Chapter 03 - Exercise 13

$f(x|\lambda)$ is a normalized probability density on the interval $[a, b]$ and depends on one parameter, λ:

$$0 \leq f \leq 1, \qquad \int_a^b f(x|\lambda)\mathrm{d}x = 1 \tag{0.1}$$

Binning: The interval [a,b] is divided into K subintervals (bins), Δ_j.

$$\Delta_j = \left[a + (j-1)\frac{b-a}{K},\, a + j\frac{b-a}{K}\right] \tag{0.2}$$

v_j: expectation of bin j.

$$v_j = \int_{\Delta_j} f(x|\lambda)\mathrm{d}x \tag{0.3}$$

n events with values x_i:

$$x_i \in [a, b] \qquad i = 1, ..., n \tag{0.4}$$

n_j is the number of events in bin j. For very fine binning:

$$n_j \in \{0, 1\} \tag{0.5}$$

$$n_j \in \{0, 1\} \Rightarrow n_j! = 1 \tag{0.6}$$

From the exercise description:
"where Δ is the size of the interval in x assumed fixed for all j":

$$\lim_{K\to\infty} \Delta_j = \Delta \qquad , \Delta \text{ very small} \tag{0.7}$$

Here, Δ_j is the width of bin j.

$$\lim_{K\to\infty} v_j = \lim_{K\to\infty} \int_{\Delta_j} f(x|\lambda)\mathrm{d}x \approx \Delta \cdot f(x_j|\lambda) \tag{0.8}$$

$$f(x_j|\lambda) = f_j \tag{0.9}$$

$$\exp(x) \approx 1 + x \qquad \forall \quad |x| \ll 1 \tag{0.10}$$

$$
\begin{aligned}
\lim_{K\to\infty} \prod_{j=1}^{K} \frac{\mathrm{e}^{-v_j} v_j^{n_j}}{n_j!} &= \lim_{K\to\infty} \prod_{j=1}^{K} \mathrm{e}^{-v_j} v_j^{n_j} && (0.11)\\
&= \prod_{j=1}^{\infty} \mathrm{e}^{-f_j\Delta} (f_j\Delta)^{n_j} \qquad \mid \exp(x) \approx 1 + x,\ f_j\Delta \ll 1 && (0.12)\\
&= \prod_{j=1}^{\infty} (1 - f_j\Delta) \cdot (f_j\Delta)^{n_j} && (0.13)\\
&= \prod_{j=1}^{\infty} (f_j\Delta)^{n_j} - (f_j\Delta)^{n_j+1} \qquad |(f_j\Delta)^{n+1} \ll (f_j\Delta)^{n} && (0.14)\\
&= \prod_{j=1}^{\infty} (f_j\Delta)^{n_j} \qquad |(f_j\Delta)^{n_j} = 1 \quad \forall\, n_j = 0 && (0.15)\\
&= \prod_{i=1}^{n} f_i\Delta = \prod_{i=1}^{n} f(x_i|\lambda)\Delta && (0.16)
\end{aligned}
$$

The iteration index, i, goes over the measured values x_i.