Proof that $\mathcal{ESB}$ subsumes $\mathcal{ES}$

We refer to the variants of $\mathcal{ESB}$ ECAI-2014 and $\mathcal{ES}$ from KR-2004. The first definitions are taken from the ECAI paper on $\mathcal{ESB}$:

Definition 1 The relation $\simeq_z$ is the least relation that satisfies

- $w' \simeq_{\langle\rangle} w$;
- $w' \simeq_{z\cdot r} w$ iff $w' \simeq_z w$, $w'[SF(r), z] = w[SF(r), z]$.

Definition 2 The translation of an $\mathcal{ES}$ formula α to an $\mathcal{ESB}$ formula α^* is defined inductively: $Know(\alpha)^*$ is $\mathbf{K}(\alpha^*)$, $OKnow(\alpha^*)$ is $\mathbf{O}(\alpha^*, \{\})$, and in all other cases α^* is the identity function.

Notation 3 We write $e, w, z \models_{\mathcal{ES}} \alpha$ for the satisfaction in $\mathcal{ES}$. We write $f, w, z \models \alpha$ for satisfaction in $\mathcal{ESB}$. We abbreviate $\mathbf{O}(\alpha, \{\})$ by $\mathbf{O}\alpha$ in $\mathcal{ESB}$ formulas.

We show that α is satisfiable in $\mathcal{ES}$ iff α^* is satisfiable in $\mathcal{ESB}$ in Corollary 6 and Corollary 9. The only-if direction is quite simple, but the if direction is a bit tricky. The reason why the latter holds is that there is an infinite supply of atoms in $\mathcal{ES}$ and $\mathcal{ESB}$.

Definition 4 For any set of worlds e we define $e_z^w = \{w' \mid w' \in e, w' \simeq_z w\}$. For any function f we define f_z^w such that $f_z^w(p) = \{w' \mid w' \in f(p), w' \simeq_z w\}$ for all $p \in \mathbb{N}$.

Theorem 5 *Suppose α is an $\mathcal{ES}$ sentence. For a given e let f be such that $f(p) = e$ for all $p \in \mathbb{N}$. Then $e, w, z \models_{\mathcal{ES}} \alpha$ iff $f, w, z \models \alpha^*$.*

Proof. By induction. The base cases $P(\vec{r})$ and $r_1 = r_2$ as well as the induction steps for $\wedge$, $\neg$, $\forall$, $[r]$, and $\Box$ are trivial because their semantic rules are equivalent in $\mathcal{ES}$ and $\mathcal{ESB}$.

Now consider $\alpha = Know(\beta)$. The following steps preserve equivalence:

1. $e, w, z \models_{\mathcal{ES}} Know(\beta)$
2. $e, w', z \models_{\mathcal{ES}} \beta$ for all $w' \in e_z^w$
3. $f, w', z \models \beta^*$ for all $w' \in e_z^w$
4. $f, w', z \models \beta^*$ for all $w' \in f_z^w(p)$ for all $p \in \mathbb{N}$
5. $f, w, z \models \mathbf{K}\beta^*$

Step 3 holds iff step 2 holds by induction hypothesis.

Now consider $\alpha = OKnow(\beta)$. The following steps preserve equivalence:

1. $e, w, z \models_{\mathcal{ES}} OKnow(\beta)$
2. $e, w', z \models_{\mathcal{ES}} \beta$ iff $w' \in e_z^w$
3. $f, w', z \models \beta^*$ iff $w' \in e_z^w$
4. $f, w', z \models \beta^*$ iff $w' \in f_z^w(p)$ for all $p \in \mathbb{N}$
5. $f, w, z \models \mathbf{O}\beta^*$

Step 3 holds iff step 2 holds by induction hypothesis. $\square$

Corollary 6 *If α is satisfiable in $\mathcal{ES}$ then α^* is satisfiable in $\mathcal{ESB}$.*

Definition 7 For an atom A we define w_i^A such that $w_i^A[A, z] = i$ and $w_i^A[B, z] = w[B, z]$ for all atoms $B \neq A$.

Theorem 8 *Suppose α is an $\mathcal{ES}$ sentence. Let A be an atom not mentioned in α. For a given f let $e = e_1 \cup \{(w')^A_0 \mid w' \in e_2, (w')^A_1 \notin e_1\}$ where $e_1 = \bigcap_{p\in\mathbb{N}} f(p)$ and $e_2 = \bigcup_{p\in\mathbb{N}} f(p)$. Then $e, w, z \models_{\mathcal{ES}} \alpha$ iff $f, w, z \models \alpha^*$.*

Proof. An atom A exists because α mentions only finitely many predicate symbols but there are countably infinitely many.

By induction. The base cases $P(\vec{r})$ and $r_1 = r_2$ as well as the induction steps for $\wedge, \neg, \forall, [r]$, and $\Box$ are trivial because their semantic rules are equivalent in $\mathcal{ES}$ and $\mathcal{ESB}$.

