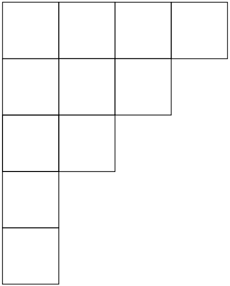
Young Tableaux: una implementazione

Candidato: Alessandro Campagni

Relatore: Marco Maggesi

Diagramma di Young

$11 = 4 + 3 + 2 + 1 + 1$



(i) lunghezza delle righe non crescente

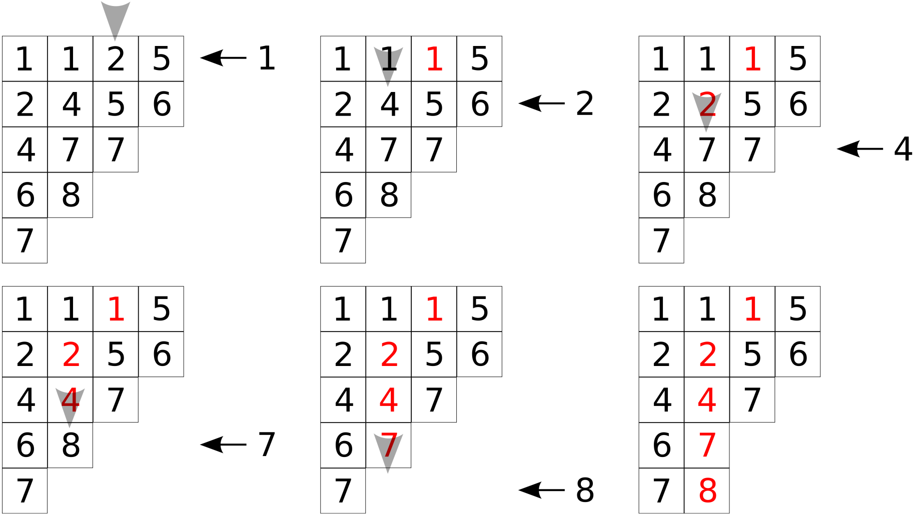
Young tableau

1	1	2	5
2	4	5	6
4	7	7	
6	8		
7			

(i) $T^i_j \leq T^i_{j+1}$

(ii) $T^i_j < T^{i+1}_j$

Row bumping


1 1 2 5 ← 1
2 4 5 6
4 7 7
6 8
7
1 1 1 5
2 4 5 6 ← 2
4 7 7
6 8
7
1 1 1 5
2 2 5 6
4 7 7 ← 4
6 8
7
1 1 1 5
2 2 5 6
4 4 7
6 8 ← 7
7
1 1 1 5
2 2 5 6
4 4 7
6 7
7 ← 8
1 1 1 5
2 2 5 6
4 4 7
6 7
7 8

Prodotto di tableaux

$$\begin{array}{cccc} 1 & 1 & 2 & 5 \\ 2 & 4 & 5 & 6 \\ 4 & 7 & 7 & \\ 6 & 8 & & \\ 7 & & & \end{array} \cdot \begin{array}{cc} 1 & 3 \\ 2 & \end{array} \qquad \begin{array}{cccc} 1 & 1 & 2 & 2 \\ 2 & 4 & 5 & 5 \\ 4 & 6 & 7 & \\ 6 & 7 & & \\ 7 & 8 & & \end{array} \cdot \begin{array}{cc} 1 & 3 \end{array}$$

$$\begin{array}{cccc} 1 & 1 & 1 & 2 \\ 2 & 2 & 5 & 5 \\ 4 & 4 & 7 & \\ 6 & 6 & & \\ 7 & 7 & & \\ 8 & & & \end{array} \cdot \begin{array}{c} 3 \end{array} \qquad \begin{array}{ccccc} 1 & 1 & 1 & 2 & 3 \\ 2 & 2 & 5 & 5 & \\ 4 & 4 & 7 & & \\ 6 & 6 & & & \\ 7 & 7 & & & \\ 8 & & & & \end{array}$$

