Subject

- B$^+$-Tree with bounded-length entries
- each node is stored in one disk block
- at least $\frac{3}{8}$ of each block is used
- only sizes of keys & values vary!

Definitions

- N is a node N with sorted entries $e \in N$
- following accessor functions: $parent(N)$, $left(N)$, $right(N)$, $key(e)$, $value(e)$ (leafs only), $count(e)$ & $child(e)$ (inner nodes only)
- $size(N)$ = fix overhead + $\sum_{e \in N} size(e)$
  $size(e)$ = fix overhead + $size(key(e)) + size(value(e))$
- B = disk block size − fix node overhead

Bounds

For root R, node $N \neq R$ and any entry e, these invariants must hold:

$$\begin{aligned} & size(e) && \leq \lfloor \tfrac{1}{4} B \rfloor \\ \lfloor \tfrac{3}{8} B \rfloor \leq \; & size(N) && \leq B \\ 0 \leq \; & size(R) && \leq B \end{aligned} \qquad (*)$$

Overflow-Theorem

A node with entries $N \cup \{e\}$ with

$$size(N) \leq B \text{ but } size(N \cup \{e\}) > B$$

can be split into two nodes such that both fulfill (*).

Underflow-Theorem

When an entry e is removed from a non-root-node with entry set L such that $size(L) \geq \lfloor \frac{3}{8} B \rfloor$ but $size(L \setminus \{e\}) < \lfloor \frac{3}{8} B \rfloor$, either entries from a neighbor can be moved or L can be merged with a neighbor.

- Someone has an idea what to put here?
- That's my name,
  ask me again and I'll tell you the same.
- San Diego here I come,
  Melmac's where I started from.
- Kate, you've got to hear this.

Proof of Overflow-Theorem

For $N \cup \{e\} = \{e_1, \ldots, e_n\}$, set $L := \{e_1, \ldots, e_i\}$ such that

$$size(L) \geq \lfloor \tfrac{3}{8} B \rfloor \text{ but } size(L \setminus \{e_i\}) < \lfloor \tfrac{3}{8} B \rfloor$$

and $R := (N \cup \{e\}) \setminus L$. Then L and R fulfill (*):

$$\begin{aligned} size(L) &\overset{\text{def}}{\geq} \lfloor \tfrac{3}{8} B \rfloor \\ size(L) &< \lfloor \tfrac{3}{8} B \rfloor + \underbrace{size(e_i)}_{\leq \lfloor \frac{1}{4} B \rfloor} \leq \lfloor \tfrac{5}{8} B \rfloor \end{aligned}$$

$$\begin{aligned} size(R) &= \overbrace{size(N \cup \{e\})}^{> B} - \overbrace{size(L)}^{< \lfloor \frac{5}{8} B \rfloor} > \lceil \tfrac{3}{8} B \rceil \\ size(R) &= \underbrace{size(N \cup \{e\})}_{\leq B + \lfloor \frac{1}{4} B \rfloor} - \underbrace{size(L)}_{\geq \lfloor \frac{3}{8} B \rfloor} \leq \lceil \tfrac{7}{8} B \rceil \end{aligned}$$

Proof of Underflow-Theorem

Let $\delta := \lfloor \frac{3}{8} B \rfloor - size(L \setminus \{e\})$ be the space in $L \setminus \{e\}$ that needs to be filled to fulfill (*). Without loss of generality, we assume that the right neighbor with entry set R of L exists; otherwise all left/right and min/max relations turn around.

1. If $size(R) \leq \lceil \frac{5}{8} B \rceil + \delta$, the nodes can be merged to a single node $E := (L \setminus \{e\}) \cup R$ that fulfills (*):

$$\begin{aligned} size(E) &= \overbrace{size(L \setminus \{e\})}^{\geq \lfloor \frac{3}{8} B \rfloor - \lfloor \frac{1}{4} B \rfloor} + \overbrace{size(R)}^{\geq \lfloor \frac{3}{8} B \rfloor} \geq \lfloor \tfrac{1}{2} B \rfloor \\ size(E) &= \underbrace{size(L \setminus \{e\})}_{\leq \lfloor \frac{3}{8} B \rfloor - \delta} + \underbrace{size(R)}_{\leq \lceil \frac{5}{8} B \rceil + \delta} \leq B \end{aligned}$$

2. If $size(R) > \lceil \frac{5}{8} B \rceil + \delta$, a set $S \subseteq R$ must be moved from R to L. Set $L' := (L \setminus \{e\}) \cup S$ and $R' := R \setminus S$. S must be chosen so that it holds

$$size(L') \geq \lfloor \tfrac{3}{8} B \rfloor \text{ but } size(L' \setminus \max S) < \lfloor \tfrac{3}{8} B \rfloor.$$

Hence, we have

$$\delta \leq size(S) < \delta + \lfloor \tfrac{1}{4} B \rfloor.$$

It follows that L' and R' fulfill (*):

$$\begin{aligned} size(L') &\overset{\text{def}}{\geq} \lfloor \tfrac{3}{8} B \rfloor \\ size(L') &= \underbrace{size(L \setminus \{e\})}_{= \lfloor \frac{3}{8} B \rfloor - \delta} + \underbrace{size(S)}_{< \delta + \lfloor \frac{1}{4} B \rfloor} < \lfloor \tfrac{5}{8} B \rfloor \\ size(R') &= \overbrace{size(R)}^{> \lceil \frac{5}{8} B \rceil + \delta} - \overbrace{size(S)}^{< \delta + \lfloor \frac{1}{4} B \rfloor} > \lfloor \tfrac{3}{8} B \rfloor \\ size(R') &< size(R) \leq B \end{aligned}$$