1. Let X be the observed sequence. Define Y as the true sequence, including the random leading nucleotides, barcode, and sample sequence, and let Y_j be the jth true nucleotide sequenced as observed nucleotide X_j. We are given that $z_{i2} \in \{0, 1, 2, 3\}$, and that this corresponds to the realization of the nucleotides at positions 35 and 42. We enumerate the possible combinations of the bases at positions 35 and 42 in the following way:

z_{i2}	Y	R
0	C	A
1	C	G
2	T	A
3	T	G

We can convert the value of z_{i2} to the true nucleotides at positions 35 and 42 as such:

$$Y_{35} = \begin{cases} C & \text{if } z_{i2} < 2, \\ T & \text{otherwise} \end{cases} \qquad Y_{42} = \begin{cases} A & \text{if } z_{i2} \,\%\, 2 = 0, \\ G & \text{otherwise} \end{cases}$$

Here, % is the modulus operator (*i.e.*, $2 \,\%\, 2 = 2 \bmod 2 = 0$). Additionally, define

$$\mathcal{I}_P \coloneqq \{1, 2, \ldots, 18, 28, \ldots, 34, 36, \ldots, 41, 43, \ldots, 52\}$$

and $\mathcal{I}_B \coloneqq \{19, 20, \ldots, 27\}$. Then, picking some sequence X so that we can drop the index i, the probability of observing base x at index j is

$$\begin{aligned} p_{jz} &= P(X_j = x | \boldsymbol{Z} = \boldsymbol{z}) = \\ &= (q_{Bx})^{\mathbb{1}\{j \le z_1\}} \times \left[\delta_j^{\mathbb{1}\{x = Y_{j-z_1}\}} \big((1-\delta_j)\gamma_{Y_{j-z_1}x}\big)^{\mathbb{1}\{x \ne Y_{j-z_1}\}} \right]^{\mathbb{1}\{j - z_1 \in \mathcal{I}_P\}} \\ &\times (q_{Bx})^{\mathbb{1}\{j - z_1 \in \mathcal{I}_B\}} \times (q_{Sx})^{\mathbb{1}\{53 \le j - z_1\}} \\ &\times \left[\left[\delta_j^{\mathbb{1}\{x=C\}} \big((1-\delta_j)\gamma_{Cx}\big)^{\mathbb{1}\{x \ne C\}} \right]^{\mathbb{1}\{z_2 < 2\}} \left[\delta_j^{\mathbb{1}\{x=T\}} \big((1-\delta_j)\gamma_{Tx}\big)^{\mathbb{1}\{x \ne T\}} \right]^{\mathbb{1}\{z_2 \ge 2\}} \right]^{\mathbb{1}\{j - z_1 = 35\}} \\ &\times \left[\left[\delta_j^{\mathbb{1}\{x=A\}} \big((1-\delta_j)\gamma_{Ax}\big)^{\mathbb{1}\{x \ne A\}} \right]^{\mathbb{1}\{z_2 \% 2 = 0\}} \left[\delta_j^{\mathbb{1}\{x=G\}} \big((1-\delta_j)\gamma_{Gx}\big)^{\mathbb{1}\{x \ne G\}} \right]^{\mathbb{1}\{z_2 \% 2 = 1\}} \right]^{\mathbb{1}\{j - z_1 = 42\}} \end{aligned} \tag{1}$$

2. Redefine $p_{jx}(\boldsymbol{z})$ derived above as $p'_{jx}(\boldsymbol{z})$. When $z_1 = 4$, then $p_{jx}(\boldsymbol{z}) = q_{Sx}$. Then our new likelihood equation is

$$p_{jx}(\boldsymbol{z}) = (q_{Sx})^{\mathbb{1}\{z_1 = 4\}} p'_{jx}(\boldsymbol{z})^{\mathbb{1}\{z_1 \ne 4\}}$$