Monomio di un tableau

1	1	2	5
2	4	5	6
4	7	7	
6	8		
7			

$$x_1^2 x_2^2 x_4^2 x_5^2 x_6^2 x_7^3 x_8$$

$$x^T = \prod_{i=1}^{m} x_i^{c(i)}$$

dove $c(i) = \#$ occorrenze di i nel tableau T

Polinomio di Schur

$$s_\lambda(x_1, \ldots, x_m) = \sum_{T \text{ tableau su } \lambda} x^T$$

(i) i polinomi di Schur sono simmetrici

(ii) i polinomi di Schur di grado n in m indeterminate formano una base per i polinomi simmetrici omogenei di grado n in m indeterminate

Polinomio di Schur

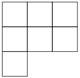
```
fl_better_yschur_polynomial_rows ([1,3,3], 3)
```

$\rightarrow x_1\, x_2^3\, x_3^3 + x_1^2\, x_2^2\, x_3^3 + x_1^3\, x_2\, x_3^3 + x_1^2\, x_2^3\, x_3^2 + x_1^3\, x_2^2\, x_3^2 + x_1^3\, x_2^3\, x_3$

Polinomio di Schur

`fl_better_yschur_polynomial_rows ([1,3,3], 3)`

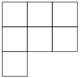
$\rightarrow x_1\, x_2^3\, x_3^3 + x_1^2\, x_2^2\, x_3^3 + x_1^3\, x_2\, x_3^3 + x_1^2\, x_2^3\, x_3^2 + x_1^3\, x_2^2\, x_3^2 + x_1^3\, x_2^3\, x_3$

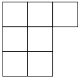


Polinomio di Schur

`fl_better_yschur_polynomial_rows ([1,3,3], 3)`

$\rightarrow x_1\, x_2^3\, x_3^3 + x_1^2\, x_2^2\, x_3^3 + x_1^3\, x_2\, x_3^3 + x_1^2\, x_2^3\, x_3^2 + x_1^3\, x_2^2\, x_3^2 + x_1^3\, x_2^3\, x_3$

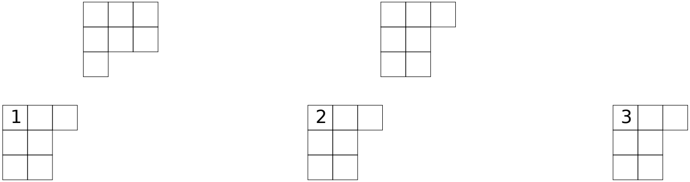




Polinomio di Schur

`fl_better_yschur_polynomial_rows ([1,3,3], 3)`

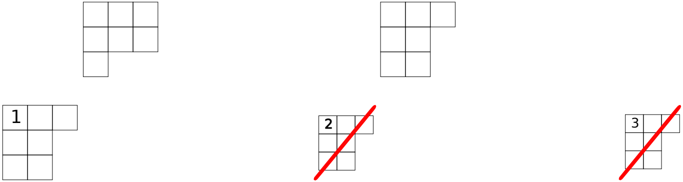
$\rightarrow x_1\, x_2^3\, x_3^3 + x_1^2\, x_2^2\, x_3^3 + x_1^3\, x_2\, x_3^3 + x_1^2\, x_2^3\, x_3^2 + x_1^3\, x_2^2\, x_3^2 + x_1^3\, x_2^3\, x_3$



