高等数学

积 分 表

公 式 推 导

目　　录

（一）含有 $ax+b$ 的积分（1~9）

1. $\int\frac{dx}{ax+b}=\frac{1}{a}\cdot ln\left|ax+b\right|+C$

证明：被积函数 $f(x)=\frac{1}{ax+b}$ 的定义域为 $\{x\mid x\neq-\frac{b}{a}\}$

令 $ax+b=t\ (t\neq0)$，则 $dt=adx$，$\therefore\ dx=\frac{1}{a}dt$

$$\therefore\int\frac{dx}{ax+b}=\frac{1}{a}\int\frac{1}{t}dt$$

$$=\frac{1}{a}\cdot ln\ \left|t\right|+C$$

将 $t=ax+b$ 代入上式得：$\int\frac{dx}{ax+b}=\frac{1}{a}\cdot ln\left|ax+b\right|+C$

2. $\int(ax+b)^{\mu}dx=\frac{1}{a(\mu+1)}\cdot(ax+b)^{\mu+1}+C\qquad(\mu\neq-1)$

证明：令 $ax+b=t$，则 $dt=adx$，$\therefore\ dx=\frac{1}{a}dt$

$$\therefore\int(ax+b)^{\mu}dx=\frac{1}{a}\int t^{\mu}dt$$

$$=\frac{1}{a(\mu+1)}\cdot t^{\mu+1}+C$$

将 $t=ax+b$ 代入上式得：$\int(ax+b)^{\mu}dx=\frac{1}{a(\mu+1)}\cdot(ax+b)^{\mu+1}+C$

3. $\int\frac{x}{ax+b}dx=\frac{1}{a^2}\left(ax+b-b\cdot ln\left|ax+b\right|\right)+C$

证明：被积函数 $f(x)=\frac{x}{ax+b}$ 的定义域为 $\{x\mid x\neq-\frac{b}{a}\}$

令 $ax+b=t\ (t\neq0)$，则 $x=\frac{1}{a}(t-b)$，$dx=\frac{1}{a}dt$

$$\therefore\int\frac{x}{ax+b}dx=\int\frac{\frac{1}{a}(t-b)}{t}\cdot\frac{1}{a}dt=\frac{1}{a^2}\int\left(1-\frac{b}{t}\right)dt$$

$$=\frac{1}{a^2}\int dt-\frac{1}{a^2}\int\frac{b}{t}dt$$

$$=\frac{t}{a^2}-\frac{b}{a^2}\cdot ln\left|t\right|+C$$

$$=\frac{1}{a^2}\left(t-b\cdot ln\left|t\right|\right)+C$$

将 $t=ax+b$ 代入上式得：$\int\frac{x}{ax+b}dx=\frac{1}{a^2}\left(ax+b-b\cdot ln\left|ax+b\right|\right)+C$

4. $\int \frac{x^2}{ax+b}dx = \frac{1}{a^3}\left[\frac{1}{2}(ax+b)^2 - 2b(ax+b) + b^2 \cdot ln\left| ax+b \right|\right] + C$

证明：$\int \frac{x^2}{ax+b}dx = \frac{1}{a^2}\int \frac{(ax+b)^2 - 2abx - b^2)}{ax+b}dx$

$$= \frac{1}{a^2}\int (ax+b)dx - \frac{1}{a^2}\int \frac{2abx}{ax+b}dx - \frac{1}{a^2}\int \frac{b^2}{ax+b}dx$$

$\because \frac{1}{a^2}\int (ax+b)dx = \frac{1}{2a^3}(ax+b)^2 + C_1$

$$\frac{1}{a^2}\int \frac{2abx}{ax+b}dx = \frac{2b}{a^3}\int \frac{ax+b-b}{ax+b}d(ax)$$

$$= \frac{2b}{a^3}\int dx - \frac{2b^2}{a^3}\int \frac{1}{ax+b}d(ax+b)$$

$$= \frac{2b}{a^3}x - \frac{2b^2}{a^3}ln\left| ax+b \right| + C_2$$

$$\frac{1}{a^2}\int \frac{b^2}{ax+b}dx = \frac{b^2}{a^3}\int \frac{1}{ax+b}d(ax+b) = \frac{b^2}{a^3}ln\left| ax+b \right| + C_3$$

由以上各式整理得：$\int \frac{x^2}{ax+b}dx = \frac{1}{a^3}\left[\frac{1}{2}(ax+b)^2 - 2b(ax+b) + b^2 \cdot ln\left| ax+b \right|\right] + C$

5. $\int \frac{dx}{x(ax+b)} = -\frac{1}{b}\cdot ln\left|\frac{ax+b}{x}\right| + C$

证明：被积函数 $f(x) = \frac{1}{x\cdot(ax+b)}$ 的定义域为 $\{x \mid x \neq -\frac{b}{a}\}$

设 $\frac{1}{x\cdot(ax+b)} = \frac{A}{x} + \frac{B}{ax+b}$，则 $1 = A(ax+b) + Bx = (Aa+B)x + Ab$

$\therefore$ 有 $\begin{cases} Aa + B = 0 \\ Ab = 1 \end{cases} \Rightarrow \begin{cases} A = \frac{1}{b} \\ B = -\frac{a}{b} \end{cases}$

于是 $\int \frac{dx}{x(ax+b)} = \int [\frac{1}{bx} - \frac{a}{b\cdot(ax+b)}]dx = \frac{1}{b}\int \frac{1}{x}dx - \frac{a}{b}\int \frac{1}{ax+b}dx$

$$= \frac{1}{b}\int \frac{1}{x}dx - \frac{1}{b}\int \frac{1}{ax+b}d(ax+b)$$

$$= \frac{1}{b}\cdot ln\left| x \right| - \frac{1}{b}\cdot ln\left| ax+b \right| + C$$

$$= \frac{1}{b}\cdot ln\left| \frac{x}{ax+b} \right| + C$$

提示：$log_a b^{-1} = -log_a b$

$$= -\frac{1}{b}\cdot ln\left| \frac{ax+b}{x} \right| + C$$

6. $\int \frac{dx}{x^2(ax+b)} = -\frac{1}{bx} + \frac{a}{b^2} \cdot ln\left|\frac{ax+b}{x}\right| + C$

证明：被积函数 $f(x) = \frac{1}{x^2 \cdot (ax+b)}$ 的定义域为 $\{x \mid x \neq -\frac{b}{a}\}$

设 $\frac{1}{x^2 \cdot (ax+b)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{ax+b}$，则 $1 = Ax(ax+b) + B(ax+b) + Cx^2$

即 $x^2(Aa + C) + x(Ab + aB) + Bb = 1$

$$\therefore \text{ 有 } \begin{cases} Aa + C = 0 \\ Ab + aB = 0 \\ Bb = 1 \end{cases} \Rightarrow \begin{cases} A = -\frac{a}{b^2} \\ B = \frac{1}{b} \\ C = \frac{a^2}{b^2} \end{cases}$$

$$\text{于是 } \int \frac{dx}{x^2(ax+b)} = -\frac{a}{b^2}\int \frac{1}{x}dx + \frac{1}{b}\int \frac{1}{x^2}dx + \frac{a^2}{b^2}\int \frac{1}{ax+b}dx$$

$$= -\frac{a}{b^2}\int \frac{1}{x}dx + \frac{1}{b}\int \frac{1}{x^2}dx + \frac{a}{b^2}\int \frac{1}{ax+b}d(ax+b)$$

$$= -\frac{a}{b^2} \cdot ln|x| - \frac{1}{bx} + \frac{a}{b^2} \cdot ln|ax+b| + C$$

$$= -\frac{1}{bx} + \frac{a}{b^2} \cdot ln\left|\frac{ax+b}{x}\right| + C$$

7. $\int \frac{x}{(ax+b)^2}dx = \frac{1}{a^2}\left(ln|ax+b| + \frac{b}{ax+b}\right) + C$

证明：被积函数 $f(x) = \frac{x}{(ax+b)^2}$ 的定义域为 $\{x \mid x \neq -\frac{b}{a}\}$

设 $\frac{x}{(ax+b)^2} = \frac{A}{ax+b} + \frac{B}{(ax+b)^2}$，则 $x = A(ax+b) + B$

即 $x \cdot Aa + (Ab + B) = x$

$$\therefore \text{ 有 } \begin{cases} Aa = 1 \\ Ab + B = 0 \end{cases} \Rightarrow \begin{cases} A = \frac{1}{a} \\ B = -\frac{b}{a} \end{cases}$$

$$\text{于是 } \int \frac{x}{(ax+b)^2}dx = \frac{1}{a}\int \frac{1}{ax+b}dx - \frac{b}{a}\int \frac{1}{(ax+b)^2}dx$$

$$= \frac{1}{a^2}\int \frac{1}{ax+b}d(ax+b) - \frac{b}{a^2}\int \frac{1}{(ax+b)^2}d(ax+b)$$

$$= \frac{1}{a^2} \cdot ln|ax+b| + \frac{b}{a^2(ax+b)} + C$$

$$= \frac{1}{a^2}\left(ln|ax+b| + \frac{b}{ax+b}\right) + C$$

8. $\int \frac{x^2}{(ax+b)^2}dx = \frac{1}{a^3}\left(ax+b-2b\cdot ln\,|\,ax+b\,|-\frac{b^2}{ax+b}\right)+C$

证明:被积函数 $f(x)=\frac{x^2}{(ax+b)^2}$ 的定义域为$\{x\,|\,x\neq -\frac{b}{a}\}$

令 $ax+b=t\quad (t\neq 0)$, 则 $x=\frac{1}{a}(t-b)$, $dx=\frac{1}{a}dt$

$$\therefore\quad \frac{x^2}{(ax+b)^2}=\frac{(b-t)^2}{a^2t^2}=\frac{b^2+t^2-2bt}{a^2t^2}$$

$$\therefore\quad \int\frac{x^2}{(ax+b)^2}dx=\int\frac{b^2+t^2-2bt}{a^3t^2}dt=\frac{b^2}{a^3}\int\frac{1}{t^2}dt+\frac{1}{a^3}\int dt-\frac{2b}{a^3}\int\frac{1}{t}dt$$

$$=-\frac{b^2}{a^3t}+\frac{1}{a^3}\cdot t-\frac{2b}{a^3}\cdot ln\,|\,t\,|+C$$

$$=\frac{1}{a^3}(t-2b\cdot ln\,|\,t\,|-\frac{b^2}{t})+C$$

将 $t=ax+b$ 代入上式得:$\int\frac{x^2}{(ax+b)^2}dx=\frac{1}{a^3}\left(ax+b-2b\cdot ln\,|\,ax+b\,|-\frac{b^2}{ax+b}\right)+C$

9. $\int\frac{dx}{x(ax+b)^2}=\frac{1}{b(ax+b)}-\frac{1}{b^2}\cdot ln\,|\,\frac{ax+b}{x}\,|+C$

证明：被积函数 $f(x)=\frac{1}{x(ax+b)^2}$ 的定义域为$\{x\,|\,x\neq -\frac{b}{a}\}$

设：$\frac{1}{x(ax+b)^2}=\frac{A}{x}+\frac{B}{ax+b}+\frac{D}{(ax+b)^2}$

则 $1=A(ax+b)^2+Bx(ax+b)+Dx$

$$=Aa^2x^2+Ab^2+2Aabx+Bax^2+Bbx+Dx$$

$$=x^2(Aa^2+Ba)+x(2Aab+Bb+D)+Ab^2$$

$$\therefore\ 有\ \begin{cases}Aa^2+Ba=0\\ 2Aab+Bb+D=0\\ Ab^2=1\end{cases}\quad\Rightarrow\quad\begin{cases}A=\frac{1}{b^2}\\ B=-\frac{a}{b^2}\\ D=-\frac{a}{b}\end{cases}$$

于是 $\int\frac{dx}{x(ax+b)}=\frac{1}{b^2}\int\frac{1}{x}dx-\frac{a}{b^2}\int\frac{1}{ax+b}dx-\frac{a}{b}\int\frac{1}{(ax+b)^2}dx$

$$=\frac{1}{b^2}\cdot ln|x|-\frac{1}{b^2}\cdot ln|ax+b|+\frac{1}{b}\cdot\frac{1}{ax+b}+C$$

$$=\frac{1}{b(ax+b)}-\frac{1}{b^2}\cdot ln|\frac{ax+b}{x}|+C$$

（二）含有 $\sqrt{ax+b}$ 的积分（10~18）

10. $\int\sqrt{ax+b}\,dx=\frac{2}{3a}\cdot\sqrt{(ax+b)^3}+C$

证明: $\int\sqrt{ax+b}\,dx=\frac{1}{a}\int(ax+b)^{\frac{1}{2}}d(ax+b)=\frac{1}{a}\cdot\frac{1}{1+\frac{1}{2}}\cdot(ax+b)^{\frac{1}{2}+1}+C$

$$=\frac{2}{3a}\cdot\sqrt{(ax+b)^3}+C$$

11. $\int x\sqrt{ax+b}\,dx=\frac{2}{15a^2}\cdot(3ax-2b)\cdot\sqrt{(ax+b)^3}+C$

证明：令 $\sqrt{ax+b}=t\quad(t\geq 0)$，则 $x=\frac{t^2-b}{a}$ ， $dx=\frac{2t}{a}dt$ ， $x\sqrt{ax+b}=\frac{t^2-b}{a}\cdot t$

$$\therefore\ \int x\sqrt{ax+b}\,dx=\int\frac{t^2-b}{a}\cdot t\cdot\frac{2t}{a}dt=\frac{2}{a^2}\int(t^4-bt^2)dt$$

$$=\frac{2}{5a^2}\int d(t^5)-\frac{2b}{3a^2}\int d(t^3)=\frac{2}{5a^2}\cdot t^5-\frac{2b}{3a^2}\cdot t^3+C$$

$$=\frac{2t^3}{15a^2}(3t^2-5b)+C$$

将 $t=\sqrt{ax+b}$ 代入上式得: $\int x\sqrt{ax+b}\,dx=\frac{2}{15a^2}[3(ax+b)-5b]\cdot\sqrt{(ax+b)^3}+C$

$$=\frac{2}{15a^2}\cdot(3ax-2b)\cdot\sqrt{(ax+b)^3}+C$$

12. $\int x^2\sqrt{ax+b}\,dx=\frac{2}{105a^3}\cdot(15a^2x^2-12abx+8b^2)\cdot\sqrt{(ax+b)^3}+C$

证明：令 $\sqrt{ax+b}=t\quad(t\geq 0)$，则 $x=\frac{t^2-b}{a}$ ， $dx=\frac{2t}{a}dt$ ，

$$x^2\sqrt{ax+b}=\frac{(t^2-b)^2}{a^2}\cdot t=\frac{t^5+b^2t-2bt^3}{a^2}$$

$$\therefore\ \int x^2\sqrt{ax+b}\,dx=\frac{2}{a^3}\int t\cdot(t^5+b^2t-2bt^3)dt$$

$$=\frac{2}{a^3}\int t^6dt-\frac{2b^2}{a^3}\int t^2dt-\frac{4b}{a^3}\int t^4dt$$

$$=\frac{2}{a^3}\cdot\frac{1}{1+6}\cdot t^{6+1}+\frac{2b^2}{a^3}\cdot\frac{1}{1+2}\cdot t^{1+2}-\frac{4b}{a^3}\cdot\frac{1}{1+4}\cdot t^{4+1}+C$$

$$=\frac{2}{7a^3}\cdot t^7+\frac{2b^2}{3a^3}\cdot t^3-\frac{4b}{5a^3}\cdot t^5+C$$

$$=\frac{2t^3}{105a^3}\cdot(15t^4+35b^2-42bt^2)+C$$

将 $t=\sqrt{ax+b}$ 代入上式得：

$$\int x^2\sqrt{ax+b}\,dx=\frac{2}{105a^3}\cdot\sqrt{(ax+b)^3}\left[15a^2x^2+15b^2+30abx+35b^2-42b\cdot(ax+b)\right]$$

$$=\frac{2}{105a^3}\cdot(15a^2x^2-12abx+8b^2)\cdot\sqrt{(ax+b)^3}+C$$

13. $\int \frac{x}{\sqrt{ax+b}}dx = \frac{2}{3a^2}\cdot(ax-2b)\cdot\sqrt{(ax+b)}+C$

