Critical Volatility Threshold for Log-Normal to Power-Law Transition

Iterated Options Model

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Abstract

Random walk models with log-normal outcomes fit local market observations remarkably well. Yet interconnected or recursive structures - layered derivatives, leveraged positions, iterative funding rounds - periodically produce power-law distributed events. We show that the transition from log-normal to power-law dynamics requires only three conditions: randomness in the underlying process, rectification of payouts, and iterative feed-forward of expected values. Using an infinite option-on-option chain as an illustrative model, we derive a critical volatility threshold at $\sigma^* = \sqrt{2\pi} \approx 250.66\%$ for the unconditional case. With selective survival - where participants require minimum returns to continue - the critical threshold drops discontinuously to $\sigma^*_{\text{th}} = \sqrt{\pi/2} \approx 125.3\%$, and can decrease further with higher survival thresholds. The resulting outcomes follow what we term the **V***

Distribution - a power-law whose exponent admits closed-form expression in terms of survival probability and conditional expected growth. The result suggests that fat tails may be an emergent property of iterative log-normal processes with selection rather than an exogenous feature.

Preface

Financial systems are built on rectified payoffs. An investment in a high-risk project returns either something or nothing - you cannot lose more than you put in. An option pays $\max(S - K, 0)$. Even limited liability is a form of rectification.

These rectified structures often feed into one another. A successful project enables others built on top of it. A successful trade becomes the capital for the next trade. Derivative products reference other derivative products. It would be useful to know how such iterations behave - whether they remain stable or exhibit qualitatively different dynamics.

To answer this, we analyze the limiting case: an infinite chain of options, each written on the expected payout of the one before. The result depends on three conditions - randomness in the underlying process, rectification of payouts, and feed-forward of expected values. These are sufficient to produce a critical threshold at $\sigma^* = \sqrt{2\pi} \approx 250.66\%$. Below this, cumulative optionality remains bounded. Above it, the system diverges. With selective survival - participants requiring minimum returns to continue - the threshold drops to $\sigma^*_{\text{th}} = \sqrt{\pi/2} \approx 125.3\%$. The divergent outcomes follow what we term the V* Distribution - a power-law whose exponent depends on the specific volatility and participants' willingness to make the next bet.

We also identify a self-similar regime at exactly the critical threshold, where each iteration reproduces the statistical structure of the previous one.

The conditions are minimal: randomness, bounded downside, and iteration. These are not exotic assumptions, suggesting the mechanism may apply broadly to dynamic systems with compounding behavior.

As for the infinite derivative tower itself: to our knowledge, no one has built one. This is probably wise. But should financial engineering continue its march toward increasingly layered products, at least the location of the cliff is now known.

1. Introduction and Motivation

The Black-Scholes framework provides a foundational model for pricing European options. Under risk-neutral valuation, the price of a call option reflects the expected value of its payoff $\max(S_T - K, 0)$, discounted appropriately. This “rectification” - the maximum of a potentially negative quantity and zero - is the essential nonlinearity that gives options their asymmetric payoff structure.

A natural question arises: what happens when we write an option on an option? And then an option on that? In principle, one could construct an arbitrarily deep tower of such instruments, each layer deriving its value from the expected payout of the layer below.

This paper analyzes the mathematical structure of such iterated rectified expectations. We find that:

1. The system exhibits a phase transition at a critical volatility of $\sigma^* = \sqrt{2\pi} \approx 250.66\%$ annualized for the unconditional ATM case. This assumes the perfect case, where pricing has no errors and volatility doesn't amplify between derivative layers but stays perfectly correlated to asset price at a constant ratio. Real systems, if built, will diverge much faster.

2. Below criticality ($\sigma < \sigma^*$), the total value of an infinite option chain converges to a finite sum, meaning optionality is “bounded” no matter how many layers are added.

3. Above criticality ($\sigma > \sigma^*$), the chain diverges - the cumulative value of optionality exceeds the underlying asset itself. This is not merely a mathematical curiosity; it implies that in extreme volatility regimes, the optionality can dominate the fundamental value of the product. This leads to amplification of expected payout at each consecutive step, making the expected payouts follow power-law dynamics.

4. At criticality ($\sigma = \sigma^*$), the system becomes self-similar, with each iteration reproducing the statistical structure of the previous one.

5. With selective survival - where participants require minimum returns to continue - the critical threshold drops to $\sigma^*_{\text{th}} = \sqrt{\pi/2} \approx 125.3\%$. Power-law dynamics (the V* Distribution) emerge when $\beta_{\text{eff}} > 1$, where β_{eff} is the conditional expected growth given survival.

These findings have implications for understanding volatility regimes during market stress, the pricing of compound options, and the theoretical limits of derivative layering.

2. The Black-Scholes Setup

2.1 Standard Framework

Under the Black-Scholes model, the underlying asset follows geometric Brownian motion:

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t$$

The Black-Scholes formula for a European call option is:

$$C(S_t, t) = N(d_+) S_t - N(d_-) K e^{-r(T-t)}$$

where:

$$d_+ = \frac{1}{\sigma\sqrt{T-t}} \left[\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t) \right]$$

$$d_- = d_+ - \sigma\sqrt{T-t}$$

and $N(\cdot)$ is the standard normal CDF.

2.2 ATM Special Case

For an at-the-money option where $S_t = K$, we have $\ln(S_t/K) = 0$, so:

$$d_+ = \frac{r + \sigma^2/2}{\sigma}\sqrt{T-t}$$

$$d_- = \frac{r - \sigma^2/2}{\sigma}\sqrt{T-t}$$

In the $r \ll \sigma$ limit (which holds for high-volatility regimes where $\sigma > 100\%$ and $r \approx 5\%$):

$$d_+ \approx \frac{\sigma^2/2}{\sigma}\sqrt{T-t} = \frac{\sigma}{2}\sqrt{T-t}$$

$$d_- \approx \frac{-\sigma^2/2}{\sigma}\sqrt{T-t} = -\frac{\sigma}{2}\sqrt{T-t}$$

Since $d_- \approx -d_+$, we have $N(d_-) = N(-d_+) = 1 - N(d_+)$, and the option price simplifies to:

$$C = K\left[N(d_+) - N(d_-)\right] = K\left[2N\left(\frac{\sigma\sqrt{T-t}}{2}\right) - 1\right]$$

The ATM option price reduces to a single Gaussian CDF minus a constant. Let us analyze its behavior.

2.3 The Gaussian Structure

The Gaussian distribution appears explicitly in Black-Scholes through $N(d_+)$ and $N(d_-)$, and simplifies to a single Gaussian CDF under the high-volatility ATM assumption. We observe that the option payoff $\max(S_T - K, 0)$ cannot be negative by definition - the option structure rectifies the underlying returns at zero.

The expected value of a rectified Gaussian is the mathematical core of option pricing. We now analyze the general case of $\mathbb{E}[\max(X, 0)]$ for $X \sim \mathcal{N}(\mu, \sigma^2)$.

In the following mathematical derivation, σ denotes the standard deviation of the distribution $X \sim \mathcal{N}(\mu, \sigma^2)$, following standard statistical convention, not the scale-less volatility.

