Additive pulse mode locking

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A new principle of mode locking is analyzed: additive pulse mode locking. It is shown to be operative in two-cavity soliton lasers, but it also permits mode locking with fibers in the positive dispersion regime. A simple model is developed that displays the pulse-shortening mechanism. Parameter ranges, within which this principle can be exploited, are given. Comparisons with experiments are made.

INTRODUCTION

Recently a unifying, time-domain description for ultrashort pulse generation in lasers with auxiliary nonlinear cavities was presented.[1] Called additive pulse mode locking (APM), it appears to explain experimental results obtained in both soliton lasers[2] and those coupled to fibers with normal, pulse spreading dispersion.[3–5] In this paper we expand on the theory, develop an analysis in terms of convenient operators, and investigate the parameter relationships required for stable mode locking in more detail.

Additive pulse mode locking utilizes self-phase modulation in an auxiliary cavity to produce pulse shortening in the main laser by coherent interference at the coupler mirror. The APM shortening mechanism is affected by, but does not rely on, dispersion in either cavity. Mollenauer and Stolen's concept of the soliton laser,[2] on the other hand, was based on soliton pulse shaping in the auxiliary, fiber resonator. In subsequent work Belanger[6] considered the soliton laser as a laser injection locked by short pulses from a compressor cavity, but Blow and Wood[7] were the first to study the soliton laser by computer simulation and to discover that negative dispersion was not always necessary for improved mode locking. Previously Oullette and Piché[8] had studied the shaping of pulses by nonlinear resonators but under conditions of relatively large changes per transit. The use of self-phase modulation in a single laser cavity to induce[9,10] or enhance[11] mode locking has also been described.

In this paper we focus on the auxiliary compressor cavity as a termination of the mode-locked laser, a termination that returns a pulse shorter than the one incident upon it. In this way one gains an understanding of the operation of the system that provides one with insight for the design of the system for optimum pulse shaping.

An auxiliary cavity viewed as a nonlinear dispersive termination transforms an incident pulse into a reshaped reflected pulse. The laser pulses incident upon, and reflected from, the auxiliary cavity and the pulses internal to the auxiliary cavity add coherently at the mirror. If the length of the auxiliary cavity is properly chosen or the loss in the auxiliary cavity is large for a given carrier frequency of the incident pulse, then the energies in the incident and reflected pulses can be much larger than the energy of the pulses traveling within the auxiliary cavity. When this is the case, then the change of the pulse on reflection can be small, as suited for proper mode-locking operation, even though the change of the pulse in the auxiliary cavity within one round trip may be large.

1. TERMINATION IN THE ABSENCE OF FIBER DISPERSION

Consider, with reference to Fig. 1, the incident and reflected waves a_i and b_i at the mirror separating the laser cavity from the auxiliary cavity. The auxiliary cavity is on side 2 of the mirror. The equations for the wave amplitudes at the mirror are[12]

$$b_1 = ra_1 + \sqrt{1-r^2}a_2, \tag{1.1}$$

$$b_2 = \sqrt{1-r^2}a_1 - ra_2. \tag{1.2}$$

Wave $b_2(t)$, the time envelope of the reflected wave on side 2, travels through the nonlinear medium, is attenuated by the factor L (<1), and returns delayed as wave a_2. We ignore dispersion at first. If the travel distance is properly adjusted so that the time delay coincides with the period of the periodically excited system, then this round-trip delay can be suppressed, and one may write

$$\begin{aligned} a_2(t) &= L \exp -j\{\phi + \kappa[|a_2(t)|^2 - |a_2(0)|^2]\}b_2(t) \\ &= L \exp[-j(\phi + \Phi)]b_2(t), \end{aligned} \tag{1.3}$$

where $t = 0$ corresponds to the peak of the pulse and Φ is the phase shift induced by the nonlinearity

$$\Phi \equiv \kappa[|a_2(t)|^2 - |a_2(0)|^2]. \tag{1.4}$$

The parameter κ is proportional to the length of the fiber and the nonlinear index. The variable ϕ is the phase shift of the carrier on reflection that occurs when the (auxiliary) cavity is detuned from antiresonance. Antiresonance is the adjustment at which the ratio of energy densities outside and inside the auxiliary cavity is a maximum. Note that the peak phase shift $\kappa|a_2(0)|^2$ has been subtracted from Φ. Thus ϕ includes the phase bias due to the peak nonlinear phase shift. This definition has been chosen because it standardizes some of the important plots later in this paper. Expressing $b_2(t)$ as a function of $a_2(t)$ and introducing the result into Eqs. (1) and (2), we obtain

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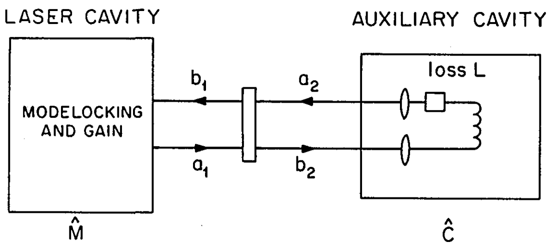


Fig. 1. Schematic of laser cavity and auxiliary cavity.

$$b_1 = \frac{1}{\sqrt{1-r^2}}\left\{1 + \frac{r}{L}\exp[j(\phi + \Phi)]\right\}a_2, \tag{1.5}$$

$$a_1 = \frac{1}{\sqrt{1-r^2}}\left\{r + \frac{1}{L}\exp[j(\phi + \Phi)]\right\}a_2. \tag{1.6}$$

Note the significance of Eqs. (1.5) and (1.6). The a_i's and b_i's are the pulse envelopes, functions of time, and so is Φ. We have treated the linear phase shift ϕ as a constant. In fact, ϕ is a function of frequency. The pulse train has spectral components spaced by the Fabry–Perot mode separation. The different components experience slightly different phase shifts. However, if the pulse is many optical cycles long (is narrow band), this variation of ϕ can be disregarded. In summary, within this approximation the pulse train is reflected distortion free, when $\Phi = 0$.

Under the approximations just stated, when $\Phi \neq 0$ one may think of the pulse composed of a distribution of impulse functions in time, where each function is phase-shifted by the phase $\Phi(t)$. If the pulse $a_2(t)$ has different phase shifts in the wings from those in the center, it can interfere with a_1 and produce a reflected pulse b_1 that has a different temporal envelope from a_1. We look at this phenomenon in greater detail.

We assume that a_2 is a Gaussian function of time, $a_2(t) = \exp[-(t^2/2\tau_0^2)]$. The question then is about the relation between the incident and reflected pulses on side 1 of the mirror. As examples, we compute the amplitudes and shapes of a_1 and b_1 for different values of ϕ and Φ, assuming that $r = 0.9$ and $L = 0.2$. Figure 2(a) shows the results for the case when the maximum nonlinear phase shift is $\kappa|a_2(0)|^2 \equiv \Phi_0 = \pi$ and the phase in the auxiliary cavity is adjusted so as to give coherent addition of a_2 to b_1 at the peak of the pulse and subtraction in the wings of the pulse. This occurs at $\phi = 0$, the case of antiresonance. In Fig. 2(b), where we have replotted the pulses normalized to the same amplitude, we see more clearly that the reflected pulse b_1 is shorter than the incident pulse a_1. However, $\phi = 0$ is not the case of best pulse shortening. Figure 3 shows the case of $\phi = -\pi/2$ and $\Phi_0 = \pi/2$. Here the shortening is truly dramatic. In fact, we may observe that the pulse experiences shortening even near the peak of the pulse, which is not observable in the case of Fig. 2. Actually, the improved shortening is easy to understand. If the nonlinearly phase-shifted pulse is added with a phase shift of $\pi/2$ at the peak, then the addition can start to be in antiphase as soon as one deviates from the peak. This is not the case when the pulses are added in phase at their peaks.

