FINAL REPORT FOR THE DATA ANALYSIS MODULE

Chapter 1:

Exercise 1.1:

- $p1 = P(the\ 2nd\ is\ a\ girl \mid the\ first\ is\ a\ girl)$

$= \frac{P(the\ 2nd\ is\ a\ girl,\ \ the\ first\ is\ a\ girl)}{P(the\ first\ is\ a\ girl)}$

$= \frac{P(She\ has\ two\ girls)}{P(the\ first\ is\ a\ girl)}$

$= \frac{0.5*0.5}{0.5}$

$= 0.5$

- Variation :

$p2 = P(the\ first\ is\ a\ boy, the\ 2nd\ is\ a\ girl) + P(the\ first\ is\ a\ girl, the\ 2nd\ is\ a\ girl)$

$= P(she\ has\ a\ boy\ and\ a\ girl) + P(all\ her\ children\ are\ girls)$

$= \left(\frac{1}{2} * \frac{1}{2} + \frac{1}{2} * \frac{1}{2}\right) + \frac{1}{2} * \frac{1}{2}$

$= 0.75$

Exercise 1.2:

Exercise 1.3:

The exact energy will be in the interval $[E - 0.1E, E + 0.1E] = [0.9E, 1.1E]$

Exercise 1.4:

a) With all the symptoms:

$P(Swampfever|Spots, Lethargic, Thirst, Sneezing) = \frac{P(Sp,L,T,S|SF) \cdot P(SF)}{P(Sp,L,T,S)}$

$= \frac{P(Sp|SF) \cdot P(L|SF) \cdot P(T|SF) \cdot P(S|SF) \cdot P(SF)}{P(Sp|SF) \cdot P(L|SF) \cdot P(T|SF) \cdot P(S|SF) \cdot P(SF) + (1 - P(SF)) \cdot P(Sp|\bar{SF}) \cdot P(L|\bar{SF}) \cdot P(T|\bar{SF}) \cdot P(S|\bar{SF})}$

$= \mathbf{0.8}$

b) With three out of four symptoms:

$P(Swampfever|Spots, Lethargic, Thirst, WithoutSneezing) = \frac{P(Sp,L,T,\bar{S}|SF) \cdot P(SF)}{P(Sp,L,T,\bar{S})}$

$= \frac{P(Sp|SF) \cdot P(L|SF) \cdot P(T|SF) \cdot P(\bar{S}|SF) \cdot P(SF)}{P(Sp|SF) \cdot P(L|SF) \cdot P(T|SF) \cdot P(\bar{S}|SF) \cdot P(SF) + (1 - P(SF)) \cdot P(Sp|\bar{SF}) \cdot P(L|\bar{SF}) \cdot P(T|\bar{SF}) \cdot P(\bar{S}|\bar{SF})}$

$= \mathbf{0.46}$

Chapter 2:

Exercise 2.8:

a) $E[X] = \int_{-\infty}^{+\infty} xP(x)dx = \int_{0}^{+\infty} x^2e^{-x}dx$

$= [-x^2e^{-x}]_0^{+\infty} + \int_0^{+\infty} 2xe^{-x}dx$

$= 0 + [-2xe^{-x}]_0^{+\infty} + \int_0^{+\infty} 2e^{-x}dx$

$= 0 + 0 + 2[-e^{-x}]_0^{+\infty}$

$= 2$

$Var(X) = \int_{-\infty}^{+\infty} x^2P(x)dx - \ 2^2$

$= \int_0^{+\infty} x^3e^{-x}dx - \ 4$

$= [-x^3e^{-x}]_0^{+\infty} + \int_0^{+\infty} 3x^2e^{-x}dx - 4$

$= 0 + 3 \times 2 - 4$

$= 2$

$std(X) = \ \sqrt{2}$

$p = \int_{2-\sqrt{2}}^{2+\sqrt{2}} xe^{-x}dx = [-xe^{-x}]_{2-\sqrt{2}}^{2+\sqrt{2}} + \int_{2-\sqrt{2}}^{2+\sqrt{2}} e^{-x}dx$

$= -(2+\sqrt{2})e^{-2-\sqrt{2}} + (2-\sqrt{2})e^{-2+\sqrt{2}} - e^{-2-\sqrt{2}} + e^{-2+\sqrt{2}}$

$= \left[(3-\sqrt{2})e^{\sqrt{2}} - (3+\sqrt{2})e^{-\sqrt{2}}\right]e^{-2}$

$= \left[3 * 2sh(\sqrt{2}) - 2\sqrt{2}sh(\sqrt{2})\right]e^{-2}$

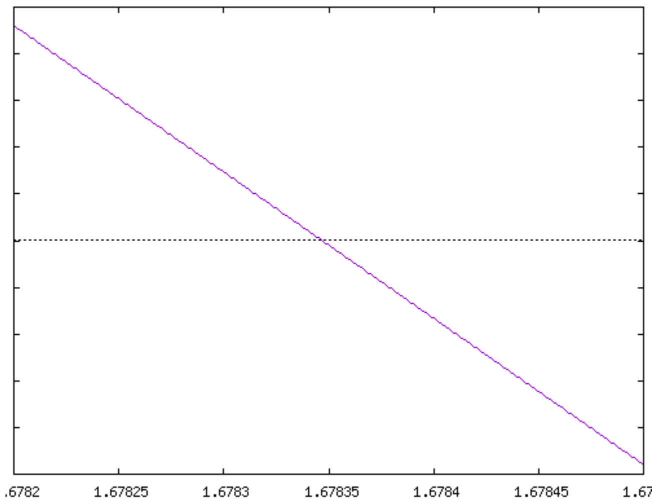
$= (6-2\sqrt{2})sh(\sqrt{2})e^{-2}$

b) $P(X \leq x) = 1 - \int_x^{+\infty} te^{-t}dt = 1 - \left([-te^{-t}]_x^{+\infty} + \int_x^{+\infty} e^{-t}dt\right)$

