

# Critical Volatility Threshold for Log-Normal to Power-Law Transition

## Iterated Options Model

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## Critical Volatility Threshold for Log-Normal to Power-Law Transition: Iterated Options Model

### Abstract

Random walk models with log-normal outcomes fit local market observations remarkably well. Yet interconnected, recursive structures - layered derivatives, leveraged positions, iterative funding rounds - periodically produce power-law distributed events. We show that the transition from log-normal to power-law dynamics requires only three conditions: randomness in the underlying process, rectification of payouts, and iterative feed-forward of expected values. Using an infinite option-on-option chain as an illustrative model, we derive a critical volatility threshold at  $\sigma^* = \sqrt{2\pi} \approx 250.66\%$  that separates convergent from divergent regimes. The result suggests that fat tails may be an emergent property of iterative log-normal processes rather than an exogenous feature.

## Preface

Financial systems are built on rectified payoffs. An investment in a high-risk project returns either something or nothing - you cannot lose more than you put in. An option pays  $\max(S - K, 0)$ . Even limited liability is a form of rectification.

These rectified structures often feed into one another. A successful project enables others built on top of it. A successful trade becomes the capital for the next trade. Derivative products reference other derivative products. It would be useful to know how such iterations behave - whether they remain stable or exhibit qualitatively different dynamics.

To answer this, we analyze the limiting case: an infinite chain of options, each written on the expected payout of the one before. The result depends on three conditions - randomness in the underlying process, rectification of payouts, and feed-forward of expected values. These are sufficient to produce a critical threshold at  $\sigma^* = \sqrt{2\pi} \approx 250.66\%$ . Below this, cumulative optionality remains bounded. Above it, the system diverges and begins exhibiting power-law dynamics.

We also identify a self-similar regime at exactly the critical threshold, where each iteration reproduces the statistical structure of the previous one.

The conditions are minimal: randomness, bounded downside, and iteration. These are not exotic assumptions, suggesting the mechanism may apply broadly to dynamic systems with compounding behavior.

As for the infinite derivative tower itself: to our knowledge, no one has built one. This is probably wise. But should financial engineering continue its march toward increasingly layered products, at least the location of the cliff is now known.

## 1. Introduction and Motivation

The Black-Scholes framework provides a foundational model for pricing European options. Under risk-neutral valuation, the price of a call option reflects the expected value of its payoff  $\max(S_T - K, 0)$ , discounted appropriately. This “rectification” - the maximum of a potentially negative quantity and zero - is the essential nonlinearity that gives options their asymmetric payoff structure.

A natural question arises: what happens when we write an option on an option? And then an option on that? In principle, one could construct an arbitrarily deep tower of such instruments, each layer deriving its value from the expected payout of the layer below.

This paper analyzes the mathematical structure of such iterated rectified expectations. We find that:

1. The system exhibits a phase transition at a critical volatility of  $\sigma^* = \sqrt{2\pi} \approx 250.66\%$  annualized. This assumes the perfect case, where pricing has no errors and volatility doesn’t amplify between derivative layers but stays perfectly correlated to asset price at a constant ratio. Real systems, if built, will diverge much faster.
2. Below criticality ( $\sigma < \sigma^*$ ), the total value of an infinite option chain converges to a finite sum, meaning optionality is “bounded” no matter how many layers are added.
3. Above criticality ( $\sigma > \sigma^*$ ), the chain diverges - the cumulative value of optionality exceeds the underlying asset itself. This is not merely a mathematical curiosity; it implies that in extreme volatility regimes, the optionality can dominate the fundamental value of the product. This leads to amplification of expected payout at each consecutive step, making the expected payouts follow power-law dynamics.
4. At criticality ( $\sigma = \sigma^*$ ), the system becomes self-similar, with each iteration reproducing the statistical structure of the previous one.

These findings have implications for understanding volatility regimes during market stress, the pricing of compound options, and the theoretical limits of derivative layering.

## 2. The Black-Scholes Setup

### 2.1 Standard Framework

Under the Black-Scholes model, the underlying asset follows geometric Brownian motion:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

The Black-Scholes formula for a European call option is:

$$C(S_t, t) = N(d_+)S_t - N(d_-)Ke^{-r(T-t)}$$

where:

$$d_+ = \frac{1}{\sigma\sqrt{T-t}} \left[ \ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t) \right]$$

$$d_- = d_+ - \sigma\sqrt{T-t}$$

and  $N(\cdot)$  is the standard normal CDF.