Define $\boldsymbol{X} = \{X_1, X_2, \ldots, X_M\}$, and let m_i be the length of sequence X_i. Let $\boldsymbol{\theta} = \{\boldsymbol{q}_B, \boldsymbol{q}_S, \boldsymbol{\gamma}, \boldsymbol{\delta}\}$. Then the complete data likelihood function is

$$\mathcal{L}_c(\theta \mid \boldsymbol{X}, \boldsymbol{z}) = \prod_{i=1}^{M} \prod_{j=1}^{m_i} \prod_{k=0}^{4} \prod_{l=0}^{3} [P(X_{ij} \mid \theta, \boldsymbol{z_i}) P(z_{i1} = k) P(z_{i2} = l)]^{\mathbb{1}\{z_{i1} = k, z_{i2} = l\}} \tag{2}$$

Since z_2 is undefined when $z_1 = 4$, we rearrange the above equation by factoring out the $z_1 = 4$ case:

$$\begin{aligned} \mathcal{L}_c(\theta \mid \boldsymbol{X}, \boldsymbol{z}) &= \prod_{i=1}^{M} \left[\left[\left(\prod_{j=1}^{m_i} P(X_{ij} \mid \boldsymbol{\theta}, z_{i1} = 4) \right) P(z_{i1} = 4) \right]^{\mathbb{1}\{z_{i1} = 4\}} \right. \\ &\quad \times \left. \left[\prod_{k=0}^{3} \prod_{l=0}^{3} \left(\prod_{j=1}^{m_i} P(X_{ij} \mid \boldsymbol{\theta}, \boldsymbol{z_i}) \right) P(z_{i1} = k) P(z_{i2} = l) \right]^{\mathbb{1}\{z_{i1} = k, z_{i2} = l\}} \right] \\ &= \prod_{i=1}^{M} \left[\left[\left(\prod_{j=1}^{m_i} P_{jX_{ij}}(4, \text{na}) \right) \xi_{1,4} \right]^{\mathbb{1}\{z_{i1} = 4\}} \left[\prod_{k=0}^{3} \prod_{l=0}^{3} \left[\left(\prod_{j=1}^{m_i} P_{jX_{ij}}(k, l) \right) \xi_{1,k} \xi_{2,l} \right] \right]^{\mathbb{1}\{z_{i1} = k, z_{i2} = l\}} \right] \end{aligned} \tag{3}$$

where $\xi_{1,k}$ the probability of $z_1 = k$, and $\xi_{2,l}$ is the probability of $z_2 = l$. Then, the complete data log-likelihood is

$$\begin{aligned} l_c(\theta \mid \boldsymbol{X}, \boldsymbol{z}) &= \sum_{i=1}^{M} \Bigg[\mathbb{1}\{z_{i1} = 4\} \Bigg(\ln \xi_{1,4} + \sum_{j=1}^{m_i} \Big[\ln P_{jX_{ij}}(4, \text{na}) \Big] \Bigg) \\ &\quad + \sum_{k=0}^{3} \sum_{l=0}^{3} \mathbb{1}\{z_{i1} = k, z_{i2} = l\} \Bigg(\ln \xi_{1,k} + \ln \xi_{2,l} + \sum_{j=1}^{m_i} \Big[\ln P_{jX_{ij}}(k, l) \Big] \Bigg) \Bigg] \\ &= \sum_{i=1}^{M} \Bigg[\mathbb{1}\{z_{i1} = 4\} \big(\ln \xi_{1,4} + P_1 \big) + \sum_{k=0}^{3} \sum_{l=0}^{3} \mathbb{1}\{z_{i1} = k, z_{i2} = l\} \Bigg(\ln \xi_{1,k} + \ln \xi_{2,l} + P_2 \Bigg) \Bigg] \end{aligned} \tag{4}$$

where $P_1 = \sum_{j=1}^{m_i} \ln P_{jX_{ij}}(4, \text{na})$ and $P_2 = \sum_{j=1}^{m_i} \ln P_{jX_{ij}}(k, l)$.