$= 1 - xe^{-x} - e^{-x}$

$= 1 - (1+x)e^{-x}$

$P(X \leq m) = 0.5 \Rightarrow (1+m)e^{-m} = 0.5 \Rightarrow m \approx 1.67835$



$f(x) = (1+x)e^{-x} - 0.5$

r	$P(r)$	F(r)	Rank
0	0	0	7
1	0.36788	0.36788	1
2	0.27067	0.63855	2
3	0.14936	0.78791	3
4	0.07326	0.86117	4
5	0.03369	0.89486	5

6	0.01487	0.90973	6

$1-\alpha = 0.68 \;=>\; \alpha = 0.32 \;=>\; {}^{\alpha}/_{2} = 0.16$

R1=1

R2=4

Then the Central Interval is {1,2,3,4}.

c) $P'(x^*) = 0 => -x^*e^{-x^*} + e^{-x^*} = 0$

$=> x^* = 1$ then the mode is 1

The Smallest Interval is {1,2,3}.

Exercise 2.10:

a) For each energy, $\hat{p} = \frac{Successes}{Trials}$

$E = 0.5 => \hat{p} = 0;\ 1-\hat{p} = 1$
$E = 1 \quad => \hat{p} = 0.04;\ 1-\hat{p} = 0.96$
$E = 1.5 => \hat{p} = 0.2;\ 1-\hat{p} = 0.8$
$E = 2 \quad => \hat{p} = 0.58;\ 1-\hat{p} = 0.42$
$E = 2.5 => \hat{p} = 0.92;\ 1-\hat{p} = 0.08$
$E = 3 \quad => \hat{p} = 0.987;\ 1-\hat{p} = 0.013$
$E = 3.5 => \hat{p} = 0.995;\ 1-\hat{p} = 0.005$
$E = 4 \quad => \hat{p} = 0.998;\ 1-\hat{p} = 0.002$

b) 68 % probability range:

Exercise 2.11:

$P(r|N,p) = \frac{N!}{r!(N-r)!}p^r(1-p)^{N-r}$

$Energie = 0.5$	r	$P(r\|N=100, p=0)$	$F(r\|N=100, p=0)$	Rank	CLI
	0	1	1	1	{0}
$Energie = 1$	r	$P(r\|N=100, p=0.04)$	$F(r\|N=100, p=0.04)$	Rank	CLI
	0	0.017	0.017	8	{4,5,6,7,8}
	1	0.07	0.087	6	
	2	0.145	0.232	4	
	3	0.197	0.429	2	
	4	0.199	0.628	1	
	5	0.159	0.787	3	
	6	0.105	0.892	5	
	7	0.058	0.95	7	
$Energie = 1.5$	r	$P(r\|N=100, p=0.2)$	$F(r\|N=100, p=0.2)$	Rank	CLI
	0	2e-10	2e-10		[20,…,28]
	13	0.021	0.021	15	
	14	0.033	0.054	12	

	15	0.048	0.102	10	
	16	0.064	0.166	8	
	17	0.079	0.245	6	
	18	0.09	0.335	4	
	19	0.098	0.433	2	
	20	0.099	0.532	1	
	21	0.094	0.627	3	
	22	0.085	0.712	5	
	23	0.072	0.784	7	
	24	0.057	0.841	9	
	25	0.044	0.885	11	
	26	0.032	0.917	13	
	27	0.022	0.939	14	
$Energie = 2$	r	$P(r\|N = 100, p = 0.58)$	$F(r\|N = 100, p = 0.58)$	Rank	CLI
	0	0	0		[58,…,67]
	52	0.038	0.038	13	
	53	0.048	0.086	11	
	54	0.057	0.143	9	
	55	0.066	0.209	7	
	56	0.074	0.283	5	
	57	0.078	0.361	3	
	58	0.08	0.441	1	
	59	0.079	0.52	2	
	60	0.075	0.595	4	
	61	0.068	0.663	6	
	62	0.058	0.721	8	
	63	0.049	0.77	10	
	64	0.039	0.809	12	
	65	0.03	0.839	14	
	51	0.029	0.0868	15	
	66	0.022	0.89	16	
	50	0.0217	0.911	17	