Polinomio di Schur

```
fl_better_yschur_polynomial_rows ([1,3,3], 3)
```

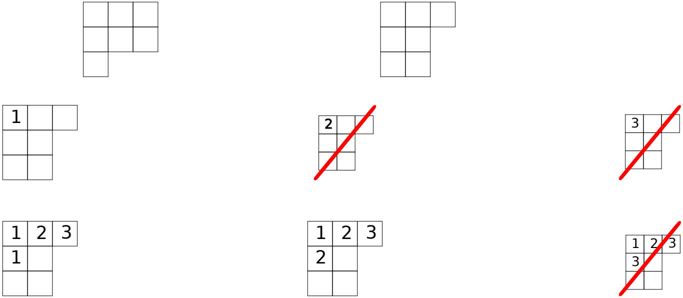
$$\rightarrow x_1\, x_2^3\, x_3^3 + x_1^2\, x_2^2\, x_3^3 + x_1^3\, x_2\, x_3^3 + x_1^2\, x_2^3\, x_3^2 + x_1^3\, x_2^2\, x_3^2 + x_1^3\, x_2^3\, x_3$$



Polinomio di Schur

`fl_better_yschur_polynomial_rows ([1,3,3], 3)`

$\rightarrow x_1\,x_2^3\,x_3^3 + x_1^2\,x_2^2\,x_3^3 + x_1^3\,x_2\,x_3^3 + x_1^2\,x_2^3\,x_3^2 + x_1^3\,x_2^2\,x_3^2 + x_1^3\,x_2^3\,x_3$



Polinomio di Schur

1	2	3
1	2	
1	2	

1	2	3
1	2	
2	3	

1	2	3
1	3	
2	3	

1	2	3
1	2	
1	3	

1	2	3
1	3	
1	3	

1	2	3
2	3	
2	3	

$$s_{(1,3,3)}(x_1, x_2, x_3) =$$
$$x_1\, x_2^3\, x_3^3 + x_1^2\, x_2^2\, x_3^3 + x_1^3\, x_2\, x_3^3 + x_1^2\, x_2^3\, x_3^2 + x_1^3\, x_2^2\, x_3^2 + x_1^3\, x_2^3\, x_3$$

"Contare" i prodotti di tabelle

$$s_\lambda(x_1,\ldots,x_m)\cdot s_\mu(x_1,\ldots,x_m)=\sum_\nu c^\nu_{\lambda\mu}s_\nu(x_1,\ldots,x_m)$$

dove $c^\nu_{\lambda\mu} = card\left(\{(T,U) \mid V = T\cdot U; T,U \text{ tableaux su } \lambda \text{ e } \mu \text{ risp.}\}\right)$

Problema

Metodo per calcolare i numeri di Littlewood-Richardson.

Skew tableaux di Littlewood-Richardson

			1
		1	1
1	2	2	
2	3		
4			

(i) skew tableau T

(ii) $w(T)$ reverse lattice word

skew tableau su
$\nu/\lambda = (4,4,3,2,1)/(3,2)$
contenuto $\mu = (3,3,1,1)$

Calcolo dei numeri di Littlewood-Richardson

Proposizione

Il numero di skew tableaux di Littlewood-Richardson che riempiono lo skew diagram ν/λ con contenuto μ è $c^{\nu}_{\lambda\mu}$.