### 2.2 ATM Special Case

For an at-the-money option where  $S_t = K$ , we have  $\ln(S_t/K) = 0$ , so:

$$d_+ = \frac{r + \sigma^2/2}{\sigma} \sqrt{T-t}$$

$$d_- = \frac{r - \sigma^2/2}{\sigma} \sqrt{T-t}$$

In the  $r \ll \sigma$  limit (which holds for high-volatility regimes where  $\sigma > 100\%$  and  $r \approx 5\%$ ):

$$d_+ \approx \frac{\sigma^2/2}{\sigma} \sqrt{T-t} = \frac{\sigma}{2} \sqrt{T-t}$$

$$d_- \approx \frac{-\sigma^2/2}{\sigma} \sqrt{T-t} = -\frac{\sigma}{2} \sqrt{T-t}$$

Since  $d_- \approx -d_+$ , we have  $N(d_-) = N(-d_+) = 1 - N(d_+)$ , and the option price simplifies to:

$$C = K [N(d_+) - N(d_-)] = K \left[ 2N\left(\frac{\sigma\sqrt{T-t}}{2}\right) - 1 \right]$$

The ATM option price reduces to a single Gaussian CDF minus a constant. Let us analyze its behavior.

### 2.3 The Gaussian Structure

The Gaussian distribution appears explicitly in Black-Scholes through  $N(d_+)$  and  $N(d_-)$ , and simplifies to a single Gaussian CDF under the high-volatility ATM assumption. We observe that the option payoff  $\max(S_T - K, 0)$  cannot be negative by definition - the option structure rectifies the underlying returns at zero.

The expected value of a rectified Gaussian is the mathematical core of option pricing. We now analyze the general case of  $\mathbb{E}[\max(X, 0)]$  for  $X \sim \mathcal{N}(\mu, \sigma)$ .

### 2.4 Rectified Gaussian Expectations

Let  $Y = \max(X, 0)$  where  $X \sim \mathcal{N}(\mu, \sigma)$ . We seek  $\mathbb{E}[Y]$ .

The expectation splits into two regions:

$$\mathbb{E}[Y] = \mathbb{E}[X \cdot \mathbf{1}_{X>0}] = \int_0^\infty x \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} dx$$

Substituting  $u = (x - \mu)/\sigma$ :

$$\mathbb{E}[Y] = \int_{-\mu/\sigma}^\infty (\mu + \sigma u) \cdot \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du$$

This separates into:

$$\mathbb{E}[Y] = \mu \int_{-\mu/\sigma}^\infty \phi(u) du + \sigma \int_{-\mu/\sigma}^\infty u \cdot \phi(u) du$$

The first integral is  $\mu \cdot \Phi(\mu/\sigma)$ . For the second, note that  $u \cdot \phi(u) = -\phi'(u)$ , so:

$$\int_{-\mu/\sigma}^\infty u \cdot \phi(u) du = [-\phi(u)]_{-\mu/\sigma}^\infty = \phi(-\mu/\sigma)$$

Therefore:

$$\mathbb{E}[Y] = \mu \Phi\left(\frac{\mu}{\sigma}\right) + \sigma \phi\left(\frac{\mu}{\sigma}\right)$$

where  $\Phi(\cdot)$  is the standard normal CDF and  $\phi(\cdot)$  is the standard normal PDF.

### 2.5 The Function $g(z)$

Normalizing by  $\sigma$  and letting  $z = \mu/\sigma$ :

$$\frac{\mathbb{E}[Y]}{\sigma} = z\Phi(z) + \phi(z)$$

Expanding using the integral forms of  $\Phi$  and  $\phi$ :

$$\frac{\mathbb{E}[Y]}{\sigma} = \frac{1}{\sqrt{2\pi}} \left( z \int_{-\infty}^z e^{-t^2/2} dt + e^{-z^2/2} \right)$$

We define:

$$g(z) = z \int_{-\infty}^z e^{-t^2/2} dt + e^{-z^2/2}$$

so that the expected value of the rectified Gaussian becomes:

$$\mathbb{E}[Y] = \frac{\sigma}{\sqrt{2\pi}} g(z)$$

This form will be essential for analyzing iterations.

### 3. Iterated Options: The Mathematical Structure

#### 3.1 The Iteration Scheme

Consider a chain of options where each option is written on the expected payout of the previous one. In a simplified model of such chain, we would be putting a new derivative instrument with the price of expected payout of the previous one, and calculate new parameters. Since the strike price from previous option would be just a constant multiplier, we can focus on analyzing the rectified Gaussian behavior as a simplified model.

