pi}^{\pi} = \frac{5}{2} \left(\frac{1}{5} \pi^5 + \frac{1}{5} \pi^5 \right)$$

$$= \frac{5}{2} \cdot \frac{2}{5} \pi^5$$

$$= \underline{\underline{\pi^5}}$$

8.1

$$\int_1^2 \frac{1}{x}\,dx \qquad h = 1, \tfrac{1}{2}, \tfrac{1}{4}$$

$$h=1: \quad \frac{1}{2}\left(f(1)+f(2)\right)\cdot(2-1) = \frac{1}{2}\left(1+\frac{1}{2}\right) = \underline{\underline{\frac{3}{4}}} = \frac{18}{24} = \frac{1260}{1680}$$

$$h=\frac{1}{2}: \quad \frac{1}{2\cdot 2}\cdot\left(f(1)+2\cdot\left(f(1+\tfrac{1}{2}(2-1))\right)+f(2)\right)\cdot(2-1)$$

$$= \frac{1}{4}\cdot\left(1+2\cdot\frac{2}{3}+\frac{1}{2}\right)\cdot 1$$

$$= \frac{1}{4}\cdot\frac{17}{6} = \underline{\underline{\frac{17}{24}}} = \frac{1190}{1680}$$

$$h=\frac{1}{4}: \quad \frac{1}{2\cdot 4}\cdot\left(f(1)+2\cdot\left(f(\tfrac{5}{4})+f(\tfrac{3}{2})+f(\tfrac{7}{4})\right)+f(2)\right)$$

$$= \frac{1}{8}\cdot\left(1+2\cdot\left(\frac{4}{5}+\frac{2}{3}+\frac{4}{7}\right)+\frac{1}{2}\right)$$

$$= \frac{1}{8}\cdot\left(\frac{3}{2}+2\cdot\left(\frac{84}{105}+\frac{70}{105}+\frac{60}{105}\right)\right)$$

$$= \frac{1}{8}\cdot\left(\frac{315}{210}+\frac{856}{210}\right)$$

$$= \frac{1}{8}\cdot\frac{1171}{210}$$

$$= \frac{1171}{1680}$$

h:	1	$\frac{1}{2}$	$\frac{1}{4}$
$\int$:	$\frac{1260}{1680}$	$\frac{1190}{1680}$	$\frac{1171}{1680}$
	$\approx 0{,}75$	$0{,}70833$	$0{,}69702$

8.2

$$\int_0^1 \sqrt{1+x^4}\, dx \qquad n=2$$

$$A \approx \frac{1}{6\cdot 2} \cdot \left[f(0) + 4 \cdot \sum_{i=0}^{1} \{ f(\tfrac{1}{2}\cdot(a_i + a_{i+1}))\} + 2 \cdot \sum_{i=1}^{1} f(a_i) + f(1)\right] \cdot 1$$

$$= \frac{1}{12} \cdot \left[1 + 4\cdot\left(f(\tfrac{1}{2}\cdot\tfrac{1}{2}) + f(\tfrac{1}{2}\cdot\tfrac{3}{2})\right) + 2\cdot\left(f(\tfrac{1}{2})\right) + f(1)\right]$$

$$= \frac{1}{12} \cdot \left[1 + 4\cdot\left(\frac{\sqrt{257}}{16} + \frac{\sqrt{337}}{16}\right) + 2\cdot\frac{\sqrt{17}}{4} + \sqrt{2}\right]$$

$$= \frac{1}{12} \cdot \left[1 + \frac{\sqrt{257}+\sqrt{337}}{4} + \frac{\sqrt{17}}{2} + \sqrt{2}\right]$$