In the steady state, the pulse-broadening mechanism at work in the laser cavity would balance the shortening effect. As the steady state is approached, however, the shortening within one pass can be considered to be a pulse shortening per pass.[13] The pulse-shortening action of the termination is most easily understood if one makes certain approximations so as to show explicitly that the reflection at the peak of the pulse is maximized and decreases in the wings. The reflection coefficient Γ is, from Eqs. (1.4) and (1.5)

$$\Gamma = \frac{b_1}{a_1} = \frac{1 + \frac{r}{L}\exp[j(\phi + \Phi)]}{r + \frac{1}{L}\exp[j(\phi + \Phi)]}. \tag{1.7}$$

If we assume that $L \ll 1$, we may expand the above equation to obtain

$$\Gamma = \{r + L(1 - r^2)\exp[-j(\phi + \Phi)]\}. \tag{1.8}$$

Suppose that Φ is small so that we can expand $\exp(-j\Phi)$. Then

$$\Gamma = r + L(1 - r^2)e^{-j\phi}(1 - j\Phi). \tag{1.9}$$

To make $|\Gamma|$ change maximally with a change of Φ we need $\phi = \pm\pi/2$. It is clear that $\phi = -\pi/2$ causes the reflection to decrease if Φ goes negative. Remember that $\Phi = 0$ at the peak and goes negative in the wings. Thus, for $\phi = -\pi/2$,

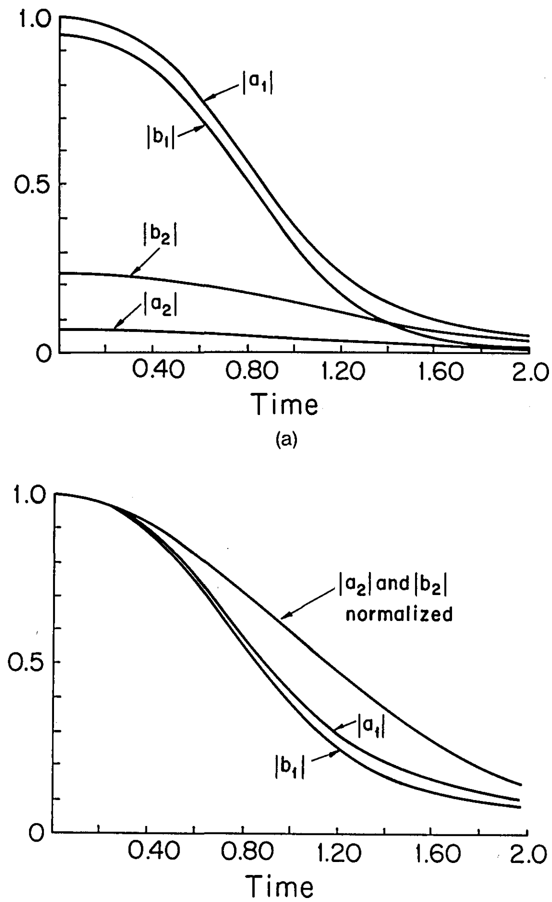
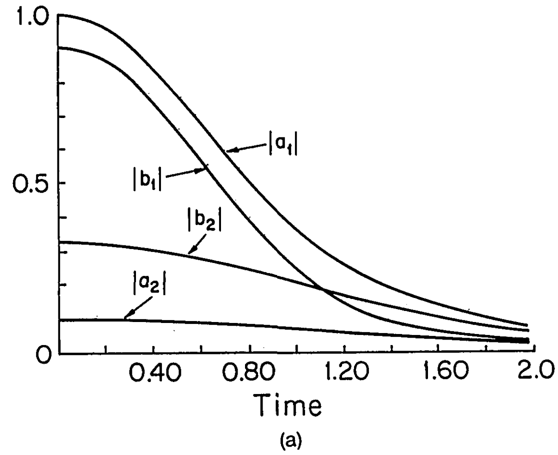


Fig. 2. The pulse shortening: $\phi = 0$, $\Phi_0 = \pi$, $r = 0.8$, $L = 0.3$, $\tau_0 = 1$, no dispersion; (a) all pulses normalized to unity peak amplitude; (b) a_1 normalized to unity peak amplitude.



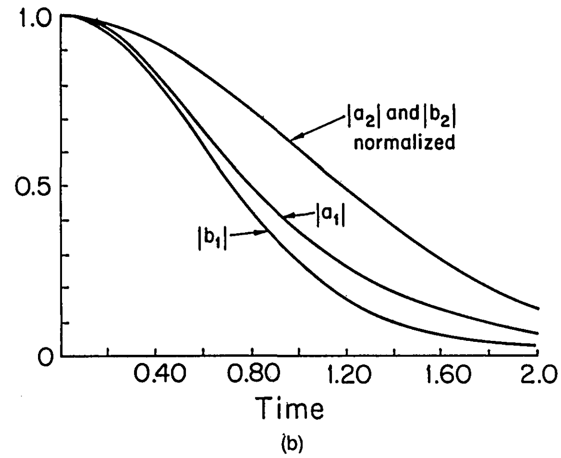


Fig. 3. The pulse shortening: $\phi = -\pi/2$, $\Phi_0 = \pi/2$, $r = 0.8$, $L = 0.3$, $\tau_0 = 1$, no dispersion; (a) all pulses normalized to unity peak amplitude; (b) a_1 normalized to unity peak amplitude.

$$|\Gamma| \simeq r + L(1 - r^2)\Phi. \tag{1.10}$$

This shows how the reflection is maximum at the pulse center and then decreases, a simple explanation of the pulse shortening. It also shows clearly why the phase-shift bias of $-\pi/2$ is important. Note, however, that for larger excursions of Φ the optimum value of ϕ may deviate from $-\pi/2$.

2. THEORY IN THE PRESENCE OF DISPERSION

We shall represent dispersion as the operator $\hat{D}$:

$$\hat{D} \equiv \left(1 + jD \frac{\mathrm{d}^2}{\mathrm{d}t^2}\right), \tag{2.1}$$

where $D \equiv {}^1\!/\!_2\beta'' l_f$, with β'' the second derivative with respect to frequency of the propagation constant l_f the length of the fiber. This operator results from the expansion to first order of the exponential containing the frequency-dependent phase shift due to group-velocity dispersion and the replacement of the square of the frequency (in the frequency domain) with the second time derivative (in the time domain). Thus the pulse envelope $b_2(t)$ is written in tems of the phase envelope $a_2(t)$:

$$b_2(t) = \frac{e^{j\phi}}{L}\, \hat{O}^{-1} a_2(t), \tag{2.2}$$

where the operator $\hat{O}$ is defined as the operator of the cascade of the operator $\hat{D}$ and $\exp(-j\Phi)$. In the spirit of weak action, on one transit we shall still write Φ as in Eq. (1.3) because when the perturbation is small the operators commute. Thus

$$\hat{O}^{-1} = e^{j\Phi}\left(1 - jD \frac{\mathrm{d}^2}{\mathrm{d}t^2}\right). \tag{2.3}$$

When Figure 2 is reevaluated with this dispersion taken into account, one obtains Fig. 4 for positive dispersion and Fig. 5 for negative dispersion. Figures 6 and 7 show the case of $\phi = -\pi/2$ and $\kappa|a_2(0)|^2 \equiv \Phi_0 = \pi/2$, for both negative and positive dispersion. It is clear that, in this limit of perturbational dispersion, positive dispersion is actually helpful, while negative dispersion reduces the shortening. For more details see Section 7.

We shall now develop an analytic theory of mode locking that characterizes the action of the auxiliary pulse-compression cavity in terms of convenient operators. The same must be done for the cavity containing the laser medium.