2.4 Rectified Gaussian Expectations

Let $Y = \max(X, 0)$ where $X \sim \mathcal{N}(\mu, \sigma^2)$. We seek $\mathbb{E}[Y]$.

The expectation splits into two regions:

$$\mathbb{E}[Y] = \mathbb{E}[X \cdot \mathbf{1}_{X>0}] = \int_0^\infty x \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} dx$$

Substituting $u = (x - \mu)/\sigma$:

$$\mathbb{E}[Y] = \int_{-\mu/\sigma}^\infty (\mu + \sigma u) \cdot \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du$$

This separates into:

$$\mathbb{E}[Y] = \mu \int_{-\mu/\sigma}^\infty \phi(u)\, du + \sigma \int_{-\mu/\sigma}^\infty u \cdot \phi(u)\, du$$

The first integral is $\mu \cdot \Phi(\mu/\sigma)$. For the second, note that $u \cdot \phi(u) = -\phi'(u)$, so:

$$\int_{-\mu/\sigma}^\infty u \cdot \phi(u)\, du = [-\phi(u)]_{-\mu/\sigma}^\infty = \phi(\mu/\sigma)$$

Therefore:

$$\mathbb{E}[Y] = \mu\Phi\left(\frac{\mu}{\sigma}\right) + \sigma\phi\left(\frac{\mu}{\sigma}\right)$$

where $\Phi(\cdot)$ is the standard normal CDF and $\phi(\cdot)$ is the standard normal PDF.

2.5 The Function g(z)

Normalizing by σ and letting $z = \mu/\sigma$:

$$\frac{\mathbb{E}[Y]}{\sigma} = z\Phi(z) + \phi(z)$$

Expanding using the integral forms of Φ and ϕ:

$$\frac{\mathbb{E}[Y]}{\sigma} = \frac{1}{\sqrt{2\pi}} \left(z \int_{-\infty}^{z} e^{-t^2/2} dt + e^{-z^2/2} \right)$$

We define:

$$g(z) = z \int_{-\infty}^{z} e^{-t^2/2} dt + e^{-z^2/2}$$

so that the expected value of the rectified Gaussian becomes:

$$\mathbb{E}[Y] = \frac{\sigma}{\sqrt{2\pi}} g(z)$$

This form will be essential for analyzing iterations.

3. Iterated Options: The Mathematical Structure

3.1 The Iteration Scheme

Consider a chain of options where each option is written on the expected payout of the previous one. In a simplified model of such chain, we would be putting a new derivative instrument with the price of expected payout of the previous one, and calculate new parameters. Since the strike price from previous option would be just a constant multiplier, we can focus on analyzing the rectified Gaussian behavior as a simplified model.

Let μ_n denote the expected value at stage n, and suppose each stage has volatility σ_n.

From Section 2.5, the expected value of the rectified Gaussian at stage n is:

$$\mathbb{E}_n[Y] = \frac{\sigma_n}{\sqrt{2\pi}} g(z_n)$$

where $z_n = \mu_n/\sigma_n$.

If we let the output of one rectification become the mean of the next (i.e., $\mu_{n+1} = \mathbb{E}_n[Y]$), we obtain:

$$\mu_{n+1} = \frac{\sigma_n}{\sqrt{2\pi}} g(z_n)$$

3.2 General Recursion

From Section 2.5:

$$\mathbb{E}_1[Y] = \frac{\sigma_1}{\sqrt{2\pi}} g(z_1)$$

Let the output become the mean of the next stage: $\mu_2 = \mathbb{E}_1[Y]$.

The next price is:

$$z_2 = \frac{\mu_2}{\sigma_2} = \frac{\sigma_1}{\sigma_2} \cdot \frac{1}{\sqrt{2\pi}} g(z_1)$$

Let $r = \sigma_1/\sigma_2$ and $w = g(z_1)$. Then:

$$z_2 = \frac{r}{\sqrt{2\pi}} w$$

The next expectation:

$$\mathbb{E}_2[Y] = \frac{\sigma_2}{\sqrt{2\pi}} g(z_2)$$

More generally, letting $w_n = g(z_n)$ and $\alpha = \frac{r}{\sqrt{2\pi}}$:

$$w_{n+1} = g(\alpha \cdot w_n)$$

Explicitly:

$$w_{n+1} = \alpha w_n \int_{-\infty}^{\alpha w_n} e^{-t^2/2}\, dt + e^{-\alpha^2 w_n^2/2}$$

This is a nonlinear recursion whose behavior depends critically on α.

3.3 The Self-Similar Case

When $\alpha = 1$ (equivalently, $r = \sqrt{2\pi}$, i.e., $\sigma_1 = \sigma_2\sqrt{2\pi}$), the recursion simplifies to:

$$w_{n+1} = g(w_n)$$

This is pure iteration of g - the process becomes self-similar. The ratio:

$$\frac{w_n}{w_{n+1}} = \frac{w_n}{g(w_n)}$$

suggests that the sequence will have its own convergence/divergence behavior depending on w_n (or equivalently α). The parameter α controls how the recursion scales, determining whether iterated expectations grow, shrink, or stabilize.

4. The Recentered (ATM) Case

4.1 Introducing the Shift

In practice, options are often struck at-the-money (ATM), where the strike equals the current expected value. We model this by introducing a shift parameter s_n that recenters the distribution at each step:

$$z_n^s = \frac{\mu_n - s_n}{\sigma_n} = z_n - \frac{s_n}{\sigma_n}$$

Setting $s_n = \mu_n$ (the ATM condition) forces $z_n^s = 0$ at every iteration.

The intuition for this shift is the following: each new option is written ATM at inception, with strike equal to the current underlying price (which is the expected value from the previous stage). The underlying then fluctuates around this strike with its own volatility over the holding period. Since the strike equals the mean, the payoff-relevant distribution is centered at zero.

4.2 Evaluation at Zero

Since $g(0) = 0 + e^0 = 1$, the shifted expectation simplifies dramatically:

$$\mathbb{E}_n^s[Y] = \frac{\sigma_n}{\sqrt{2\pi}}$$

This is the well-known result that an ATM option's expected payout (before discounting and without market price multiplier) is proportional to volatility. It connects directly to the well-known practitioner's approximation:

$$C \approx \frac{S \cdot \sigma\sqrt{T}}{\sqrt{2\pi}} \approx 0.4 \cdot S \cdot \sigma\sqrt{T}$$

4.3 The Geometric Regime

In financial contexts, we frequently assume that volatility is a percentage related to the price - a stock with higher price has proportionally higher absolute volatility. We use a similar definition which scales with expected values.

We now return to the finance convention where σ denotes percentage volatility (coefficient of variation, σ_n/μ_n). Assuming constant percentage volatility (absolute volatility scales proportionally with price):

$$\mu_{n+1} = \frac{\sigma \cdot \mu_n}{\sqrt{2\pi}} = \beta \cdot \mu_n$$

where $\beta = \sigma/\sqrt{2\pi}$.

This yields the closed form:

$$\mu_n = \mu_1 \cdot \beta^{n-1}$$

The expected value at each stage forms a geometric sequence.