证明：令$\sqrt{ax+b}=t \quad (t>0)$，则$x=\frac{t^2-b}{a}$ ，$dx=\frac{2t}{a}dt$ ，

$$\therefore \int \frac{x}{\sqrt{ax+b}}dx = \int \frac{t^2-b}{at}\cdot\frac{2t}{a}dt$$

$$=\frac{2}{a^2}\int t^2dt-\frac{2}{a^2}\int bdt$$

$$=\frac{2}{a^2}\cdot\frac{1}{1+2}\cdot t^{2+1}-\frac{2b}{a^2}\cdot t+C$$

$$=\frac{2}{3a^2}\cdot t^3-\frac{2b}{a^2}\cdot t+C$$

将$t=\sqrt{ax+b}$代入上式得：$\int \frac{x}{\sqrt{ax+b}}dx=\frac{2}{3a^2}\cdot(ax+b)\cdot\sqrt{(ax+b)}-\frac{2b}{a^2}\cdot\sqrt{(ax+b)}+C$

$$=\frac{2}{3a^2}\cdot(ax-2b)\cdot\sqrt{(ax+b)}+C$$

14. $\int \frac{x^2}{\sqrt{ax+b}}dx = \frac{2}{15a^3}\cdot(3a^2x^2-4abx+8b^2)\cdot\sqrt{(ax+b)}+C$

证明：令$\sqrt{ax+b}=t \quad (t>0)$，则$x=\frac{t^2-b}{a}$ ，$dx=\frac{2t}{a}dt$ ，

$$\therefore \int \frac{x^2}{\sqrt{ax+b}}dx = \int (\frac{t^2-b}{a})^2\cdot\frac{1}{t}\cdot\frac{2t}{a}dt$$

$$=\frac{2}{a^3}\int (t^4+b^2-2bt^2)dt$$

$$=\frac{2}{a^3}\int t^4dt+\frac{2}{a^3}\int b^2dt-\frac{4b}{a^3}\int t^2dt$$

$$=\frac{2}{a^3}(\frac{1}{5}t^5+b^2t-\frac{2b}{3}t^3)+C$$

$$=\frac{2t}{15a^3}\cdot(3t^4+15b^2-10bt^2)+C$$

将$t=\sqrt{ax+b}$代入上式得：

$$\int \frac{x^2}{\sqrt{ax+b}}dx=\frac{2}{15a^3}\cdot\sqrt{(ax+b)}\cdot\left[3(a^2x^2+b^2+2abx)+15b^2-10b\cdot(ax+b)\right]\cdot\sqrt{(ax+b)}+C$$

$$=\frac{2}{15a^3}\cdot(3a^2x^2-4abx+8b^2)\cdot\sqrt{(ax+b)}+C$$

15. $$\int \frac{dx}{x\sqrt{ax+b}} = \begin{cases} \frac{1}{\sqrt{b}} \cdot ln\left|\frac{\sqrt{ax+b}-\sqrt{b}}{\sqrt{ax+b}+\sqrt{b}}\right| + C & (b>0) \\ \frac{2}{\sqrt{-b}} \cdot arctan\sqrt{\frac{ax+b}{-b}} + C & (b<0) \end{cases}$$

证明：令$\sqrt{ax+b}=t \quad (t>0)$，则$x=\frac{t^2-b}{a}$ ，$dx=\frac{2t}{a}dt$ ，

$$\therefore \int \frac{dx}{x\sqrt{ax+b}} = \int \frac{1}{\frac{t^2-b}{a}\cdot t}\cdot\frac{2t}{a}dt$$

$$= \int \frac{2}{t^2-b}dt$$

1.当b>0时，$\int \frac{2}{t^2-b}dt = 2\int \frac{1}{t^2-(\sqrt{b})^2}dt$

$$= \frac{1}{\sqrt{b}}\cdot ln\left|\frac{t-\sqrt{b}}{t+\sqrt{b}}\right| + C$$

公式21：$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\cdot ln\left|\frac{x-a}{x+a}\right| + C$

将$t=\sqrt{ax+b}$代入上式得：$\int \frac{dx}{x\sqrt{ax+b}} = \frac{1}{\sqrt{b}}\cdot ln\left|\frac{\sqrt{ax+b}-\sqrt{b}}{\sqrt{ax+b}+\sqrt{b}}\right| + C$

2.当$b<0$时，$\int \frac{2}{t^2-b}dt = 2\int \frac{1}{t^2+(\sqrt{-b})^2}dt$

公式19：$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\cdot arctan\frac{x}{a} + C$

$$= \frac{2}{\sqrt{-b}}\cdot arctan\frac{\mathrm{t}}{\sqrt{-b}} + C$$

将$t=\sqrt{ax+b}$代入上式得：$\int \frac{dx}{x\sqrt{ax+b}} = \frac{2}{\sqrt{-b}}\cdot arctan\sqrt{\frac{ax+b}{-b}} + C$

综合讨论1，2得：$$\int \frac{dx}{x\sqrt{ax+b}} = \begin{cases} \frac{1}{\sqrt{b}} \cdot ln\left|\frac{\sqrt{ax+b}-\sqrt{b}}{\sqrt{ax+b}+\sqrt{b}}\right| + C & (b>0) \\ \frac{2}{\sqrt{-b}} \cdot arctan\sqrt{\frac{ax+b}{-b}} + C & (b>0) \end{cases}$$

16. $\int \frac{dx}{x^2\sqrt{ax+b}} = -\frac{\sqrt{ax+b}}{bx} - \frac{a}{2b}\int \frac{dx}{x\sqrt{ax+b}}$

证明：设 $\frac{1}{x^2\cdot\sqrt{ax+b}} = \frac{A}{x\sqrt{ax+b}} + \frac{B\sqrt{ax+b}}{x^2}$，则 $1 = Ax + B(ax+b)$

$$\therefore \text{有} \begin{cases} A + Ba = 0 \\ Bb = 1 \end{cases} \Rightarrow \begin{cases} A = -\frac{a}{b} \\ B = \frac{1}{b} \end{cases}$$

于是 $\int \frac{dx}{x^2\sqrt{ax+b}} = -\frac{a}{b}\int \frac{1}{x\sqrt{ax+b}}dx + \frac{1}{b}\int \frac{\sqrt{ax+b}}{x^2}dx$

$$= -\frac{a}{b}\int \frac{1}{x\sqrt{ax+b}}dx - \frac{1}{b}\int \sqrt{ax+b}\, d\frac{1}{x}$$

$$= -\frac{a}{b}\int \frac{1}{x\sqrt{ax+b}}dx - \frac{\sqrt{ax+b}}{bx} + \frac{1}{b}\int \frac{1}{x} d\sqrt{ax+b}$$

$$= -\frac{a}{b}\int \frac{1}{x\sqrt{ax+b}}dx - \frac{\sqrt{ax+b}}{bx} + \frac{1}{b}\int \frac{1}{x}\cdot\frac{a}{2}(ax+b)^{-\frac{1}{2}}dx$$

$$= -\frac{a}{b}\int \frac{1}{x\sqrt{ax+b}}dx - \frac{\sqrt{ax+b}}{bx} + \frac{a}{2b}\int \frac{1}{x\sqrt{ax+b}}dx$$

$$= -\frac{\sqrt{ax+b}}{bx} - \frac{a}{2b}\int \frac{dx}{x\sqrt{ax+b}}$$

17. $\int \frac{\sqrt{ax+b}}{x}dx = 2\sqrt{ax+b} + b\int \frac{dx}{x\sqrt{ax+b}}$

证明：令 $\sqrt{ax+b} = t \quad (t \ge 0)$，则 $x = \frac{t^2-b}{a}$ ， $dx = \frac{2t}{a}dt$

$$\therefore \int \frac{\sqrt{ax+b}}{x}dx = \int \frac{at}{t^2-b}\cdot\frac{2t}{a}dt = 2\int \frac{t^2}{t^2-b}dt$$

$$= 2\int \frac{t^2-b^2+b^2}{t^2-b}dt = 2\int dt + 2b\int \frac{1}{t^2-b}dt$$

$$= 2t + 2b\int \frac{1}{t^2-b}dt$$

$\because$ b取值为R，符号可正可负 $\therefore$ $\int \frac{1}{t^2-b}dt$ 不能明确积分

$$\therefore \int \frac{\sqrt{ax+b}}{x}dx = 2t + 2b\int \frac{1}{t^2-b}dt$$

$$= 2t + 2b\int \frac{1}{t^2-b}\cdot\frac{a}{2t}dx$$

将$t = \sqrt{ax+b}$代入上式得：$\int \frac{\sqrt{ax+b}}{x}dx = 2\sqrt{(ax+b)} + 2b\int \frac{1}{ax+b-b}\cdot\frac{a}{2\sqrt{ax+b}}dx$

$$= 2\sqrt{ax+b} + b\int \frac{dx}{x\sqrt{ax+b}}$$

18. $\int \frac{\sqrt{ax+b}}{x^2}dx = -\frac{\sqrt{ax+b}}{x} + \frac{a}{2}\int \frac{dx}{x\sqrt{ax+b}}$

证明：$\int \frac{\sqrt{ax+b}}{x^2}dx = -\int \sqrt{ax+b}\, d\frac{1}{x}$

$$= -\frac{\sqrt{ax+b}}{x} + \int \frac{1}{x} d\sqrt{ax+b}$$

$$= -\frac{\sqrt{ax+b}}{x} + \int \frac{1}{x} \cdot (ax+b)^{-\frac{1}{2}} \cdot \frac{a}{2} dx$$

$$= -\frac{\sqrt{ax+b}}{x} + \frac{a}{2}\int \frac{dx}{x\sqrt{ax+b}}$$

（三）含有 $x^2 \pm a^2$ 的积分（19~21）

19. $\int \frac{dx}{x^2+a^2} = \frac{1}{a} \cdot arctan\frac{x}{a} + C$

证明：令 $x = a \cdot tant \quad (-\frac{\pi}{2} < t < \frac{\pi}{2})$，则 $dx = d(a \cdot tant) = a \cdot sec^2 t\, dt$

$$\frac{1}{x^2+a^2} = \frac{dx}{a^2 \cdot (1+tan^2 t)} = \frac{1}{a^2 sec^2 t}$$

$$\therefore \int \frac{dx}{x^2+a^2} = \int \frac{1}{a^2 sec^2 t} \cdot a \cdot sec^2 t\, dt$$

$$= \frac{1}{a}\int dt$$

$$= \frac{1}{a} \cdot t + C$$

$\because\ x = a \cdot tant \qquad \therefore\ t = arctan\frac{x}{a}$

将 $t = arctan\frac{x}{a}$ 代入上式得：$\int \frac{dx}{x^2+a^2} = \frac{1}{a} \cdot arctan\frac{x}{a} + C$

20. $\int \frac{dx}{(x^2+a^2)^n} = \frac{x}{2(n-1)\cdot a^2\cdot(x^2+a^2)^{n-1}} + \frac{2n-3}{2(n-1)\cdot a^2}\int \frac{dx}{(x^2+a^2)^{n-1}}$

证明：$\int \frac{dx}{(x^2+a^2)^n} = \frac{x}{(x^2+a^2)^n} - \int x\, d\,\frac{1}{(x^2+a^2)^n}$

$$= \frac{x}{(x^2+a^2)^n} - \int x\cdot(-n)\cdot(x^2+a^2)^{-n-1}\cdot 2x\,dx$$

$$= \frac{x}{(x^2+a^2)^n} + 2n\int \frac{x^2}{(x^2+a^2)^{n+1}}dx$$

$$= \frac{x}{(x^2+a^2)^n} + 2n\int \frac{x^2+a^2-a^2}{(x^2+a^2)^{n+1}}dx$$

$$= \frac{x}{(x^2+a^2)^n} + 2n\int \frac{1}{(x^2+a^2)^n}dx - 2na^2\int \frac{1}{(x^2+a^2)^{n+1}}dx$$

移项并整理得：$(1-2n)\int \frac{dx}{(x^2+a^2)^n} = \frac{x}{(x^2+a^2)^n} - 2na^2\int \frac{1}{(x^2+a^2)^{n+1}}dx$

$$\therefore \int \frac{1}{(x^2+a^2)^{n+1}}dx = \frac{1}{2na^2}\left[\frac{x}{(x^2+a^2)^n} + (2n-1)\int \frac{dx}{(x^2+a^2)^n}\right]$$

令$n+1=n$，则$\int \frac{dx}{(x^2+a^2)^n} = \frac{1}{2(n-1)\cdot a^2}\left[\frac{x}{(x^2+a^2)^{n-1}} + (2n-3)\int \frac{dx}{(x^2+a^2)^{n-1}}\right]$

$$= \frac{x}{2(n-1)\cdot a^2\cdot(x^2+a^2)^{n-1}} + \frac{2n-3}{2(n-1)\cdot a^2}\int \frac{dx}{(x^2+a^2)^{n-1}}$$

21. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\cdot ln\left|\frac{x-a}{x+a}\right| + C$

证明：$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\int [\frac{1}{x-a} - \frac{1}{x+a}]\,dx$

$$= \frac{1}{2a}\int \frac{1}{x-a}dx - \frac{1}{2a}\int \frac{1}{x+a}dx$$

$$= \frac{1}{2a}\cdot ln\,|\,x-a\,| - \frac{1}{2a}\cdot ln\,|\,x+a\,| + C$$

$$= \frac{1}{2a}\cdot ln\left|\frac{x-a}{x+a}\right| + C$$

（四）含有 $ax^2+b\ (a>0)$ 的积分（22~28）

22. $$\int\frac{dx}{ax^2+b}=\begin{cases}\frac{1}{\sqrt{ab}}\cdot arctan\sqrt{\frac{a}{b}}\cdot x+C & (b>0)\\ \frac{1}{2\sqrt{-ab}}\cdot ln\left|\frac{\sqrt{a}\cdot x-\sqrt{-b}}{\sqrt{a}\cdot x+\sqrt{-b}}\right|+C & (b<0)\end{cases}\qquad (a>0)$$

证明：

1.当 $b>0$ 时，$\frac{1}{ax^2+b}=\frac{1}{x^2+\frac{b}{a}}\cdot\frac{1}{a}=\frac{1}{x^2+(\sqrt{\frac{b}{a}})^2}\cdot\frac{1}{a}$

$$\therefore\ \int\frac{dx}{ax^2+b}=\frac{1}{a}\int\frac{1}{x^2+(\sqrt{\frac{b}{a}})^2}dx$$

$$=\frac{1}{a}\cdot\sqrt{\frac{a}{b}}\cdot arctan\sqrt{\frac{a}{b}}\cdot x+C$$

$$=\frac{1}{\sqrt{ab}}\cdot arctan\sqrt{\frac{a}{b}}\cdot x+C$$

2.当 $b<0$ 时，$\frac{1}{ax^2+b}=\frac{1}{x^2-(-\frac{b}{a})}\cdot\frac{1}{a}=\frac{1}{x^2-(\sqrt{-\frac{b}{a}})^2}\cdot\frac{1}{a}$

$$\therefore\ \int\frac{dx}{ax^2+b}=\frac{1}{a}\int\frac{1}{x^2-(\sqrt{-\frac{b}{a}})^2}dx$$

$$=\frac{1}{2\sqrt{\frac{-b}{a}}}\cdot\frac{1}{a}\cdot ln\left|\frac{x-\sqrt{\frac{-b}{a}}}{x+\sqrt{\frac{-b}{a}}}\right|+C$$

$$=\frac{1}{2\sqrt{-ab}}\cdot ln\left|\frac{\sqrt{a}\cdot x-\sqrt{-b}}{\sqrt{a}\cdot x+\sqrt{-b}}\right|+C$$