Exercise 2.13:

a)

b) Expectation value and variance:

$$P_n(p|N,r) = \frac{(nN+1)!}{(nr)!(nN-nr)!} p^{nr}(1-p)^{nN-nr}$$

$$E = \int_0^1 \frac{(nN+1)!}{(nr)!(nN-nr)!} p^{nr+1}(1-p)^{nN-nr} dp$$

$$= \frac{(nN+1)!}{(nr)!(nN-nr)!} \int_0^1 p^{nr+2-1}(1-p)^{nN-nr+1-1} dp$$

$$= \frac{(nN+1)!}{(nr)!(nN-nr)!} Beta(nr+2, nN-nr+1)$$

$$= \frac{(nN+1)!}{(nr)!(nN-nr)!} \frac{\Gamma(nr+2)\Gamma(nN-nr+1)}{\Gamma(nN+3)}$$

$$= \frac{(nN+1)!}{(nr)!(nN-nr)!}\frac{(nr+1)!\,(nN-nr)!}{(nN+2)!}$$
$$= \boldsymbol{\frac{nr+1}{nN+2} \xrightarrow[n->\infty]{} \frac{r}{N}}$$

$$Var = E[p^2] - (E[p])^2$$
$$= \int_0^1 \frac{(nN+1)!}{(nr)!(nN-nr)!} p^{nr+2}(1-p)^{nN-nr}dp - \left(\frac{nr+1}{nN+2}\right)^2$$
$$= \frac{(nN+1)!}{(nr)!(nN-nr)!} Beta(nr+3, nN-nr+1) - \left(\frac{nr+1}{nN+2}\right)^2$$
$$= \frac{(nN+1)!}{(nr)!(nN-nr)!}\frac{\Gamma(nr+3)\Gamma(nN-nr+1)}{\Gamma(nN+3)} - \left(\frac{nr+1}{nN+2}\right)^2$$
$$= \frac{(nN+1)!}{(nr)!(nN-nr)!}\frac{(nr+2)!\,(nN-nr)!}{(nN+2)!} - \left(\frac{nr+1}{nN+2}\right)^2$$
$$= \frac{(nr+1)(nr+2)}{nN+2} - \left(\frac{nr+1}{nN+2}\right)^2$$
$$= \frac{(nr+1)[(nr+2)(nN+2)-(nr+1)]}{(nN+2)^2} = \frac{(nr+1)\left[n^2rN+2nr+2nN+4-nr-1\right]}{(nN+2)^2}$$
$$= \boldsymbol{\frac{(nr+1)\left(n^2rN+nr+2nN+3\right)}{(nN+2)^2} \sim \frac{n^3r^2N}{n^2N^2} \xrightarrow[n->\infty]{}} +\infty$$

Chapter 3:

Exercise 3.4:

a) $E[X] = \int_{-\infty}^{+\infty} xP(x)dx = \frac{1}{2}\int_{-\infty}^{0} xe^{x}dx + \frac{1}{2}\int_{0}^{+\infty} xe^{-x}dx$

$= \frac{1}{2}\left([xe^{x}]_{-\infty}^{0} - \int_{-\infty}^{0} e^{x}dx + [-xe^{-x}]_{0}^{+\infty} + \int_{0}^{+\infty} e^{-x}dx\right)$

$= \frac{1}{2}\left(0 - [e^{x}]_{-\infty}^{0} + 0 + [-e^{-x}]_{0}^{+\infty}\right)$

$= \frac{1}{2}(-1 + 1)$

$= 0$

$Var(X) = \int_{-\infty}^{+\infty} x^2P(x)dx - 0^2$

$= \frac{1}{2}\int_{-\infty}^{0} x^2e^{x}dx + \frac{1}{2}\int_{0}^{+\infty} x^2e^{-x}dx$

$= \frac{1}{2}\left([x^2e^{x}]_{-\infty}^{0} + \int_{-\infty}^{0} 2xe^{x}dx + [-x^2e^{-x}]_{0}^{+\infty} + \int_{0}^{+\infty} 2xe^{-x}dx\right)$

$= \frac{1}{2}\left(0 + [2xe^{x}]_{-\infty}^{0} + \int_{-\infty}^{0} 2e^{x}dx + 0 + [-2xe^{-x}]_{0}^{+\infty} + \int_{0}^{+\infty} 2e^{-x}dx\right)$

$= \frac{1}{2}\left(0 + 0 + 2[e^{x}]_{-\infty}^{0} + 0 + 0 + 2[-e^{-x}]_{0}^{+\infty}\right)$

$= \frac{1}{2}(2 + 2)$

$= 2$

$std(X) = \sqrt{2}$

b) We can see that $P_{max} = \frac{1}{2}$

$P(x) = \frac{1}{4} \Rightarrow \frac{1}{2}e^{-|x|} = \frac{1}{4}$

$\Rightarrow -|x| = \ln\left(\frac{1}{2}\right)$

$\Rightarrow x = -\ln(2)\ or\ x = \ln(2)$

$\Rightarrow \mathrm{FWHM} = 2\ln(2)$

$\Rightarrow std(X) > \mathrm{FWHM}$

c) $p = \int_{-\sqrt{2}}^{\sqrt{2}} e^{-|x|}dx = 2\int_{0}^{\sqrt{2}} e^{-x}dx$

$= 2[-e^{-x}]_{0}^{\sqrt{2}}$

$= 2(1 - e^{-\sqrt{2}})$

Exercise 3.7:

a) $Flat\ prior\ then\ p(\nu|X = 9) \propto p(X = 9|\nu)$

$p(X = 9|\nu) = \frac{\nu^9}{9!}e^{-\nu}$

ν	$P(\nu\|X = 9)$	$\mathrm{F}(\nu\|X = 9)$
0	0	0
1	1.013e-6	1.013e-6
2	1.091e-4	1.101e-4