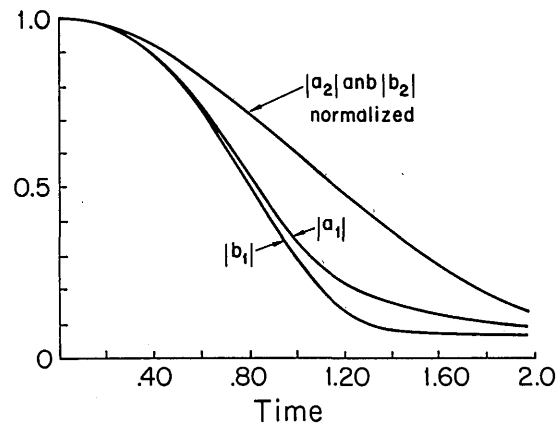


Fig. 4. The pulse shortening: $\phi = 0$, $\Phi_0 = \pi$, $r = 0.8$, $L = 0.3$, $D = 0.2$, $\tau_0 = 1$.

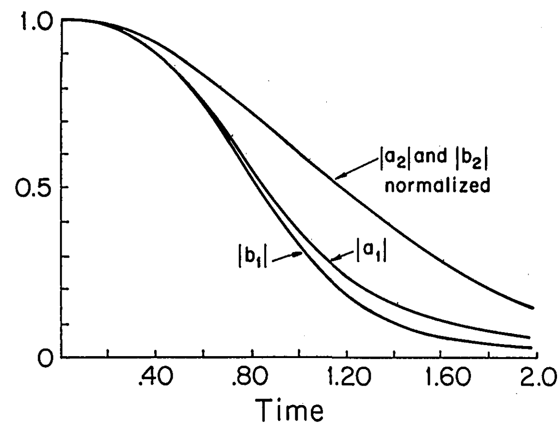


Fig. 5. The pulse shortening: $\phi = 0$, $\Phi_0 = \pi$, $r = 0.8$, $L = 0.3$, $D = -0.2$, $\tau_0 = 1$.

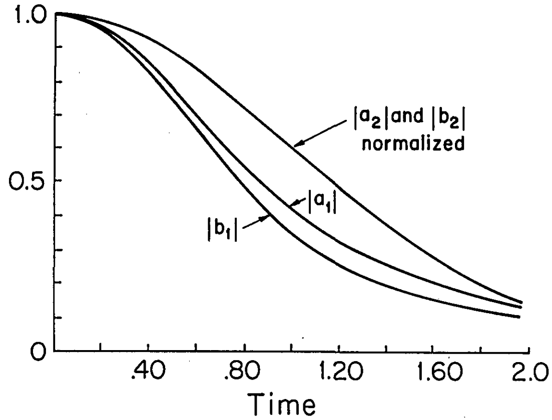


Fig. 6. The pulse shortening: $\phi = -\pi/2$, $\Phi_0 = \pi/2$, $r = 0.8$, $L = 0.3$, $D = 0.2$, $\tau_0 = 1$.

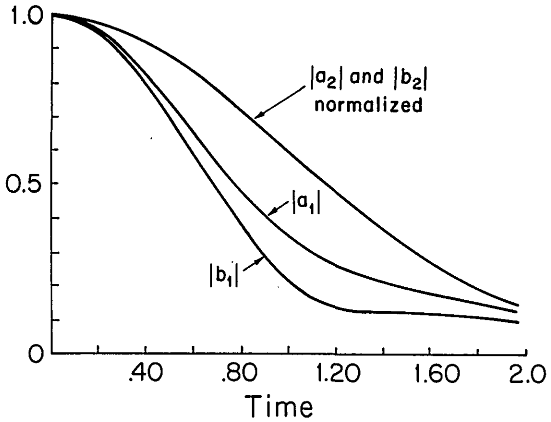


Fig. 7. The pulse shortening: $\phi = -\pi/2$, $\Phi_0 = \pi/2$, $r = 0.8$, $L = 0.3$, $D = -0.2$, $\tau_0 = 1$.

3. THE GAIN CAVITY

The gain cavity contains the laser medium of gain g and gain bandwidth ω_g. The gain is pulse-energy dependent:

$$g = \frac{g_0}{1 + (E/E_s)}, \tag{3.1}$$

where E is the pulse energy, E_s is the saturation energy, and g_0 is the small-signal gain. The gain has, also, a frequency dependence

$$\frac{g}{1 + \dfrac{(\omega - \omega_0 + \Delta\omega)^2}{{\omega_g}^2}} \simeq g\left(1 - \frac{(\omega - \omega_0 + \Delta\omega)^2}{{\omega_g}^2}\right), \tag{3.2}$$

where ω_0 is the carrier frequency and $\Delta\omega$ is the deviation of the carrier frequency from the frequency of peak gain. This effect, if small, is additive to Eq. (3.1). Now, in the time domain we identify $j(\omega - \omega_0)$ as the time derivative

$$j(\omega - \omega_0) \leftrightarrow \frac{\mathrm{d}}{\mathrm{d}t}, \tag{3.3}$$

and thus expression (3.2) becomes in the time domain, in the limit when the pulse-broadening effect is small,[14]

$$g\left(1 + \frac{1}{{\omega_g}^2}\frac{\mathrm{d}^2}{\mathrm{d}t^2} - \frac{\Delta\omega^2}{{\omega_g}^2} + 2j\frac{\Delta\omega}{{\omega_g}^2}\frac{\mathrm{d}}{\mathrm{d}t}\right). \tag{3.4}$$

Active modulation of the gain, or cavity loss, is assumed to initiate pulse formation. In synchronous mode locking the gain is turned off by the pulse, and the modulation depends on pulse shape and energy. This is rather complicated, and to simplify the analysis we shall assume that we may represent the temporal modulation by the simple, forced mode-locking equations[11,13]

$$\begin{aligned} \text{loss} &\propto l + M{\omega_M}^2(t - \Delta T)^2 \\ &= l + M{\omega_M}^2t^2 - 2M{\omega_M}^2\Delta Tt + M{\omega_M}^2\Delta T^2, \end{aligned} \tag{3.5}$$

where l represents the time-independent loss, M is the loss modulation coefficient, and ω_M is the modulation frequency. We have assumed that the pulse occurs at the time ΔT displaced forward from the instant of the transmission peak, when $\Delta T > 0$ (see Fig. 8). We assume further that the pulse experiences an overall phase delay ψ on one pass through the gain cavity. Thus the transformation of the pulse in one passage through the gain medium is represented by the operator $\hat{M}$ (see Fig. 1):

$$a_1 = \hat{M}b_1, \tag{3.6}$$

where the operator $\hat{M}$ is defined by

$$\begin{aligned} \hat{M} &= e^{-j\psi}(1 + g - l + \hat{m}), \\ &= e^{-j\psi}\Bigg[1 + g - l + g\left(\frac{1}{{\omega_g}^2}\frac{\mathrm{d}^2}{\mathrm{d}t^2} - \frac{\Delta\omega^2}{{\omega_g}^2} + 2j\frac{\Delta\omega}{{\omega_g}^2}\frac{\mathrm{d}}{\mathrm{d}t}\right) \\ &\quad - M{\omega_M}^2(t^2 - 2\Delta Tt + \Delta T^2) + T_L\frac{\mathrm{d}}{\mathrm{d}t}\Bigg]. \end{aligned} \tag{3.7}$$

We assume that the change of the pulse per pass is small, so that the operator $\hat{m}$ causes only small changes of the pulse per pass. Further, we allowed for a desynchronization per pass, T_L, due to a cavity travel time different from the period of the modulation; $T_L > 0$ when the pulse is advanced with respect to the modulation period.