$$\overset{TR}{=} 1{,}08941$$

$$a_0 = 0 + \frac{0}{2}\cdot 1 = 0$$

$$a_1 = 0 + \frac{1}{2}\cdot 1 = \frac{1}{2}$$

$$a_2 = 0 + \frac{2}{2}\cdot 1 = 1$$

$$f(\tfrac{1}{4}) = \sqrt{1 + (\tfrac{1}{4})^4} = \sqrt{1 + \tfrac{1}{256}}$$

$$f(\tfrac{3}{4}) = \sqrt{1 + \tfrac{81}{256}}$$

$$f(\tfrac{1}{2}) = \sqrt{1 + \tfrac{1}{16}}$$

8.3

$$y'(x) = \sqrt{1 - y^2(x)} \qquad y(0) = 0$$

ges: $y(1)$ mit $\Delta x = 0{,}1$

$$y_{0,1} = y_0 + h \cdot f(x_0, y_0)$$

$$y_{0,1} = 0 + \frac{1}{10}\cdot 1$$

$$y_{0,1} = 0{,}1$$

$$y_{0,2} = y_{0,1} + h\cdot f(x_{0,1}, y_{0,1})$$

$$= 0{,}1 + \frac{1}{10}\cdot\sqrt{0{,}99}$$

$$\cong 0{,}199499$$

$$y_{0,3} = y_{0,2} + \frac{1}{10}\cdot f(x_{0,2}\ y_{0,2})$$

$$y_{0,3} = 0{,}199499 + \frac{1}{10}\cdot\sqrt{1 - 0{,}199499^2}$$

$$\cong 0{,}297489$$

$$y_{0,4} = y_{0,3} + \frac{1}{10}\cdot f(x_{0,3}\ y_{0,3})$$

$$= 0{,}297489 + \frac{1}{10}\cdot\sqrt{1 - 0{,}297489}$$

$$\cong 0{,}392962$$

Übungsaufgabe 8.3: Gegeben sei das Anfangswertproblem $y'(x) = \sqrt{1 - y^2(x)}$ mit $y(0) = 0$. Bestimmen Sie den Wert $y(1)$ mittels der Näherungslösung mit Hilfe des Eulerschen Polygonzugverfahrens (Schrittweite $\Delta x = 0.1$) und vergleichen Sie diesen mit der exakten Lösung.

8.4

$y'(x) = 2x \qquad y(0) = 0$

$\int 2x$

$$y_1 = y_0 + h \cdot y'(x_0)$$
$$= 0 + \frac{x}{n} \cdot 0 \quad = 0$$

$$y_2 = 0 + \frac{x}{n} \cdot \frac{2x}{n}$$
$$= 0 + \frac{2x^2}{n^2}$$

$$y_3 = 0 + \frac{2x^2}{n^2} + \frac{1}{n} \cdot \frac{4x^2}{n}$$
$$= 0 + \frac{2x^2}{n^2} + \frac{4x^2}{n^2} + \frac{6x^2}{n^2}$$

$$y_n = \frac{\sum_{i=0}^{n-1} 2i}{n^2} \; x^2$$

mit $n=1$: $\quad 0 \cdot x^2 = 0 \qquad x^2$

$n=2$: $\quad \frac{2}{4} x^2 = \frac{1}{2} x^2 \qquad \frac{1}{2} x^2$

$n=4$: $\quad \frac{12}{16} x^2 = \frac{3}{4} x^2 \qquad \frac{1}{4} x^2$

Übungsaufgabe 8.4: Gegeben sei das Anfangswertproblem $y'(x) = 2x$ mit $y(0) = 0$. Bestimmen Sie den Wert y(x) numerisch mittels

1. Euler-Verfahren (1, 2 und 4 Schritte),
2. Mittelpunktsregel (1 Schritt) und
3. Runge-Kutta-Verfahren 4. Ordnung (1 Schritt).

Geben Sie das Ergebnis der Näherungsverfahren jeweils als analytischen Ausdruck an und berechnen Sie die Differenz zwischen Näherungslösung und exakter Lösung.

Hinweis: Hier gilt offensichtlich $f(x, y) = f(x)$.