3. Taking the expectation of equation (4), we have

$$\begin{aligned} Q(\boldsymbol{\theta} \mid \boldsymbol{\theta}^{(m)}) = \sum_{i=1}^{M} \Bigg[&\mathbb{E}\big(\mathbb{1}\{z_{i1} = 4\} \mid X_i; \boldsymbol{\theta}^{(m)}\big) \big[\ln \xi_{1,4} + P_1 \big] \\ &+ \sum_{k=0}^{3} \sum_{l=0}^{3} \mathbb{E}\big(\mathbb{1}\{z_{i1} = k, z_{i2} = l\} \mid X_i; \boldsymbol{\theta}^{(m)}\big) \Big[\ln \xi_{1,k} + \ln \xi_{2,l} + P_2 \Big] \Bigg] \end{aligned} \tag{5}$$

For some sequence X_i and nucleotide $b \in N$, define $n_{ib} = \sum_{j=1}^{m_i} \mathbb{1}\{X_{ij} = b\}$. Then, applying Bayes' rule to $\mathbb{E}\big(\mathbb{1}\{z_{i1} = 4\} \mid X_i; \boldsymbol{\theta}^{(m)}\big)$, we have

$$\mathbb{E}_{i1} = \mathbb{E}\big(\mathbb{1}\{z_{i1} = 4\} \mid X_i; \boldsymbol{\theta}^{(m)}\big) = P(z_{i1} = 4 \mid X_i, \boldsymbol{\theta}^{(m)}) = \frac{P(X_i \mid z_{i1} = 4, \boldsymbol{\theta}^{(m)}) P(z_{i1} = 4)}{P(X_i, \theta^{(m)})} \tag{6}$$

$$P(X_i \mid z_{i1} = 4, \boldsymbol{\theta}^{(m)}) P(z_{i1} = 4) = \left[\prod_{j=1}^{m_i} q_{SX_{ij}} \right] \xi_{1,4}^{(m)} = \left[\prod_{b \in N} \big(q_{Sb}\big)^{n_{ib}} \right] \xi_{1,4}^{(m)} \tag{7}$$

where:
$P(X_i, \theta^{(m)}) = P(X_i \mid z_{i1} = 4, \boldsymbol{\theta}^{(m)}) P(z_{i1} = 4) + \sum_{k=0}^{3} \sum_{l=0}^{3} P\big(X_i \mid z_{i1} = k, z_{i2} = l; \boldsymbol{\theta}^{(m)}\big) P(z_{i1} = k, z_{i2} = l)$

Using Bayes' rule on the other expectation in equation (5) yields

$$\begin{aligned} \mathbb{E}_{ikl2} = \mathbb{E}\big(\mathbb{1}\{z_{i1} = k, z_{i2} = l\} \mid X_i; \boldsymbol{\theta}^{(m)}\big) &= P\big(z_{i1} = k, z_{i2} = l \mid X_i; \boldsymbol{\theta}^{(m)}\big) \\ &= \frac{P\big(X_i \mid z_{i1} = k, z_{i2} = l; \boldsymbol{\theta}^{(m)}\big) P(z_{i1} = k, z_{i2} = l)}{P(X_i, \theta^{(m)})} \end{aligned}$$

$$\begin{aligned} P\big(X_i \mid z_{i1} = k, z_{i2} = l; \boldsymbol{\theta}^{(m)}\big) P(z_{i1} = k, z_{i2} = l) &= \left[\prod_{j=1}^{m_i} P_{jX_{ij}}(k, l; \boldsymbol{\theta}^{(m)}) \right] P(z_{i1} = k) P(z_{i2} = l) \\ &= \left[\prod_{j=1}^{m_i} P_{jX_{ij}}(k, l; \boldsymbol{\theta}^{(m)}) \right] \xi_{1,k}^{(m)} \xi_{2,l}^{(m)} \end{aligned} \tag{8}$$

Substituting (6) and (8) into equation (5), we have this Q function

$$Q(\boldsymbol{\theta} \mid \boldsymbol{\theta}) = \sum_{i=1}^{M} \left[\left(\mathbb{E}_{i1} \left(\ln \xi_{1,4} + P_1 \right) \right) + \sum_{k=0}^{3} \sum_{l=0}^{3} \left(\mathbb{E}_{ikl2} \left(\ln \xi_{1,k} + \ln \xi_{2,l} + P_2 \right) \right) \right] \tag{9}$$