5. Convergence and the Critical Threshold

5.1 Sum of the Infinite Chain

The total expected value across an infinite chain of ATM options is:

$$\sum_{n=1}^{\infty} \mu_n = \mu_1 \sum_{n=0}^{\infty} \beta^n = \frac{\mu_1}{1-\beta}$$

This converges if and only if $\beta < 1$.

5.2 The Critical Volatility

The convergence condition $\beta < 1$ translates to:

$$\frac{\sigma}{\sqrt{2\pi}} < 1 \implies \sigma < \sqrt{2\pi} \approx 2.5066$$

In percentage terms, the critical volatility is $\sigma^* \approx 250.66\%$ annualized.

5.3 Closed Form for the Sum

When $\sigma < \sqrt{2\pi}$:

$$\sum_{n=1}^{\infty} \mu_n = \frac{\mu_1 \sqrt{2\pi}}{\sqrt{2\pi} - \sigma}$$

6. Divergence and Power-Law Beyond the Critical Threshold

6.1 Exponential Growth in the Supercritical Regime

When $\sigma > \sqrt{2\pi}$, we have $\beta = \sigma/\sqrt{2\pi} > 1$, and the expected values grow exponentially:

$$\mu_n = \mu_1 \cdot \beta^{n-1}$$

Each iteration amplifies the previous expected value. The total sum diverges - there is no finite bound on cumulative optionality.

6.2 Survival Condition and Power-Law Emergence

On the real market, participants do not receive the expected value - they receive a realized draw from the distribution. The ability of players to continue playing depends on their outcomes and risk tolerance.

We can take simplified option-like model which scales payoff with the bet, and assume payoff is $\max(X, 0)$ where $X \sim \mathcal{N}(0, \sigma w)$ - a zero-centered Gaussian. Participants require a minimum return to justify continued risk-taking. Let k_{th} be the threshold multiplier: participants survive only if their payoff exceeds $k_{\text{th}} \cdot w$.

Since the payoff must be positive and exceed the threshold, survival requires $X \geq k_{\text{th}} \cdot w$. Standardizing to $Z = X/(\sigma w)$ where $Z \sim \mathcal{N}(0, 1)$:

$$P(\text{survive}) = P\left(Z \geq \frac{k_{\text{th}}}{\sigma}\right) = 1 - \Phi\left(\frac{k_{\text{th}}}{\sigma}\right)$$

For example, with $\sigma = 3$ and $k_{\text{th}} = 2.5$, we have $k_{\text{th}}/\sigma \approx 0.833$, giving $p \approx 0.20$ (20% survival rate per round).

The probability of surviving n consecutive stages is:

$$P(\text{survive } n \text{ stages}) = p^n$$

Among survivors, the expected wealth multiplier per stage is the **conditional expectation given survival**. For the zero-centered truncated normal:

$$\beta_{\text{eff}} = \mathbb{E}\left[\frac{X}{w} \,\middle|\, X \geq k_{\text{th}} w\right] = \sigma \cdot \frac{\phi(k_{\text{th}}/\sigma)}{1 - \Phi(k_{\text{th}}/\sigma)} = \sigma \cdot \frac{\phi(k_{\text{th}}/\sigma)}{p}$$

where ϕ is the standard normal PDF.

The V* Critical Threshold. The phase transition to power-law behavior occurs when $\beta_{\text{eff}} = 1$. Setting $z = k_{\text{th}}/\sigma$, this condition gives:

$$\sigma_{\text{critical}} = \frac{1 - \Phi(z)}{\phi(z)}$$

This is the inverse Mills ratio. As $z \to 0^+$ (threshold approaching zero from above):

$$\sigma^*_{\text{th}} = \lim_{z\to 0^+} \frac{1-\Phi(z)}{\phi(z)} = \frac{0.5}{1/\sqrt{2\pi}} = \sqrt{\frac{\pi}{2}} \approx 1.253 \approx 125.3\%$$

This reveals a second critical constant: the moment any positive survival threshold is introduced, the critical volatility drops from $\sigma^* = \sqrt{2\pi} \approx 250.66\%$ to $\sigma^*_{\text{th}} = \sqrt{\pi/2} \approx 125.3\%$. This happens because filtering out participants with $X < 0$ (approximately half the population) doubles the conditional growth rate of survivors.

For higher thresholds ($z > 0$), the critical volatility decreases further. The critical curve in (σ, k_{th}) space is parameterized by:

$$\sigma_{\text{critical}}(z) = \frac{1-\Phi(z)}{\phi(z)}, \quad k_{\text{th, critical}}(z) = z \cdot \frac{1-\Phi(z)}{\phi(z)}$$

The number of surviving processes decays exponentially (p^n), but when $\beta_{\text{eff}} > 1$, the value of each survivor grows exponentially (β^n_{eff}). This combination produces power-law distributed outcomes.

6.3 The V* Distribution (Critical Volatility Distribution)

If N independent processes start, after n iterations approximately $N \cdot p^n$ survive, each with value proportional to β^n_{eff}.

Setting $v = \beta^n_{\text{eff}}$ (the value), we have $n = \log(v)/\log(\beta_{\text{eff}})$, so:

$$\text{Number with value} > v \propto p^n = p^{\log(v)/\log(\beta_{\text{eff}})} = v^{\log(p)/\log(\beta_{\text{eff}})}$$

This is a power-law with exponent:

$$\alpha = -\frac{\log(p)}{\log(\beta_{\text{eff}})}$$

Since $p < 1$ (survival is not guaranteed) and $\beta_{\text{eff}} > 1$ (supercritical with conditional growth), we have $\alpha > 0$: a proper power-law tail.

The V* Distribution is thus:

$$P(V > v) \propto \left(1 - \Phi\left(\frac{k_{\text{th}}}{\sigma}\right)\right)^{\frac{\log v}{\log\left(\sigma \cdot \frac{\phi(k_{\text{th}}/\sigma)}{1-\Phi(k_{\text{th}}/\sigma)}\right)}}$$

where:

- k_{th} is the threshold multiplier (minimum payoff as multiple of wealth)
- σ is the volatility parameter
- ϕ, Φ are the standard normal PDF and CDF

This can be written as $P(V > v) \propto v^{-\alpha}$ where $\alpha = -\log(p)/\log(\beta_{\text{eff}})$.

6.4 At The Value: Extending to Negative Thresholds

The survival condition in Section 6.2 requires $X \geq k_{\text{th}} \cdot w$, where X is the underlying return (before rectification) and k_{th} is the threshold multiplier. While the payoff remains $\max(X, 0)$, the survival condition is evaluated on X itself.

For $k_{\text{th}} > 0$, participants require positive returns above a threshold - a natural constraint for investors seeking real gains. For $k_{\text{th}} = 0$, participants continue if $X > 0$; the option paid something.

Mathematically, nothing prevents $k_{\text{th}} < 0$. This models a different game: participants accept losses to their wealth to continue playing. If $X = -0.5w$, the option pays zero, but a participant with $k_{\text{th}} = -1$ survives - they absorb the loss from reserves and enter the next round.