Now consider $\alpha^* = \mathbf{K}\beta^*$. The following steps preserve equivalence:

1. $f, w, z \models \mathbf{K}\beta^*$
2. $f, w', z \models \beta^*$ for all $w' \in f^w_z(p)$ for all $p \in \mathbb{N}$
3. $e, w', z \models_{\mathcal{ES}} \beta$ for all $w' \in f^w_z(p)$ for all $p \in \mathbb{N}$
4. $e, w', z \models_{\mathcal{ES}} \beta$ for all $w' \in (e_1)^w_z$ and
   $e, w', z \models_{\mathcal{ES}} \beta$ for all $w' \in (e_2 \setminus e_1)^w_z$
5. $e, w', z \models_{\mathcal{ES}} \beta$ for all $w' \in (e_1)^w_z$ and
   $e, (w')^A_0, z \models_{\mathcal{ES}} \beta$ for all $w' \in (e_2 \setminus e_1)^w_z$ such that $(w')^A_1 \notin (e_1)^w_z$
6. $e, w', z \models_{\mathcal{ES}} \beta$ for all $w' \in (e_1)^w_z$ and
   $e, (w')^A_0, z \models_{\mathcal{ES}} \beta$ for all $w' \in (e_2)^w_z$ such that $(w')^A_1 \notin (e_1)^w_z$
7. $e, w', z \models_{\mathcal{ES}} \beta$ for all $w' \in e^w_z$ and
8. $e, w, z \models_{\mathcal{ES}} \mathit{Know}(\beta)$

Generally we have $e, w', z \models_{\mathcal{ES}} \beta$ iff $e, (w')^A_i, z \models_{\mathcal{ES}} \beta$ because β does not mention A. Step 3 follows from 2 by induction hypothesis. Step 4 follows from 5 because for all $w' \in (e_2 \setminus e_1)^w_z$ with $(w')^A_1 \in (e_1)^w_z$ we have $e, w', z \models_{\mathcal{ES}} \beta$ by the first condition of step 5. Step 6 follows from 5 because if $w' \notin (e_2 \setminus e_1)^w_z$ but $w' \in (e_2)^w_z$ then $w' \in (e_1)^w_z$ and by the first condition of step 5 follows $e, w', z \models_{\mathcal{ES}} \beta$.

Now consider $\alpha^* = \mathbf{O}\beta^*$. The following steps preserve equivalence:

1. $f, w, z \models \mathbf{O}\beta^*$
2. $f, w', z \models \beta^*$ iff $w' \in f^w_z(p)$ for all $p \in \mathbb{N}$
3. $e, w', z \models_{\mathcal{ES}} \beta$ iff $w' \in f^w_z(p)$ for all $p \in \mathbb{N}$
4. $e, w', z \models_{\mathcal{ES}} \beta$ iff $w' \in (e_1)^w_z$ and
   $(e_2)^w_z = (e_1)^w_z$
5. $e, w', z \models_{\mathcal{ES}} \beta$ iff $w' \in (e_1)^w_z$ and
   $e^w_z = (e_1)^w_z$
6. $e, w', z \models_{\mathcal{ES}} \beta$ iff $w' \in e^w_z$
7. $e, w, z \models_{\mathcal{ES}} \mathit{OKnow}(\beta)$

Step 3 follows from 2 by induction hypothesis. Step 5 holds iff step 4 holds because due to the "iff" in the first conditions of steps 4 and 5 we have $w' \in e_1$ iff $(w')^A_i \in e_1$ and therefore $(e_2)^w_z = (e_1)^w_z$ iff $\{(w')^A_0 \mid w' \in e_2, (w')^A_1 \notin e_1\}^w_z = \{\}$ iff $e^w_z = (e_1)^w_z$. $\Box$

Notice that Theorem 8 does not hold for $e = e_2$ in general. Let $f(0)$ be the set of all worlds and $f(p) = \{\}$ for all $p > 0$. Then e_2 is the set of all worlds. Thus we have $e_2 \models_{\mathcal{ES}} \mathit{OKnow}(\mathit{True})$ but $f \not\models \mathbf{O}\mathit{True}$.

Corollary 9 *If α^* is satisfiable in $\mathcal{ESB}$ then α is satisfiable in $\mathcal{ES}$.*

Corollary 10 *Let α be a sentence of $\mathcal{ES}$. Then α is valid in $\mathcal{ES}$ iff α^* is valid in $\mathcal{ESB}$.*

Proof. For the if direction, suppose $\models \alpha^*$ but $e, w \not\models_{\mathcal{ES}} \alpha$. Then $e, w \models_{\mathcal{ES}} \neg\alpha$. By Corollary 6, there is some f with $f, w \models \neg\alpha^*$ which contradicts $\models \alpha^*$.

For the only-if direction, suppose $\models_{\mathcal{ES}} \alpha$ but $f, w \not\models \alpha^*$. Then $f, w \models \neg\alpha^*$. By Corollary 9, there is some e with $e, w \models_{\mathcal{ES}} \neg\alpha$ which contradicts $\models_{\mathcal{ES}} \alpha$. $\square$