4. THE COMPRESSOR CAVITY

We now take advantage of the formalism developed above. We can define

$$b_1 = \hat{C}_b a_2, \tag{4.1}$$

$$a_1 = \hat{C}_a a_2, \tag{4.2}$$

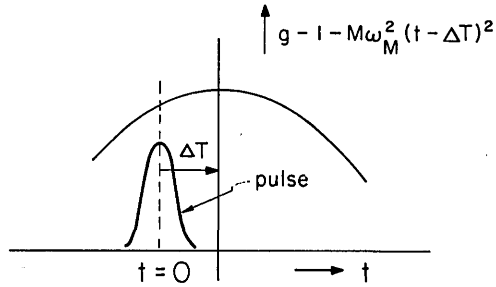


Fig. 8. The relative timing of modulation and pulse.

where the operators $\hat{C}_b$ and $\hat{C}_a$ are, according to Eqs. (1.5) and (1.6),

$$\hat{C}_b = \frac{1}{\sqrt{1-r^2}}\left(1 + \frac{r}{L}e^{j\phi}\hat{O}^{-1}\right), \tag{4.3}$$

$$\hat{C}_a = \frac{1}{\sqrt{1-r^2}}\left(r + \frac{1}{L}e^{j\phi}\hat{O}^{-1}\right). \tag{4.4}$$

We supplement the operator $\hat{O}$ of Eq. (2.3) with a time shift that would be caused by a change of the cavity transit time from that of the modulation period. Further, we expand $e^{j\Phi}$ to order t^2 around the peak of the pulse $|a_2(0)|^2 \exp[-(t^2/2\tau^2)]$ [compare Eq. (1.4)]:

$$\hat{O} = \left(1 + jD\frac{d^2}{dt^2} + j\Phi_0\frac{t^2}{\tau^2} + T_A\frac{d}{dt}\right) \equiv 1 + \hat{o}, \tag{4.5}$$

where $\Phi_0 = \kappa|a_2(0)|^2$ is the maximum nonlinear phase shift at the peak of the pulse.

It is convenient to separate the operators into a part that is c-numberlike and into an operator part, which is again assumed to effect only a small change on one transit:

$$\hat{C}_b = C_b(1 + \hat{\partial}_b), \tag{4.6}$$

where

$$C_b = \frac{1}{\sqrt{1-r^2}}\left(1 + \frac{r}{L}e^{j\phi}\right), \tag{4.7}$$

with

$$\hat{\partial}_b = -\frac{\frac{r}{L}e^{j\phi}}{1 + \frac{r}{L}e^{j\phi}}\hat{o} = -(G_b + jB_b)\hat{o}. \tag{4.8}$$

We can do the same with the operator C_a:

$$\hat{C}_a = C_a(1 + \hat{\partial}_a), \tag{4.9}$$

$$C_a \equiv \frac{1}{\sqrt{1-r^2}}\left(r + \frac{1}{L}e^{j\phi}\right), \tag{4.10}$$

$$\hat{\partial}_a = -\frac{\frac{1}{L}e^{j\phi}}{r + \frac{1}{L}e^{j\phi}}\hat{o} = -(G_a + jB_a)\hat{o}. \tag{4.11}$$

5. CLOSURE

We shall now run the system through one cycle and look for the change of the pulse in the process. From Fig. 1 we see that b_1 becomes a_1 through the operator $\hat{M}$. Then a_1 becomes $\hat{a}_2$ through the inverse of the operator $\hat{C}_a$. Then b_1 is produced from a_2 through the operator $\hat{C}_b$. The new waveform is $b_1 + \Delta b_1$:

$$b_1 + \Delta b_1 = \hat{C}_b\hat{C}_a^{-1}\hat{M}b_1. \tag{5.1}$$

In the steady state, $\Delta b_1 = 0$, and

$$\Delta b_1 = (\hat{C}_b\hat{C}_a^{-1}\hat{M} - 1)b_1 = 0. \tag{5.2}$$

Above we assumed a Gaussian pulse $\exp(-t^2/2\tau^2)$. We shall allow for the more general case of chirp by making τ^2 complex, $1/\tau^2 = 1/\tau_0^2(1 + j\gamma)$, where τ_0 and γ are real and γ is the chirp parameter. With this generalization,

$$\frac{d}{dt} = -t/\tau^2 = -(t/\tau_0^2)(1 + j\gamma) \tag{5.3a}$$

and

$$\frac{d^2}{dt^2} = t^2/\tau^4 - 1/\tau^2 = (t^2/\tau_0^4)(1 + 2j\gamma) - (1 + j\gamma)/\tau_0^2. \tag{5.3b}$$

We do not retain the term quadratic in γ. This must be done in order to conform with the observation that gain dispersion lengthens the pulse rather than shortens it. Specifically, if one retained γ^2, the operator action produced by the multiplier $g(1 - \gamma^2)/\omega_g^2(t^2/\tau_0^4)$ would reverse sign for $|\gamma^2| > 1$, an unacceptable behavior.

Equating to zero all constant terms, the terms containing t, and the terms containing t^2, one obtains three complex equations:

Constant:

$$\frac{C_b}{C_a}e^{-j\psi}\left\{1 + g\left[1 - \frac{1}{\omega_g^2\tau_0^2}(1 + j\gamma)\right] - l - g\frac{\Delta\omega^2}{\omega_g^2} - M\omega_M^2\Delta T^2\right\} - 1 = 0. \tag{5.4}$$

Here we ignore a term of order D^2/τ^4 and a term of order $g\gamma/\omega_g^2\tau_0^2$ because $|D/\tau^2|$, $g/\omega_g^2\tau_0^2$, and γ are assumed small.

Proportional to t:

$$-2j\Delta\omega\frac{g}{\omega_g^2}\frac{1}{\tau^2} + 2M\omega_M^2\Delta T - T_L\frac{1}{\tau^2} + (\Delta G + j\Delta B)T_A\frac{1}{\tau^2} = 0, \tag{5.5}$$

where $\Delta G + j\Delta B = (G_b - G_a) + j(B_b - B_a)$.

Proportional to t^2:

$$\frac{g}{\omega_g^2}\frac{1}{\tau^4} - M\omega_M^2 - (\Delta G + j\Delta B)j\left(D\frac{1}{\tau^4} + \Phi_0\frac{1}{\tau_0^2}\right) = 0. \tag{5.6}$$

Consider first Eq. (5.4). Noting that g, l, and $M\omega_M^2\Delta T^2$ are assumed small compared with unity, we satisfy the phase condition by

$$\arg(C_b/C_a) = \psi \tag{5.4a}$$

and the amplitude condition

$$\left|\frac{C_b}{C_a}\right|\left[1 + g\left(1 - \frac{\Delta\omega^2}{\omega_g^2} - \frac{1}{\omega_g^2\tau_0^2}\right) - l - M\omega_M^2\Delta T^2\right] - 1 = 0. \tag{5.4b}$$

Equation (5.4a) calls for a (slight) shift of the comb of frequencies of the Fabry–Perot resonant modes of the laser cavity. Figure 9 shows the phase of C_b/C_a as a function of ϕ. A slight shift of the comb of frequencies changes both ϕ and ψ by roughly the same amount, because both resonators are of the same optical length (within a few wavelengths). If one sets ϕ equal to $(\omega/c)nl_A$ in Fig. 9, with the auxiliary cavity