3	2.7e-3	2.81e-3
4	1.32e-2	1.701e-2
5	3.62e-2	5.321e-2
6	6.9e-2	1.222e-1
7	0.101	0.223
8	0.124	0.347
9	0.131	0.478
10	0.125	0.603
11	0.108	0.711
12	0.087	0.798
13	0.066	0.864
14	0.047	0.911
15	0.032	0.943
16	0.021	0.964
17	0.013	0.977

Then according to this table the 95 % probability lower limit on ν is **16**.

b) $P'(\nu) = 0 \Rightarrow \frac{1}{9!}(9\nu^8 e^{-\nu} - \nu^9 e^{-\nu}) = 0$

$\Rightarrow 9 - \nu = 0$

$\Rightarrow \nu = 9$ then the mode is **9.**

The Smallest Interval is **{9,10,11}**.

Exercise 3.8:

Exercise 3.13:

Exercise 3.16:

X is a Poisson distribution like N. It's mean is:

$$E[X] = E[\sum_{n=1}^{N} X_n] = E\big[E[\sum_{n=1}^{N} X_n \,|N]\big]$$

$$= E_N[\sum_{n=1}^{N} E[X_n]]$$

$$= E_N\big[\sum_{n=1}^{N}\big(1 \times p + 0 \times (1-p)\big)\big]$$

$$= E_N[p \times N]$$

$$= p \times E_N[N] = p \times \nu = \nu p$$

Then $P(X) = \frac{(\nu p)^X}{X!} e^{-\nu p}$

Chapter 4:

Exercise 4.8:

a) $p(X) = \prod_{i=1}^{n} p(x_i) = \prod_{i=1}^{n} \lambda e^{-\lambda x_i}$

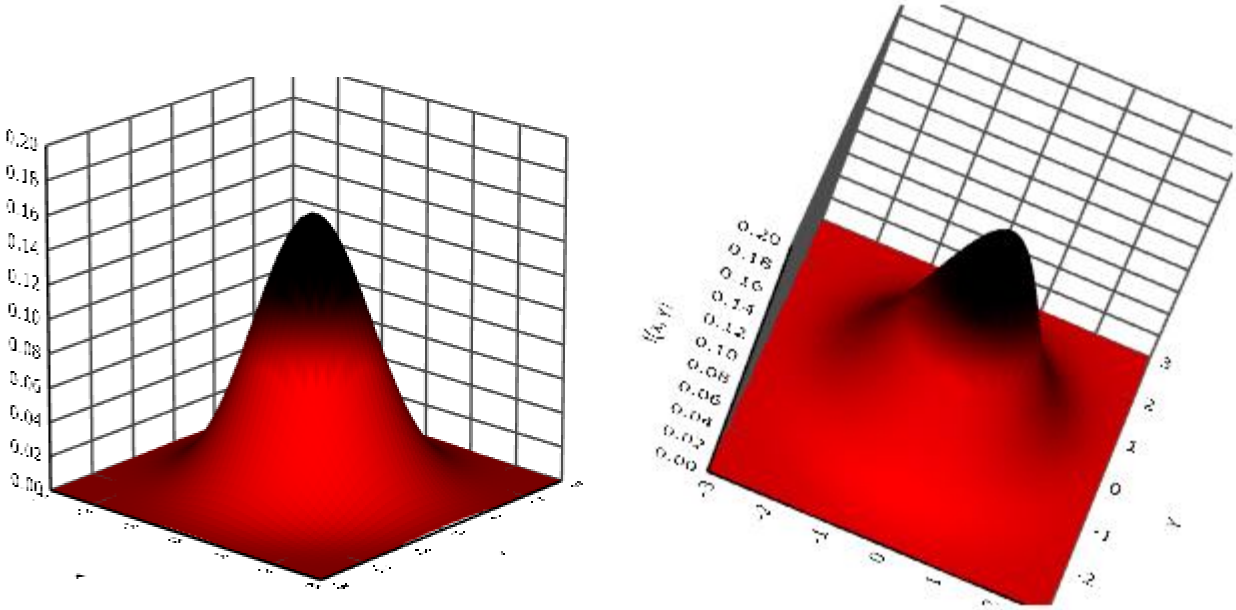
$log\big(p(X)\big) = \sum_{i=1}^{n}(-\lambda x_i + \log(\lambda)) = n\log(\lambda) - \lambda\sum_{i=1}^{n} x_i$

$0 = \frac{\partial p}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i \;\; => \;\; \boldsymbol{\hat{\lambda}} = \frac{\boldsymbol{n}}{\boldsymbol{\sum_{i=1}^{n} x_i}}$