Before taking partial derivatives of equation (9), we observe that the parameters $\boldsymbol{\delta}, \boldsymbol{q}_S, \boldsymbol{q}_B, \boldsymbol{\gamma}, \xi_{1,k}$, and $\xi_{2,l}$ only appear in the P_1 and P_2 terms and that everything else is a constant with respect to those variables. For a given sequence X_i and expanding the P_1 term in equation (9), we have

$$\sum_{j=1}^{m_i} P_1 = \sum_{j=1}^{m_i} \big[\ln q_{SX_{ij}}\big] = \sum_{b\in N} \big[n_{ib} \ln q_{Sb}\big] \tag{10}$$

Expanding the P_2 sum gives

$$\sum_{j=1}^{m_i} P_2 = \sum_{j=1}^{m_i} \big[\ln P_{jX_{ij}}(k,l)\big] \tag{11}$$

For some nucleotide $b \in N$, define $N'_b = N \setminus b$. Then, expanding $\sum_{j=1}^{m_i} \ln P_{jX_{ij}}(k,l)$, we have

$$\begin{aligned}
\sum_{j=1}^{m_i} \ln P_{jX_{ij}}(k,l) &= \sum_{j=1}^{k}\sum_{b\in N} \mathbb{1}\{x_{ij}=b\} \ln q_{Bb} \\
&+ \sum_{j-k\in\mathcal{I}_{\mathcal{P}}}\sum_{a\in N}\Big[\mathbb{1}\{X_{ij}=a\}\ln\delta_j + \sum_{b\in N'_a}\mathbb{1}\{X_{ij}=b\}\big[\ln(1-\delta_j)+\ln_{\gamma_{ab}}\big]\Big] \\
&+ \Big[\sum_{j=19+k}^{27+k}\sum_{b\in N}\mathbb{1}\{X_{ij}=N\}\ln q_{Bb}\Big] + \Big[\sum_{j=53+k}^{m_i}\sum_{b\in N}\mathbb{1}\{X_{ij}=b\}\ln q_{Sb}\Big] \\
&+ \mathbb{1}\{l<2\}\sum_{j=1}^{mi}\mathbb{1}\{j=35+k\}\Big[\mathbb{1}\{X_{ij}=C\}\ln\delta_j + \sum_{M\epsilon\{A,G,T\}}\mathbb{1}\{X_{ij=M}\}\big(\ln(1-\delta_1)+\ln\gamma_{CM}\big)\Big] \\
&+ \mathbb{1}\{l\geq 2\}\sum_{j=1}^{mi}\mathbb{1}\{j=35+k\}\Big[\mathbb{1}\{X_{ij}=T\}\ln\delta_j + \sum_{M\epsilon\{A,C,G\}}\mathbb{1}\{X_{ij=M}\}\big(\ln(1-\delta_1)+\ln\gamma_{TM}\big)\Big] \\
&+ \mathbb{1}\{l \bmod 2=0\}\sum_{j=1}^{mi}\mathbb{1}\{j=42+k\}\Big[\mathbb{1}\{X_{ij}=A\}\ln\delta_j + \sum_{M\epsilon\{C,G,T\}}\mathbb{1}\{X_{ij=M}\}\big(\ln(1-\delta_1)+\ln\gamma_{AM}\big)\Big] \\
&+ \mathbb{1}\{l \bmod 2=1\}\sum_{j=1}^{mi}\mathbb{1}\{j=42+k\}\Big[\mathbb{1}\{X_{ij}=G\}\ln\delta_j + \sum_{M\epsilon\{A,C,T\}}\mathbb{1}\{X_{ij=M}\}\big(\ln(1-\delta_1)+\ln\gamma_{GM}\big)\Big] \\
&+ \lambda_B(1-q_{BA}-q_{BC}-q_{BG}-q_{BT}) + \lambda_S(1-q_{SA}-q_{SC}-q_{SG}-q_{ST}) \\
&+ \lambda_{\gamma_A}(1-\sum_{b\in N'_A}\gamma_{Ab}) + \lambda_{\gamma_C}(1-\sum_{b\in N'_C}\gamma_{Cb}) + \lambda_{\gamma_G}(1-\sum_{b\in N'_G}\gamma_{Gb}) + \lambda_{\gamma_T}(1-\sum_{b\in N'_T}\gamma_{Tb}) \\
&+ \lambda_{\xi_{1,k}}(1-\xi_{1,0}-\xi_{1,1}-\xi_{1,2}-\xi_{1,3}-\xi_{1,4}) + \lambda_{\xi_{2,l}}(1-\xi_{2,0}-\xi_{2,1}-\xi_{2,2}-\xi_{2,3})
\end{aligned}$$