This is no longer option-like behavior. With negative thresholds, participants have linear exposure to losses up to $|k_{\text{th}}| \cdot w$. We call this regime **At The Value (ATV)**: participants commit to continue through adverse outcomes, accepting wealth destruction for the chance to remain in the game.

The ATV regime characterizes patient capital:

- Venture funds that continue supporting portfolio companies through down rounds
- Strategic investors with long time horizons
- Any participant with reserves who values continuation over immediate returns

The phase diagrams that follow extend into this negative k_{th} region, revealing how the critical boundary behaves when participants tolerate losses.

6.5 Phase Space Characterization

The V* Distribution exists in a two-dimensional parameter space (σ, k_{th}). Figure 1 shows this space with the critical boundary $\beta_{\text{eff}} = 1$ separating two regimes:

- **Subcritical region** (upper-left, blue): $\beta_{\text{eff}} < 1$. Conditional growth does not compensate for attrition. Outcomes are thin-tailed.
- **Supercritical region** (lower-right, red/orange): $\beta_{\text{eff}} > 1$. Conditional growth exceeds attrition. Outcomes follow the V* Distribution with power-law tails.

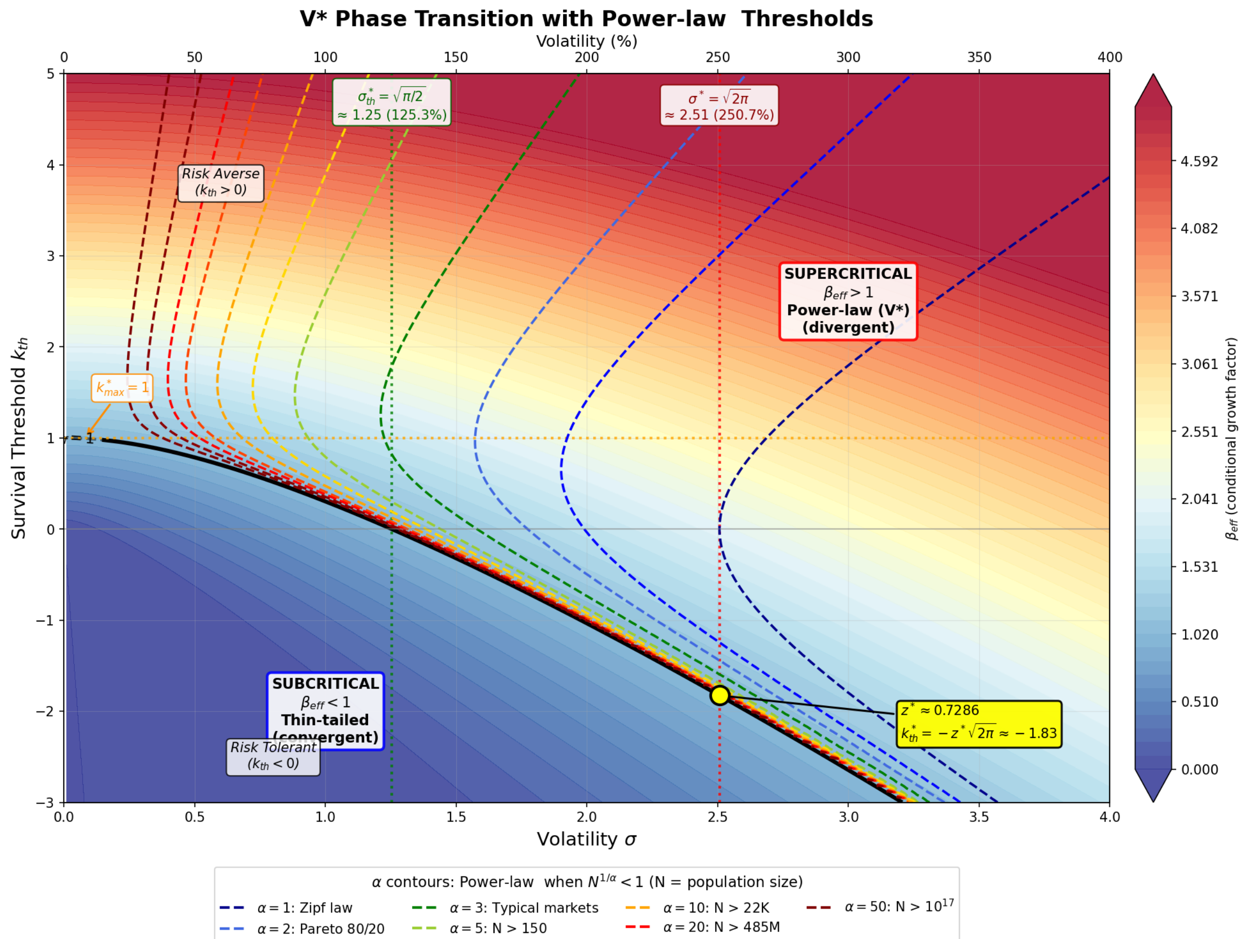

Figure 1: V* Phase Transition in (σ, k_{th}) space. Color indicates conditional growth factor β_{eff}. The critical boundary (solid curve) where $\beta_{\text{eff}} = 1$ separates subcritical (thin-tailed) from supercritical (V* power-law) regimes. Vertical lines mark $\sigma^*_{\text{th}} = \sqrt{\pi/2}$ and $\sigma^* = \sqrt{2\pi}$.

Figure 2 decomposes the phase space into its constituent quantities. The left panel shows survival probability $p = 1 - \Phi(k_{\text{th}}/\sigma)$, which decreases as the threshold becomes more selective (higher k_{th}) or volatility decreases (lower σ). The right panel shows the power-law exponent $\alpha = -\log(p)/\log(\beta_{\text{eff}})$ in the supercritical region, with lower α indicating heavier tails.

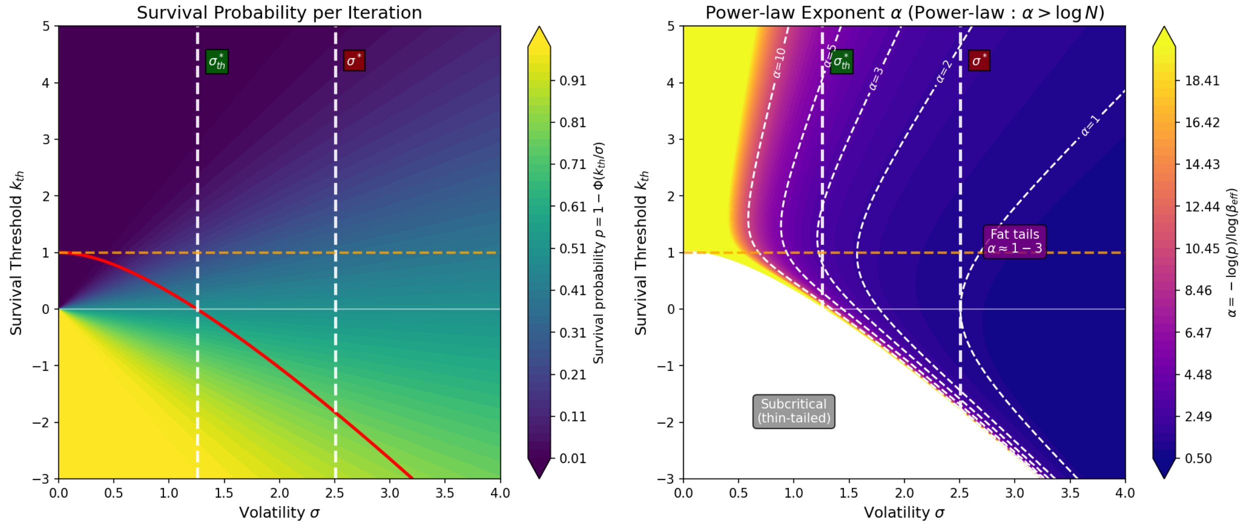

Figure 2: Phase space components. Left: Survival probability p per iteration. Right: V* power-law exponent α in the supercritical region. Lower α corresponds to heavier tails and more extreme wealth concentration.