Let  $\mu_n$  denote the expected value at stage  $n$ , and suppose each stage has volatility  $\sigma_n$ .

From Section 2.5, the expected value of the rectified Gaussian at stage  $n$  is:

$$\mathbb{E}_n[Y] = \frac{\sigma_n}{\sqrt{2\pi}} g(z_n)$$

where  $z_n = \mu_n/\sigma_n$ .

If we let the output of one rectification become the mean of the next (i.e.,  $\mu_{n+1} = \mathbb{E}_n[Y]$ ), we obtain:

$$\mu_{n+1} = \frac{\sigma_n}{\sqrt{2\pi}} g(z_n)$$

#### 3.2 General Recursion

From Section 2.5:

$$\mathbb{E}_1[Y] = \frac{\sigma_1}{\sqrt{2\pi}} g(z_1)$$

Let the output become the mean of the next stage:  $\mu_2 = \mathbb{E}_1[Y]$ .

The next price is:

$$z_2 = \frac{\mu_2}{\sigma_2} = \frac{\sigma_1}{\sigma_2} \cdot \frac{1}{\sqrt{2\pi}} g(z_1)$$

Let  $r = \sigma_1/\sigma_2$  and  $w = g(z_1)$ . Then:

$$z_2 = \frac{r}{\sqrt{2\pi}} w$$

The next expectation:

$$\mathbb{E}_2[Y] = \frac{\sigma_2}{\sqrt{2\pi}} g(z_2)$$

More generally, letting  $w_n = g(z_n)$  and  $\alpha = \frac{r}{\sqrt{2\pi}}$ :

$$w_{n+1} = g(\alpha \cdot w_n)$$

Explicitly:

$$w_{n+1} = \alpha w_n \int_{-\infty}^{\alpha w_n} e^{-t^2/2} dt + e^{-\alpha^2 w_n^2 / 2}$$

This is a nonlinear recursion whose behavior depends critically on  $\alpha$ .

### 3.3 The Self-Similar Case

When  $\alpha = 1$  (equivalently,  $r = \sqrt{2\pi}$ , i.e.,  $\sigma_1 = \sigma_2 \sqrt{2\pi}$ ), the recursion simplifies to:

$$w_{n+1} = g(w_n)$$

This is pure iteration of  $g$  - the process becomes self-similar. The ratio:

$$\frac{w_n}{w_{n+1}} = \frac{w_n}{g(w_n)}$$

suggests that the sequence will have its own convergence/divergence behavior depending on  $w_n$  (or equivalently  $\alpha$ ). The parameter  $\alpha$  controls how the recursion scales, determining whether iterated expectations grow, shrink, or stabilize.

## 4. The Recentered (ATM) Case

### 4.1 Introducing the Shift

In practice, options are often struck at-the-money (ATM), where the strike equals the current expected value. We model this by introducing a shift parameter  $s_n$  that recenters the distribution at each step:

$$z_n^s = \frac{\mu_n - s_n}{\sigma_n} = z_n - \frac{s_n}{\sigma_n}$$

Setting  $s_n = \mu_n$  (the ATM condition) forces  $z_n^s = 0$  at every iteration.

The intuition for this shift is the following: each new option is written ATM at inception, with strike equal to the current underlying price (which is the expected value from the previous stage). The underlying then fluctuates around this strike with its own volatility over the holding period. Since the strike equals the mean, the payoff-relevant distribution is centered at zero.

### 4.2 Evaluation at Zero

Since  $g(0) = 0 + e^0 = 1$ , the shifted expectation simplifies dramatically:

$$\mathbb{E}_n^s[Y] = \frac{\sigma_n}{\sqrt{2\pi}}$$

This is the well-known result that an ATM option's expected payout (before discounting and without market price multiplier) is proportional to volatility. It connects directly to the well-known practitioner's approximation:

$$C \approx \frac{S \cdot \sigma \sqrt{T}}{\sqrt{2\pi}} \approx 0.4 \cdot S \cdot \sigma \sqrt{T}$$

### 4.3 The Geometric Regime

In financial contexts, we frequently assume that volatility is a percentage related to the price - a stock with higher price has proportionally higher absolute volatility. We use a similar definition which scales with expected values.

Assuming constant percentage volatility  $\sigma_n = \sigma \cdot \mu_n$  (volatility scales with the underlying):

$$\mu_{n+1} = \frac{\sigma \cdot \mu_n}{\sqrt{2\pi}} = \beta \cdot \mu_n$$

where  $\beta = \sigma / \sqrt{2\pi}$ .