where each of the λ terms are Lagrange multipliers with corresponding constraints. We proceed to take partial derivatives of Q with respect to each of the variables: $q_{Bb}, q_{Sb}, \gamma_{MN}, \delta_j, \xi_1k, \xi_2l$. Lets start with q_{Bb}, q_{Sb}. For some nucleotide $b \in N$, we have:

$$\begin{aligned}
\frac{\partial Q}{\partial q_{Bb}} &= \sum_{i=1}^{M}\sum_{k=0}^{3}\sum_{l=0}^{3}\mathbb{E}_{ikl2}\Big[\sum_{j=1}^{k}\frac{\mathbb{1}\{x_{ij}=b\}}{q_{Bb}} + \sum_{j=k+19}^{k+27}\frac{\mathbb{1}\{x_{ij}=b\}}{q_{Bb}}\Big] - \lambda_B = 0 \\
\Rightarrow q_{Bb} &= \frac{1}{\lambda_B}\sum_{i=1}^{M}\sum_{k=0}^{3}\sum_{l=0}^{3}\mathbb{E}_{ikl2}\Big[\sum_{j=1}^{k}\mathbb{1}\{x_{ij}=b\} + \sum_{j=k+19}^{k+27}\mathbb{1}\{x_{ij}=b\}\Big] \qquad (12)
\end{aligned}$$

Similarly, for Q_{Sb}, we have:

$$\begin{aligned}
\frac{\partial Q}{\partial q_{Sb}} &= \sum_{i=1}^{M}\Big[\mathbb{E}_{i1}\frac{n_{ib}}{q_{Sb}} + \sum_{k=0}^{3}\sum_{l=0}^{3}\mathbb{E}_{ikl2}\Big[\sum_{j=53+k}^{m_i}\frac{\mathbb{1}\{x_{ij}=b\}}{q_{Sb}}\Big]\Big] - \lambda_S = 0 \\
\Rightarrow q_{Sb} &= \frac{1}{\lambda_S}\sum_{i=1}^{M}\Big[\mathbb{E}_{i1}n_{ib} + \sum_{k=0}^{3}\sum_{l=0}^{3}\mathbb{E}_{ikl2}\Big[\sum_{j=53+k}^{m_i}\mathbb{1}\{x_{ij}=b\}\Big]\Big] \qquad (13)
\end{aligned}$$

For $k = 0, 1, 2$, or 3, the following holds:

$$\frac{\partial Q}{\partial \xi_{1,k}} = \sum_{i=1}^{M}\sum_{l=0}^{3}\frac{\mathbb{E}_{ikl2}}{\xi_{1,k}} - \lambda_{\xi_1} = 0 \qquad \Longrightarrow \qquad \xi_{1,k} = \frac{1}{\lambda_{\xi_1}}\sum_{i=1}^{M}\sum_{l=0}^{3}\mathbb{E}_{ikl2} \tag{14}$$

For $k = 4$, we have the following:

$$\frac{\partial Q}{\partial \xi_{1,4}} = \sum_{i=0}^{M}\left[\frac{\mathbb{E}_{i1}}{\xi_{1,4}}\right] - \lambda_{\xi_1} = 0 \qquad \Longrightarrow \qquad \xi_{1,4} = \sum_{i=1}^{M}\frac{\mathbb{E}_{i1}}{\lambda_{\xi_1}} \tag{15}$$