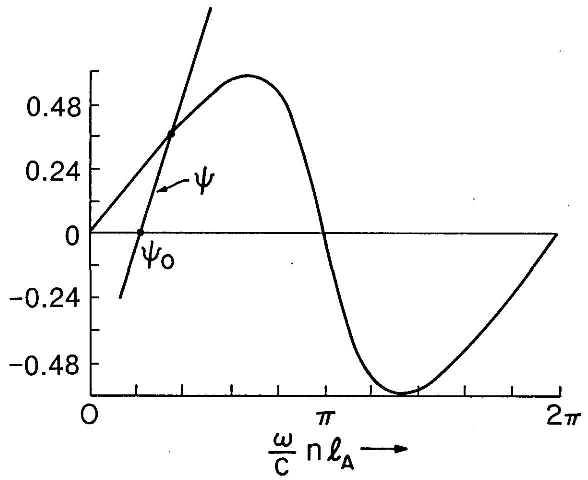


Fig. 9. Phase of C_b/C_a as function of ϕ and construction of equilibrium point: $r = 0.8, L = 0.3$. The phase shifts ϕ and ψ are assumed proportional to frequency with equal proportionality constants.

length l_A fixed, then the abscissa may be identified with frequency. Now ψ is roughly the same function of frequency and may be represented by a straight line at 45 deg. If the cavity has phase ψ_0 at one frequency, then the intersection point gives the frequency deviation from the reference necessary for cw operation. A change of length of the compressor cavity slides the periodic curve along the abscissa. Up to this point, the chirp parameter γ has been ignored. In fact, the term $-j\gamma g/\omega_g^2\tau_0^2$ introduces a slight correction to Eq. (5.4a) and thus to Fig. 9.

Equation (5.4b) gives the balance of gain and loss. The gain is reduced by the detuning $\Delta\omega$ and by the fact that a pulse of width τ utilizes the gain less effectively than a spectral line at gain center. The loss is increased when the pulse does not pass the mode-locking modulator at the instant of zero loss. Equation (5.4b) is satisfied by an adjustment of pulse energy, because g decreases with increasing pulse energy owing to gain saturation [see Eq. (3.1)]. In summary, the two equations obtained from the time-independent terms determine the frequency shift of the comb of frequencies with respect to the comb of the cavity transmission peaks and the pulse energy from the gain saturation.

Separating Eq. (5.5) into its real and imaginary parts, one finds an equation for the pulse timing and one for the frequency shift:

$$2M\omega_M^2\Delta T - \frac{T_L}{\tau_0^2} + \Delta G\frac{T_A}{\tau_0^2} = 0, \tag{5.5a}$$

$$-2\Delta\omega\frac{g}{\omega_g^2}\frac{1}{\tau_0^2} + \Delta B\frac{T_A}{\tau_0^2} = 0. \tag{5.5b}$$

In Eq. (5.5a) we have dropped products of γ with $\Delta\omega$ and T_A, because γ must be assumed small compared with unity if the interpretation of the dispersion operator is to be self-consistent, and T_A and $\Delta\omega$ are also small perturbation parameters. We see that the timing of the pulse is controlled, in part, by the temporal slip ΔT from the peak of the mode-locking drive. This slip causes temporal pushing of the pulse, because the pulse is shaved off asymmetrically. The consequence is a change in transit time in an attempt to compensate for a change of the round-trip time in the laser cavity, T_L. The auxiliary cavity can also change the pulse round-trip time if the parameter ΔG is nonzero. Equation (5.5a) is to be interpreted as the equation governing deviation from synchronism. As the length of the laser cavity is changed to change the period, the mode locking pushes the pulse forward (or back) to compensate for the change in round-trip time and to maintain the period equal to the mode-locking period. Changes of the round-trip time in the auxiliary cavity do not affect the pulse-repetition period if ΔG is zero.

In Eq. (5.5b) we have neglected products of T_L and T_A with γ. We note that a change of the length of the auxiliary cavity causes a change of the carrier frequency. This effect is analogous to the effect of temporal shifts under a temporal index modulation (imaginary M). Such a temporal shift causes frequency pushing or pulling (rather than time pulling or pushing when M is real). This frequency pushing or pulling must be compensated for by the gain profile, which also pulls or pushes the frequency when the signal spectrum is off the center of the gain profile. The frequency shift $\Delta\omega$ must be distinguished from the frequency shift responsible for the adjustment of ψ discussed in connection with Eq. (5.4a). Whereas the latter represented a slight shift of the comb of the Fourier components of the pulse within the frequency response of an individual Fabry–Perot mode, $\Delta\omega$ represents a shift of the comb envelope within the gain spectrum.

In summary, Eqs. (5.5a) and (5.5b) determine the time deviation ΔT from the transmission peak of the modulator and the frequency deviation of the spectral envelope of the pulse from the peak of the (frequency-dependent) gain.

Separating Eq. (5.6) into real and imaginary parts, we find

$$\frac{g}{\omega_g^2}\frac{1}{\tau_0^4} - M\omega_M^2 + \Delta B\left(D\frac{1}{\tau_0^4} + \frac{\Phi_0}{\tau_0^2}\right) + \Delta G D\frac{1}{\tau_0^4}2\gamma = 0, \tag{5.6a}$$

$$\frac{g}{\omega_g^2}\frac{1}{\tau_0^4}2\gamma - \Delta G\left(D\frac{1}{\tau_0^4} + \frac{\Phi_0}{\tau_0^2}\right) + \Delta B D\frac{1}{\tau_0^4}2\gamma = 0. \tag{5.6b}$$

We note, first, that pulse shortening that compensates for the pulse broadening of gain dispersion requires a negative value of ΔB. Figure 10 shows both ΔB and ΔG as functions of phase. We find that ΔB is negative for negative phase and positive for positive phase. This is consistent with the

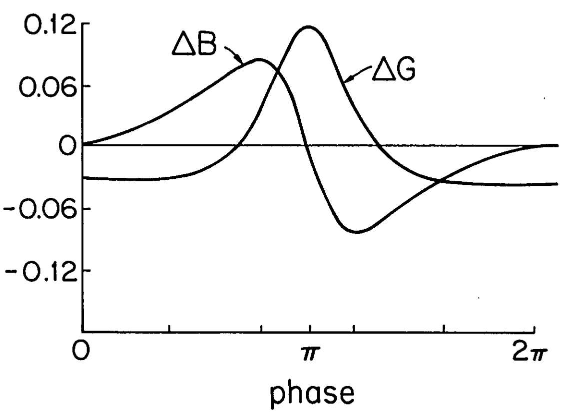


Fig. 10. Plot of ΔG and ΔB as functions of ϕ: $r = 0.9, L = 0.3$.

preceding study of pulse shortening by the auxiliary cavity termination.

The forced mode-locking term proportional to M is usually unimportant in Eq. (5.6a) when successful APM operation is achieved and can be omitted. It is necessary, in general, to initiate the pulse formation by forced, or synchronous, mode locking. Once pulsed operation is initiated, operation can be maintained by a cw pump without synchronous mode locking, as observed by Mitschke and Mollenauer[15] and also by us. When this occurs, one must reexamine the meaning of the terms in Eq. (5.5a) with $M = 0$. The parameter T_L, as originally defined, indicated deviation of the laser-cavity round-trip time from the mode-locking-drive period. In the absence of such a drive, T_L must be set to zero. How, then, can one balance Eq. (5.5a) as a function of the deviation of the auxiliary-cavity round-trip time from the laser-cavity round-trip time? A more careful look shows that Eq. (5.5a) must be reconsidered. In Eq. (5.2) we assumed that the pulse returns unchanged at the repetition period imposed by the mode-locking drive. When such a drive is missing, the pulse can return time shifted. Such a time shift can be expressed by a term $T_s \mathrm{d}/\mathrm{d}t$ in the equations. It is such a term that is then balanced against the contribution of $T_A \mathrm{d}/\mathrm{d}t$ in Eq. (5.5a).

In summary, Eqs. (5.6a) and (5.6b) determine the pulse width and the chirp of the pulse. It is gratifying that the perturbation analysis gives such a clear separation of the various effects.