This yields the closed form:

$$\mu_n = \mu_1 \cdot \beta^{n-1}$$

The expected value at each stage forms a geometric sequence.

## 5. Convergence and the Critical Threshold

### 5.1 Sum of the Infinite Chain

The total expected value across an infinite chain of ATM options is:

$$\sum_{n=1}^{\infty} \mu_n = \mu_1 \sum_{n=0}^{\infty} \beta^n = \frac{\mu_1}{1 - \beta}$$

This converges if and only if  $\beta < 1$ .

### 5.2 The Critical Volatility

The convergence condition  $\beta < 1$  translates to:

$$\frac{\sigma}{\sqrt{2\pi}} < 1 \implies \sigma < \sqrt{2\pi} \approx 2.5066$$

In percentage terms, the critical volatility is  $\sigma^* \approx 250.66\%$  annualized.

### 5.3 Closed Form for the Sum

When  $\sigma < \sqrt{2\pi}$ :

$$\sum_{n=1}^{\infty} \mu_n = \frac{\mu_1 \sqrt{2\pi}}{\sqrt{2\pi} - \sigma}$$

## 6. Divergence and Power-Law Beyond the Critical Threshold

### 6.1 Exponential Growth in the Supercritical Regime

When  $\sigma > \sqrt{2\pi}$ , we have  $\beta = \sigma/\sqrt{2\pi} > 1$ , and the expected values grow exponentially:

$$\mu_n = \mu_1 \cdot \beta^{n-1}$$

Each iteration amplifies the previous expected value. The total sum diverges - there is no finite bound on cumulative optionality.

### 6.2 Survival Condition and Power-Law Emergence

On the real market, participants do not receive the expected value - they receive a realized draw from the distribution. The ability of players to continue playing depends on their outcomes and risk tolerance.

Let us assume there exists a risk threshold  $R$  for each participant: after losing more than  $R$ , they stop playing (bankruptcy, margin call, fund redemption, etc.). At each stage  $n$ , the realized value  $X_n$  is drawn from  $\mathcal{N}(\mu_n, \sigma_n)$ . The participant survives to the next round if:

$$X_n > \mu_n - R$$

For simplicity, express  $R$  in units of standard deviation:  $R = k \cdot \sigma_n$  for some constant  $k$ . The survival probability at each stage is:

$$p = P(X_n > \mu_n - k\sigma_n) = \Phi(k)$$

where  $\Phi$  is the standard normal CDF. For example,  $k = 1$  gives  $p \approx 0.84$ ;  $k = 2$  gives  $p \approx 0.98$ . Note that  $k$  can be negative, imposing a “minimum gains” condition rather than loss tolerance:  $k = -0.43$  gives  $p \approx 0.33$ , meaning participants must achieve returns above the 33rd percentile to continue.

The probability of surviving  $n$  consecutive stages is:

$$P(\text{survive } n \text{ stages}) = p^n = \Phi(k)^n$$

The number of surviving processes decays exponentially, but in the supercritical regime, the value of each survivor grows exponentially (by factor  $\beta^n$ ). This combination produces power-law distributed outcomes.

### 6.3 The V\* Distribution (Critical Volatility Distribution)

If  $N$  independent processes start, after  $n$  iterations approximately  $N \cdot p^n$  survive, each with value proportional to  $\beta^n$ .

Setting  $v = \beta^n$  (the value), we have  $n = \log(v)/\log(\beta)$ , so:

$$\text{Number with value } > v \propto p^n = p^{\log(v)/\log(\beta)} = v^{\log(p)/\log(\beta)}$$

This is a power-law with exponent:

$$\alpha = -\frac{\log(p)}{\log(\beta)} = -\frac{\log(p)}{\log(\sigma/\sqrt{2\pi})}$$

Since  $p < 1$  (survival is not guaranteed) and  $\beta > 1$  (supercritical), we have  $\alpha > 0$ : a proper power-law tail.

Unpacking  $p^n$  fully, we obtain the  $V^*$  Distribution:

$$P(V > v) \propto \Phi(k)^{\frac{\log v}{\log(\sigma/\sqrt{2\pi})}}$$

where:

- $\Phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^k e^{-t^2/2} dt$  is the survival probability per stage
- $k = R/\sigma_n$  is the risk tolerance in standard deviations
- $\sigma$  is the volatility parameter
- $\sqrt{2\pi}$  is the critical threshold ( $\sigma^*$  or  $V^*$ )

## 6.4 Implications

The power-law exponent depends on both:

- The survival probability  $p$  (how selective each iteration is, or how much risk participants can tolerate)
- The growth factor  $\beta$  (how much survivors amplify)

Near criticality ( $\beta \approx 1$ ), even modest selection pressure produces heavy tails. Deep in the supercritical regime ( $\beta \gg 1$ ), the distribution becomes increasingly extreme - a few massive winners, many losers.