For $l = 0, 1, 2$, or 3, the following holds:

$$\frac{\partial Q}{\partial \xi_{2,l}} = \sum_{i=1}^{M}\sum_{k=0}^{3}\frac{\mathbb{E}_{ikl2}}{\xi_{2,l}} - \lambda_{\xi_2} = 0 \qquad \Longrightarrow \qquad \xi_{20} = \frac{1}{\lambda_{\xi_2}}\sum_{i=1}^{M}\sum_{k=0}^{3}\mathbb{E}_{ikl2} \tag{16}$$

For δ_j:

$$\begin{aligned}\frac{\partial Q}{\partial \delta_j} = \sum_{i=1}^{M}\sum_{k=0}^{3}\sum_{l=0}^{3}\mathbb{E}_{ikl2}\Bigg[&\mathbb{1}\{j - z_1 \in \mathcal{I}_P\}\sum_{M\in N}\left[\frac{\mathbb{1}\{X_{ij} = M\}}{\delta_j} - \sum_{N\neq M}\frac{\mathbb{1}\{X_{ij} = N\}}{1-\delta_j}\right] \\ &+ \mathbb{1}\{l < 2\}\Bigg[\mathbb{1}\{j = 35 + k\}\Bigg[\frac{\mathbb{1}\{X_{ij} = C\}}{\delta_j} - \sum_{M\in(A,C,T)}\frac{\mathbb{1}\{X_{ij} = M\}}{1-\delta_j}\Bigg]\Bigg] \\ &+ \mathbb{1}\{l \geq 2\}\Bigg[\mathbb{1}\{j = 35 + k\}\Bigg[\frac{\mathbb{1}\{X_{ij} = T\}}{\delta_j} - \sum_{M\in(A,C,G)}\frac{\mathbb{1}\{X_{ij} = M\}}{1-\delta_j}\Bigg]\Bigg] \\ &+ \mathbb{1}\{l \bmod 2 = 0\}\Bigg[\mathbb{1}\{j = 42 + k\}\Bigg[\frac{\mathbb{1}\{X_{ij} = A\}}{\delta_j} - \sum_{M\in(C,G,T)}\frac{\mathbb{1}\{X_{ij} = M\}}{1-\delta_j}\Bigg]\Bigg] \\ &+ \mathbb{1}\{l \bmod 2 = 1\}\Bigg[\mathbb{1}\{j = 42 + k\}\Bigg[\frac{\mathbb{1}\{X_{ij} = G\}}{\delta_j} - \sum_{M\in(A,C,T)}\frac{\mathbb{1}\{X_{ij} = M\}}{1-\delta_j}\Bigg]\Bigg]\Bigg] = 0\end{aligned} \tag{17}$$

Define A as the following:

$$\begin{aligned}A = \sum_{i=1}^{M}\sum_{k=0}^{3}\sum_{l=0}^{3}\mathbb{E}_{ikl2}\Bigg[&\mathbb{1}\{j - k \in \mathcal{I}_P\}\sum_{M\in N}\mathbb{1}\{x_{ij} = M\} \\ &+ \mathbb{1}\{j = 35 + k\}\Big[\mathbb{1}\{l < 2\}\mathbb{1}\{x_{ij} = C\} + \mathbb{1}\{l \geq 2\}\mathbb{1}\{x_{ij} = T\}\Big] \\ &+ \mathbb{1}\{j = 42 + k\}\Big[\mathbb{1}\{l \bmod 2 = 0\}\mathbb{1}\{x_{ij} = A\} + \mathbb{1}\{l \bmod 2 = 1\}\mathbb{1}\{x_{ij} = G\}\Big]\Bigg]\end{aligned}$$