Mittelpunktregel:

$$y_{\frac{1}{2}} = y_0 + \frac{x}{2n} \cdot y'(0) = 0$$

$$y_1 = y_0 + \frac{x}{n} \cdot y'\left(y_{\frac{1}{2}}, x_0 + \frac{h}{2}\right)$$
$$= 0 + \frac{x}{n} \cdot \cancel{2} \cdot \frac{x}{\cancel{2}n}$$
$$= \frac{x^2}{n^2}$$

mit $n=1$: $\quad y_1 = x^2 \qquad 0$

Runge-Kutta: 4. Ordnung

$h = \frac{x}{n}$

$$K_1 = \frac{x}{n} \cdot 2 \cdot 0 = 0$$
$$K_2 = \frac{x}{n} \cdot 2 \cdot \left(0 + \frac{x}{2n}\right) = \frac{x^2}{n^2}$$
$$K_3 = \frac{x}{n} \cdot 2 \cdot \left(\frac{x}{2n}\right) = \frac{x^2}{n^2}$$
$$K_4 = \frac{x}{n} \cdot 2 \cdot \left(\frac{x}{n}\right) = \frac{2x^2}{n^2}$$

$$y_1 = 0 + \frac{1}{6} \cdot \left(0 + \frac{2x^2}{n^2} + \frac{2x^2}{n^2} + \frac{2x^2}{n^2}\right)$$
$$= \frac{1}{6} \cdot \frac{6x^2}{n^2} = \frac{x^2}{n^2}$$

$\Rightarrow$ mit $n=1$: $\quad y_1 = x^2 \qquad 0$

8.5

$$y'(x) = e^x = y(x)$$

$$h = \frac{1}{n}$$

$$K_1 = \frac{1}{n} \cdot f(0) = \frac{1}{n}$$

$$K_2 = \frac{1}{n} \cdot \left(y_0 + \frac{K_1}{2}\right) = \frac{1}{n} \cdot \left(1 + \frac{1}{2n}\right) = \frac{1}{n} + \frac{1}{2n^2}$$

$$K_3 = \frac{1}{n} \cdot \left(y_0 + \frac{K_2}{2}\right) = \frac{1}{n} \cdot \left(1 + \frac{1}{2n} + \frac{1}{4n^2}\right) = \frac{1}{n} + \frac{1}{2n^2} + \frac{1}{4n^3}$$

$$K_4 = \frac{1}{n} \cdot \left(y_0 + K_3\right) = \frac{1}{n} \cdot \left(1 + \frac{1}{n} + \frac{1}{2n^2} + \frac{1}{4n^3}\right) = \frac{1}{n} + \frac{1}{n^2} + \frac{1}{2n^3} + \frac{1}{4n^4}$$

für $n=1$: $y_1 = 1 + \frac{1}{6}\left(1 + 2 \cdot 1{,}5 + 2 \cdot \frac{7}{4} + 2{,}75\right)$

$$= 1 + \frac{1}{6} \cdot (10{,}25)$$

$$= \frac{65}{24}$$

$$h = I/n = (x_n - x_0)/n$$
$$x_{k-1} = x_0 + h \cdot (k-1)$$
$$= x_{k-2} + h$$
$$K_1 = h \cdot f(y_{k-1}, x_{k-1})$$
$$K_2 = h \cdot f(y_{k-1} + \frac{K_1}{2}, x_{k-1} + \frac{h}{2})$$
$$K_3 = h \cdot f(y_{k-1} + \frac{K_2}{2}, x_{k-1} + \frac{h}{2})$$
$$K_4 = h \cdot f(y_{k-1} + K_3, x_{k-1} + h)$$
$$y_k = y_{k-1} + \frac{1}{6} \cdot (K_1 + 2 \cdot K_2 + 2 \cdot K_3 + K_4)$$
$$= y_{k-1} + \Delta y_{k-1} .$$

die Eulersche Zahl e unter Verwendung von ei- … -erfahrens vierter Ordnung. Wieviele Schritte des Euler-Verfahrens wären jeweils mindestens nötig, um eine mindestens gleichgute Näherung zu berechnen?