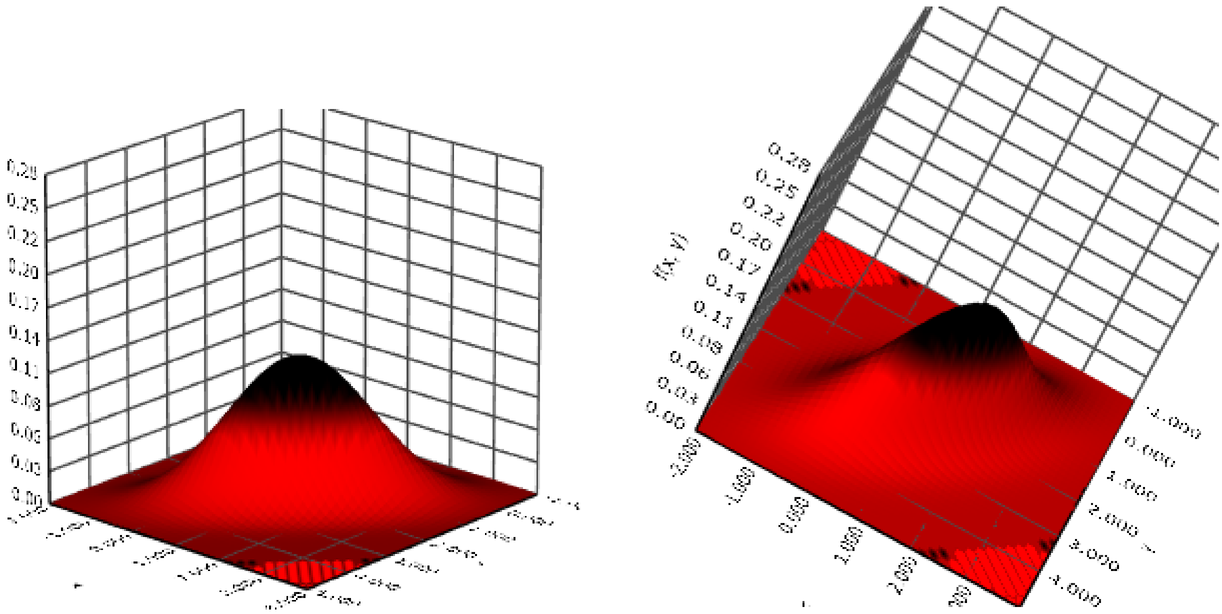
b)

Exercise 4.11:

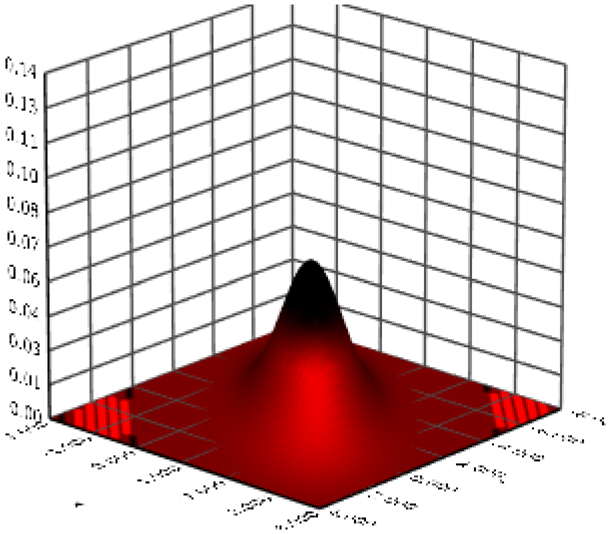
a)

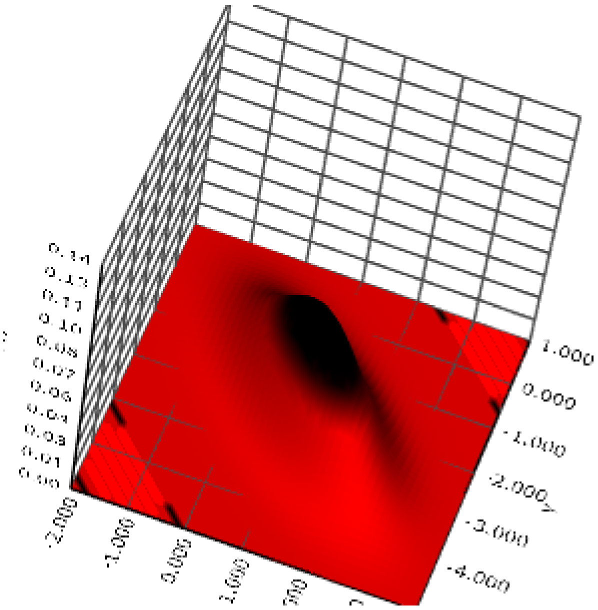


b)



c)





<u>Exercise 4.12:</u>

a) $p(x,y) = \frac{1}{2\pi\sqrt{|\Sigma|}} e^{\frac{-1}{2}X^T\Sigma^{-1}X}$

$$|\Sigma| = \sigma_x^2\sigma_y^2 - \sigma_{xy}^2 = \sigma_x^2\sigma_y^2\left(1 - \frac{\sigma_{xy}^2}{\sigma_x^2\sigma_y^2}\right) = \sigma_x^2\sigma_y^2(1-\rho^2)$$

$$\begin{aligned} X^T\Sigma^{-1}X &= \frac{1}{|\Sigma|}\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} \sigma_y^2 & -\sigma_{xy} \\ -\sigma_{xy} & \sigma_x^2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \\ &= \frac{1}{|\Sigma|}\begin{pmatrix} x\sigma_y^2 - y\sigma_{xy} & -x\sigma_{xy} + y\sigma_x^2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \\ &= \frac{1}{|\Sigma|}\left(x^2\sigma_y^2 - xy\sigma_{xy} - xy\sigma_{xy} + y^2\sigma_x^2\right) \\ &= \frac{1}{(1-\rho^2)}\frac{x^2\sigma_y^2 + y^2\sigma_x^2 - 2xy\sigma_{xy}}{\sigma_x^2\sigma_y^2} \\ &= \frac{1}{(1-\rho^2)}\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} - \frac{2xy\rho\sigma_x\sigma_y}{\sigma_x^2\sigma_y^2}\right) = \frac{1}{1-\rho^2}\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} - \frac{2xy\rho}{\sigma_x\sigma_y}\right) \end{aligned}$$