综合讨论1,2得：$$\int\frac{dx}{ax^2+b}=\begin{cases}\frac{1}{\sqrt{ab}}\cdot arctan\sqrt{\frac{a}{b}}\cdot x+C & (b>0)\\ \frac{1}{2\sqrt{-ab}}\cdot ln\left|\frac{\sqrt{a}\cdot x-\sqrt{-b}}{\sqrt{a}\cdot x+\sqrt{-b}}\right|+C & (b<0)\end{cases}$$

23. $$\int\frac{x}{ax^2+b}dx=\frac{1}{2a}\cdot ln\ \left|ax^2+b\right|+C\qquad (a>0)$$

证明：$$\int\frac{x}{ax^2+b}dx=\frac{1}{2}\int\frac{1}{ax^2+b}d(x^2)$$

$$=\frac{1}{2a}\int\frac{1}{ax^2+b}d(ax^2+b)$$

$$=\frac{1}{2a}\cdot ln\ \left|ax^2+b\right|+C$$

24. $\int \frac{x^2}{ax^2+b}dx = \frac{x}{a} - \frac{b}{a}\int \frac{dx}{ax^2+b} \qquad (a>0)$

证明：$\int \frac{x^2}{ax^2+b}dx = \frac{b}{a}\int \frac{ax^2}{ax^2+b}\cdot\frac{1}{b}dx$

$$= \frac{b}{a}\int (\frac{1}{b} - \frac{1}{ax^2+b})dx$$

$$= \frac{b}{a}\int \frac{1}{b}dx - \frac{b}{a}\int \frac{1}{ax^2+b}dx$$

$$= \frac{x}{a} - \frac{b}{a}\int \frac{dx}{ax^2+b}$$

25. $\int \frac{dx}{x(ax^2+b)} = \frac{1}{2b}\cdot ln\frac{x^2}{|ax^2+b|} + \mathrm{C} \qquad (a>0)$

证明：$\int \frac{dx}{x(ax^2+b)} = \int \frac{x}{x^2(ax^2+b)}dx$

$$= \frac{1}{2}\int \frac{1}{x^2(ax^2+b)}d(x^2)$$

设： $\frac{1}{x^2(ax^2+b)} = \frac{A}{x^2} + \frac{B}{ax^2+b}$

则 $1 = A(ax^2+b) + Bx^2 = x^2(Aa+B) + Ab$

$\therefore$ 有 $\begin{cases} Aa+B=0 \\ Ab=1 \end{cases} \Rightarrow \begin{cases} A=\frac{1}{b} \\ B=-\frac{a}{b} \end{cases}$

于是 $\int \frac{dx}{x(ax^2+b)} = \frac{1}{2}\int [\frac{1}{bx^2} - \frac{a}{b(ax^2+b)}]d(x^2)$

$$= \frac{1}{2b}\int \frac{1}{x^2}dx^2 - \frac{a}{2b}\int \frac{1}{ax^2+b}d(x^2)$$

$$= \frac{1}{2b}\int \frac{1}{x^2}dx^2 - \frac{1}{2b}\int \frac{1}{ax^2+b}d(ax^2+b)$$

$$= \frac{1}{2b}\cdot ln\ |x^2| - \frac{1}{2b}?ln|ax^2+b| + C$$

$$= \frac{1}{2b}\cdot ln\frac{x^2}{|ax^2+b|} + \mathrm{C}$$

26. $\int \frac{dx}{x^2(ax^2+b)} = -\frac{1}{bx} - \frac{a}{b}\int \frac{dx}{ax^2+b} \qquad (a>0)$

证明：设： $\frac{1}{x^2(ax^2+b)} = \frac{A}{x^2} + \frac{B}{ax^2+b}$

则 $1 = A(ax^2+b) + Bx^2 = x^2(Aa+B) + Ab$

$\therefore$ 有 $\begin{cases} Aa+B=0 \\ Ab=1 \end{cases} \Rightarrow \begin{cases} A=\frac{1}{b} \\ B=-\frac{a}{b} \end{cases}$

于是 $\int \frac{dx}{x^2(ax^2+b)} = \int [\frac{1}{bx^2} - \frac{a}{b(ax^2+b)}]dx$

$$= \frac{1}{b}\int \frac{1}{x^2}dx - \frac{a}{b}\int \frac{1}{ax^2+b}dx$$

$$= -\frac{1}{bx} - \frac{a}{b}\int \frac{dx}{ax^2+b}$$

27. $\int \frac{dx}{x^3(ax^2+b)} = \frac{a}{2b^2} ln \frac{|ax^2+b|}{x^2} - \frac{1}{2bx^2} + C \qquad (a>0)$

证明:$\int \frac{dx}{x^3(ax^2+b)} = \int \frac{x}{x^4(ax^2+b)}dx$

$$= \frac{1}{2}\int \frac{1}{x^4(ax^2+b)}d(x^2)$$

设： $\frac{1}{x^4(ax^2+b)} = \frac{A}{x^2} + \frac{B}{x^4} + \frac{C}{ax^2+b}$

则 $1 = Ax^2(ax^2+b) + B(ax^2+b) + Cx^4$

$$= (Aa+C)x^4 + (Ab+Ba)x^2 + Bb$$

$\therefore$ 有 $\begin{cases} Aa+C=0 \\ Ab+Ba=0 \\ Bb=1 \end{cases} \Rightarrow \begin{cases} B=\frac{1}{b} \\ A=-\frac{a}{b^2} \\ C=\frac{a^2}{b^2} \end{cases}$

于是 $\int \frac{dx}{x^3(ax^2+b)} = -\frac{a}{2b^2}\int \frac{1}{x^2}d(x^2) + \frac{1}{2b}\int \frac{1}{x^4}d(x^2) + \frac{a^2}{2b^2}\int \frac{1}{ax^2+b}d(x^2)$

$$= -\frac{a}{2b^2}\cdot ln\ |x^2| - \frac{1}{2bx^2} + \frac{a}{2b^2} ?ln|ax^2+b| + C$$

$$= \frac{a}{2b^2} ln \frac{|ax^2+b|}{x^2} - \frac{1}{2bx^2} + C$$

28. $\int \frac{dx}{(ax^2+b)^2} = \frac{x}{2b(ax^2+b)} + \frac{1}{2b}\int \frac{dx}{ax^2+b} \qquad (a>0)$

证明:$\int \frac{dx}{(ax^2+b)^2} = -\int \frac{1}{2ax} d\frac{1}{ax^2+b} = -\frac{1}{2ax}\cdot\frac{1}{ax^2+b} + \int \frac{1}{ax^2+b} d\frac{1}{2ax}$

$$= -\frac{1}{2ax}\cdot\frac{1}{ax^2+b} - \int \frac{1}{ax^2+b}\cdot\frac{1}{2ax^2}dx$$

设: $\frac{1}{2ax^2(ax^2+b)} = \frac{A}{2ax^2} + \frac{B}{ax^2+b}$, 则 $1 = A(ax^2+b) + 2Bax^2 = (Aa+2Ba)x^2 + Ab$

$$\therefore \text{ 有 } \begin{cases} Aa+2Ba=0 \\ Ab=1 \end{cases} \Rightarrow \begin{cases} A=\frac{1}{b} \\ B=-\frac{1}{2b} \end{cases}$$

于是上式 $= -\frac{1}{2ax(ax^2+b)} - \int (\frac{1}{2abx^2} - \frac{1}{2b(ax^2+b)})dx$

$$= -\frac{1}{2ax(ax^2+b)} - \frac{1}{2ab}\int \frac{1}{x^2}dx + \frac{1}{2b}\int \frac{1}{2b(ax^2+b)}dx$$

$$= -\frac{1}{2ax(ax^2+b)} + \frac{1}{2abx} + \frac{1}{2b}\int \frac{1}{2b(ax^2+b)}dx = \frac{ax^2+b-b}{2abx(ax^2+b)} + \frac{1}{2b}\int \frac{1}{2b(ax^2+b)}dx$$

$$= \frac{x}{2b(ax^2+b)} + \frac{1}{2b}\int \frac{dx}{ax^2+b}$$

（五）含有 ax^2+bx+c $(a>0)$ 的积分（29~30）

29. $$\int \frac{dx}{ax^2+bx+c} = \begin{cases} \frac{2}{\sqrt{4ac-b^2}}\cdot \arctan\frac{2ax+b}{\sqrt{4ac-b^2}} + C & (b^2<4ac) \\ \frac{1}{\sqrt{b^2-4ac}}\cdot \ln\left|\frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right| + C & (b^2>4ac) \end{cases} \qquad (a>0)$$

证明: $\because ax^2+bx+c = \frac{1}{4a}\left[(2ax+b)^2 + (4ac-b^2)\right]$

$$\therefore \int \frac{dx}{ax^2+bx+c} = 4a\int \frac{1}{(2ax+b)^2+(4ac-b^2)}dx$$

1.当 $b^2<4ac$ 时, $\int \frac{dx}{ax^2+bx+c} = 4a\int \frac{1}{(2ax+b)^2+(\sqrt{4ac-b^2})^2}dx$

$\boxed{\text{公式 19: } \int \frac{dx}{x^2+a^2} = \frac{1}{a}\cdot\arctan\frac{x}{a} + C}$

$$= \frac{4a}{2a}\int \frac{1}{(2ax+b)^2+(\sqrt{4ac-b^2})^2}d(2ax+b)$$

$$= \frac{2}{\sqrt{4ac-b^2}}\cdot\arctan\frac{2ax+b}{\sqrt{4ac-b^2}} + C$$

2.当 $b^2>4ac$ 时, $\int \frac{dx}{ax^2+bx+c} = 4a\int \frac{1}{(2ax+b)^2+(4ac-b^2)}dx$

$$= 4a\int \frac{1}{(2ax+b)^2-(b^2-4ac)}dx$$

$\boxed{\text{公式21: } \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\cdot\ln\left|\frac{x-a}{x+a}\right| + C}$

$$= \frac{4a}{2a}\int \frac{1}{(2ax+b)^2-(\sqrt{b^2-4ac})^2}d(2ax+b)$$

$$= \frac{1}{\sqrt{b^2-4ac}}\cdot\ln\left|\frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right| + C$$

综合讨论1,2得: $$\int \frac{dx}{ax^2+bx+c} = \begin{cases} \frac{2}{\sqrt{4ac-b^2}}\cdot \arctan\frac{2ax+b}{\sqrt{4ac-b^2}} + C & (b^2<4ac) \\ \frac{1}{\sqrt{b^2-4ac}}\cdot \ln\left|\frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right| + C & (b^2>4ac) \end{cases}$$

30. $\int \frac{x}{ax^2+bx+c}dx = \frac{1}{2a}\cdot ln\left| ax^2+bx+c \right| - \frac{b}{2a}\int \frac{dx}{ax^2+bx+c} \qquad (a>0)$

证明:$\int \frac{x}{ax^2+bx+c}dx = \int \frac{1}{2a}\cdot\frac{2ax+b-b}{ax^2+bx+c}dx$

$$= \frac{1}{2a}\int \frac{2ax+b}{ax^2+bx+c}dx + \frac{1}{2a}\int \frac{-b}{ax^2+bx+c}dx$$

$$= \frac{1}{2a}\int \frac{1}{ax^2+bx+c}d(ax^2+bx+c) - \frac{b}{2a}\int \frac{1}{ax^2+bx+c}dx$$

$$= \frac{1}{2a}\cdot ln\left| ax^2+bx+c \right| - \frac{b}{2a}\int \frac{dx}{ax^2+bx+c}$$

（六）含有 $\sqrt{x^2+a^2}$ $(a>0)$ 的积分（31~44）

31. $\int \frac{dx}{\sqrt{x^2+a^2}} = arsh\frac{x}{a} + C_1 = ln\,(x+\sqrt{x^2+a^2}) + \mathrm{C} \qquad (a>0)$

证明:被积函数 $f(x) = \frac{1}{\sqrt{x^2+a^2}}$ 的定义域为$\{x \mid x \in R\}$

可令$x = a\,tant \quad (-\frac{\pi}{2} < t < \frac{\pi}{2})$, 则$dx = d(a\,tant) = a\,sec^2tdt$, $\sqrt{x^2+a^2} = |a\,sect|$

$\because -\frac{\pi}{2} < t < \frac{\pi}{2}, sect = \frac{1}{cost} > 0, \therefore \sqrt{x^2+a^2} = a\,sect$

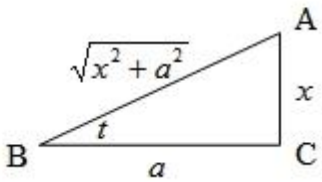
$\therefore \int \frac{dx}{\sqrt{x^2+a^2}} = \int \frac{1}{a\,sect}\cdot a\,sec^2t\,dt$ $\boxed{公式\ 87:\int sect dt = ln|sect+tant|+C}$

$$= \int sect\,dt$$

$$= ln\left| sect+tant \right| + C_2$$

在Rt$\triangle ABC$中，设$\angle \mathrm{B} = t$,$|BC| = a$,则$|\mathrm{AC}| = x$, $|\mathrm{AB}| = \sqrt{x^2+a^2}$

$\therefore\ sect = \frac{1}{cost} = \frac{\sqrt{x^2+a^2}}{a}$, $tant = \frac{x}{a}$



$\therefore \int \frac{dx}{\sqrt{x^2+a^2}} = ln\left| sect+tant \right| + C_2$

$$= ln\left| \frac{\sqrt{x^2+a^2}+x}{a} \right| + C_2$$

$$= ln\left| \sqrt{x^2+a^2}+x \right| - lna + C_2$$

$$= ln\left| \sqrt{x^2+a^2}+x \right| + C_3$$

$\because \sqrt{x^2+a^2} + x > 0$

$\therefore \int \frac{dx}{\sqrt{x^2+a^2}} = ln\,(x+\sqrt{x^2+a^2}) + \mathrm{C}$

32. $\int \frac{dx}{\sqrt{(x^2+a^2)^3}} = \frac{x}{a^2\sqrt{x^2+a^2}} + C \qquad (a>0)$

证明:被积函数 $f(x) = \frac{1}{\sqrt{(x^2+a^2)^3}}$ 的定义域为$\{x \mid x \in R\}$

可令$x = a\,tant \quad (-\frac{\pi}{2} < t < \frac{\pi}{2})$, 则$dx = d(a\,tant) = a\,sec^2 tdt$, $\sqrt{(x^2+a^2)^3} = |a^3\ sec^3 t|$

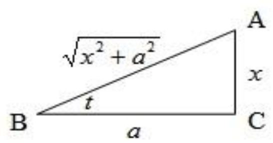
$\because -\frac{\pi}{2} < t < \frac{\pi}{2}, sect = \frac{1}{cost} > 0$, $\therefore \sqrt{(x^2+a^2)^3} = a^3\ sec^3 t$

$$\therefore \int \frac{dx}{\sqrt{(x^2+a^2)^3}} = \int \frac{1}{a^3\ sec^3 t} \cdot a\,sec^2 t\, dt = \frac{1}{a^2}\int \frac{1}{sect} dt$$

$$= \frac{1}{a^2}\int \cos t dt = \frac{1}{a^2} sint + C$$

在RtΔABC中, 设$\angle B = t$, $|BC| = a$,则$|AC| = x$, $|AB| = \sqrt{x^2+a^2}$

$$\therefore \sin t = \frac{|AC|}{|AB|} = \frac{x}{\sqrt{x^2+a^2}}$$



$$\therefore \int \frac{dx}{\sqrt{(x^2+a^2)^3}} = \frac{1}{a^2} \cdot sint + C = \frac{x}{a^2\sqrt{x^2+a^2}} + C$$