This mechanism requires no exotic assumptions: just iterated rectification of a Gaussian process with feed-forward of expected values. The fat tails emerge from the mathematics itself.

## 7. Numerical Simulations

To validate the theoretical predictions, we simulate a simplified ATM model where participants repeatedly bet their entire wealth on an at-the-money option with payoff  $\max(X, 0)$  where  $X \sim \mathcal{N}(0, \sigma w)$ .

### 7.1 Simulation Setup

We simulate  $N = 10,000,000$  participants over  $T = 15$  periods, each starting with wealth  $w_0 = \$20,000$ . At each period, participants in the high-risk game receive payoff  $\max(X, 0)$  where  $X \sim \mathcal{N}(0, \sigma w)$ . Participants drop out and switch to a safe alternative (10% volatility) if their payoff falls below  $2.5 \times$  their current wealth - representing the requirement that returns must justify continued risk-taking.

The model tests volatilities ranging from  $\sigma = 0.1$  (10%) to  $\sigma = 4.0$  (400%), spanning the critical threshold at  $\sigma^* = \sqrt{2\pi} \approx 2.507$  (251%).

### 7.2 Results

Figure 1 shows the rank-wealth distribution (Zipf plot) across all volatility regimes on a log-log scale. The transition from curved (log-normal) to linear (power-law) behavior is clearly visible as volatility crosses the critical threshold. Figure 2 compares the wealth distributions in subcritical and supercritical regimes, showing both histogram and complementary cumulative distribution function (CCDF) representations. Table 1 presents detailed statistics for each volatility level.