Then:

$$p(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} - \frac{2xy\rho}{\sigma_x\sigma_y}\right)\right)$$

b) $z = x - y$

$E[z] = E[x-y] = E[x] - E[y]$ because of the linearity of E

$= \mu_x - \mu_y$

$$\begin{aligned} Var(z) &= Var(x) + Var(y) + 2cov(x,-y) \\ &= Var(x) + Var(y) - 2cov(x,y) \\ &= \sigma_x^2 + \sigma_y^2 - 2\rho\sigma_x\sigma_y \end{aligned}$$

Exercise 4.13:

$$
\begin{aligned}
p(y) &= \int_{-\infty}^{+\infty} p(y|x)p(x)dx \\
&= \frac{1}{2\pi\sigma_x\sigma_y}\int_{-\infty}^{+\infty} \exp\left(-\frac{(x-x_0)^2}{2\sigma_x^2}-\frac{(y-x)^2}{2\sigma_y^2}\right)dx \\
&= \frac{1}{2\pi\sigma_x\sigma_y}\int_{-\infty}^{+\infty} \exp\left(-\frac{x^2\sigma_y^2+{x_0}^2\sigma_y^2-2xx_0\sigma_y^2}{2\sigma_x^2\sigma_y^2}-\frac{y^2\sigma_x^2+x^2\sigma_x^2-2xy\sigma_x^2}{2\sigma_x^2\sigma_y^2}\right)dx \\
&= \frac{1}{2\pi\sigma_x\sigma_y}\exp\left(-\frac{{x_0}^2\sigma_y^2+y^2\sigma_x^2}{2\sigma_x^2\sigma_y^2}\right)\int_{-\infty}^{+\infty} \exp\left(-\frac{x^2\sigma_y^2-2xx_0\sigma_y^2+x^2\sigma_x^2-2xy\sigma_x^2}{2\sigma_x^2\sigma_y^2}\right)dx \\
&= \frac{1}{2\pi\sigma_x\sigma_y}\exp\left(-\frac{{x_0}^2\sigma_y^2+y^2\sigma_x^2}{2\sigma_x^2\sigma_y^2}\right)\int_{-\infty}^{+\infty} \exp\left(-\frac{(\sigma_x^2+\sigma_y^2)x^2-2(x_0\sigma_y^2+y\sigma_x^2)x}{2\sigma_x^2\sigma_y^2}\right)dx \\
&= \frac{1}{2\pi\sigma_x\sigma_y}\exp\left(-\frac{{x_0}^2\sigma_y^2+y^2\sigma_x^2}{2\sigma_x^2\sigma_y^2}\right)\int_{-\infty}^{+\infty} \exp\left(-\frac{\sigma_x^2+\sigma_y^2}{2\sigma_x^2\sigma_y^2}\left(x^2-2\frac{x_0\sigma_y^2+y\sigma_x^2}{\sigma_x^2+\sigma_y^2}x\right)\right)dx \\
&= \frac{1}{2\pi\sigma_x\sigma_y}\exp\left(-\frac{{x_0}^2\sigma_y^2+y^2\sigma_x^2}{2\sigma_x^2\sigma_y^2}\right)\int_{-\infty}^{+\infty} \exp\left(-\frac{\sigma_x^2+\sigma_y^2}{2\sigma_x^2\sigma_y^2}(\left(x-\frac{x_0\sigma_y^2+y\sigma_x^2}{\sigma_x^2+\sigma_y^2}\right)^2-(\frac{x_0\sigma_y^2+y\sigma_x^2}{\sigma_x^2+\sigma_y^2})^2)\right)dx \\
&= \frac{1}{2\pi\sigma_x\sigma_y}\exp\left(-\frac{{x_0}^2\sigma_y^2+y^2\sigma_x^2}{2\sigma_x^2\sigma_y^2}\right)\exp\left(\frac{(x_0\sigma_y^2+y\sigma_x^2)^2}{2\sigma_x^2\sigma_y^2(\sigma_x^2+\sigma_y^2)}\right)\int_{-\infty}^{+\infty} \exp\left(-\frac{\sigma_x^2+\sigma_y^2}{2\sigma_x^2\sigma_y^2}\left(x-\frac{x_0\sigma_y^2+y\sigma_x^2}{\sigma_x^2+\sigma_y^2}\right)^2\right)dx \\
&= \frac{1}{2\pi\sigma_x\sigma_y}\exp\left(\frac{(x_0\sigma_y^2+y\sigma_x^2)^2}{2\sigma_x^2\sigma_y^2(\sigma_x^2+\sigma_y^2)}-\frac{{x_0}^2\sigma_y^2+y^2\sigma_x^2}{2\sigma_x^2\sigma_y^2}\right)\int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2\frac{\sigma_x^2\sigma_y^2}{\sigma_x^2+\sigma_y^2}}\left(x-\frac{x_0\sigma_y^2+y\sigma_x^2}{\sigma_x^2+\sigma_y^2}\right)^2\right)dx \\
&= \frac{1}{2\pi\sigma_x\sigma_y}\exp\left(\frac{(x_0\sigma_y^2+y\sigma_x^2)^2-({x_0}^2\sigma_y^2+y^2\sigma_x^2)(\sigma_x^2+\sigma_y^2)}{2\sigma_x^2\sigma_y^2(\sigma_x^2+\sigma_y^2)}\right)\sqrt{2\pi\frac{\sigma_x^2\sigma_y^2}{\sigma_x^2+\sigma_y^2}} \\
&= \frac{1}{\sqrt{2\pi(\sigma_x^2+\sigma_y^2)}}\exp\left(\frac{{x_0}^2\sigma_y^4+y^2\sigma_x^4+2x_0y\sigma_y^2\sigma_x^2-({x_0}^2\sigma_y^2\sigma_x^2+{x_0}^2\sigma_y^4+y^2\sigma_x^4+y^2\sigma_x^2\sigma_y^2)}{2\sigma_x^2\sigma_y^2(\sigma_x^2+\sigma_y^2)}\right) \\
&= \frac{1}{\sqrt{2\pi(\sigma_x^2+\sigma_y^2)}}\exp\left(\frac{2x_0y\sigma_y^2\sigma_x^2-({x_0}^2\sigma_y^2\sigma_x^2+y^2\sigma_x^2\sigma_y^2)}{2\sigma_x^2\sigma_y^2(\sigma_x^2+\sigma_y^2)}\right) \\
&= \frac{1}{\sqrt{2\pi(\sigma_x^2+\sigma_y^2)}}\exp\left(\frac{-(x_0-y)^2\sigma_x^2\sigma_y^2}{2\sigma_x^2\sigma_y^2(\sigma_x^2+\sigma_y^2)}\right) \\
&= \frac{1}{\sqrt{2\pi(\sigma_x^2+\sigma_y^2)}}\exp\left(-\frac{(y-x_0)^2}{2(\sigma_x^2+\sigma_y^2)}\right) \qquad \text{then} \quad y\sim N(x_0,\sigma_x^2+\sigma_y^2)
\end{aligned}
$$