Define B as the following:

$$\begin{aligned}B = \sum_{i=1}^{M}\sum_{k=0}^{3}\sum_{l=0}^{3}\mathbb{E}_{ikl2}\Bigg[&\mathbb{1}\{j - k \in \mathcal{I}_P\}\sum_{M\in(A,C,G,T)}\sum_{N\neq M}\mathbb{1}\{x_{ij} = N\} \\ &+ \mathbb{1}\{j = 35 + k\}\Bigg[\mathbb{1}\{l < 2\}\sum_{M\in(A,G,T)}\mathbb{1}\{x_{ij} = M\} + \mathbb{1}\{l \geq 2\}\sum_{M\in(A,C,G)}\mathbb{1}\{x_{ij} = M\}\Bigg] \\ &+ \mathbb{1}\{j = 42 + k\}\Bigg[\mathbb{1}\{l \bmod 2 = 0\}\Big(\sum_{M\in(C,G,T)}\mathbb{1}\{x_{ij} = M\} \\ &+ \mathbb{1}\{l \bmod 2 = 1\}\Bigg(\sum_{M\in(A,C,T)}\mathbb{1}\{x_{ij} = M\}\Bigg)\Bigg]\Bigg]\end{aligned} \tag{18}$$

Then, from (17)

$$\frac{A}{\delta} - \frac{B}{1-\delta} = 0 \qquad \Rightarrow \qquad A - \delta A = B\delta = A = \delta(A+B) \qquad \Rightarrow \qquad \delta_j = \frac{A}{A+B} \tag{19}$$

And finally for γ we will take two examples, one for each value of l, which will consider one of the positions 35+k or 42+k. Then for γ_{AC}

$$\frac{\partial Q}{\partial \gamma_{AC}} = \sum_{i=1}^{M} \Bigg[\sum_{k=0}^{3} \sum_{l=0}^{3} \mathbb{E}_{ikl2} \Big[\sum_{j-k_1 \in \mathcal{I}_P} \Big[\mathbb{1}\{x_{ij} = C\} \Big(\frac{1}{\gamma_{AC}} \Big) \Big] \Big]$$
$$+ \mathbb{1}\{l \bmod 2 = 0\} \sum_{j=1}^{m_i} \mathbb{1}\{j = 42 + k\} \Big[\mathbb{1}\{x_{ij} = C\} \frac{1}{\gamma_{AC}} \Big] \Bigg] - \lambda_{\gamma_A} = 0$$

Rearranging the above, we have:

$$\gamma_{AC} = \frac{\sum_{i=1}^{M} \Big[\sum_k \sum_l \mathbb{E}_{ikl2} \Big[\sum_{j-k_1 \in \mathcal{I}_P} \Big[\mathbb{1}\{x_{ij} = C\} \Big] + \mathbb{1}\{l \bmod 2 = 0\} \sum_{j=1}^{m_i} \mathbb{1}\{j = 42 + k\} \Big[\mathbb{1}\{x_{ij} = C\} \Big] \Big] \Big]}{\lambda_{\gamma_A}} \tag{20}$$

Similarly for γ_{TG}

$$\frac{\partial Q}{\partial \gamma_{TG}} = \sum_{i=1}^{M} \Bigg[\sum_{k=0}^{3} \sum_{l=0}^{3} \mathbb{E}_{ikl2} \Big[\sum_{j-k_1 \in \mathcal{I}_P} \Big[\mathbb{1}\{x_{ij} = G\} \Big(\frac{1}{\gamma_{TG}} \Big) \Big] \Big]$$
$$+ \mathbb{1}\{l \geq 2\} \sum_{j=1}^{m_i} \mathbb{1}\{j = 35 + k\} \Big[\mathbb{1}\{x_{ij} = G\} \frac{1}{\gamma_{AC}} \Big] \Bigg] - \lambda_{\gamma_T} = 0$$

Rearranging the above, we have:

$$\gamma_{TG} = \frac{\sum_{i=1}^{M} \Big[\sum_k \sum_l \mathbb{E}_{ikl2} \Big[\sum_{j-k_1 \in \mathcal{I}_P} \Big[\mathbb{1}\{x_{ij} = G\} \Big] + \mathbb{1}\{l \geq 2\} \sum_{j=1}^{m_i} \mathbb{1}\{j = 35 + k\} \Big[\mathbb{1}\{x_{ij} = G\} \Big] \Big] \Big]}{\lambda_{\gamma_T}} \tag{21}$$

With this we can continue with the programming part of the assignment