$$y_1 = 1 + \frac{1}{n} + \frac{7}{12n^2} + \frac{1}{8n^3}$$

$$K_1 = \frac{1}{n} \cdot \left(1 + \frac{1}{n} + \frac{7}{12n^2} + \frac{1}{8n^3}\right)$$

$$= \frac{1}{n} + \frac{1}{n^2} + \frac{7}{12n^3} + \frac{1}{8n^4}$$

$$K_2 = \frac{1}{n} \cdot \left(\cancel{1} + \cancel{\frac{1}{n}} + \cancel{\frac{7}{12n^2}} + \cancel{\frac{3}{24n^3}} + \cancel{\frac{1}{2n}} + \cancel{\frac{1}{2n^2}} + \cancel{\frac{7}{24n^3}} + \frac{1}{16n^4}\right)$$

$$= \frac{1}{n} + \frac{3}{2n^2} + \frac{13}{12n^3} + \frac{5}{12n^4} + \frac{1}{16n^5}$$

$$K_3 = \frac{1}{n} \cdot \left(1 + \cancel{\frac{2}{2n}} + \cancel{\frac{7}{12n^2}} + \cancel{\frac{3}{24n^3}} + \cancel{\frac{1}{2n}} + \cancel{\frac{9}{12n^2}} + \cancel{\frac{13}{24n^3}} + \frac{5}{24n^4} + \frac{1}{32n^5}\right)$$

$$= \frac{1}{n} + \frac{3}{2n^2} + \frac{16}{12n^3} + \frac{16}{24n^4} + \frac{5}{24n^5} + \frac{1}{32n^6}$$

$$K_4 = \frac{1}{n} \cdot \left(\cancel{1} + \cancel{\frac{1}{n}} + \cancel{\frac{7}{12n^2}} + \cancel{\frac{3}{24n^3}} + \cancel{\frac{1}{n}} + \cancel{\frac{18}{12n^2}} + \cancel{\frac{32}{24n^3}} + \frac{2}{3n^4} + \frac{5}{24n^5} + \frac{1}{32n^6}\right)$$

$$= \frac{1}{n} + \frac{2}{n^2} + \frac{25}{12n^3} + \frac{35}{24n^4} + \frac{2}{3n^5} + \frac{5}{24n^6} + \frac{1}{32n^7}$$

$$y_2 = 1 + \frac{1}{n} + \frac{7}{12n^2} + \frac{1}{8n^3} + \frac{1}{6}\left(\cancel{\frac{1}{n}} + \cancel{\frac{1}{n^2}} + \cancel{\frac{7}{12n^3}} + \cancel{\frac{3}{24n^4}} + \cancel{\frac{2}{n}} + \cancel{\frac{3}{n^2}} + \cancel{\frac{26}{12n^3}} + \cancel{\frac{20}{24n^4}} + \cancel{\frac{3}{24n^5}}\right.$$
$$+ \cancel{\frac{2}{n}} + \cancel{\frac{3}{n^2}} + \cancel{\frac{32}{12n^3}} + \cancel{\frac{32}{24n^4}} + \cancel{\frac{10}{24n^5}} + \cancel{\frac{2}{48n^6}} + \cancel{\frac{1}{n}} + \cancel{\frac{2}{n^2}} + \cancel{\frac{25}{12n^3}} + \cancel{\frac{35}{24n^4}}$$
$$\left. + \cancel{\frac{16}{24n^5}} + \cancel{\frac{10}{48n^6}} + \frac{1}{32n^7}\right)$$

$$= 1 + \frac{1}{n} + \frac{7}{12n^2} + \frac{1}{8n^3} + \frac{1}{6}\left(\frac{6}{n} + \frac{9}{n^2} + \frac{90}{12n^3} + \frac{90}{24n^4} + \frac{29}{24n^5} + \frac{13}{48n^6} + \frac{1}{32n^7}\right)$$

$$= \cancel{1} + \cancel{\frac{1}{n}} + \cancel{\frac{7}{12n^2}} + \cancel{\frac{1}{8n^3}} + \cancel{\frac{1}{n}} + \cancel{\frac{18}{12n^2}} + \cancel{\frac{15}{8n^3}} + \frac{15}{24n^4} + \frac{29}{144n^5} + \frac{13}{288n^6} + \frac{1}{192n^7}$$

$$= 1 + \frac{2}{n} + \frac{25}{12n^2} + \frac{11}{8n^3} + \frac{15}{24n^4} + \frac{29}{144n^5} + \frac{13}{288n^6} + \frac{1}{192n^7}$$

für $n = 2$

$$y_2 = 2{,}739$$

Euler: $\left(1 + \frac{1}{n}\right)^n$

10.4

$$\frac{d^2y}{dx} \approxeq \frac{y_{k+1}-2y_k+y_{k-1}}{\Delta x^2} \approxeq y_{k+1}-2y_k+y_{k-1}$$

