This document sketches out the calculation of autocorrelation functions.

First, suppose we have calculate the left eigenvectors and eigenvalues:

$$T^T \phi_i = \lambda \phi_i$$

Suppose that π is the equilibrium population. Then, we can normalize the eigenvectors such that:

$$\phi_i^T \pi^{-1} \phi_j = \delta_{ij}$$

Above, we denote π^{-1} to be a diagonal matrix with elements π_i^{-1}.

The autocorrelation function of the observable f_i can be denoted:

$$E(f(z_t)f(z_0)) = \sum_{i,j} f_i P(z_0 = i) f_j P(z_t = j | z_0 = i) = \sum_{i,j} f_i f_j \pi_i T_{ij} =$$

We know that

$$T_{ab}(t) = \sum_k \lambda_k(t) (\psi_k)_a (\phi_k)_b = \sum_k \lambda_k(t) (\pi_a)^{-1} (\phi_k)_a (\phi_k)_b$$

Thus,

$$E(f(z_t)f(z_0)) = \sum_{i,j,k} f_i f_j \lambda_k(t) (\phi_k)_i (\phi_k)_j = \sum_k \lambda_k(t) s_k^2$$

Where

$$s_k = \sum_i f_i (\phi_k)_i$$

Finally, note that $\lambda_i(\infty) = \delta_{i0}$, so the long-timescale behavior is simply:

$$E(f(z_\infty)f(z_0)) = s_0^2$$

For most applications, one is interested in the zero-centered ACF, so we simply skip the $k = 0$ term in the summation.