The shape of the critical boundary reflects a fundamental tradeoff. High volatility generates large values among survivors; high selection pressure concentrates growth among fewer participants. Both mechanisms can produce $\beta_{\text{eff}} > 1$:

- At low σ, strong selection (high k_{th}) is required to achieve supercriticality
- At high σ, weaker selection (lower k_{th}) suffices because volatility alone drives growth

The boundary curves through the parameter space accordingly, extending into the risk-tolerant region ($k_{\text{th}} < 0$) at sufficiently high volatility.

6.6 Divergence versus Power-Law: The Role of Selection

It is important to distinguish two phenomena that the framework reveals.

Divergence of expected values occurs when $\beta = \sigma/\sqrt{2\pi} > 1$, i.e., when $\sigma > \sqrt{2\pi}$. In this regime, the expected value at each iteration exceeds the previous: $\mu_{n+1} = \beta \cdot \mu_n$. The sum of an infinite chain diverges. This is the result derived in Section 5 for the unconditional ATM case.

Power-law distribution requires selection. The V* Distribution emerges from the combination of two exponential processes:

- The number of surviving participants decays as p^n
- The value of each survivor grows as β_{eff}^n

The power-law exponent $\alpha = -\log(p)/\log(\beta_{\text{eff}})$ is well-defined only when both $p < 1$ (selection occurs) and $\beta_{\text{eff}} > 1$ (conditional growth exceeds unity). Without selection ($p = 1$), there is no distribution of outcomes - all participants follow the same trajectory.

These phenomena are related but distinct:

- Divergence concerns the total expected value of the system
- Power-law concerns the shape of the outcome distribution under selection

The critical volatility $\sigma^* = \sqrt{2\pi}$ marks where expected values begin to diverge. But power-law behavior can emerge at any volatility, provided selection is strong enough to achieve $\beta_{\text{eff}} > 1$. Conversely, even above σ^*, insufficient selection can fail to produce power-law tails if the survival pool is too large.

6.7 The Critical Intersection Point

The phase diagram reveals where these two thresholds intersect: the point at which the unconditional divergence threshold $\sigma = \sqrt{2\pi}$ meets the critical boundary $\beta_{\text{eff}} = 1$.

At $\sigma = \sqrt{2\pi}$, how much selection is required to produce power-law behavior?

Setting $\beta_{\text{eff}} = 1$:

$$\sigma \cdot \frac{\phi(z)}{1 - \Phi(z)} = 1 \quad \Rightarrow \quad \sigma = \frac{1 - \Phi(z)}{\phi(z)} = M(z)$$

where $M(z)$ is the Mills ratio and $z = k_{\text{th}}/\sigma$. Substituting $\sigma = \sqrt{2\pi}$:

$$M(z^*) = \sqrt{2\pi}$$

With $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$, this simplifies to:

$$1 - \Phi(z^*) = e^{-z^{*2}/2}$$

The solution is $z^* \approx 0.7286$, corresponding to $k^*_{\text{th}} = -z^* \sqrt{2\pi} \approx -1.83$.

Interpretation. At $\sigma = \sqrt{2\pi}$, the system exhibits:

Selection	**Regime**	**Behavior**
$k_{\text{th}} > -1.83$	$\beta_{\text{eff}} > 1$	Power-law (V*)
$k_{\text{th}} = -1.83$	$\beta_{\text{eff}} = 1$	Critical
$k_{\text{th}} < -1.83$	$\beta_{\text{eff}} < 1$	Thin-tailed

At the unconditional divergence threshold, expected values grow without bound. Yet this divergence alone does not guarantee power-law outcomes. If selection pressure is too weak ($k_{\text{th}} < -1.83$), the survival pool includes too many participants, diluting conditional growth below unity. The distribution remains thin-tailed despite divergent expectations.

The constant z^* thus marks the minimum selection pressure required to convert divergent growth into power-law distribution at $\sigma = \sqrt{2\pi}$. Above this point, selection concentrates growth sufficiently for V* dynamics to emerge. Below it, dilution dominates.

6.8 Four Constants of Gaussian Rectification

The framework yields four characteristic constants:

Constant	Value	Interpretation
σ^*	$\sqrt{2\pi} \approx 2.507$	Divergence threshold: expected values unbounded
σ^*_{th}	$\sqrt{\pi/2} \approx 1.253$	Power-law threshold at $k_{\text{th}} \to 0^+$
z^*	≈ 0.7286	Standardized selection threshold at $\sigma = \sigma^*$
k^*_{th}	$-z^* \sqrt{2\pi} \approx -1.83$	Selection threshold in parameter space

The ratio $\sigma^*/\sigma^*_{\text{th}} = 2$ reflects the doubling of conditional growth when the survival filter excludes negative outcomes. The constant z^*, defined by $1 - \Phi(z^*) = e^{-z^{*2}/2}$, is the standardized selection parameter at which power-law behavior first emerges when volatility reaches the divergence threshold; k^*_{th} expresses this in the units of the phase diagram.

These constants arise from the geometry of Gaussian rectification - the interplay of tail probability, local density, and the normalization factor $\sqrt{2\pi}$.

6.9 Implications

The power-law exponent $\alpha = -\log(p)/\log(\beta_{\text{eff}})$ depends on both survival probability and conditional growth. Near criticality ($\beta_{\text{eff}} \approx 1$), even modest selection pressure produces heavy tails. Deep in the supercritical regime ($\beta_{\text{eff}} \gg 1$), the distribution becomes increasingly extreme - a few massive winners among many losers.

The phase diagrams reveal that V* dynamics are accessible across a wide range of volatilities, provided selection is appropriately tuned. Systems with moderate volatility ($\sigma \approx 100 - 200\%$) can exhibit power-law behavior if participants impose sufficient selectivity on continuation. Systems with extreme volatility can exhibit power-law behavior even with weak selection.

This mechanism requires no exotic assumptions: iterated rectification of a Gaussian process with selective continuation based on outcomes. The fat tails emerge from the mathematics itself - specifically, from the tension between exponential attrition and exponential conditional growth that selection creates.