**$V^*$  Distribution: Phase Transition at  $\sigma^* = \sqrt{2\pi} \approx 2.507$**

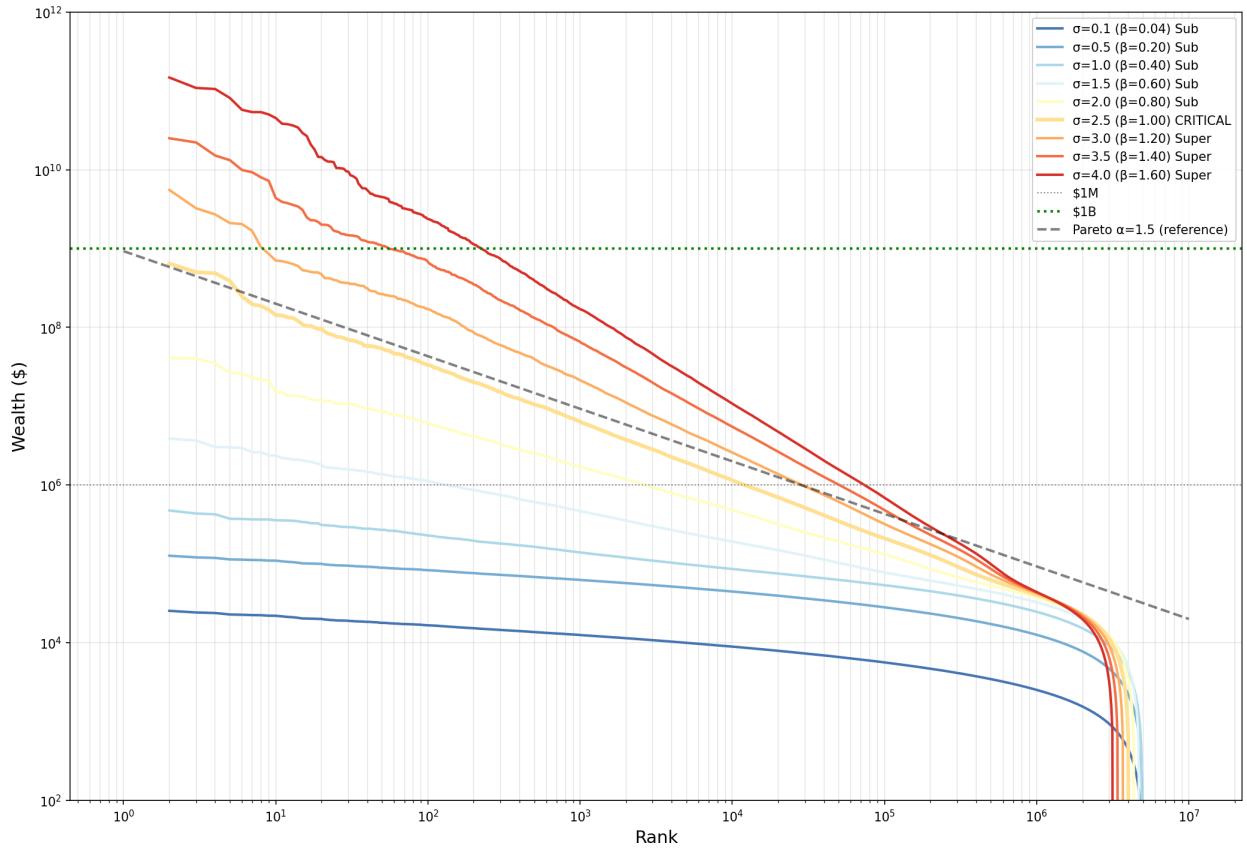


Figure 1: Phase transition in rank-wealth distribution across volatility regimes. Each colored line represents a different volatility level from  $\sigma=0.1$  (blue) to  $\sigma=4.0$  (red). The critical threshold at  $\sigma^* \approx 2.507$  (thick orange line) marks the transition from log-normal to power-law behavior. Above criticality, the lines become straighter on the log-log plot, indicating power-law tails.

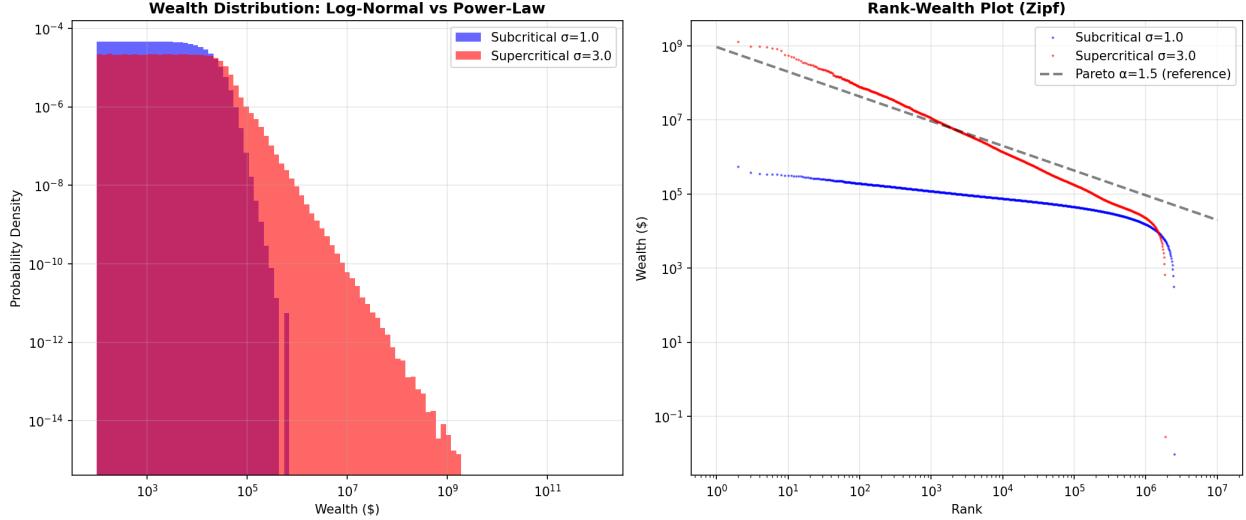


Figure 2: Comparison of subcritical ( $\sigma=2.0$ , blue) vs supercritical ( $\sigma=3.0$ , red) wealth distributions. Left: Probability density on log-log scale. Right: Complementary CDF ( $P(V>v)$ ) showing power-law behavior (straight line) in the supercritical regime.