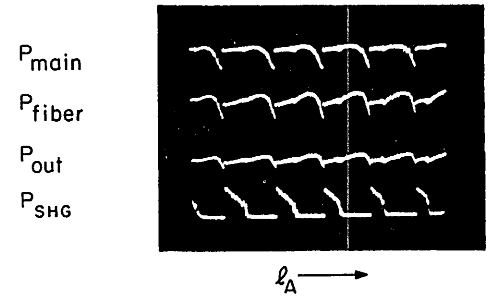


Fig. 11. (Top) Power in laser cavity, (second trace) power in the fiber, (third trace) output power, (bottom trace) second harmonic.

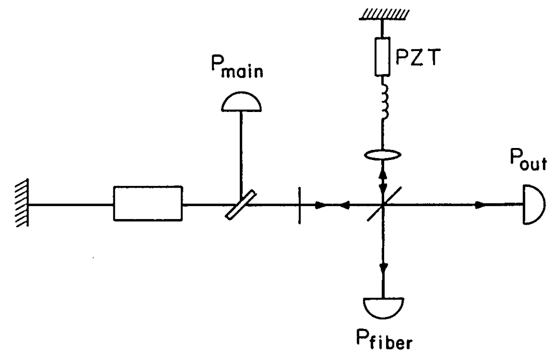


Fig. 12. Schematic of measurement. The length of the fiber is varied with a piezoelectric transducer (PZT).

6. SOME EXPERIMENTAL RESULTS

Figure 11 was obtained by carefully monitoring the power in the laser cavity and the output power as a function of length of the compressor cavity. The detectors were arranged as shown in Fig. 12. The power in the laser cavity is a maximum when $\phi = 0$. This can serve as a reference. As the fiber cavity shortens, the phase goes negative. It is clear that short-pulse formation exists only on the portions of the curves with negative phase. It should be pointed out, however, that the phase as defined in the theory refers to the peak of the pulse, not to the unbiased phase of the fiber resonator. The unbiased phase is more negative than ϕ defined here.

From experiments with synchronous mode locking it is known that short-pulse operation tends to get disrupted more easily when the pulse is pushed forward in time (the laser cavity is shortened) away from synchronism, as opposed to being moved backward in time (laser cavity lengthened). The reason for this is that when the pulse is pushed toward the rising gain curve it is deprived of the necessary gain. We shall now argue that the same effect works when the fiber cavity is lengthened but is less effective when the fiber cavity is shortened. For this purpose we look at Eq. (5.5a). In this equation, for $T_L = 0$,

$$2M\omega_M^2 \Delta T + \Delta G \frac{T_A}{\tau_0^2} = 0. \tag{6.1}$$

Thus the sign of ΔT depends on the sign of ΔG as well as on the sign of T_A. Figure 10 shows that ΔG is small when negative but much larger when positive. Now, ΔT positive means that the pulse is pushed forward with respect to the modulation (i.e., the bad case). A longer fiber cavity gives a negative T_A. Thus the bad case may be expected to occur for a longer fiber cavity.

Another observation is also worth reporting. This is the fact that the compressor cavity could be detuned by many micrometers (30 μm to be precise) and the system would still be mode locking. This is in contrast with the observations on synchronous mode locking in which only slight detunings of the laser cavity disrupt the mode locking. This observation may be explained by the fact that the effect of T_A on the mode locking is multiplied by ΔG in Eq. (5.5a) and ΔG is of order 0.1 according to Fig. 10. Thus the auxiliary cavity detuning is sensed by the mode-locking process much less than is the detuning of the laser cavity itself.

Suppose that there is no chirp, $\gamma = 0$, no dispersion, $D = 0$, and the phase is set so that $\Delta G = 0$ as shown in Fig. 10. Then Eq. (5.6b) is satisfied, and Eq. (5.6a) becomes

$$\frac{g}{\omega_g^2}\frac{1}{\tau_0^4} - M\omega_M^2 + \Delta B \frac{\Phi_0}{\tau_0^2} = 0. \tag{6.2}$$

Now, Φ_0 is the phase shift at the peak of the pulse and thus is pulse-energy dependent. For a particular operating level, the intensity at the peak of the pulse is proportional to $1/\tau_0$, and thus Φ_0/τ_0^2 is proportional to K/τ_0^3, where K is now an energy-independent constant. Figure 13 plots the pulse-shortening effect

$$M\omega_M^2 - \Delta B \frac{K}{\tau_0^3} \tag{6.3}$$

and the pulse-lengthening effect of the gain dispersion proportional to $1/\tau_0^4$. This operating point can be maintained stably, even after the mode locking is turned off, when $M = 0$; of course it gets shifted slightly to longer pulse widths. Self-starting, with $M = 0$, is also possible but may be more difficult to achieve in practice because of the longer pulse-formation time required.

The above analysis considers stabilization that is due to dispersion in the laser medium alone. In some cases, however, pulse-width limiting and stabilization may occur as a result of group-velocity dispersion in the fiber. Analysis of that case is complicated by the fact that pulse-shape changes in the fiber also affect the shaping that is due to pulse addition at the coupling mirror.

One particularly dramatic experimental observation in support of the theory was provided by the frequency shift of the pulse spectrum as a function of auxiliary cavity length. An effect not explicitly included in the formalism is the group-velocity dependence on frequency. Indeed, the fiber dispersion has been taken into account only by the diffusion operator $jD\mathrm{d}^2/\mathrm{d}t^2$ in Eq. (4.5). When $D > 0$, then the group velocity decreases with increasing frequency. If this effect were indeed important, then one would expect that lengthening of the auxiliary cavity would lead to a frequency decrease, so that the round-trip time in the auxiliary cavity would be maintained equal to that in the laser cavity. Experiments indicate that this prediction is erroneous. Figure 14 shows two spectra for two different cavity lengths. The larger cavity length leads to an increase of frequency. The preceding theory indeed predicts such a behavior, confirming the fact that the group-velocity changes due to frequency changes are less important than the frequency pushing due to the APM modulation predicted by Eq. (5.5b). In order to predict the sign of the frequency shift, we note that an increase of auxiliary cavity length from a match with the laser cavity leads to a negative $-T_A$, according to the definition in Eq. (4.5). ΔB is negative when successful APM operation occurs. Thus Eq. (5.5b) predicts a positive $\Delta\omega$, in agreement with observation.

The magnitude of the frequency shift can also be predicted. The pulse is unchirped, according to Eq. (5.6b), when ΔG is equal to zero. Concentrating on that case, we may evaluate $g/(\omega_g\tau_0)^2$ from Eq. (5.6a), ignoring the forced mode-

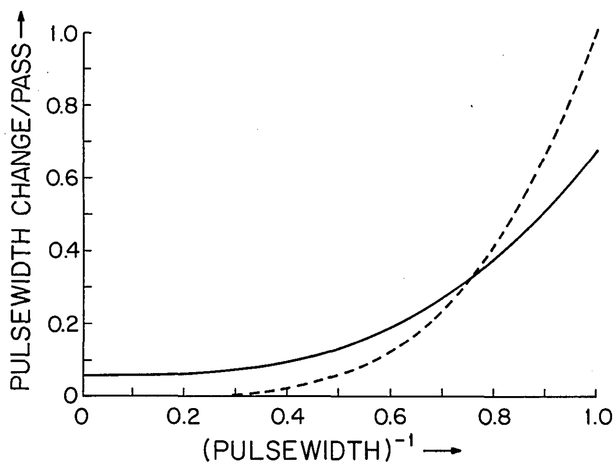


Fig. 13. Solid curve, pulse shortening per pass. Dashed curve, pulse lengthening per pass. Both in arbitrary units. See text and Eq. (6.2).