Exercise 4.14:

Chapter 5:

Exercise 5.1:

1) We see that our efficiency is about 50 % at E = 2. Then $E_0 = 2$.

$s(E|A, E_0) = \frac{1}{1+e^{-A(E-E_0)}}$

We can also use the fact that the efficiency changes by about 25 % when we move away from E = 2 by roughly 0.5 units.

$\frac{ds}{dE} = \frac{Ae^{-A(E-E_0)}}{(1+e^{-A(E-E_0)})^2} = \frac{A}{4}$ $for\ E = E_0$. Then: $0.5\ \frac{A}{4} = 0.25 => A = 2$.

$$P(r|A, E_0) = \prod_{i=1}^{8} \binom{N_i}{r_i} s(E_i|A, E_0)^{r_i}(1 - s(E_i|A, E_0))^{N_i - r_i}$$
$$= \prod_{i=1}^{8} \binom{N_i}{r_i} \left(\frac{1}{1+e^{-A(E-E_0)}}\right)^{r_i} \left(\frac{1}{1+e^{A(E-E_0)}}\right)^{N_i - r_i}$$
$$= \prod_{i=1}^{8} \binom{N_i}{r_i} \left(1 + e^{-A(E-E_0)}\right)^{-r_i} \left(1 + e^{A(E-E_0)}\right)^{-N_i + r_i}$$
$$= \binom{100}{0} \left(\frac{1}{1+e^{A(0.5-E_0)}}\right)^{100} \binom{100}{4} \left(\frac{1}{1+e^{-A(1-E_0)}}\right)^{4} \left(\frac{1}{1+e^{A(1-E_0)}}\right)^{96} \binom{100}{22} \left(\frac{1}{1+e^{-A(1.5-E_0)}}\right)^{22} \left(\frac{1}{1+e^{A(1.5-E_0)}}\right)^{78} \cdots$$
$$= \binom{100}{0} (\frac{1}{1+e^{A(0.5-E_0)}})^{100} \binom{100}{4} (\frac{1}{1+e^{-A(1-E_0)}})^{4} (\frac{1}{1+e^{A(1-E_0)}})^{96} \binom{100}{22} (\frac{1}{1+e^{-A(1.5-E_0)}})^{22} (\frac{1}{1+e^{A(1.5-E_0)}})^{78} \cdots$$

2)

Exercise 5.2:

1) We see that our efficiency is about 50 % at E = 2. Then $E_0 = 2$.

$s(E|A, E_0) = sin\big(A(E - E_0)\big)$

We can also use the fact that the efficiency changes by about 25 % when we move away from E = 2 by roughly 0.5 units.