7. Numerical Simulations for V*

To validate the theoretical predictions, we simulate a simplified model which we call ATV (At The Value) where participants repeatedly bet their entire wealth on an at-the-money option with payoff $\max(X, 0)$ where $X \sim \mathcal{N}(0, \sigma w)$.

7.1 Simulation Setup

We simulate N = 10,000,000 participants over T = 15 periods, each starting with wealth w_0 = \$20,000. At each period, participants in the high-risk game receive payoff $\max(X, 0)$ where $X \sim \mathcal{N}(0, \sigma w)$. Participants drop out and switch to a safe alternative (10% volatility with in the money structure) if their payoff falls below 2.5× their current wealth - representing the requirement that returns must justify continued risk-taking in situations where bankruptcy risk is ~50% per turn.

The model tests volatilities ranging from $\sigma = 0.1$ (10%) to $\sigma = 4.0$ (400%), spanning the critical threshold at $\sigma^* = \sqrt{2\pi} \approx 2.507$ (251%).

The simulation algorithm:

```python
def simulate_atv_model(n=10_000_000, t=15, w0=20_000, sigma=2.5, threshold_k=2.5):
    w = np.full(n, w0, dtype=float)
    in_high_risk = np.ones(n, dtype=bool)
    sigma_low = 0.1

    for year in range(t):
        x_high = np.random.randn(n) * sigma * w
        payoff_high = np.maximum(x_high, 0)

        payoff_low = w * (1 + np.random.randn(n) * sigma_low)
        payoff_low = np.maximum(payoff_low, 0)

        threshold = threshold_k * w
        dropout = in_high_risk & (payoff_high < threshold)

        w = np.where(in_high_risk, payoff_high, payoff_low)
        in_high_risk = in_high_risk & ~dropout

    return w
```

7.2 Results

Figure 3 shows the rank-wealth distribution from simulation across all volatility regimes on a log-log scale, with the V* theoretical prediction overlaid (purple dashed line). The transition from curved (log-normal) to linear (power-law) behavior is clearly visible as volatility crosses the critical threshold. Figure 4 compares the wealth distributions in subcritical and supercritical regimes, with the V* theoretical power-law slope shown for comparison. Table 1 presents detailed statistics for each volatility level.

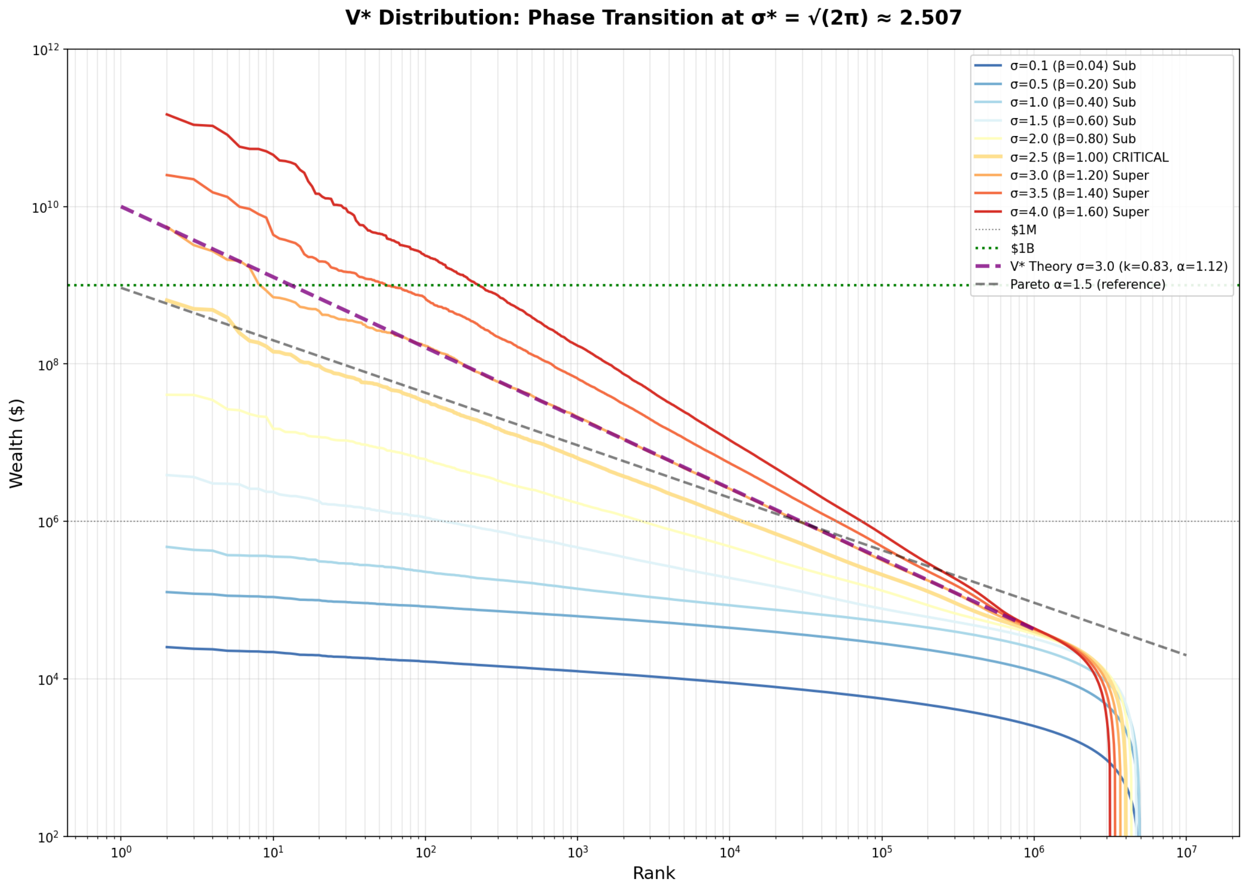

Figure 3: Simulation vs V* Theory: Rank-wealth distribution across volatility regimes. Colored lines show simulation results from σ=0.1 (blue) to σ=4.0 (red). The purple dashed line shows the V* theoretical prediction ($\alpha = -\log(p)/\log(\beta_{\text{eff}})$) for σ=3.0. Above criticality ($\sigma^* \approx 2.507$), simulated distributions converge to the theoretical power-law slope.