33. $\int \frac{x}{\sqrt{x^2+a^2}} dx = \sqrt{x^2+a^2} + C \qquad (a>0)$

证明: 令$\sqrt{x^2+a^2} = t \quad (t>0)$, 则$x = \sqrt{t^2-a^2}$

$$\therefore dx = \frac{1}{2}(t^2-a^2)^{-\frac{1}{2}} \cdot 2tdt = \frac{t}{\sqrt{t^2-a^2}} dt$$

$$\therefore \int \frac{x}{\sqrt{x^2+a^2}} dx = \int \frac{\sqrt{t^2-a^2}}{t} \cdot \frac{t}{\sqrt{t^2-a^2}} dt$$

$$= \int dt = t + C$$

将$t = \sqrt{x^2+a^2}$代入上式得: $\int \frac{x}{\sqrt{x^2+a^2}} dx = \sqrt{x^2+a^2} + C$

34. $\int \frac{x}{\sqrt{(x^2+a^2)^3}} dx = -\frac{1}{\sqrt{x^2+a^2}} + C \qquad (a>0)$

证明: $\int \frac{x}{\sqrt{(x^2+a^2)^3}} dx = \int x \cdot (x^2+a^2)^{-\frac{3}{2}} dx = \frac{1}{2}\int (x^2+a^2)^{-\frac{3}{2}} dx^2$

$$= \frac{1}{2}\int (x^2+a^2)^{-\frac{3}{2}} d(x^2+a^2)$$

$$= \frac{1}{2} \times \frac{1}{1-\frac{3}{2}} \cdot (x^2+a^2)^{1-\frac{3}{2}} + C$$

$$= -\frac{1}{\sqrt{x^2+a^2}} + C$$

35. $\int \frac{x^2}{\sqrt{x^2+a^2}}dx = \frac{x}{2}\cdot\sqrt{x^2+a^2} - \frac{a^2}{2} ln\,(x+\sqrt{x^2+a^2}) + C \qquad (a>0)$

证明: $\int \frac{x^2}{\sqrt{x^2+a^2}}dx = \int \frac{x^2+a^2-a^2}{\sqrt{x^2+a^2}}dx$

$$= \int \sqrt{x^2+a^2}\,dx - a^2\int \frac{1}{\sqrt{x^2+a^2}}dx$$

$\because \int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\cdot\sqrt{x^2+a^2} + \frac{a^2}{2}\cdot ln\,(x+\sqrt{x^2+a^2}) + C \qquad$ (公式39)

$\int \frac{1}{\sqrt{x^2+a^2}}dx = ln\,(x+\sqrt{x^2+a^2}) + C \qquad$ (公式31)

$\therefore \int \frac{x^2}{\sqrt{x^2+a^2}}dx = \frac{x}{2}\cdot\sqrt{x^2+a^2} + \frac{a^2}{2} ln\,(x+\sqrt{x^2+a^2}) - a^2\cdot ln\,(x+\sqrt{x^2+a^2}) + C$

$$= \frac{x}{2}\cdot\sqrt{x^2+a^2} - \frac{a^2}{2}\cdot ln\,(x+\sqrt{x^2+a^2}) + C$$

36. $\int \frac{x^2}{\sqrt{(x^2+a^2)^3}}dx = -\frac{x}{\sqrt{x^2+a^2}} + ln\,(x+\sqrt{x^2+a^2}) + C \qquad (a>0)$

证明:被积函数 $f(x) = \frac{x^2}{\sqrt{(x^2+a^2)^3}}$ 的定义域为 $\{x\,|\,x\in R\}$

可令 $x = a\,tant \quad (-\frac{\pi}{2} < t < \frac{\pi}{2})$, 则 $dx = d(a\,tant) = a\,sec^2 t dt$, $\frac{x^2}{\sqrt{(x^2+a^2)^3}} = \frac{a^2 tan^2 t}{|a^3\,sec^3 t|}$

$\because -\frac{\pi}{2} < t < \frac{\pi}{2}$, $sect = \frac{1}{cost} > 0$, $\therefore \frac{x^2}{\sqrt{(x^2+a^2)^3}} = \frac{tan^2 t}{a\,sec^3 t}$

$\therefore \int \frac{x^2}{\sqrt{(x^2+a^2)^3}}dx = \int \frac{tan^2 t}{a\,sec^3 t}\cdot a\,sec^2 t dt = \int \frac{tan^2 t}{sect}dt = \int \frac{sec^2 t - 1}{sect}dt$

$$= \int sectdt - \int \frac{1}{sect}dt = \int sectdt - \int \cos tdt$$

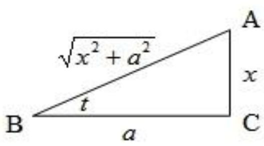
$$= ln\left|\,sect + tant\,\right| - sint + C_1$$

$\boxed{\text{公式 } 87: \int \sec t\,dt = ln\,|\,sect + tant\,\,| + C}$

在Rt⊿ABC中, 设$\angle B = t$, $|\,BC\,| = a$,则 $|\,AC\,| = x$, $|\,AB\,| = \sqrt{x^2+a^2}$

$\therefore\ sint = \frac{x}{\sqrt{x^2+a^2}}$, $tant = \frac{x}{a}$, $sect = \frac{1}{cost} = \frac{\sqrt{x^2+a^2}}{a}$

$\therefore \int \frac{x^2}{\sqrt{(x^2+a^2)^3}}dx = ln\left|\,sect + tant\,\right| - sint + C_1$



$$= ln\left|\frac{\sqrt{x^2+a^2}+x}{a}\right| - \frac{x}{\sqrt{x^2+a^2}} + C_1$$

$$= ln\left|\sqrt{x^2+a^2}+x\right| - \frac{x}{\sqrt{x^2+a^2}} - lna + C_1$$

$\because \sqrt{x^2+a^2} + x > 0$

$\therefore \int \frac{x^2}{\sqrt{(x^2+a^2)^3}}dx = -\frac{x}{\sqrt{x^2+a^2}} + ln\,(x+\sqrt{x^2+a^2}) + C$

37. $\int \frac{dx}{x\cdot\sqrt{x^2+a^2}}=\frac{1}{a}\cdot ln\frac{\sqrt{x^2+a^2}-a}{|x|}+C \qquad (a>0)$

证明：令$\sqrt{x^2+a^2}=t \quad (t>0)$，则$x=\sqrt{t^2-a^2}$

$$\therefore\ dx=\frac{1}{2}(t^2-a^2)^{-\frac{1}{2}}\cdot 2tdt=\frac{t}{\sqrt{t^2-a^2}}dt$$

$$\therefore \int\frac{dx}{x\cdot\sqrt{x^2+a^2}}=\int\frac{1}{t\cdot\sqrt{t^2-a^2}}\cdot\frac{t}{\sqrt{t^2-a^2}}dt$$

$$=\int\frac{1}{t^2-a^2}dt$$

$\boxed{\text{公式21:}\int\frac{dx}{x^2-a^2}=\frac{1}{2a}\cdot ln\left|\frac{x-a}{x+a}\right|+C}$

$$=\frac{1}{2a}\cdot ln\left|\frac{t-a}{t+a}\right|+C$$

$$=\frac{1}{2a}\cdot ln\left|\frac{(t-a)^2}{t^2-a^2}\right|+C$$

将$t=\sqrt{x^2+a^2}$代入上式得：$\int\frac{dx}{x\cdot\sqrt{x^2+a^2}}=\frac{1}{2a}\cdot ln\left|\frac{(\sqrt{x^2+a^2}-a)^2}{x^2+a^2-a^2}\right|+C$

$\boxed{\text{提示:}log_a b^n=nlog_a b}$

$$=\frac{1}{2a}\cdot ln\left|\frac{(\sqrt{x^2+a^2}-a)^2}{x^2}\right|+C$$

$$=\frac{1}{a}\cdot ln\frac{\sqrt{x^2+a^2}-a}{|x|}+C$$

38. $\int\frac{dx}{x^2\cdot\sqrt{x^2+a^2}}=-\frac{\sqrt{x^2+a^2}}{a^2x}+C \qquad (a>0)$

证明：$\int\frac{dx}{x^2\cdot\sqrt{x^2+a^2}}=-\int\frac{1}{\sqrt{x^2+a^2}}d\frac{1}{x}$

令$t=\frac{1}{x} \quad (t\neq 0)$，则$x=\frac{1}{t}$

$$\therefore\ -\int\frac{1}{\sqrt{x^2+a^2}}d\frac{1}{x}=-\int\frac{1}{\sqrt{\frac{1}{t^2}+a^2}}dt=-\int\frac{t}{\sqrt{1+a^2t^2}}dt$$

$$=-\frac{1}{2a^2}\int\frac{2a^2t}{\sqrt{1+a^2t^2}}dt$$

$$=-\frac{1}{2a^2}\int\frac{1}{\sqrt{1+a^2t^2}}d(1+a^2t^2)$$

$$=-\frac{1}{2a^2}\cdot\frac{1}{1-\frac{1}{2}}(1+a^2t^2)^{1-\frac{1}{2}}+C$$

$$=-\frac{1}{a^2}\cdot\sqrt{1+a^2t^2}+C$$

将$t=\frac{1}{x}$代入上式得：$\int\frac{dx}{x^2\cdot\sqrt{x^2+a^2}}=-\frac{\sqrt{x^2+a^2}}{a^2x}+C$

39. $\int\sqrt{x^2+a^2}\,dx=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\cdot ln\,(x+\sqrt{x^2+a^2})+C \qquad (a>0)$

证法1: $\because \int\sqrt{x^2+a^2}\,dx=x\sqrt{x^2+a^2}-\int x\,d\sqrt{x^2+a^2}$

$$=x\sqrt{x^2+a^2}-\int\frac{x^2}{\sqrt{x^2+a^2}}dx$$

$$\therefore \int\sqrt{x^2+a^2}\,dx+\int\frac{x^2}{\sqrt{x^2+a^2}}dx=x\sqrt{x^2+a^2} \qquad ①$$

$$又\int\sqrt{x^2+a^2}\,dx-\int\frac{x^2}{\sqrt{x^2+a^2}}dx=\int\frac{a^2}{\sqrt{x^2+a^2}}dx \qquad ②$$

$\boxed{公式31: \int\frac{1}{\sqrt{x^2+a^2}}dx=ln(x+\sqrt{x^2+a^2})+C \quad (a>0)}$

$$=a^2\cdot ln\,(x+\sqrt{x^2+a^2})+C_1$$

由①+②得，$2\int\sqrt{x^2+a^2}\,dx=x\sqrt{x^2+a^2}+a^2\cdot ln\,(x+\sqrt{x^2+a^2})$

即 $\int\sqrt{x^2+a^2}\,dx=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\cdot ln\,(x+\sqrt{x^2+a^2})+C$

39. $\int\sqrt{x^2+a^2}\,dx=\frac{x}{2}\cdot\sqrt{x^2+a^2}+\frac{a^2}{2}\cdot ln\,(x+\sqrt{x^2+a^2})+C \qquad (a>0)$

证法 2: 令 $x=a\cdot tant \quad (-\frac{\pi}{2}<t<\frac{\pi}{2})$，则 $\sqrt{x^2+a^2}=a\sqrt{1+tan^2t}=|a\cdot sect|$,

$$\because -\frac{\pi}{2}<t<\frac{\pi}{2}, sec\,t=\frac{1}{cos\,t}>0, \therefore \sqrt{x^2+a^2}=a\cdot sect$$

$\boxed{提示: 1+tan^2t=sec^2t}$

$$\therefore \int\sqrt{x^2+a^2}\,dx=\int a\cdot sectd\,(a\cdot tant)=a^2\int sect\,dtant$$

$$=a^2sect\cdot tant-a^2\int tantdsect \qquad ①$$

$$又\int tantdsect=\int tant\cdot sect\cdot tantdt=\int\frac{sin^2t}{cos^3t}dt$$

$$=\int\frac{1-cos^2t}{cos^3t}dt=\int\frac{1}{cost}\cdot\frac{1}{cos^2t}dt-\int\frac{1}{cost}dt$$

$$=\int sect\,dtant-\int sectdt \qquad ②$$

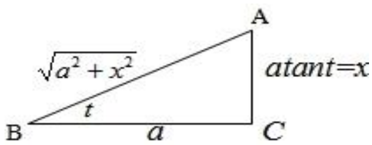
联立①②有 $a^2\int sect\,dtant=\frac{1}{2}(a^2sect\cdot tant+a^2\int sectdt)$ ③

又 $\int sectdt=ln\,|\,sect+tant\,|+C_1$ （公式 87） ④

联立③④有 $a^2\int sect\,dtant=\frac{1}{2}a^2sect\cdot tant+\frac{1}{2}a^2ln\,|\,sect+tant\,|+C_2$ ⑤

$\because x=a\cdot tant$,$\therefore$ 在Rt$\triangle ABC$中，可设 $\angle B=t,|\,BC\,|=a$,

则 $|\,AC\,|=a\cdot tant=x,|\,AB\,|=\sqrt{a^2+x^2}$



$$\therefore sect=\frac{1}{cost}=\frac{\sqrt{a^2+x^2}}{a}, tant=\frac{x}{a}$$

$$\therefore \frac{1}{2}a^2sect\cdot tant+\frac{1}{2}a^2ln\,|\,sect+tant\,|=\frac{x}{2}\cdot\sqrt{x^2+a^2}+\frac{a^2}{2}\cdot ln\,|\frac{x+\sqrt{x^2+a^2}}{a}|$$

$$=\frac{x}{2}\cdot\sqrt{x^2+a^2}+\frac{a^2}{2}\cdot ln\,(x+\sqrt{x^2+a^2})-\frac{a^2}{2}\cdot lna$$

综合①②③④⑤得 $\int\sqrt{x^2+a^2}\,dx=\frac{x}{2}\cdot\sqrt{x^2+a^2}+\frac{a^2}{2}\cdot ln\,(x+\sqrt{x^2+a^2})+C$

40. $\int \sqrt{(x^2+a^2)^3}\,dx = \frac{x}{8}\cdot(2x^2+5a^2)\sqrt{x^2+a^2} + \frac{3}{8}\cdot a^4 \cdot ln\,(x+\sqrt{x^2+a^2})+C \qquad (a>0)$

证明：被积函数 $f(x)=\sqrt{(x^2+a^2)^3}$ 的定义域为 $\{x\,|\,x\in R\}$

可令 $x = a\,tant \quad (-\frac{\pi}{2}<t<\frac{\pi}{2})$，则 $\sqrt{(x^2+a^2)^3} = |a^3\ sec^3t|$

$\because -\frac{\pi}{2}<t<\frac{\pi}{2},\ sect=\frac{1}{cost}>0$，$\therefore \sqrt{(x^2+a^2)^3} = a^3\cdot sec^3t$

$$\begin{aligned}
\therefore \int\sqrt{(x^2+a^2)^3}\,dx &= \int a^3\cdot sec^3t\ d\,(a\,tant) = a^4\int sec^3t\ d\,tant \\
&= a^4sec^3t\cdot tant - a^4\int tant\ d\,sec^3t \\
&= a^4sec^3t\cdot tant - a^4\int tant\cdot 3\cdot sec^2t\cdot sect\,tant\ dt \\
&= a^4sec^3t\cdot tant - 3a^4\int tan^2t\cdot sec^3t\ dt \\
&= a^4sec^3t\cdot tant - 3a^4\int tan^2t\cdot sect\ d\,tant \\
&= a^4sec^3t\cdot tant - 3a^4\int (sec^2t-1)\cdot sect\ d\,tant \\
&= a^4sec^3t\cdot tant - 3a^4\int sec^3t\ d\,tant + 3a^4\int sect\ d\,tant
\end{aligned}$$