$\sigma$	$\beta$	Bankrupt	Heavy Loss	\$2k-20k	>\$20k	>\$50k	>\$100k	>\$1M	>\$10M	>\$100M	>\$1B	Ratio
0.10	0.04	5,229,613	3,313,805	1,456,562	20	0	0	0	0	0	0	$\infty$
0.50	0.20	5,045,879	862,558	3,759,334	332,229	4,878	20	0	0	0	0	$\infty$
1.00	0.40	5,054,141	436,565	3,093,118	1,416,176	134,222	4,702	0	0	0	0	$\infty$
1.50	0.60	5,267,191	295,832	2,421,761	2,015,216	360,722	53,688	134	0	0	0	$\infty$
2.00	0.80	5,603,627	225,395	1,960,344	2,210,634	562,968	158,609	2,672	34	1	0	78.6
2.51	1.00	5,957,643	181,984	1,632,383	2,227,990	707,131	275,089	12,076	553	17	1	21.8
3.00	1.20	6,277,439	153,424	1,398,570	2,170,567	794,856	371,871	28,760	2,234	175	8	12.9
3.50	1.40	6,564,826	132,271	1,217,426	2,085,477	845,480	444,779	50,506	5,647	674	54	8.9
4.00	1.60	6,818,493	116,154	1,075,393	1,989,960	870,073	497,072	73,912	10,648	1,602	224	6.9

**Table 1:** Simulation results showing wealth distribution across volatility regimes. “Bankrupt” = wealth < \$100, “Heavy Loss” = \$100-\$2,000, “Ratio” = count(\$1M+) / count(\$10M+).

### 7.3 Key Observations

**The Low-Volatility Trap.** At  $\sigma = 0.1$ , where  $\beta = 0.04$ , an astonishing 85% of participants are either bankrupt or suffer heavy losses. This occurs because the expected payoff per period is only  $0.04w$  - a 96% loss rate per iteration. Low volatility provides insufficient upside to compensate for the inherent bankruptcy risk of the ATM structure. Zero millionaires emerge from 10 million participants.

**Critical Transition.** At  $\sigma = 2.507 \approx \sigma^*$ , we observe  $\beta = 1.00$  - the break-even point where expected payoff equals current wealth. The ratio of millionaires to decamillionaires drops sharply to 21.8, indicating the emergence of heavy tails. The first billionaire appears in the simulation.

**Supercritical Power-Law.** For  $\sigma > \sigma^*$ , the ratio stabilizes ( $21.8 \rightarrow 12.9 \rightarrow 8.9 \rightarrow 6.9$ ), the hallmark of power-law behavior where the proportion between consecutive magnitude classes becomes constant. At  $\sigma = 4.0$ , despite 70% bankruptcy or heavy loss, 224 billionaires emerge - a clear demonstration of the few-massive-winners, many-losers distribution characteristic of power laws.

## 8. Interpretation and Market Implications

### 8.1 Three Regimes

The parameter  $\sigma$  relative to  $\sqrt{2\pi}$  defines three distinct regimes:

Regime	Condition	Behavior
<b>Subcritical</b>	$\sigma < \sqrt{2\pi} \approx 250.66\%$	Convergent: Each payoff expected value is lower than previous and the total payoff is bounded. Produces log-normal distribution of outcomes.
<b>Critical</b>	$\sigma \approx \sqrt{2\pi}$	Self-similar: Each layer reproduces the previous. Power-law behavior begins emerging as we approach from either direction.
<b>Supercritical</b>	$\sigma > \sqrt{2\pi}$	Divergent: Each payoff expected value is higher than previous and the total payoff goes to infinity. Produces power-law distribution of outcomes.

### 8.2 Volatility of Options on Options

An important real-world consideration: the volatility of an option's value is generally *higher* than the volatility of the underlying. This is due to the convexity (gamma) of the option payoff. For a compound option (option on an option), this effect compounds.

If we denote the volatility of the  $n$ -th layer as  $\sigma_n$ , empirically we observe:

$$\sigma_n > \sigma_{n-1}$$

This means that in practice, iterated option structures tend to *accelerate* toward the supercritical regime. The constant-percentage-volatility assumption in our geometric regime is thus conservative; real compound structures may diverge faster than our model predicts.

### 8.3 Time to Criticality

In Black-Scholes, the relevant volatility parameter is  $\sigma\sqrt{T}$ , where  $\sigma$  is the annualized volatility and  $T$  is time to expiration in years. The critical threshold  $\sigma\sqrt{T} = \sqrt{2\pi}$  can be rewritten as:

$$T^* = \frac{2\pi}{\sigma^2}$$

This gives the time horizon at which a given annualized volatility reaches criticality:

Annualized Vol $\sigma$	Time to Criticality $T^*$
10%	628 years
20%	157 years
50%	25 years
100%	6.3 years
150%	2.8 years
200%	1.6 years
250%	1.0 year
300%	8.4 months
400%	4.7 months
500%	3.0 months
800%	1.2 months

For typical equity volatilities (15-30%), criticality is centuries away - irrelevant for any practical instrument. But during crisis periods when implied volatility spikes to 100%+, the critical horizon shrinks to single-digit years. For meme stocks and distressed names exhibiting 400-800% implied volatility, **criticality occurs within months**.