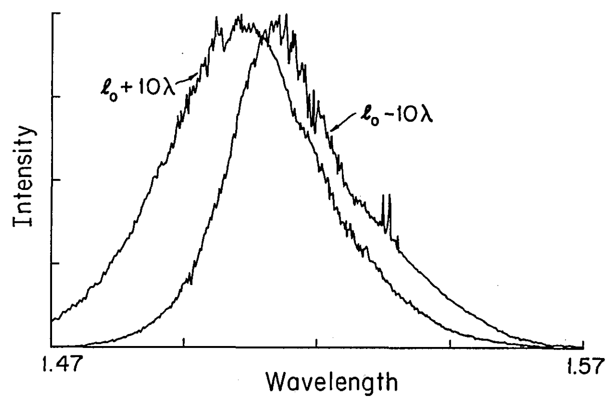


Fig. 14. Spectrum of the APM output for $l_0 + 10\lambda$ ($T_A = -50$ fsec) and $l_0 - 10\lambda$ ($T_A = +50$ fsec).

locking term because it is usually negligible compared with the APM term:

$$\frac{g}{\omega_g{}^2\tau_0{}^2} = -\Delta B\left(\frac{D}{\tau_0{}^2} + \Phi_0\right). \tag{6.4}$$

When we introduce this expression into Eq. (5.5b) we find that

$$\Delta\omega\tau_0 = \frac{T_A/\tau_0}{2\left(\dfrac{D}{\tau_0{}^2} + \Phi_0\right)}. \tag{6.5}$$

In our experiments $D = 3 \times 10^{-27}\ \mathrm{sec}^2$ and $\tau_0 = 100$ fsec. Thus, with a measured $\Delta\omega = 6 \times 10^{12}\ \mathrm{sec}^{-1}$ for $T_A = 100$ fsec, Eq. (6.5) implies a $\Phi \approx 0.5$. This is somewhat less than that calculated by using our estimate of peak intensity in the fiber but nevertheless is a reasonable confirmation of our model's prediction, considering experimental uncertainties and the theoretical approximations made.

7. STABILITY ANALYSIS

We have already investigated stability in a simple way by identifying as the stable operating point the point in Fig. 13 at which pulse shortening and pulse lengthening balance. A temporary deviation from this point toward shorter pulses makes the pulse-lengthening mechanism predominate and pushes the pulse width back toward the steady-state operating point. Conversely, a temporary deviation toward longer pulses causes the pulse-shortening mechanism to predominate and again returns the operation to the steady state. A full-fledged stability analysis requires the study of the equations of motion of the six pulse parameters: the phase perturbation $\delta\psi$; the amplitude perturbation $\delta|b_1|$; the frequency deviation from the peak of the gain curve, $\delta\omega$; the pulse timing δT, the pulse-width parameter $\delta(1/\tau_0{}^2)$; and the chirp parameter $\delta\gamma$. The pulse amplitude is, of course, determined by the gain adjustment. Excessive energy in the pulse decreases the gain by gain saturation and becomes suppressed thereby. The phase ψ is the phase in the laser cavity that adjusts itself with a slight deviation of the comb of frequencies constituting the spectrum of the pulse, from the comb of frequencies constituting the Fabry–Perot resonances of the cavity. The frequency deviation $\delta\omega$ is the deviation of the spectral peak of the pulse from the peak of the gain spectrum. It adjusts itself by frequency pulling or pushing owing to gain dispersion, chirp, and pulse timing δT. This timing in turn is affected by temporal pulse pushing or pulling. A change of δT is accompanied by a change of the spacing of the frequency components of the pulse spectrum, as opposed to the shift of the comb causing a change of ψ. Then, of course, the pulse width and chirp are governed by the fiber nonlinearity and dispersion. The six parameters call for six equations. These six equations are obtained by evaluating the change per pass of the pulse as determined by the generalization of the three complex equations (5.4)–(5.6). In one pass ψ changes by $\delta\psi$, ω by $\delta\omega$, etc. We determine the change per pass of b_1 by writing

$$\begin{aligned}\Delta b_1 &= (B_1 + \delta B_1)\exp[-j(\psi + \delta\psi)]\exp[j(\omega_0 + \delta\omega)t] \\ &\quad\times \exp\left(-\left\{\frac{1}{2}(t + \delta T)^2\left[\frac{1}{{\tau_0}^2} + \delta\left(\frac{1}{{\tau_0}^2}\right)\right][1 + j(\gamma + \delta\gamma)]\right\}\right) \\ &\quad - B_1 e^{j\psi}\exp(-j\omega_0 t)\exp[-(t^2/2{\tau_0}^2)(1 + j\gamma)] \\ &\simeq \left\{\delta B_1 - j\delta\psi B_1 + \left[j\delta\omega t - \frac{t\delta T}{{\tau_0}^2}(1 + j\gamma)\right.\right. \\ &\quad \left.\left. - \frac{t^2}{2}\delta\left(\frac{1}{{\tau_0}^2}\right)(1 + j\gamma) - \frac{t^2}{2{\tau_0}^2}j\delta\gamma\right]B_1\right\} \\ &\quad\times \exp(-j\psi)\exp(j\omega_0 t)\exp\left[-\frac{t^2}{2{\tau_0}^2}(1 + j\gamma)\right], \end{aligned} \tag{7.1}$$

where we have used B_1 for the amplitude of the pulse.

Now a change of b_1 in one transit can be identified as the derivative $T_R \cdot \mathrm{d}/\mathrm{d}T$, where T_R is the round-trip time and $\mathrm{d}/\mathrm{d}T$ is the long-term derivative of the pulse evolution (as opposed to the short-term time derivative used thus far, $\mathrm{d}/\mathrm{d}t$, operating on an individual pulse). We recognize that Eqs. (5.4)–(5.6), which have zeros on their right-hand sides, describe the change per pass if they do not balance. By equating time-dependent terms, terms proportional to t, and terms proportional to t^2, we obtain the three complex equations. From the time-independent terms we find that

$$\begin{aligned}&T_R\frac{\mathrm{d}}{\mathrm{d}t}(\delta B_1 + j\delta\psi B_1) \\ &= \left\{\frac{C_B}{C_a}e^{-j\psi}\left(1 - \frac{1}{{\omega_g}^2\tau^2}\right) - M{\omega_M}^2\Delta T^2 - \frac{g\Delta\omega^2}{{\omega_g}^2} - 1\right\}B_1. \end{aligned}\tag{7.2}$$

The gain g in the above equation is a function of the energy as stated in Eq. (3.1) [compare Eq. (5.4)]. The terms proportional to t give [compare Eq. (5.5)]

$$\begin{aligned}T_R\frac{\mathrm{d}}{\mathrm{d}t}\left[-\frac{\delta T}{{\tau_0}^2}(1 + j\gamma) + j\delta\omega\right] &= -2j\Delta\omega\frac{g}{{\omega_g}^2}\frac{1}{\tau^2} + 2M{\omega_M}^2\Delta T \\ &\quad - \frac{T_L}{\tau^2} + (\Delta G + j\Delta B)\frac{T_A}{\tau^2}. \end{aligned}\tag{7.3}$$

The terms proportional to t^2 yield [compare Eq. (5.6)]

$$\begin{aligned}T_R\frac{\mathrm{d}}{\mathrm{d}T}\left[-j\frac{\delta\gamma}{2{\tau_0}^2} - \frac{1}{2}\delta\left(\frac{1}{{\tau_0}^2}\right)(1 + j\gamma)\right] &= \frac{g}{{\omega_g}^2}\frac{1}{\tau^4} - M{\omega_M}^2 \\ &\quad - (\Delta G + j\Delta B)j\left(\frac{D}{\tau^4} + \frac{\Phi_0}{{\tau_0}^2}\right). \end{aligned}\tag{7.4}$$

These complex equations must be linearized in terms of the perturbations and separated into real and imaginary parts, yielding six coupled differential equations. An additional complication is that most terms on the right-hand sides of the equations also depend on the independent variables. Thus ΔG and ΔB depend on $\delta\psi$, etc. We shall not write them down explicitly but refer back to their steady-state counterparts, the left-hand sides of which give the change of one of the six independent variables per pass. Stability is proved when it is shown that the six eivenvalues of the six equations describe decaying exponentials. A necessary, but not sufficient, condition for stability is that all diagonal matrix elements of the coupling matrix be negative. This means, physically, that the perturbation of any one of the six independent variables by itself, in the absence of all the other, must decay. We shall consider only these restricted stability conditions.