移项并整理的：$a^4\int sec^3t\ d\,tant = \frac{1}{4}(a^4sec^3t\,tant + 3a^4\int sect\ d\,tant)$ ①

$$\begin{aligned}
\because \int sect\ d\,tant &= sect\cdot tant - \int tant\ d\,sect \\
&= sect\cdot tant - \int tan^2t\cdot sectdt \\
&= sect\cdot tant - \int (sec^2t-1)\cdot sectdt \\
&= sect\cdot tant - \int sec^3t\ dt + \int sect\ dt
\end{aligned}$$ ②

又 $\because \int sect\ d\,tant = \int sec^3t\ dt$ ③

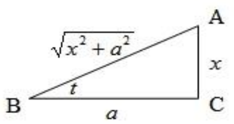
公式 87：$\int sec\,t\ dt = ln\,|\,sect+tant\,|+C$

联立②③得：$a^4\int sec^3t\ d\,tant = \frac{1}{2}\cdot sect\cdot tant + \frac{1}{2}\int sect\ dt$

$= \frac{1}{2}\cdot sect\cdot tant + \frac{1}{2}ln\,|\,sect+tant\,|+C_1$ ④

联立①④得 $a^4\int sec^3t\ d\,tant = \frac{1}{4}a^4sec^3t\cdot tant + \frac{3}{8}a^4sect\cdot tant + \frac{3}{8}a^4\cdot ln\,|\,sect+tant\,|+C_1$

在 Rt$\triangle ABC$ 中，设 $\angle B = t$，$|BC|=a$，则 $|AC|=x$，$|AB|=\sqrt{x^2+a^2}$



$\therefore\ tant = \frac{x}{a},\ sect = \frac{1}{\cos t} = \frac{\sqrt{x^2+a^2}}{a}$

$$\therefore\ a^4\int sec^3t\ d\,tant = \frac{a^4}{4}\cdot\frac{x}{a}\cdot\frac{x^2+a^2}{a^3}\cdot\sqrt{x^2+a^2} + \frac{3a^4}{8}\cdot\frac{\sqrt{x^2+a^2}}{a}\cdot\frac{x}{a} + \frac{3}{8}a^4\cdot ln\left|\frac{\sqrt{x^2+a^2}+x}{a}\right| + C_1$$

$$= \frac{x}{4}(x^2+a^2)\sqrt{x^2+a^2} + \frac{3a^2\cdot x}{8}\cdot\sqrt{x^2+a^2} + \frac{3a^4}{8}\cdot ln\left|\sqrt{x^2+a^2}+x\right| + C$$

$$\therefore \int\sqrt{(x^2+a^2)^3}\,dx = \frac{x}{8}\cdot(2x^2+5a^2)\sqrt{x^2+a^2} + \frac{3}{8}\cdot a^4\cdot ln\,(x+\sqrt{x^2+a^2}) + C$$

41. $\int x\cdot\sqrt{x^2+a^2}dx=\frac{1}{3}\sqrt{(x^2+a^2)^3}+C \qquad (a>0)$

证明：$\int x\cdot\sqrt{x^2+a^2}dx=\frac{1}{2}\int(x^2+a^2)^{\frac{1}{2}}dx^2$

$$=\frac{1}{2}\int(x^2+a^2)^{\frac{1}{2}}d(x^2+a^2)$$

$$=\frac{1}{2}\times\frac{1}{1+\frac{1}{2}}\cdot(x^2+a^2)^{1+\frac{1}{2}}+C$$

$$=\frac{1}{3}\sqrt{(x^2+a^2)^3}+C$$

42. $\int x^2\cdot\sqrt{x^2+a^2}\;dx=\frac{x}{8}\cdot(2x^2+a^2)\sqrt{x^2+a^2}-\frac{a^4}{8}\cdot ln\,(x+\sqrt{x^2+a^2})+C \qquad (a>0)$

证明:被积函数 $f(x)=x^2\cdot\sqrt{x^2+a^2}$的定义域为$\{x\,|\,x\in R\}$

可令$x=a\,tant \quad (-\frac{\pi}{2}<t<\frac{\pi}{2})$, 则$x^2\cdot\sqrt{(x^2+a^2)}=a^2tan^2t\,|\,a\,sect\,|$

$\because\ -\frac{\pi}{2}<t<\frac{\pi}{2},\ sect=\frac{1}{cost}>0$, $\therefore\ x^2\cdot\sqrt{(x^2+a^2)}=a^3tan^2t\cdot sect$

$$\therefore\ \int x^2\cdot\sqrt{(x^2+a^2)}\;dx=\int a^3tan^2t\cdot sect\;d\,(a\,tant)=a^4\int tan^2t\cdot sect\;d\,tant=a^4\int tan^2t\cdot sec^3t\;dt$$

$$=a^4\int tant\cdot sec^2t\;dsect$$

$$=a^4\int tant\cdot(1+tan^2t)\;dsect=a^4\int tant\;dsect+a^4\int tan^3t\;dsect$$

$$=a^4\int tant\;dsect+a^4\cdot tan^3t\cdot sect-a^4\int sect\;dtan^3t$$

$$=a^4\int tant\;dsect+a^4\cdot tan^3t\cdot sect-3a^4\int sec^3t\;tan^2t\;dt$$

$$=a^4\int tant\;dsect+a^4\cdot tan^3t\cdot sect-3a^4\int sec^2t\;tant\;dsect$$

移项并整理的: $a^4\int tant\cdot sec^2t\;dsect=\frac{1}{4}(a^4\int tant\;dsect+a^4\cdot tan^3t\cdot sect)$

$$=\frac{a^4}{4}\int tant\;dsect+\frac{a^4}{4}\cdot tan^3t\cdot sect \qquad ①$$

$$\because\ \int tant\;d\;sect=sect\cdot tant-\int sect\;dtant\ =sect\cdot tant-\int sec^3tdt$$

$$=sect\cdot tant-\int(1+tan^2t)\cdot sectdt$$

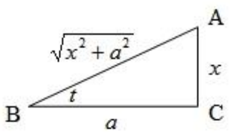
$$=sect\cdot tant-\int sect\;dt-\int tan^2t\;sect\;dt$$

$$=sect\cdot tant-\int sect\;dt-\int tant\;d\;sect$$

移项并整理得: $\int tant\;d\;sect=\frac{1}{2}\cdot sect\cdot tant-\frac{1}{2}\int sect\;dt$ 　 $\boxed{公式\ 87:\int\sec t\;dt=ln\,|\,sect+tant\ \,|+C}$

$$=\frac{1}{2}\cdot sect\cdot tant-\frac{1}{2}ln\,|\,sect+tant\,|+C_1 \qquad ②$$

联立①②得: $a^4\int tant\cdot sec^2t\;dsect=\frac{a^4}{8}sect\cdot tant-\frac{a^4}{8}ln\,|\,sect+tant\,|+\frac{a^4}{4}tan^3t\cdot sect+C_1$



在Rt⊿ABC中, 设$\angle B=t$, $|BC|=a$, 则$|AC|=x$, $|AB|=\sqrt{x^2+a^2}$

$\therefore\ tant=\frac{x}{a},\ sect=\frac{1}{\cos t}=\frac{\sqrt{x^2+a^2}}{a}$

$$\therefore\ a^4\int tant\cdot sec^2t\;dsect=\frac{a^4}{8}\cdot\frac{x}{a}\cdot\frac{\sqrt{x^2+a^2}}{a}-\frac{a^4}{8}\cdot ln\left|\frac{\sqrt{x^2+a^2}+x}{a}\right|+\frac{a^4}{4}\cdot\frac{x^3}{a^3}\cdot\frac{\sqrt{x^2+a^2}}{a}+C_1$$

$$=\frac{a^4x}{8}\cdot\sqrt{x^2+a^2}-\frac{a^4}{8}\cdot ln\left|\sqrt{x^2+a^2}+x\right|+\frac{x^3}{4}\cdot\sqrt{x^2+a^2}+C_2$$

$$=\frac{x}{8}\cdot(2x^2+a^2)\sqrt{x^2+a^2}-\frac{a^4}{8}\cdot ln\left|\sqrt{x^2+a^2}+x\right|+C$$

$\because\ \sqrt{x^2+a^2}+x>0$, $\therefore\ \frac{a^4}{8}\cdot ln\left|\sqrt{x^2+a^2}+x\right|=\frac{a^4}{8}\cdot ln\,(x+\sqrt{x^2+a^2})$

$$\therefore\ \int x^2\cdot\sqrt{x^2+a^2}\;dx=\frac{x}{8}\cdot(2x^2+a^2)\sqrt{x^2+a^2}-\frac{a^4}{8}\cdot ln\,(x+\sqrt{x^2+a^2})+C$$

43. $\int \frac{\sqrt{x^2+a^2}}{x}dx=\sqrt{x^2+a^2}+a\cdot ln\frac{\sqrt{x^2+a^2}-a}{|x|}+C \qquad (a>0)$

证明：被积函数 $f(x)=\frac{\sqrt{x^2+a^2}}{x}$ 的定义域为$\{x\,|\,x\neq 0\}$

令$\sqrt{x^2+a^2}=t \quad (t\geq 0$且$t\neq a)$，则$x=\sqrt{t^2-a^2}$

$$\therefore\ dx=\frac{1}{2}(t^2-a^2)^{-\frac{1}{2}}\cdot 2tdt=\frac{t}{\sqrt{t^2-a^2}}dt$$

$$\therefore \int\frac{\sqrt{x^2+a^2}}{x}dx=\int\frac{t}{\sqrt{t^2-a^2}}\cdot\frac{t}{\sqrt{t^2-a^2}}dt=\int\frac{t^2}{t^2-a^2}dt$$

$$=\int\frac{t^2-a^2+a^2}{t^2-a^2}dt=\int dt+a^2\int\frac{1}{t^2-a^2}dt$$

$\boxed{\text{公式21:}\int\frac{dx}{x^2-a^2}=\frac{1}{2a}\cdot ln\left|\frac{x-a}{x+a}\right|+C}$

$$=t+a^2\cdot\frac{1}{2a}\cdot ln\left|\frac{t-a}{t+a}\right|+C=t+\frac{a}{2}\cdot ln\left|\frac{(t-a)^2}{t^2-a^2}\right|+C$$

将$t=\sqrt{x^2+a^2}$代入上式得：$\int\frac{\sqrt{x^2+a^2}}{x}dx=\sqrt{x^2+a^2}+\frac{a}{2}\cdot ln\left|\frac{(\sqrt{x^2+a^2}-a)^2}{x^2+a^2-a^2}\right|+C$

$$=\sqrt{x^2+a^2}+a\cdot ln\left|\frac{(\sqrt{x^2+a^2}-a)}{x}\right|+C$$

$$=\sqrt{x^2+a^2}+a\cdot ln\frac{\sqrt{x^2+a^2}-a}{|x|}+C$$

44. $\int\frac{\sqrt{x^2+a^2}}{x^2}dx=-\frac{\sqrt{x^2+a^2}}{x}+ln\,(x+\sqrt{x^2+a^2})+\mathrm{C} \qquad (a>0)$

证明：被积函数 $f(x)=\frac{\sqrt{x^2+a^2}}{x^2}$ 的定义域为$\{x\,|\,x\neq 0\}$

1.当$x>0$时，可令$x=a\,tant \quad (0<t<\frac{\pi}{2})$， 则$dx=d(a\,tant)=a\,sec^2tdt$ ，

$$\frac{\sqrt{x^2+a^2}}{x^2}=\frac{|a\,sect|}{a^2\,tan^2t}\ ,\ \because\ 0<t<\frac{\pi}{2},sect=\frac{1}{cost}>0,\ \therefore\ \frac{\sqrt{x^2+a^2}}{x^2}=\frac{sect}{a\,tan^2t}$$

$$\therefore\ \int\frac{\sqrt{x^2+a^2}}{x^2}dx=\int\frac{sect}{a\,tan^2t}\cdot a\,sec^2tdt=\int\frac{sect}{tan^2t}\cdot(1+tan^2t)dt$$

$$=\int sectdt+\int\frac{sect}{tan^2t}dt=\int sectdt+\int\frac{1}{cost}\cdot\frac{cos^2t}{sin^2t}dt$$

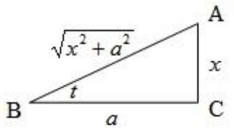
$$=\int sectdt+\int\frac{cost}{sin^2t}dt=\int sectdt+\int\frac{1}{sin^2t}dsint$$

$\boxed{\text{公式 87:}\int sec\,t\,dt=ln\,|\,sect+tant\,|+C}$

$$=ln\,|\,sect+tant\,|-\frac{1}{sint}+C_1$$

在Rt$\triangle ABC$中， 设$\angle B=t$，$|BC|=a$ ，则$|AC|=x$，$|AB|=\sqrt{x^2+a^2}$

$$\therefore\ sint=\frac{x}{\sqrt{x^2+a^2}},tant=\frac{x}{a},\ sect=\frac{1}{cost}=\frac{\sqrt{x^2+a^2}}{a}$$



$$\therefore\ \int\frac{\sqrt{x^2+a^2}}{x^2}dx=ln\left|\frac{\sqrt{x^2+a^2}+x}{a}\right|-\frac{\sqrt{x^2+a^2}}{x}+C_1$$

$$=-\frac{\sqrt{x^2+a^2}}{x}+ln\left|\sqrt{x^2+a^2}+x\right|-lna+C_1$$

$$\because\ \sqrt{x^2+a^2}+x>0 \qquad \therefore\ \int\frac{\sqrt{x^2+a^2}}{x^2}dx=-\frac{\sqrt{x^2+a^2}}{x}+ln\,(\sqrt{x^2+a^2}+x)+\mathrm{C}$$

2.当$x<0$时，同理可证得：$\int\frac{\sqrt{x^2+a^2}}{x^2}dx=-\frac{\sqrt{x^2+a^2}}{x}+ln\,(\sqrt{x^2+a^2}+x)+\mathrm{C}$

综合讨论1, 2 得：$\int\frac{\sqrt{x^2+a^2}}{x^2}dx=-\frac{\sqrt{x^2+a^2}}{x}+ln\,(\sqrt{x^2+a^2}+x)+\mathrm{C}$

（七）含有 $\sqrt{x^2-a^2}\ (a>0)$ 的积分（45~58）

45. $\displaystyle\int\frac{dx}{\sqrt{x^2-a^2}}=\frac{x}{|x|}\cdot arsh\frac{|x|}{a}+C_1=ln\,|x+\sqrt{x^2-a^2}\,|+C \qquad (a>0)$

证法1：被积函数 $f(x)=\dfrac{1}{\sqrt{x^2-a^2}}$ 的定义域为 $\{x\,|\,x>a\text{或}x<-a\}$

1. 当 $x>a$ 时，可设 $x=a\cdot sect\quad (0<t<\dfrac{\pi}{2})$，则 $dx=a\cdot sect\cdot tantdt$

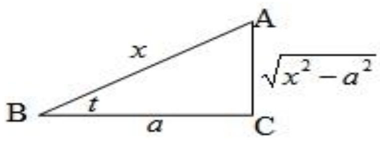
$\sqrt{x^2-a^2}=a\sqrt{sec^2t-1}=a\cdot|tant\,|\ \because 0<t<\dfrac{\pi}{2}$，$\sqrt{x^2-a^2}=a\cdot tant$

$\displaystyle\therefore\int\frac{dx}{\sqrt{x^2-a^2}}=\int\frac{a\cdot sect\cdot tant}{a\cdot tant}dt=\int sectdt$ $\boxed{\text{公式 87：}\int sec\,tdt=ln\,|\,sec\,t+tan\,t\,|+C}$