This means a 3-month ATM option on a 500% vol underlying is already at the critical regime - its expected payoff structure exhibits the self-similar properties described in Section 3.3. A 6-month option on the same underlying is supercritical.

**An interesting observation:** The 200-250% volatility range, with its 1-1.6 year critical horizon, closely matches the expected valuation volatility and funding round timing of venture-backed startups. Early-stage startups exhibit annual valuation volatility in the 150-250% range, with funding rounds occurring every 12-24 months. This places startup equity precisely in the critical regime - perhaps not coincidentally, venture capital returns are famously power-law distributed rather than log-normally distributed. The framework developed here suggests this may be a natural consequence of iterated optionality operating near the critical threshold - the consecutive high-volatility bets start to accumulate in power-law mode, with successful funding stages representing the next iteration of “option”.

#### 8.4 Connection to Real Instruments

Several existing instruments exhibit related dynamics:

- **Compound options** (options on options): Used in corporate finance for staged investments and in FX markets.
- **Volatility derivatives**: VIX options are options on a volatility index, which is itself derived from option prices - a form of second-order optionality.
- **Leveraged ETFs**: Daily rebalancing creates path-dependent compounding effects related to iterated expectations.
- **Convertible bonds with call provisions**: Multiple embedded options create layered optionality.

## 9. Conclusion

We have analyzed the behavior of iterated rectified Gaussian expectations, illustrated by the theoretical construct of an infinite chain of options-on-options. Our main findings:

1. A critical volatility threshold exists at  $\sigma^* = \sqrt{2\pi} \approx 250.66\%$ . Below this threshold, the cumulative value of an infinite option chain converges; above it, the chain diverges. This is the upper bound of stability under the idealized assumption of the system being maximally stable.
2. The supercritical regime implies that optionality can exceed underlying value. This is practically relevant during market stress events when implied volatilities spike above 250%.
3. Real compound structures tend toward supercriticality because option volatility exceeds underlying volatility due to convexity effects. Real-world volatility amplification, leverage, or imperfect pricing would result in a lower critical bound.

The threshold  $\sigma^* = \sqrt{2\pi}$  emerges purely from the geometry of Gaussian rectification - a non-obvious boundary that separates fundamentally different economic regimes. Whether anyone should actually construct an infinite derivative tower remains, we maintain, inadvisable. But at least we now know where it would break.

One might wonder if this model could help in predicting instability in less exotic cases. Could black swans, fat tails, unexpected VC returns, and volatility smiles have been predicted by feeding the random walk back into itself and checking if it converges?

**A final observation:** While literal towers of derivatives-on-derivatives are rare, our mathematical framework requires a much weaker assumption - merely that expected returns propagate through some iterative structure. Many common financial arrangements satisfy this condition without being explicit derivative chains: loans and credit facilities (where the borrower's ability to repay depends on asset values), margin accounts and leveraged positions (where maintenance requirements create recursive dependencies), and tightly coupled instrument prices (where one instrument's value serves as collateral or reference for another). The interaction of these structures during stress events - when correlations spike and volatilities exceed normal ranges - may exhibit dynamics similar to those analyzed here, even without any formal options being written.

Furthermore, the framework does not require that all layers of the derivative structure exist simultaneously. Consecutive dependent instruments unfolding over time - where each stage's payout becomes the underlying for the next - satisfy the same mathematical recursion. The critical threshold we identify may thus be relevant not just for exotic derivatives, but for understanding systemic behavior in leveraged, interconnected financial systems evolving through time.

## Appendix: Notation Summary

Symbol	Definition
$\Phi(z)$	Standard normal CDF
$\phi(z)$	Standard normal PDF
$g(z)$	$z \int_{-\infty}^z e^{-t^2/2} dt + e^{-z^2/2}$ , unnormalized expected rectified value
$\mu_n$	Mean at stage $n$
$\sigma_n$	Volatility at iteration $n$
$\beta$	$\sigma/\sqrt{2\pi}$ , geometric ratio
$\sigma^*$	$\sqrt{2\pi} \approx 2.5066 \approx 250.66\%$ , critical volatility
$k$	Risk tolerance in standard deviations
$p$	Survival probability per stage, $\Phi(k)$
$\alpha$	Power-law exponent, $-\log(p)/\log(\beta)$
$V^*$	$V^*$ Distribution: $P(V > v) \propto \Phi(k)^{\frac{\log v}{\log(\sigma/\sqrt{2\pi})}}$

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