$\frac{ds}{dE} = A\ cos\big(A(E - E_0)\big) = A$ $for\ E = E_0$. Then: $0.5\ A = 0.25 => A = 0.5$.

$$P(r|A, E_0) = \prod_{i=1}^{8} \binom{N_i}{r_i} s(E_i|A, E_0)^{r_i}(1 - s(E_i|A, E_0))^{N_i - r_i}$$
$$= \prod_{i=1}^{8} \binom{N_i}{r_i} \Big(sin\big(A(E_i - E_0)\big)\Big)^{r_i} \Big(1 - sin\big(A(E_i - E_0)\big)\Big)^{N_i - r_i}$$

2)

Exercise 5.3:

$f(\chi^2) = \frac{1}{\sqrt{2\pi\chi^2}} e^{\frac{-\chi^2}{2}}$

$$E[\chi^2] = \int_0^{+\infty} \frac{1}{\sqrt{2\pi\chi^2}} \chi^2 e^{\frac{-\chi^2}{2}} d\chi^2 = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} \sqrt{\chi^2} e^{\frac{-\chi^2}{2}} d\chi^2$$
$$= \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} u^{1/2} e^{\frac{-u}{2}} du = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} (2t)^{\frac{3}{2}-1} e^{\frac{-2t}{2}} \times 2dt$$
$$= \frac{2}{\sqrt{\pi}} \int_0^{+\infty} t^{\frac{3}{2}-1} e^{-t} dt = \frac{2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right) = \frac{2}{\sqrt{\pi}} \times \frac{\sqrt{\pi}}{2}$$
$$= \mathbf{1}$$

$$Var(\chi^2) = E[(\chi^2)^2] - (E[\chi^2])^2$$

$$= \int_0^{+\infty} \frac{1}{\sqrt{2\pi\chi^2}} \chi^4 e^{\frac{-\chi^2}{2}} d\chi^2 - 1^2$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} (\chi^2)^{\frac{3}{2}} e^{\frac{-\chi^2}{2}} d\chi^2 - 1 = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} u^{\frac{3}{2}} e^{\frac{-u}{2}} du - 1$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} (2t)^{\frac{5}{2}-1} e^{\frac{-2t}{2}} \times 2dt - 1$$

$$= \frac{2^{\frac{3}{2}+1-\frac{1}{2}}}{\sqrt{\pi}} \int_0^{+\infty} t^{\frac{5}{2}-1} e^{\frac{-2t}{2}} dt - 1$$

$$= \frac{4}{\sqrt{\pi}} \Gamma\left(\frac{5}{2}\right) - 1 = \frac{4}{\sqrt{\pi}} \frac{3\sqrt{\pi}}{4} - 1$$

$$= \mathbf{2}$$

$$f(u) = \frac{1}{\sqrt{2\pi}} u^{\frac{-1}{2}} e^{\frac{-u}{2}} \;\; => f'(u) = \frac{1}{\sqrt{2\pi}} \left(-\frac{1}{2} u^{\frac{-3}{2}} e^{\frac{-u}{2}} - \frac{1}{2} u^{\frac{-1}{2}} e^{\frac{-u}{2}} \right) = -\frac{1}{2\sqrt{2\pi}} (1+u) u^{\frac{-3}{2}} e^{\frac{-u}{2}} \; < 0$$

Then f is decreasing. Moreover $\lim_{u \to 0} f(u) = +\infty$. Then $\max_{u>0} f(u) = +\infty$ and **then** $\boldsymbol{u^* = 0}$.

Last set of exercises:

Exercise 1:

a) $\ln\big(P(x|p)\big) = x ln(p) + (1-x)\ln(1-p)$

$$\frac{\partial^2 \ln\big(P(x|p)\big)}{\partial p^2} = \frac{\partial}{\partial p}\left(\frac{x}{p} - \frac{1-x}{1-p}\right) = \frac{-x}{p^2} + \frac{1-x}{(1-p)^2}$$

$$I(p) = -(0 + 1 \times \ln(p))$$

$$\boldsymbol{I(p) = -ln(p)}$$

b) $L(p) = \ln(\prod_{i=1}^n P(x_i|p))$

$$= \sum_{i=1}^n \ln\big(P(x_i|p)\big) = \sum_{i=1}^n x_i ln(p) + (1-x_i)\ln(1-p)$$

$$= ln(p) \sum_{i=1}^n x_i + \ln(1-p) \sum_{i=1}^n (1-x_i)$$

$$= ln(p) \sum_{i=1}^n x_i + \ln(1-p)(n - \sum_{i=1}^n x_i)$$

$$\frac{\partial L(p)}{\partial p} = \frac{1}{p} \sum_{i=1}^n x_i - \frac{1}{1-p}(n - \sum_{i=1}^n x_i)$$

$$= \left(\frac{1}{p} + \frac{1}{1-p}\right) \sum_{i=1}^n x_i - \frac{n}{1-p}$$

$$= \frac{1}{p(1-p)} \sum_{i=1}^n x_i - \frac{np}{p(1-p)}$$

$$\frac{\partial L(p)}{\partial p} = 0 => \sum_{i=1}^n x_i - n\hat{p} = 0$$

$$=> \boldsymbol{\hat{p} = \frac{1}{n} \sum_{i=1}^n x_i}$$

c) $E[\hat{p} - p_0] = E\left[\frac{1}{n} \sum_{i=1}^n x_i\right] - p_0 = \frac{1}{n} \sum_{i=1}^n E[x_i] - p_0$

$$= \frac{1}{n} \mathrm{n} \times p_0 - p_0$$

$$= \mathbf{0}$$