Un esempio

(i) $\lambda = (1, 2, 3), \mu = (1, 3, 3)$

(ii) ν diagramma di Young costruito a partire da λ

(iii) numero di skew tableaux di Littlewood-Richardson su ν/λ con contenuto μ

Verifica

```
lr_calc([3,2,1],[3,3,1]);
```
$\rightarrow$ [[3, 3, 3, 2, 1, 1], 1], [[3, 3, 3, 2, 2], 1], [[3, 3, 3, 3, 1], 1], [[4, 3, 2, 2, 1, 1], 1], [[4, 3, 2, 2, 2], 1], [[4, 3, 3, 1, 1, 1], 1], [[4, 3, 3, 2, 1], 3], [[4, 3, 3, 3], 1], [[4, 4, 2, 1, 1, 1], 1], [[4, 4, 2, 2, 1], 2], [[4, 4, 3, 1, 1], 2], [[4, 4, 3, 2], 2], [[4, 4, 4, 1], 1], [[5, 3, 2, 1, 1, 1], 1], [[5, 3, 2, 2, 1], 2], [[5, 3, 3, 1, 1], 2], [[5, 3, 3, 2], 2], [[5, 4, 1, 1, 1, 1], 1], [[5, 4, 2, 1, 1], 3], [[5, 4, 2, 2], 2], [[5, 4, 3, 1], 3], [[5, 4, 4], 1], [[5, 5, 1, 1, 1], 1], [[5, 5, 2, 1], 2], [[5, 5, 3], 1], [[6, 3, 2, 1, 1], 1], [[6, 3, 2, 2], 1], [[6, 3, 3, 1], 1], [[6, 4, 1, 1, 1], 1], [[6, 4, 2, 1], 2], [[6, 4, 3], 1], [[6, 5, 1, 1], 1], [[6, 5, 2], 1]

$$
\begin{aligned}
& s_{(1,2,3)}(x_1,x_2,x_3,x_4)\cdot s_{(1,3,3)}(x_1,x_2,x_3,x_4)= \\
& = s_{(3,3,3,2,1,1)}(x_1,x_2,x_3,x_4)+s_{(3,3,3,2,2)}(x_1,x_2,x_3,x_4)+s_{(3,3,3,3,1)}(x_1,x_2,x_3,x_4)+s_{(4,3,2,2,1,1)}(x_1,x_2,x_3,x_4)+ \\
& + s_{(4,3,2,2,2)}(x_1,x_2,x_3,x_4)+s_{(4,3,3,1,1,1)}(x_1,x_2,x_3,x_4)+3s_{(4,3,3,2,1)}(x_1,x_2,x_3,x_4)+s_{(4,3,3,3)}(x_1,x_2,x_3,x_4)+ \\
& + s_{(4,4,2,1,1,1)}(x_1,x_2,x_3,x_4)+2s_{(4,4,2,2,1)}(x_1,x_2,x_3,x_4)+2s_{(4,4,3,1,1)}(x_1,x_2,x_3,x_4)+2s_{(4,4,3,2)}(x_1,x_2,x_3,x_4)+ \\
& + s_{(4,4,4,1)}(x_1,x_2,x_3,x_4)+s_{(5,3,2,1,1,1)}(x_1,x_2,x_3,x_4)+2s_{(5,3,2,2,1)}(x_1,x_2,x_3,x_4)+2s_{(5,3,3,1,1)}(x_1,x_2,x_3,x_4)+ \\
& + 2s_{(5,3,3,2)}(x_1,x_2,x_3,x_4)+s_{(5,4,1,1,1,1)}(x_1,x_2,x_3,x_4)+3s_{(5,4,2,1,1)}(x_1,x_2,x_3,x_4)+2s_{(5,4,2,2)}(x_1,x_2,x_3,x_4)+ \\
& + 3s_{(5,4,3,1)}(x_1,x_2,x_3,x_4)+s_{(5,4,4)}(x_1,x_2,x_3,x_4)+s_{(5,5,1,1,1)}(x_1,x_2,x_3,x_4)+2s_{(5,5,2,1)}(x_1,x_2,x_3,x_4)+ \\
& + s_{(5,5,3)}(x_1,x_2,x_3,x_4)+s_{(6,3,2,1,1)}(x_1,x_2,x_3,x_4)+s_{(6,3,2,2)}(x_1,x_2,x_3,x_4)+s_{(6,3,3,1)}(x_1,x_2,x_3,x_4)+ \\
& + s_{(6,4,1,1,1)}(x_1,x_2,x_3,x_4)+2s_{(6,4,2,1)}(x_1,x_2,x_3,x_4)+s_{(6,4,3)}(x_1,x_2,x_3,x_4)+s_{(6,5,1,1)}(x_1,x_2,x_3,x_4)+ \\
& + s_{(6,5,2)}(x_1,x_2,x_3,x_4).
\end{aligned}
$$