mit $\Delta x = 1$

$f(y) = y^2$

$f'(y) = 2y$

$f''(y) = 2$

$y_{k+1}-2y_k+y_{k-1} = 2$ mit $k+1=k$

$y_k = 2+2y_{k-1}-y_{k-2}$

Einsetzen: $y_{k-2} = 0 = 0^2$

$y_{k-1} = 1 = 1^2$

$y_k = 2+2-0 = 4 = 2^2$

$y_{k+1} = 2+2\cdot 4-1 = 9 = 3^2$

$y_{k+2} = 2+2\cdot 9-4 = 16 = 4^2$

10.1

$$y_n = 3y_{n-1} - 4$$

$$y_6 = -1456$$

$$y_{n-1} = \frac{y_n + 4}{3}$$

$$y_5 = -\frac{1452}{3} = -484$$

$$y_4 = -\frac{480}{3} = -160$$

$$y_3 = -\frac{156}{3} = -52$$

$$y_2 = -\frac{48}{3} = -16$$

$$y_1 = -\frac{12}{3} = -4$$

$$y_0 = -\frac{0}{3} = 0$$

$$L_{a \neq -1}(k) = (-a)^k \cdot C + \frac{b}{1+a}$$

mit $a = -3$, $b = -4$

$$L(k) = 3^k \cdot C + 2$$

$$L(1) = 3C + 2$$

$$-4 = 3C + 2$$

$$-6 = 3C$$

$$C = -2$$

$$\Rightarrow L(k) = 3^k(-2) + 2$$

10.2

S_0 = Kredit zu $t=0$
p Zinssatz

$S_n = S_{n-1}(1+p) - A$

=> Allg Lösung

$L(k) = (-(1+p))^k \cdot C + \frac{A}{1+p}$

$p = 0{,}1 \quad A = 0{,}1 \cdot S_0$

$L(k) = (-1{,}1)^k \cdot C + \frac{0{,}1 S_0}{1{,}1}$

$S_1 = S_0(1{,}1) - S_0 \cdot 0{,}1$

$S_1 = S_0(1{,}1 - 0{,}1) = S_0$

$\Rightarrow S_{10} = S_0$

10.3

$x_n = 3x_{n-1} - 2x_{n-2} + y_{n-1}$ $\qquad$ $y_{n-1} = x_n - 3x_{n-1} + 2x_{n-2}$

$y_n = y_{n-2} - x_{n-1}$ $\qquad$ $y_n = -3x_{n-2} + 2x_{n-3}$

$x_{n-1} = y_{n-2} - y_n$
$x_n = y_{n-1} - y_{n+1}$

$y_{n-1} - y_{n+1} = 3(y_{n-2} - y_n) - 2(y_{n-3} - y_{n-1}) + y_{n-1}$

$y_{n-1} - y_{n+1} = 3y_{n-2} - 3y_n - 2y_{n-3} + 2y_{n-1} + y_{n-1}$

$-y_{n+1} = -2y_{n-3} + 3y_{n-2} + 2y_{n-1} - 3y_n$

$y_n = 2y_{n-4} - 3y_{n-3} - 2y_{n-2} + 3y_{n-1}$

=> 4 Anfangswerte

10.4

$f_y = y^2$

$y' = 2y$

$y'' = 2$

$y'' - 2 = 0$

$\frac{y_n - 2y_{n-1} + y_{n-2}}{(\Delta x)^2} - 2 = 0$

$y_n - 2y_{n-1} + y_{n-2} = 2$

$y_n = 2 + 2y_{n-1} - y_{n-2}$

$y_1 = 1 \quad y_2 = 4$

$y_n = e^{\lambda t}$

10.6

$S_n = S_{n-1} \cdot (1+p) - A \qquad S_n - S_{n-1}(1+p) = -A$

$y' = \frac{y_n - y_{n-1}}{\Delta x} \qquad +(1+p) y_n - (1+p) y_{n-1}$

$y = y_n$

$(1+p) y' - \quad y = -A \qquad$ mit $C = p$