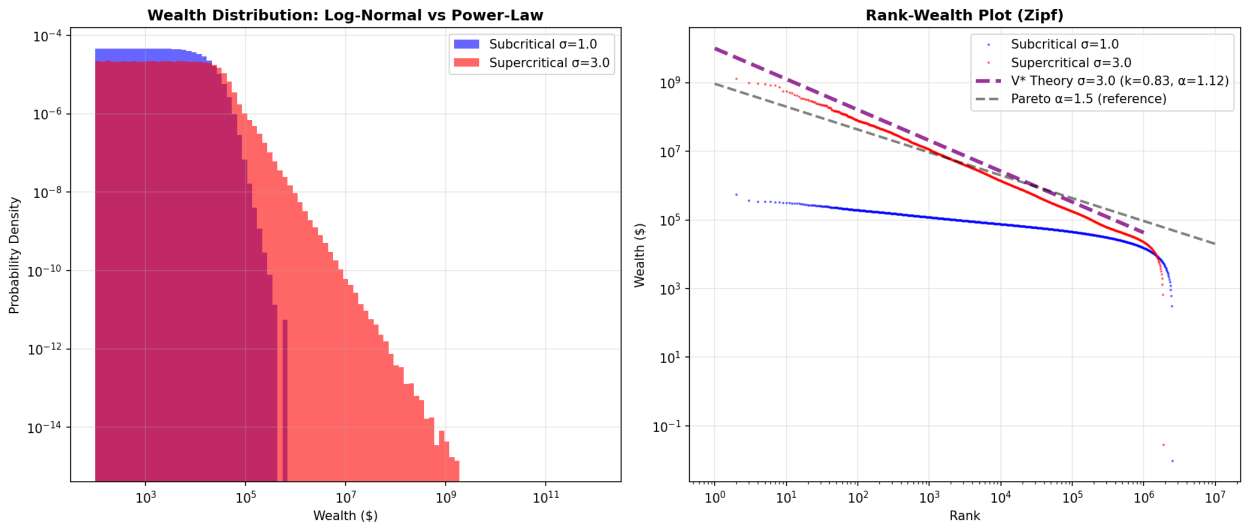

Figure 4: Simulation vs V* Theory: Subcritical (σ=2.0, blue) vs supercritical (σ=3.0, red) wealth distributions. Left: Probability density on log-log scale. Right: Rank-wealth plot with V* theoretical prediction (purple dashed) showing close agreement with supercritical simulation.

σ	β	**Bankrupt**	**Heavy Loss**	**$2k-20k**	**>$20k**	**>$50k**	**>$100k**	**>$1M**	**>$10M**	**>$100M**	**>$1B**	**Ratio**
0.10	0.04	5,229,613	3,313,805	1,456,562	20	0	0	0	0	0	0	∞
0.50	0.20	5,045,879	862,558	3,759,334	332,229	4,878	20	0	0	0	0	∞
1.00	0.40	5,054,141	436,565	3,093,118	1,416,176	134,222	4,702	0	0	0	0	∞
1.50	0.60	5,267,191	295,832	2,421,761	2,015,216	360,722	53,688	134	0	0	0	∞
2.00	0.80	5,603,627	225,395	1,960,344	2,210,634	562,968	158,609	2,672	34	1	0	78.6
2.51	1.00	5,957,643	181,984	1,632,383	2,227,990	707,131	275,089	12,076	553	17	1	21.8
3.00	1.20	6,277,439	153,424	1,398,570	2,170,567	794,856	371,871	28,760	2,234	175	8	12.9
3.50	1.40	6,564,826	132,271	1,217,426	2,085,477	845,480	444,779	50,506	5,647	674	54	8.9
4.00	1.60	6,818,493	116,154	1,075,393	1,989,960	870,073	497,072	73,912	10,648	1,602	224	6.9

Table 1: Simulation results showing wealth distribution across volatility regimes. "Bankrupt" = wealth < \$100, "Heavy Loss" = \$100-\$2,000, "Ratio" = count(\$1M+) / count(\$10M+).

7.3 Key Observations

The Low-Volatility Trap. At $\sigma = 0.1$, where $\beta = 0.04$, an astonishing 85% of participants are either bankrupt or suffer heavy losses. This occurs because the expected payoff per period is only $0.04w$ - a 96% loss rate per iteration. Low volatility provides insufficient upside to compensate for the inherent bankruptcy risk of the ATM structure. Zero millionaires emerge from 10 million participants.

Critical Transition. At $\sigma = 2.507 \approx \sigma^*$, we observe $\beta = 1.00$ - the break-even point where expected payoff equals current wealth. The ratio of millionaires to decamillionaires drops sharply to

21.8, indicating the emergence of heavy tails. The first billionaire appears in the simulation.

Supercritical Power-Law. For $\sigma > \sigma^*$, the ratio stabilizes ($21.8 \to 12.9 \to 8.9 \to 6.9$), the hallmark of power-law behavior where the proportion between consecutive magnitude classes becomes constant. At $\sigma = 4.0$, despite 70% bankruptcy or heavy loss, 224 billionaires emerge - a clear demonstration of the few-massive-winners, many-losers distribution characteristic of power laws.

8. Interpretation and Market Implications

8.1 Three Regimes

Unconditional ATM Case. When all participants continue regardless of outcomes, the parameter $\beta = \sigma/\sqrt{2\pi}$ defines three regimes:

Regime	Condition	Behavior
Subcritical	$\sigma < \sqrt{2\pi} \approx 250.66\%$	Convergent: Each payoff expected value is lower than previous and the total payoff is bounded. Produces thin-tailed distribution of outcomes.
Critical	$\sigma \approx \sqrt{2\pi}$	Self-similar: Each layer reproduces the previous.
Supercritical	$\sigma > \sqrt{2\pi}$	Divergent: Each payoff expected value is higher than previous and the total payoff goes to infinity.

V* Case with Survival Threshold. When participants require minimum returns k_{th} to continue, the conditional growth factor $\beta_{\text{eff}} = \sigma \cdot \phi(k_{\text{th}}/\sigma)/p$ determines the regime:

Regime	Condition	Behavior
Subcritical	$\beta_{\text{eff}} < 1$	Convergent: Survivors do not grow fast enough to compensate for attrition. Produces thin-tailed distribution.
Critical	$\beta_{\text{eff}} \approx 1$	Self-similar: Conditional growth exactly balances survival probability. V* behavior emerges.
Supercritical	$\beta_{\text{eff}} > 1$	Divergent: Survivors grow faster than the population decays. Produces V* Distribution with exponent $\alpha = -\log(p)/\log(\beta_{\text{eff}})$.

The V* framework generalizes the ATM result: with any positive survival threshold, the critical volatility drops to $\sigma^*_{\text{th}} = \sqrt{\pi/2} \approx 125.3\%$, and decreases further as k_{th} increases. Power-law dynamics can thus emerge at volatilities far below σ^*.

8.2 Volatility of Options on Options

An important real-world consideration: the volatility of an option's value is generally *higher* than the volatility of the underlying. This is due to the convexity (gamma) of the option payoff. For a compound option (option on an option), this effect compounds.

If we denote the volatility of the n-th layer as σ_n, empirically we observe:

$$\sigma_n > \sigma_{n-1}$$

This means that in practice, iterated option structures tend to *accelerate* toward the supercritical regime. The constant-percentage-volatility assumption in our geometric regime is thus conservative; real compound structures may diverge faster than our model predicts.