The perturbation of amplitude and phase is described by Eq. (7.2). The real part expresses the relaxation of the amplitude if it deviates from the steady-state value. Since the gain decreases with increasing amplitude, deviations away from the steady-state amplitude decay (in the absence of any other perturbations). The imaginary part of Eq. (7.2) governs the relaxation of phase. This is dominated by the phase shift in the laser cavity [$\exp(-j\psi)$] and slightly modified by the dependence of C_b/C_a on ϕ. Indeed, suppose that we have a sudden change of phase $\delta\psi > 0$. This causes a negative $\mathrm{d}(\delta\psi)/\mathrm{d}T$ according to Eq. (6.2) because $|C_b/C_a|(1 + g - l) > 0$, only slightly modified by the compressor cavity if L is small, as it usually is. A negative $\mathrm{d}(\delta\psi)/\mathrm{d}T$ corresponds to a decrease of frequency (a shift of the Fourier comb to slightly lower frequencies) because the time derivative of $\delta\psi$ is a frequency shift. A decrease of frequency decreases the phase delay and restores ψ to the original value.

The perturbation of the pulse width as described by Eq. (5.6a) has been shown to be stable already. The perturbation of the chirp parameter, according to Eq. (7.4) [compare Eq. (5.6b)] will decay if

$$\frac{g}{{\omega_g}^2} + \Delta BD > 0. \tag{7.5}$$

Since $\Delta B < 0$, we see that negative dispersion guarantees stability, whereas positive dispersion must be balanced by narrowing the gain dispersion to compensate for it. Thus the pulse shortening attributable to positive dispersion is illusory since it must be balanced out to provide stability.

The real part of the left-hand side of Eq. (7.3) [compare Eq. (5.5a)] gives the change of the pulse timing per pass. The negative coefficient $-2M{\omega_M}^2$ in the equation for δT guarantees stability. Of course, when the mode locking (or gain modulation) is turned off, the timing is unstable, and the pulse can occur at any time. Yet that does not mean that short-pulse generation is impossible but only that the timing of the pulse train is indeterminate. Turning to the imaginary part of Eq. (7.3) [compare Eq. (5.5b)], the left-hand side of which gives the change of the carrier frequency, one also finds stability because of the negative coefficient $-g(\omega_g{}^2\tau_0{}^2)$ in front of $\delta\omega$.

8. CONCLUSIONS

We have developed the theory of a mode-locking process that we called additive pulse mode locking (APM). It utilizes the nonlinear phase shift and spectral broadening produced by a Kerr medium. By adding the phase-shifted pulse to an unshifted version of itself, pulse shortening is achieved directly. This mechanism explains the operation of the soliton laser and also explains the mode-locking behavior induced by fibers in the normal dispersion regime. This form of mode locking has great potential. Indeed, more conventional mode locking with a slow saturable absorber requires (a) a saturable absorber of adequate optical cross section and relaxation time and (b) a gain medium with a relaxation time comparable with that of the absorber. These two conditions have been achieved only with a limited number of laser systems. The APM mechanism promises to be applicable over a much broader wavelength range and with a greater variety of gain media.

The compressor cavity was analyzed as a termination that returns an incident pulse shortened and only slightly attenuated. The full analysis of mode locking revealed several features that were confirmed by experiment. Mode locking can occur only within a portion of the 2π phase shift of the compressor cavity. The system can self-pulse even after the mode-locking modulation is turned off. The wide range over which the compressor cavity can be detuned is understandable from the analytic model. The analytic model also shows that positive dispersion does not prevent mode locking. At first sight, it even seems to enhance it. However, the stability analysis showed that positive dispersion can drive the system unstable and must be compensated for by narrowing of the gain bandwidth. This does not necessarily lead to longer pulses because of the intrinsic pulse-shortening effect of positive dispersion. Negative dispersion appears to hinder mode locking at first, an observation that seems surprising because of the soliton image that adheres to this kind of mode-locking mechanism. However, the simple observation that a soliton formation itself relies on the cancellation of nonlinear phase shifts supports this notion. Finally, we note that the sign and the magnitude of the APM laser frequency shift as a function of auxiliary cavity length were observed to vary experimentally as predicted by the theory.

ACKNOWLEDGMENTS

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REFERENCES

1. J. Mark, L. Y. Liu, K. L. Hall, H. A. Haus, and E. P. Ippen, "Femtosecond pulse generation in a laser with a nonlinear external resonator," Opt. Lett. **14**, 48–50 (1989).
2. L. F. Mollenauer and R. H. Stolen, "The soliton laser," Opt. Lett. **9**, 13–15 (1984).
3. P. N. Kean, R. S. Grant, X. Zhu, D. W. Crust, D. Burns, and W. Sibbett, "Enhanced mode locking of colour-centre lasers by coupled-cavity feedback control," in *Digest of Conference on Lasers and Electro-Optics* (Optical Society of America, Washington, D.C., 1988), paper PD7.
4. P. N. Kean, X, Zhu, D. W. Crust, R. S. Grant, N. Langford, and W. Sibbett, "Enhanced mode locking of color-center lasers," Opt. Lett. **14**, 39–41 (1989).
5. K. J. Blow and B. P. Nelson, "Non-soliton mode locking of an F-center laser with nonlinear external resonator," in *Ultrafast Phenomena VI*, T. Yajima, K. Yoshihara, C. B. Harris, and S. Shionoya, eds., Vol. 48 of Springer Series in Chemical Physics (Springer-Verlag, Berlin, 1988), pp. 67–99.
6. P. A. Belanger, "Soliton laser. I: A simplified model," J. Opt. Soc. Am. B **5**, 793–798 (1988).
7. K. J. Blow and D. Wood, "Mode-locked lasers with nonlinear external cavities," J. Opt. Soc. Am. B **5**, 629–632 (1988).
8. F. Oullette and M. Piché, "Ultrashort pulse reshaping with a nonlinear Fabry–Perot cavity matched to a train of short pulses," J. Opt. Soc. Am. B **5**, 1228–1236 (1988).
9. L. E. Dahlstrom, "Passive mode locking and Q-switching of high power lasers by beams of the optical Kerr effect," Opt. Commun. **5**, 157–162 (1972).
10. F. Oullette and M. Piché, "Pulse shaping and passive mode-locking with a nonlinear Michelson interferometer," Opt. Commun. **60**, 99–103 (1986).
11. H. A. Haus and Y. Silberberg, "Laser mode locking with addition of nonlinear index," IEEE J. Quantum Electron. **QE-22**, 325–331 (1986).
12. H. A. Haus, *Waves and Fields in Optoelectronics* (Prentice-Hall, Englewood Cliffs, N.J., 1984). We use a more convenient phase reference for the scattering matrix of the mirror.
13. M. S. Stix and E. P. Ippen, "Pulse-shaping in passively mode-locked ring dye lasers," IEEE J. Quantum Electron. **QE-19**, 520–525 (1983).
14. H. A. Haus, "A theory of forced mode locking," IEEE J. Quantum Electron. **QE-11**, 323–330 (1975).
15. F. M. Mitschke and L. F. Mollenauer, "Stabilizing the soliton laser," IEEE J. Quantum Electron. **QE-22**, 2242–2250 (1986).