$=ln\,|\,sect+tant\,|+C_2$

在Rt ΔABC 中，可设 $\angle B=t$，$|BC|=a$，则 $|AB|=x$，$|AC|=\sqrt{x^2-a^2}$

$\therefore\ sec\,t=\dfrac{1}{cos\,t}=\dfrac{x}{a}$，$tan\,t=\dfrac{|AC|}{|BC|}=\dfrac{\sqrt{x^2-a^2}}{a}$



$\displaystyle\therefore\int\frac{dx}{\sqrt{x^2-a^2}}=ln\,|\,sec\,t+tan\,t\,|=ln\,|\,\frac{x+\sqrt{x^2-a^2}}{a}\,|$

$=ln\,|\,x+\sqrt{x^2-a^2}\,|+C_3$

2. 当 $x<-a$，即 $-x>a$ 时，令 $\mu=-x$，即 $x=-\mu$

由讨论 1可知 $\displaystyle\int\frac{dx}{\sqrt{x^2-a^2}}=-\int\frac{d\mu}{\sqrt{\mu^2-a^2}}=-ln\,|\,\mu+\sqrt{\mu^2-a^2}\,|+C_4$

$\displaystyle=-ln\,|-x+\sqrt{x^2-a^2}\,|+C_4=ln\frac{1}{|-x+\sqrt{x^2-a^2}\,|}+C_4$

$\displaystyle=ln\frac{|-x+\sqrt{x^2-a^2}\,|}{a^2}+C_4$

$=ln\,|-x-\sqrt{x^2-a^2}\,|+C_5$

综合讨论 1,2，可写成 $\displaystyle\int\frac{dx}{\sqrt{x^2-a^2}}=\frac{x}{|x|}\cdot arsh\frac{|x|}{a}+C_1=ln\,|\,x+\sqrt{x^2-a^2}\,|+C$

45. $\int\frac{dx}{\sqrt{x^2-a^2}}=\frac{x}{|x|}\cdot arsh\frac{|x|}{a}+C_1=ln|x+\sqrt{x^2-a^2}|+C \qquad (a>0)$

证法2: 被积函数 $f(x)=\frac{1}{\sqrt{x^2-a^2}}$的定义域为 $\{x\,|\,x>a或x<-a\}$

1. 当$x>a$时,可设$x=a\cdot cht\ \ (t>0)$, 则$t=arch\frac{x}{a}$

$$\sqrt{x^2-a^2}=\sqrt{a^2ch^2t-a^2}=a\cdot sht\ \ ,dx=a\cdot shtdt$$

$$\therefore\int\frac{dx}{\sqrt{x^2-a^2}}=\int\frac{a\cdot sht}{a\cdot sht}dt=\int dt=t+C_1$$

$$=arch\frac{x}{a}+C=ln\left[\frac{x}{a}+\sqrt{\left(\frac{x}{a}\right)^2-1}\right]+C_2$$

$$=ln|x+\sqrt{x^2-a^2}|+C_3$$

2.当$x<-a$,即$-x>a$时, 令 $\mu=-x$,即$x=-\mu$

由讨论 1可知 $\int\frac{dx}{\sqrt{x^2-a^2}}=-\int\frac{d\mu}{\sqrt{\mu^2-a^2}}=-ln|\mu+\sqrt{\mu^2-a^2}|+C_4$

$$=-ln(-x+\sqrt{x^2-a^2})+C_4=ln\frac{1}{|-x+\sqrt{x^2-a^2}|}+C_4$$

$$=ln\frac{|-x+\sqrt{x^2-a^2}|}{a^2}+C_4$$

$$=ln|-x-\sqrt{x^2-a^2}|+C_5$$

综合讨论 1,2, 可写成 $\int\frac{dx}{\sqrt{x^2-a^2}}=\frac{x}{|x|}\cdot arsh\frac{|x|}{a}+C_1=ln|x+\sqrt{x^2-a^2}|+C$

46. $\int \frac{dx}{\sqrt{(x^2-a^2)^3}} = -\frac{x}{a^2 \cdot \sqrt{x^2-a^2}} + C \qquad (a>0)$

证明:被积函数 $f(x) = \frac{1}{\sqrt{(x^2-a^2)^3}}$ 的定义域为$\{x \mid x > a 或 x < -a\}$

1. 当$x>a$时,可设$x = a \cdot sect \quad (0<t<\frac{\pi}{2})$,则$dx = a \cdot sect \cdot tantdt$

$\sqrt{(x^2-a^2)^3} = \left|a^3 \cdot tan^3 t\right| \quad \because 0<t<\frac{\pi}{2}$,$tant>0$,$\sqrt{(x^2-a^2)^3} = a^3 \cdot tan^3 t$

$$\therefore \int \frac{dx}{\sqrt{(x^2-a^2)^3}} = \int \frac{a \cdot sect \cdot tant}{a^3 \cdot tan^3 t} dt = \frac{1}{a^2} \int \frac{sect}{tan^3 t} dt$$

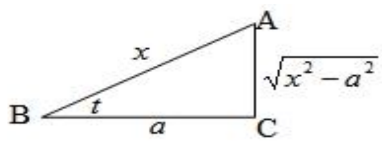
$$= \frac{1}{a^2} \int \frac{1}{cos\, t} \cdot \frac{cos^2 t}{sin^2 t} dt = \frac{1}{a^2} \int \frac{cos\, t}{sin^2 t} dt$$

$$= \frac{1}{a^2} \int \frac{1}{sin^2 t} d\, sint$$

$$= -\frac{1}{a^2\, sin t} + C$$

在Rt ΔABC中,可设 $\angle B = t$,$|BC| = a$,则$|AB| = x$,$|AC| = \sqrt{x^2-a^2}$

$$\therefore sin t = \frac{\sqrt{x^2-a^2}}{x}$$



$$\therefore \int \frac{dx}{\sqrt{(x^2-a^2)^3}} = -\frac{x}{a^2 \cdot \sqrt{x^2-a^2}} + C$$

2 .当$x<-a$,即$-x>a$时,令 $\mu = -x$,即$x = -\mu$

$$\therefore \int \frac{dx}{\sqrt{(x^2-a^2)^3}} = -\int \frac{d\mu}{\sqrt{(\mu^2-a^2)^3}}$$

由讨论 1可知 $-\int \frac{d\mu}{\sqrt{(\mu^2-a^2)^3}} = \frac{\mu}{a^2 \cdot \sqrt{(\mu^2-a^2)}} + C$

将$\mu = -x$代入得:$\int \frac{dx}{\sqrt{(x^2-a^2)^3}} = -\frac{x}{a^2 \cdot \sqrt{x^2-a^2}} + C$

综合讨论 1,2 得: $\int \frac{dx}{\sqrt{(x^2-a^2)^3}} = -\frac{x}{a^2 \cdot \sqrt{x^2-a^2}} + C$

47. $\int \frac{x}{\sqrt{x^2-a^2}} dx = \sqrt{x^2-a^2} + C \qquad (a>0)$

证明:$\int \frac{x}{\sqrt{x^2-a^2}} dx = \frac{1}{2} \int (x^2-a^2)^{-\frac{1}{2}} d(x^2)$

$$= \frac{1}{2} \int (x^2-a^2)^{-\frac{1}{2}} d(x^2-a^2)$$

$$= \frac{1}{2} \times \frac{1}{1-\frac{1}{2}} (x^2-a^2)^{1-\frac{1}{2}} + C$$

$$= \sqrt{x^2-a^2} + C$$

48. $\int \frac{x}{\sqrt{(x^2-a^2)^3}}dx=-\frac{1}{\sqrt{x^2-a^2}}+C \qquad (a>0)$

证明:被积函数 $f(x)=\frac{x}{\sqrt{(x^2-a^2)^3}}$ 的定义域为 $\{x\mid x>a或x<-a\}$

1. 当 $x>a$ 时,可设 $x=a\cdot sect \quad (0<t<\frac{\pi}{2})$,则 $dx=a\cdot sect\cdot tantdt$

$$\frac{x}{\sqrt{(x^2-a^2)^3}}=\frac{a\cdot sect}{|a^3\cdot tan^3 t|} \quad \because 0<t<\frac{\pi}{2}, \quad \frac{x}{\sqrt{(x^2-a^2)^3}}=\frac{sect}{a^2\cdot tan^3 t}$$

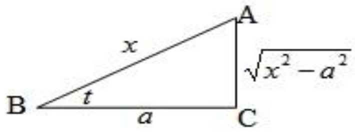
$$\therefore \int \frac{x}{\sqrt{(x^2-a^2)^3}}dx=\int \frac{sect}{a^2\cdot tan^3 t}\cdot a\cdot sect\cdot tant\, dt$$

$$=\frac{1}{a}\int \frac{sec^2 t}{tan^2 t}dt=\frac{1}{a}\int \frac{1}{sin^2 t}dt$$

$$=-\frac{1}{a}\int -csc^2 tdt=-\frac{1}{a}\cdot cot\, t+C$$

在Rt $\varDelta ABC$ 中, 可设 $\angle B=t$, $|BC|=a$, 则 $|AB|=x$, $|AC|=\sqrt{x^2-a^2}$

$$\therefore \ cot\, t=\frac{a}{\sqrt{x^2-a^2}}$$



$$\therefore \int \frac{x}{\sqrt{(x^2-a^2)^3}}dx=-\frac{1}{a}\cdot \frac{a}{\sqrt{x^2-a^2}}+C=-\frac{1}{\sqrt{x^2-a^2}}+C$$

2 .当 $x<-a$,即 $-x>a$ 时, 令 $\mu=-x$,即 $x=-\mu$

$$\therefore \int \frac{x}{\sqrt{(x^2-a^2)^3}}dx=\int \frac{\mu}{\sqrt{(\mu^2-a^2)^3}}d\mu$$

由讨论 1可知 $\int \frac{\mu}{\sqrt{(\mu^2-a^2)^3}}d\mu=-\frac{1}{\sqrt{\mu^2-a^2}}+C$

将 $\mu=-x$ 代入得: $\int \frac{x}{\sqrt{(x^2-a^2)^3}}dx=-\frac{1}{\sqrt{x^2-a^2}}+C$

综合讨论 1,2 得: $\int \frac{x}{\sqrt{(x^2-a^2)^3}}dx=-\frac{1}{\sqrt{x^2-a^2}}+C$

49. $\int \frac{x^2}{\sqrt{x^2-a^2}}dx=\frac{x}{2}\sqrt{x^2-a^2}+\frac{a^2}{2}\cdot ln\left| x+\sqrt{x^2-a^2}\right|+C \qquad (a>0)$

证明: $\int \frac{x^2}{\sqrt{x^2-a^2}}dx=\int \frac{x^2-a^2+a^2}{\sqrt{x^2-a^2}}dx$

$$=\int (\sqrt{x^2-a^2}+\frac{a^2}{\sqrt{x^2-a^2}})dx$$

$$=\int \sqrt{x^2-a^2}dx+a^2\int \frac{1}{\sqrt{x^2-a^2}}dx$$

$\because \int \sqrt{x^2-a^2}\, dx=\frac{x}{2}\cdot\sqrt{x^2-a^2}-\frac{a^2}{2}\cdot ln\left| x+\sqrt{x^2-a^2}\right|+C$ ① (公式53)

$a^2\int \frac{dx}{\sqrt{x^2-a^2}}=a^2\cdot ln\left| x+\sqrt{x^2-a^2}\right|+C$ ② (公式45)

$\therefore$ 由①+②得: $\int \frac{x^2}{\sqrt{x^2-a^2}}dx=\frac{x}{2}\sqrt{x^2-a^2}+\frac{a^2}{2}\cdot ln\left| x+\sqrt{x^2-a^2}\right|+C$

50. $\int \frac{x^2}{\sqrt{(x^2-a^2)^3}}dx = -\frac{x}{\sqrt{x^2-a^2}} + ln\left| x+\sqrt{x^2-a^2}\right| + C \qquad (a>0)$

证明:被积函数 $f(x)=\frac{x^2}{\sqrt{(x^2-a^2)^3}}$的定义域为$\{x\,|\,x>a或x<-a\}$

1. 当$x>a$时,可设$x=a\cdot sect \quad (0<t<\frac{\pi}{2})$,则$dx=a\cdot sect\cdot tantdt$

$$\frac{x^2}{\sqrt{(x^2-a^2)^3}}=\frac{a^2\cdot sec^2 t}{\left|a^3\cdot tan^3 t\right|} \because 0<t<\frac{\pi}{2}, \therefore \frac{x^2}{\sqrt{(x^2-a^2)^3}}=\frac{sec^2 t}{a\cdot tan^3 t}$$

$$\therefore \int \frac{x^2}{\sqrt{(x^2-a^2)^3}}dx=\int \frac{sec^2 t}{a\cdot tan^3 t}\cdot a\cdot sect\cdot tant\,dt=\int \frac{sec^3 t}{tan^2 t}dt=\int \frac{1}{cos^3 t}\cdot\frac{cos^2 t}{sin^2 t}dt=\int \frac{1}{sin^2 t\cdot cos\,t}dt$$

$$=\int \frac{cos\,t}{sin^2 t\cdot cos^2 t}dt=\int \frac{1}{sin^2 t\,(1-sin^2 t)}d\,sin t=\int (\frac{1}{sin^2 t}+\frac{1}{1-sin^2 t})d\,sin t$$

$$=\int \frac{1}{sin^2 t}d\,sin t+\int \frac{1}{1-sin^2 t}d\,sin t=\int \frac{1}{sin^2 t}d\,sin t-\int \frac{1}{sin^2 t-1}d\,sin t$$

$$=\int \frac{1}{sin^2 t}d\,sin t+\frac{1}{2}\int (\frac{1}{sin t+1}-\frac{1}{sin t-1})d\,sin t$$

$$=\int \frac{1}{sin^2 t}d\,sin t+\frac{1}{2}\int \frac{1}{sin t+1}d(sin t+1)-\frac{1}{2}\int \frac{1}{sin t-1}d(sin t-1)$$

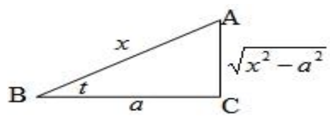
$$=-\frac{1}{sin t}+\frac{1}{2}ln\,|\,sin t+1\,|-\frac{1}{2}ln\,|\,sin t-1\,|+C_1$$

$$=-\frac{1}{sin t}+\frac{1}{2}ln\left|\frac{sin t+1}{sin t-1}\right|+C_1=-\frac{1}{sin t}+\frac{1}{2}ln\left|\frac{(sin t+1)^2}{sin^2 t-1}\right|+C_1$$

$$=-\frac{1}{sin t}+\frac{1}{2}ln\left|\frac{(sin t+1)^2}{cos^2 t}\cdot(-1)\right|+C_1=-\frac{1}{sin t}+ln\,|\,tan t+sec t\,|+C_2$$

在Rt $\varDelta ABC$中, 可设 $\angle B=t$,$|BC|=a$,则$|AB|=x$,$|AC|=\sqrt{x^2-a^2}$

$$\therefore \quad sin t=\frac{\sqrt{x^2-a^2}}{x},\ tan t=\frac{\sqrt{x^2-a^2}}{a},\ sec t=\frac{x}{a}$$



$$\therefore \int \frac{x^2}{\sqrt{(x^2-a^2)^3}}dx=-\frac{x}{\sqrt{x^2-a^2}}+ln\left|\frac{x+\sqrt{x^2-a^2}}{a}\right|+C_2=-\frac{x}{\sqrt{x^2-a^2}}+ln\left|x+\sqrt{x^2-a^2}\right|+C$$

2 .当$x<-a$,即$-x>a$时, 令 $\mu=-x$,即$x=-\mu$

$$\therefore \int \frac{x^2}{\sqrt{(x^2-a^2)^3}}dx=-\int \frac{\mu^2}{\sqrt{(\mu^2-a^2)^3}}d\mu$$