8.3 Time to Criticality

In Black-Scholes, the relevant volatility parameter is $\sigma\sqrt{T}$, where σ is the annualized volatility and T is time to expiration in years. For the unconditional case, the critical threshold $\sigma\sqrt{T} = \sqrt{2\pi}$ gives:

$$T^* = \frac{2\pi}{\sigma^2}$$

For the V* case with survival threshold, the critical threshold drops to $\sigma\sqrt{T} = \sqrt{\pi/2}$, giving:

$$T^*_{\text{th}} = \frac{\pi/2}{\sigma^2}$$

Annualized Vol σ	T^* (unconditional)	T^*_{th} (V* with threshold)
10%	628 years	157 years
20%	157 years	39 years
50%	25 years	6.3 years
100%	6.3 years	1.6 years
150%	2.8 years	8.4 months
200%	1.6 years	4.7 months
250%	1.0 year	3.0 months
300%	8.4 months	2.1 months
400%	4.7 months	1.2 months
500%	3.0 months	3.3 weeks
800%	1.2 months	1.3 weeks

For typical equity volatilities (15-30%), the unconditional critical threshold is centuries away. But with survival thresholds - which are ubiquitous in real markets - criticality arrives four times faster. For meme stocks and distressed names exhibiting 400-800% implied volatility, **V* criticality occurs within weeks**.

This means a 3-month ATM option on a 500% vol underlying is already at the critical regime - its expected payoff structure exhibits the self-similar properties described in Section 3.3. A 6-month option on the same underlying is supercritical.

An interesting observation: Early-stage startups exhibit annual valuation volatility in the 100-250% range, with funding rounds occurring every 12-24 months. Under the unconditional model, the critical horizon is 1-6 years - placing startups below criticality for typical funding cycles. However, with survival thresholds ($\sigma^*_{\text{th}} \approx 125\%$), the critical horizon drops to 3-19 months - squarely within typical funding cycles.

Venture capitalists impose implicit survival thresholds: startups must demonstrate sufficient progress to secure the next funding round. This selective continuation - where only companies exceeding some return threshold k_{th} survive to the next stage - creates conditional growth $\beta_{\text{eff}} > 1$ even when unconditional $\beta < 1$. The famously power-law distributed VC returns may thus be a natural consequence of the V* mechanism: iterated optionality with selective survival, where each funding stage represents both a survival filter and a growth multiplier for those who pass.

8.4 Connection to Real Instruments

Several existing instruments exhibit related dynamics:

- **Compound options** (options on options): Used in corporate finance for staged investments and in FX markets.
- **Volatility derivatives**: VIX options are options on a volatility index, which is itself derived from option prices - a form of second-order optionality.
- **Leveraged ETFs**: Daily rebalancing creates path-dependent compounding effects related to iterated expectations.
- **Convertible bonds with call provisions**: Multiple embedded options create layered optionality.

Instruments involving averaging over multiple options (such as VIX derivatives) present an interesting direction for potential extension of the framework. The aggregation may produce lower volatility compared to individual instruments, which warrants additional modelling not covered in this paper.

9. Conclusion

We have analyzed the behavior of iterated rectified Gaussian expectations, illustrated by the theoretical construct of an infinite chain of options-on-options. Our main findings:

1. A critical volatility threshold exists at $\sigma^* = \sqrt{2\pi} \approx 250.66\%$. Below this threshold, the cumulative value of an infinite option chain converges; above it, the chain diverges. This is the upper bound of stability under the idealized assumption of the system being maximally stable.

2. The supercritical regime implies that optionality can exceed underlying value. This is practically relevant during market stress events when implied volatilities spike above 250%.

3. Real compound structures tend toward supercriticality because option volatility exceeds underlying volatility due to convexity effects. Real-world volatility amplification, leverage, or imperfect pricing would result in a lower critical bound.

4. With selective survival, the critical threshold drops discontinuously to $\sigma^*_{\text{th}} = \sqrt{\pi/2} \approx 125.3\%$. We term the resulting power-law the V* Distribution, characterized by survival probability $p = 1 - \Phi(k_{\text{th}}/\sigma)$ and conditional expected growth $\beta_{\text{eff}} = \sigma \cdot \phi(k_{\text{th}}/\sigma)/p$. The power-law exponent $\alpha = -\log(p)/\log(\beta_{\text{eff}})$ admits closed-form expression.

The thresholds $\sigma^* = \sqrt{2\pi}$ and $\sigma^*_{\text{th}} = \sqrt{\pi/2}$ emerge purely from the geometry of Gaussian rectification - non-obvious boundaries that separate fundamentally different economic regimes.

The V* Distribution provides a mechanism for power-law emergence that requires no exotic assumptions. In repeated games with rectified Gaussian payoffs, the number of surviving participants decays exponentially as p^n, while the wealth of each survivor grows exponentially as β^n_{eff}. The conditional nature of β_{eff} is essential: it measures the expected growth *given survival*, not the unconditional expected payoff. This interplay between exponential attrition and exponential conditional growth produces fat tails with predictable exponents.

A final observation: While literal towers of derivatives-on-derivatives are rare, our mathematical framework requires a much weaker assumption - merely that expected returns propagate through some iterative structure. Many common financial arrangements satisfy this condition without being explicit derivative chains: loans and credit facilities (where the borrower's ability to repay depends on asset values), margin accounts and leveraged positions (where maintenance requirements create recursive dependencies), and tightly coupled instrument prices (where one instrument's value serves as collateral or reference for another). The interaction of these structures during stress events - when correlations spike and volatilities exceed normal ranges - may exhibit dynamics similar to those analyzed here, even without any formal options being written.

Furthermore, the framework does not require that all layers of the derivative structure exist simultaneously. Consecutive dependent instruments unfolding over time - where each stage's payout becomes the underlying for the next - satisfy the same mathematical recursion. As we have shown in simulation, the V* Distribution emerges reliably from this process, with power-law exponents matching theoretical predictions. The critical threshold we identify may thus be relevant not just for exotic derivatives, but for understanding systemic behavior in leveraged, interconnected financial systems evolving through time.

One might wonder if this model could help in predicting instability in less exotic cases. Could black swans, fat tails, unexpected VC returns, and volatility smiles have been predicted by feeding the random walk back into itself and checking if it converges?

As a last note, we would like to emphasize that whether anyone should actually construct an infinite derivative tower remains, we maintain, inadvisable. But at least we now know where it would break.

Appendix: Notation Summary

Symbol	Definition
$\Phi(z)$	Standard normal CDF
$\phi(z)$	Standard normal PDF
$g(z)$	$z\int_{-\infty}^{z} e^{-t^2/2}dt + e^{-z^2/2}$, unnormalized expected rectified value
σ	Volatility parameter
β	$\sigma/\sqrt{2\pi}$, unconditional geometric ratio
σ^*	$\sqrt{2\pi} \approx 2.5066 \approx 250.66\%$, critical volatility (unconditional)
σ^*_{th}	$\sqrt{\pi/2} \approx 1.2533 \approx 125.3\%$, critical volatility (with any survival threshold)
k_{th}	Threshold multiplier (minimum payoff as multiple of wealth)
p	Survival probability per stage, $1 - \Phi(k_{\text{th}}/\sigma)$
β_{eff}	Conditional expected growth factor, $\sigma \cdot \phi(k_{\text{th}}/\sigma)/p$
α	Power-law exponent, $-\log(p)/\log(\beta_{\text{eff}})$
V^*	V* Distribution: $P(V > v) \propto v^{-\alpha}$

Code Availability

Simulation code and supplementary materials are available at: github.com/sci2sci-opensource/research

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