由讨论 1可知 $-\int \frac{\mu^2}{\sqrt{(\mu^2-a^2)^3}}d\mu=\frac{\mu}{\sqrt{\mu^2-a^2}}-ln\left|\mu+\sqrt{\mu^2-a^2}\right|+C$

将$\mu=-x$代入得:$\int \frac{x^2}{\sqrt{(x^2-a^2)^3}}dx=-\frac{x}{\sqrt{x^2-a^2}}-ln\left|-x+\sqrt{x^2-a^2}\right|+C$

$$=\int \frac{x^2}{\sqrt{(x^2-a^2)^3}}dx=-\frac{x}{\sqrt{x^2-a^2}}-ln\left|\frac{(\sqrt{x^2-a^2}-x)(\sqrt{x^2-a^2}+x)}{\sqrt{x^2-a^2}+x}\right|+C$$

$$=-\frac{x}{\sqrt{x^2-a^2}}-ln\left|\frac{x^2-a^2-x^2}{(\sqrt{x^2-a^2}+x)}\right|+C=-\frac{x}{\sqrt{x^2-a^2}}+2\,ln\,a-ln\left|\frac{1}{\sqrt{x^2-a^2}+x}\right|+C$$

提示:$log_a b^n=n\,log_a b$

$$=-\frac{x}{\sqrt{x^2-a^2}}+ln\left|x+\sqrt{x^2-a^2}\right|+C$$

综合讨论 1,2得: $\int \frac{x^2}{\sqrt{(x^2-a^2)^3}}dx=-\frac{x}{\sqrt{x^2-a^2}}+ln\left|x+\sqrt{x^2-a^2}\right|+C$

51. $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\cdot arccos\ \frac{a}{|x|} + C \qquad (a>0)$

证法 1:被积函数 $f(x) = \frac{1}{x\sqrt{x^2-a^2}}$的定义域为 $\{x \mid x>a或x<-a\}$

1. 当$x>a$时,可设$x = a\cdot sect \quad (0<t<\frac{\pi}{2})$, 则

$x\sqrt{x^2-a^2} = a^2\cdot sec\,t\sqrt{sec^2 t-1} = a^2 sect\cdot tant \ , dx = a\cdot sect\cdot tant\ d\,t$

$\therefore \int \frac{dx}{x\sqrt{x^2-a^2}} = \int \frac{a\cdot sec\,t\cdot tant}{a^2 sect\cdot tant}dt = \int \frac{1}{a}dt$

$= \frac{1}{a}t + C_1$

$\because x = a\cdot sect, \therefore cost = \frac{a}{x}, \therefore \ t = arccos\frac{a}{x}$

$\therefore \int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\cdot arccos\frac{a}{x} + C$

2 .当$x<-a$,即$-x>a$时,令 $\mu = -x$,即$x = -\mu$

由讨论 1可知 $\int \frac{dx}{x\sqrt{x^2-a^2}} = \int \frac{d\mu}{\mu\sqrt{\mu^2-a^2}} = \frac{1}{a}\cdot arccos\frac{a}{\mu} + C_2$

$= \frac{1}{a}\cdot arccos\ \frac{a}{-x} + C$

综合讨论 1,2,可写成 $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\cdot arccos\ \frac{a}{|x|} + C$

51. $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a} \cdot arccos\ \frac{a}{|x|} + C \qquad (a>0)$

证法2：被积函数 $f(x) = \frac{1}{x\sqrt{x^2-a^2}}$ 的定义域为 $\{x \mid x>a或x<-a\}$

1. 当 $x>a$ 时，可设 $x = a \cdot ch\,t \quad (0<t)$，则

$$x\sqrt{x^2-a^2} = a\cdot cht \cdot a \cdot sht = a^2 cht \cdot sht\ , dx = a \cdot sht\, dt$$

$$\therefore \int \frac{dx}{x\sqrt{x^2-a^2}} = \int \frac{a\cdot sht}{a\cdot cht \cdot sht} dt = \int \frac{1}{a}\cdot\frac{1}{cht} dt$$

$$= \frac{1}{a}\int \frac{cht}{ch^2 t} dt = \frac{1}{a}\int \frac{1}{1+sh^2 t} dsht$$

$$= \frac{1}{a}\cdot arctan\,(sht) + C$$

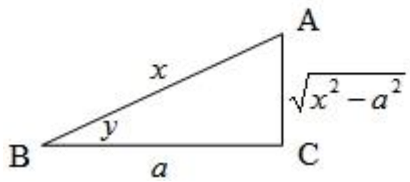
$\boxed{公式\ 19：\int \frac{dx}{x^2+a^2} = \frac{1}{a} arctan\ \frac{x}{a} + C}$

$$\because\ x = a\cdot cht,\ \therefore\ cht = \frac{x}{a},\ \therefore\ \ sht = \sqrt{1-ch^2 t} = \frac{\sqrt{x^2-a^2}}{a}$$

在Rt$\triangle ABC$中，设 $tany = sht = \frac{\sqrt{x^2-a^2}}{a}, \angle B = y, |BC| = a$

$$\therefore\ y = arctan\,(sht), |AC| = \sqrt{x^2-a^2}, |AB| = \sqrt{|AC|^2 + |BC|^2} = x$$

$$\therefore\ cosy = \frac{|BC|}{|AB|} = \frac{a}{x}$$



即 $cosy = cos\ arctan\,(sht) = \frac{a}{x}$

$$\therefore\ arctan(sht\ ) = arccos\frac{a}{x} + C$$

$$\therefore\ \int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\cdot arctan\,(sht) + C = \frac{1}{a}\cdot arccos\ \frac{a}{x} + C$$

2 . 当 $x<-a$，即 $-x>a$ 时，令 $\mu = -x$，即 $x = -\mu$

由讨论 1可知 $\int \frac{dx}{x\sqrt{x^2-a^2}} = \int \frac{d\mu}{\mu\sqrt{\mu^2-a^2}} = \frac{1}{a}\cdot arccos\frac{a}{\mu} + C_2$

$$= \frac{1}{a}\cdot arccos\frac{a}{-x} + C$$

综合讨论 1,2，可写成 $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\cdot arccos\ \frac{a}{|x|} + C$

52. $\int \frac{dx}{x^2\sqrt{x^2-a^2}}=\frac{\sqrt{x^2-a^2}}{a^2x}+C \qquad (a>0)$

证明：被积函数 $f(x)=\frac{1}{x^2\sqrt{x^2-a^2}}$ 的定义域为 $\{x\,|\,x>a或x<-a\}$

1．当 $x>a$时，可设$x=\frac{1}{t} \quad (0<t<\frac{1}{a})$，则 $dx=-\frac{1}{t^2}dt$ ， $\frac{1}{x^2\sqrt{x^2-a^2}}=\frac{t^3}{\sqrt{1-a^2t^2}}$

$$\therefore \int \frac{dx}{x^2\sqrt{x^2-a^2}}=\int \frac{t^3}{\sqrt{1-a^2t^2}}\cdot(-\frac{1}{t^2})dt$$

$$=-\int \frac{t}{\sqrt{1-a^2t^2}}dt \quad =-\frac{1}{2}\int (1-a^2t^2)^{-\frac{1}{2}}d(t^2)$$

$$=\frac{1}{2a^2}\int (1-a^2t^2)^{-\frac{1}{2}}d(1-a^2t^2) \quad =\frac{1}{2a^2}\cdot\frac{1}{1-\frac{1}{2}}\cdot(1-a^2t^2)^{1-\frac{1}{2}}+C$$

$$=\frac{\sqrt{1-a^2t^2}}{a^2}+C$$

将$x=\frac{1}{t}$，即$t=\frac{1}{x}$代入上式得：$\int \frac{dx}{x^2\sqrt{x^2-a^2}}=\frac{1}{a^2}\cdot\sqrt{1-a^2(\frac{1}{x})^2}+C \;=\frac{1}{a^2}\cdot\sqrt{\frac{x^2-a^2}{x^2}}+C$

$$=\frac{1}{a^2}\cdot\frac{\sqrt{x^2-a^2}}{|\,x\,|}+C$$

$$\because \; x>a>0 \quad \therefore \; \int \frac{dx}{x^2\sqrt{x^2-a^2}}=\frac{\sqrt{x^2-a^2}}{a^2x}+C$$

2．当 $x<-a$,即$-x>a$时，令 $\mu=-x$,即$x=-\mu$

由讨论 1可知 $\int \frac{dx}{x^2\sqrt{x^2-a^2}}=-\int \frac{d\mu}{\mu^2\sqrt{\mu^2-a^2}}=-\frac{\sqrt{\mu^2-a^2}}{a^2\mu}+C$

将$\mu=-x$代入上式得：$\int \frac{dx}{x^2\sqrt{x^2-a^2}}=\frac{\sqrt{x^2-a^2}}{a^2x}+C$

综合讨论 1，2 得：$\int \frac{dx}{x^2\sqrt{x^2-a^2}}=\frac{\sqrt{x^2-a^2}}{a^2x}+C$

53. $\int\sqrt{x^2-a^2}\,dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\cdot ln\left|\,x+\sqrt{x^2-a^2}\,\right|+C \qquad (a>0)$

证明:被积函数 $f(x)=\sqrt{x^2-a^2}$的定义域为$\{x\,|\,x>a$或$x<-a\}$

1. 当$x>a$时,可设$x=a\cdot sect \quad (0<t<\frac{\pi}{2})$,则$\sqrt{x^2-a^2}=|\,a\cdot tan\,t\,|$

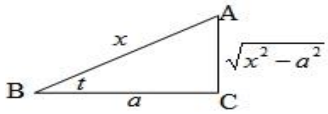
$\because 0<t<\frac{\pi}{2}$,$\therefore \sqrt{x^2-a^2}=a\cdot tan\,t$

$$\begin{aligned}\therefore \int\sqrt{x^2-a^2}\,dx&=\int a\cdot tan\,t\,d\,(a\cdot sect)=a^2\int tan\,t\,d\,sec\,t\\&=a^2\cdot tan\,t\cdot sec\,t-a^2\int sec\,t\,d\,tan\,t\\&=a^2\cdot tan\,t\cdot sec\,t-a^2\int sec^3 t\,dt\\&=a^2\cdot tan\,t\cdot sec\,t-a^2\int sec\,t\,(1+tan^2 t)\,dt\\&=a^2\cdot tan\,t\cdot sec\,t-a^2\int sec\,t\,dt-a^2\int sec\,t\,tan^2 t\,dt\\&=a^2\cdot tan\,t\cdot sec\,t-a^2\int sec\,t\,dt-a^2\int tan\,t\,d\,sec\,t\\&=a^2\cdot tan\,t\cdot sec\,t-a^2\cdot ln\left|\,sec\,t+tan\,t\,\right|-a^2\int tan\,t\,d\,sec\,t\end{aligned}$$

移项并整理得:$a^2\int tan\,t\,d\,sec\,t=\frac{a^2}{2}\cdot tan\,t\cdot sec\,t-\frac{a^2}{2}\cdot ln\left|\,sec\,t+tan\,t\,\right|+C_1$

在Rt ΔABC中, 可设 $\angle B=t$,$|\,BC\,|=a$,则$|\,AB\,|=x$,$|\,AC\,|=\sqrt{x^2-a^2}$

$\therefore \ tan\,t=\frac{\sqrt{x^2-a^2}}{a}$, $sec\,t=\frac{x}{a}$



$$\begin{aligned}\therefore \int\sqrt{x^2-a^2}\,dx&=a^2\int tan\,t\,d\,sec\,t\\&=\frac{a^2}{2}\cdot\frac{\sqrt{x^2-a^2}}{a}\cdot\frac{x}{a}-\frac{a^2}{2}\cdot ln\left|\frac{\sqrt{x^2-a^2}+x}{a}\right|+C_1\\&=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\cdot ln\left|\,x+\sqrt{x^2-a^2}\,\right|+C\end{aligned}$$

2 .当$x<-a$时, 可设$x=a\cdot sect \quad (-\frac{\pi}{2}<t<0)$ 同理可证

综合讨论 1,2 得:$\int\sqrt{x^2-a^2}\,dx==\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\cdot ln\left|\,x+\sqrt{x^2-a^2}\,\right|+C$

54. $\int\sqrt{(x^2-a^2)^3}\,dx=\frac{x}{8}\cdot(2x^2-5a^2)\sqrt{x^2-a^2}+\frac{3}{8}\cdot a^4\cdot ln\left|x+\sqrt{x^2-a^2}\right|+C \qquad (a>0)$

证明：$\int\sqrt{(x^2-a^2)^3}\,dx=x\cdot(x^2-a^2)^{\frac{3}{2}}-\int xd\,(x^2-a^2)^{\frac{3}{2}}$

$$=x\cdot(x^2-a^2)^{\frac{3}{2}}-\int x\cdot\frac{3}{2}\cdot(2x)\cdot(x^2-a^2)^{\frac{1}{2}}d\,x$$

$$=x\cdot(x^2-a^2)^{\frac{3}{2}}-3\int x^2(x^2-a^2)^{\frac{1}{2}}d\,x$$

$$=x\cdot(x^2-a^2)^{\frac{3}{2}}-3\int(x^2-a^2+a^2)(x^2-a^2)^{\frac{1}{2}}d\,x$$

$$=x\cdot(x^2-a^2)^{\frac{3}{2}}-3\int(x^2-a^2)^{\frac{3}{2}}d\,x-3a^2\int(x^2-a^2)^{\frac{1}{2}}d\,x$$

移项并整理得：$\int\sqrt{(x^2-a^2)^3}\,dx=\frac{x}{4}\cdot(x^2-a^2)^{\frac{3}{2}}-\frac{3a^2}{4}\int(x^2-a^2)^{\frac{1}{2}}d\,x$ ①

又$\int(x^2-a^2)^{\frac{1}{2}}d\,x=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\cdot ln\left|x+\sqrt{x^2-a^2}\right|+C$ （公式53） ②

联立①②得：

$$\int\sqrt{(x^2-a^2)^3}\,dx=\frac{x}{4}(x^2-a^2)^{\frac{3}{2}}-\frac{3x}{8}\cdot a^2\cdot\sqrt{x^2-a^2}+\frac{3}{8}\cdot a^4\cdot ln\left|x+\sqrt{x^2-a^2}\right|+C$$

$$=(\frac{x^3}{4}-\frac{a^2x}{4})\sqrt{x^2-a^2}-\frac{3x}{8}\cdot a^2\cdot\sqrt{x^2-a^2}+\frac{3}{8}\cdot a^4\cdot ln\left|x+\sqrt{x^2-a^2}\right|+C$$

$$=\frac{x}{8}\cdot(2x^2-5a^2)\sqrt{x^2-a^2}+\frac{3}{8}\cdot a^4\cdot ln\left|x+\sqrt{x^2-a^2}\right|+C$$

55. $\int x\sqrt{x^2-a^2}\,dx=\frac{1}{3}\sqrt{(x^2-a^2)^3}+\mathrm{C} \qquad (a>0)$

证明：$\int x\sqrt{x^2-a^2}\,dx=\frac{1}{2}\int\sqrt{x^2-a^2}\,d(x^2)$

$$=\frac{1}{2}\int(x^2-a^2)^{\frac{1}{2}}\,d\,(x^2-a^2)$$

$$=\frac{1}{2}\times\frac{1}{1+\frac{1}{2}}\cdot(x^2-a^2)^{1+\frac{1}{2}}+C$$

$$=\frac{1}{3}\sqrt{(x^2-a^2)^3}+\mathrm{C}$$

56. $\int x^2\sqrt{x^2-a^2}\,dx=\frac{x}{8}\cdot(2x^2-a^2)\sqrt{x^2-a^2}-\frac{a^4}{8}\cdot ln\left|\,x+\sqrt{x^2-a^2}\,\right|+C \qquad (a>0)$

证明：被积函数 $f(x)=x^2\sqrt{x^2-a^2}$的定义域为$\{x\,|\,x>a或x<-a\}$

1.当$x>a$时，可令$x=a\cdot sect \quad (0<t<\frac{\pi}{2})$， 则$x^2\sqrt{x^2-a^2}=a^2sec^2t\,|\,a\,tant\,|$

$\because\ 0<t<\frac{\pi}{2}, tan\,t>0$， $\therefore\ x^2\sqrt{x^2-a^2}=a^3sec^2t\cdot tant$

$$\therefore\ \int x^2\sqrt{x^2-a^2}\,dx=\int a^3sec^2t\cdot tant\ d\,(a\,sect)=a^4\int sec^3t\ d\ tan^2t\ dt \qquad ①$$

$$=\frac{a^4}{3}\int sect\cdot 3\cdot sec^2t\cdot tan^2\,t\ dt=\frac{a^4}{3}\int sect\ d\ tan^3\,t$$

$$=\frac{a^4}{3}\cdot sec\,t\cdot tan^3\,t-\frac{a^4}{3}\int tan^3\,t\ d\ sec\,t$$

$$=\frac{a^4}{3}\cdot sec\,t\cdot tan^3\,t-\frac{a^4}{3}\int tan\,t\,(sec^2\,t-1)d\ sec\,t$$

$$=\frac{a^4}{3}\cdot sec\,t\cdot tan^3\,t-\frac{a^4}{3}\int tan\,t\cdot sec^2\,td\ sec\,t+\frac{a^4}{3}\int tan\,t\ d\ sec\,t$$

$$=\frac{a^4}{3}\cdot sec\,t\cdot tan^3\,t-\frac{a^4}{3}\int tan^2\,t\cdot sec^3\,t\ dt+\frac{a^4}{3}\int tan\,t\ d\ sec\,t$$

移项并整理得：$a^4\int sec^3t\ d\ tan^2t\ dt=\frac{a^4}{4}\cdot sec\,t\cdot tan^3\,t+\frac{a^4}{4}\int tan\,t\ d\ sec\,t \qquad ②$

又$\int tant\ d\ sect=sect\cdot tant-\int sect\ dtant\ \ =sect\cdot tant-\int sec^3tdt$

$$=sect\cdot tant-\int(1+tan^2t)\cdot sectdt$$

$$=sect\cdot tant-\int sect\ dt-\int tan^2t\ sect\ dt$$

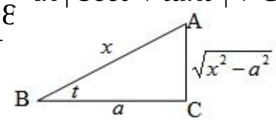
$$=sect\cdot tant-\int sect\ dt-\int tant\ d\ sect$$

移项并整理得：$\int tant\ d\ sect=\frac{1}{2}\cdot sect\cdot tant-\frac{1}{2}\int sect\ dt$

$$=\frac{1}{2}\cdot sect\cdot tant-\frac{1}{2}ln\,|\,sect+tant\,|+C_1 \qquad ③$$

将③式代入②式得： $a^4\int sec^3t\ d\ tan^2t\ dt=\frac{a^4}{4}tan^3t\cdot sect+\frac{a^4}{8}sect\cdot tant-\frac{a^4}{8}ln\,|\,sect+tant\,|+C_1$

在Rt⊿ABC中， 设$\angle B=t$，$|BC|=a$ ，则$|AB|=x$，$|AC|=\sqrt{x^2-a^2}$



$\therefore\ tant=\frac{\sqrt{x^2-a^2}}{a}, sect=\frac{1}{cos\,t}=\frac{x}{a}$

$$\therefore\ a^4\int sec^3t\ d\ tan^2t\ dt=\frac{a^4}{4}\cdot\frac{x}{a}\cdot\frac{x^2-a^2}{a^3}\cdot\sqrt{x^2-a^2}+\frac{a^4}{8}\cdot\frac{\sqrt{x^2-a^2}}{a}\cdot\frac{x}{a}-\frac{a^4}{8}\cdot ln\left|\frac{x+\sqrt{x^2-a^2}}{a}\right|+C_1$$

$$=\frac{x}{4}\cdot(x^2-a^2)\cdot\sqrt{x^2-a^2}+\frac{a^2x}{8}\cdot\sqrt{x^2-a^2}-\frac{a^4}{8}\cdot ln\left|\,x+\sqrt{x^2-a^2}\,\right|+C_2$$

$$=\frac{x}{8}\cdot(2x^2-a^2)\sqrt{x^2-a^2}-\frac{a^4}{8}\cdot ln\left|\,x+\sqrt{x^2-a^2}\,\right|+C$$

2. 当$x<-a$,即$-x>a$时， 令$\mu=-x$,则$x=-\mu$

由讨论1得：$\int x^2\sqrt{x^2-a^2}\,dx=-\int\mu^2\sqrt{\mu^2-a^2}\,d\mu$

$$=\frac{-\mu}{8}\cdot(2\mu^2-a^2)\sqrt{\mu^2-a^2}+\frac{a^4}{8}\cdot ln\left|-\mu+\sqrt{\mu^2-a^2}\,\right|+C$$

将$\mu=-x$代入上式得：$\int x^2\sqrt{x^2-a^2}\,dx=\frac{x}{8}\cdot(2x^2-a^2)\sqrt{x^2-a^2}-\frac{a^4}{8}\cdot ln\left|\,x+\sqrt{x^2-a^2}\,\right|+C$

综合讨论1,2得：$\int x^2\sqrt{x^2-a^2}\,dx=\frac{x}{8}\cdot(2x^2-a^2)\sqrt{x^2-a^2}-\frac{a^4}{8}\cdot ln\left|\,x+\sqrt{x^2-a^2}\,\right|+C$

57. $\int \frac{\sqrt{x^2-a^2}}{x}dx = \sqrt{x^2-a^2} - a\cdot arccos\ \frac{a}{|x|} + C \qquad (a>0)$

证法1:被积函数 $f(x)=\frac{\sqrt{x^2-a^2}}{x}$ 的定义域为 $\{x\,|\,x>a\text{或}x<-a\}$

1. 当 $x>a$ 时，可设 $x=a\cdot sect \quad (0<t<\frac{\pi}{2})$,

则 $\frac{\sqrt{x^2-a^2}}{x}=\frac{a\cdot tant}{a\cdot sect}$, $\quad dx=a\cdot sect\cdot tant\ d\ t$

$$\therefore \int \frac{\sqrt{x^2-a^2}}{x}dx = \int \frac{a\cdot tant\cdot a\cdot sect\cdot tant}{a\cdot sect}dt = \int a\cdot tan^2 t dt$$

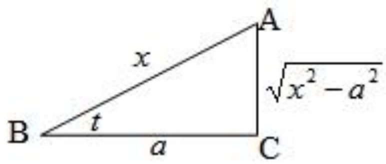
$$= a\int \frac{sin^2 t}{cos^2 t}dt = a\int \frac{1-cos^2 t}{cos^2 t}dt = a\int \frac{1}{cos^2 t}dt - \int dt$$

$$= a\cdot tant - a\cdot t + C$$

$\because\ x=a\cdot sect,\ \therefore\ cost=\frac{a}{x},\ \therefore\ t=arccos\frac{a}{x}$

在Rt$\triangle ABC$中，设$\angle$B$=t$,$|$BC$|=a$，则$|AB|=x$, $|AC|=\sqrt{x^2-a^2}$

$\therefore\ tant=\frac{|AC|}{|BC|}=\frac{\sqrt{x^2-a^2}}{a}$



$\therefore \int \frac{\sqrt{x^2-a^2}}{x}dx = a\cdot tant - a\cdot t + C$

$$= \sqrt{x^2-a^2} - a\cdot arccos\ \frac{a}{x} + C$$

2 .当 $x<-a$,即 $-x>a$ 时，令 $\mu=-x$,即 $x=-\mu$

由讨论 1可知 $\int \frac{\sqrt{x^2-a^2}}{x}dx = \int \frac{\sqrt{\mu^2-a^2}}{\mu}d\ \mu = \sqrt{\mu^2-a^2} - a\cdot arccos\ \frac{a}{\mu} + C$

$$= \sqrt{x^2-a^2} - a\cdot arccos\ \frac{a}{-x} + C$$

综合讨论 1,2，可写成: $\int \frac{\sqrt{x^2-a^2}}{x}dx = \sqrt{x^2-a^2} - a\cdot arccos\ \frac{a}{|x|} + C$

57. $\int \frac{\sqrt{x^2-a^2}}{x}dx = \sqrt{x^2-a^2} - a\cdot arccos\frac{a}{|x|} + C \qquad (a>0)$

证法2:被积函数 $f(x)=\frac{\sqrt{x^2-a^2}}{x}$ 的定义域为$\{x\,|\,x>a或x<-a\}$

1. 当$x>a$时,可设$x=a\cdot ch\,t \quad (0<t)$,

则 $\frac{\sqrt{x^2-a^2}}{x}=\frac{a\cdot sht}{a\cdot cht}=\frac{sht}{cht}$,$dx=a\cdot sht\,dt$

$$\therefore \int\frac{\sqrt{x^2-a^2}}{x}dx=\int\frac{sht}{cht}\cdot a\cdot sht\,dt = a\int\frac{sh^2t}{cht}dt$$

$$=a\int\frac{ch^2t-1}{cht}dt=a\int chtdt-a\int\frac{cht}{ch^2t}dt$$

$$=a\int chtdt-a\int\frac{1}{1+sh^2t}dsht$$

$$=a\cdot sht-a\cdot arctan(sht)+C$$

提示： $ch^2t-sh^2t=1$

$(cht)'=sht$

$(sht)'=cht$

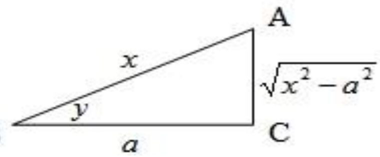
公式19: $\int\frac{dx}{x^2+a^2}=\frac{1}{a}\cdot arctan\frac{x}{a}+C$

$\because\ x=a\cdot cht,\ \therefore\ cht=\frac{x}{a},\ \therefore\ \ sht=\sqrt{1-ch^2t}=\frac{\sqrt{x^2-a^2}}{a}$

在Rt⊿ABC中， 设 $tany=sht=\frac{\sqrt{x^2-a^2}}{a}$,$\angle B=y$,$|BC|=a$

$\therefore\ \ y=arctan\,(sht),\ |AC|=\sqrt{x^2-a^2},\ |AB|=\sqrt{|AC|^2+|BC|^2}=x$

$\therefore\ \ cosy=\frac{|BC|}{|AB|}=\frac{a}{x}$



即$cosy=cos\,arctan(sht)=\frac{a}{x}$

$\therefore\ \ arctan(sht)=arccos\frac{a}{x}$

$\therefore\ \ \int\frac{\sqrt{x^2-a^2}}{x}dx=\sqrt{x^2-a^2}-a\cdot arccos\frac{a}{x}+C$

2 .当$x<-a$,即$-x>a$时， 令 $\mu=-x$,即$x=-\mu$

由讨论 1可知 $\int\frac{\sqrt{x^2-a^2}}{x}dx=\int\frac{\sqrt{\mu^2-a^2}}{\mu}d\mu=\sqrt{\mu^2-a^2}-a\cdot arccos\frac{a}{\mu}+C$

$$=\sqrt{x^2-a^2}-a\cdot arccos\frac{a}{-x}+C$$

综合讨论1,2， 可写成:$\int\frac{\sqrt{x^2-a^2}}{x}dx=\sqrt{x^2-a^2}-a\cdot arccos\frac{a}{|x|}+C$

58. $\int \frac{\sqrt{x^2-a^2}}{x^2}dx = -\frac{\sqrt{x^2-a^2}}{x} + ln\left| x+\sqrt{x^2-a^2} \right| + C \qquad (a>0)$

证明：$\int \frac{\sqrt{x^2-a^2}}{x^2}dx = -\int \sqrt{x^2-a^2}\, d\frac{1}{x}$

$$= -\frac{\sqrt{x^2-a^2}}{x} + \int \frac{1}{x} d\sqrt{x^2-a^2}$$

$$= -\frac{\sqrt{x^2-a^2}}{x} + \int \frac{1}{x}\cdot\frac{1}{2}\cdot 2x\cdot(x^2-a^2)^{-\frac{1}{2}}dx$$

$$= -\frac{\sqrt{x^2-a^2}}{x} + \int \frac{1}{\sqrt{x^2-a^2}}dx$$

$\boxed{\text{公式45：}\int \frac{dx}{\sqrt{x^2-a^2}} = ln\left| x+\sqrt{x^2-a^2} \right| + C}$

$$= -\frac{\sqrt{x^2-a^2}}{x} + ln\left| x+\sqrt{x^2-a^2} \right| + C$$

（八）含有 $\sqrt{a^2-x^2}$ $(a>0)$ 的积分（59~72）

59. $\int \frac{dx}{\sqrt{a^2-x^2}} = arcsin\frac{x}{a} + C \qquad (a>0)$

证明：被积函数 $f(x) = \frac{1}{\sqrt{a^2-x^2}}$ 的定义域为 $\{x \mid -a<x<a\}$

$\therefore$ 可设 $x = a\cdot sint \quad (-\frac{\pi}{2}<t<\frac{\pi}{2})$，则 $dx = a\cdot cost\, dt$，$\frac{1}{\sqrt{a^2-x^2}} = \frac{1}{|a\cdot cost|}$

$\because -\frac{\pi}{2}<t<\frac{\pi}{2}$，$cost>0$ $\therefore$ $\frac{1}{\sqrt{a^2-x^2}} = \frac{1}{a\cdot cost}$

$$\therefore \int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{1}{a\cdot cost}\cdot a\cdot cost\, dt$$

$$= \int dt$$

$$= t + C$$

$\because\ x = a\cdot sint \quad \therefore\ t = arcsin\frac{x}{a}$

$$\therefore \int \frac{dx}{\sqrt{a^2-x^2}} = arcsin\frac{x}{a} + C$$

60. $\int \frac{dx}{\sqrt{(a^2-x^2)^3}} = \frac{x}{a^2\cdot\sqrt{a^2-x^2}} + C \qquad (a>0)$

证明:被积函数 $f(x)=\frac{1}{\sqrt{(a^2-x^2)^3}}$ 的定义域为 $\{x\,|\,-a<x<a\}$

$\therefore$ 可设 $x=a\cdot sint \quad (-\frac{\pi}{2}<t<\frac{\pi}{2})$，则 $dx=a\cdot cos\,t\,dt$ ，$\frac{1}{\sqrt{(a^2-x^2)^3}}=\frac{1}{|a^3\cdot cos^3 t|}$

$\because -\frac{\pi}{2}<t<\frac{\pi}{2}$ ，$cos\,t>0$ $\therefore$ $\frac{1}{\sqrt{(a^2-x^2)^3}}=\frac{1}{a^3\cdot cos^3 t}$

$$\therefore \int \frac{dx}{\sqrt{(a^2-x^2)^3}} = \int \frac{1}{a^3\cdot cos^3 t}\cdot a\cdot cos\,t\,dt$$

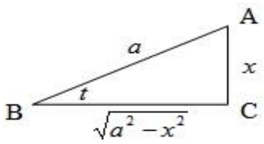
$$= \int \frac{1}{a^2\cdot cos^2 t}\,dt$$

$$= \int \frac{1}{a^2}\cdot sec^2 t\,dt$$

$$= \frac{1}{a^2}\cdot tan\,t + C$$

在Rt $\varDelta ABC$ 中，设 $\angle B=t$,$|AB|=a$,则 $|AC|=x$,$|BC|=\sqrt{a^2-x^2}$

$$\therefore\ tan\,t=\frac{x}{\sqrt{a^2-x^2}}$$



$$\therefore\ \int \frac{dx}{\sqrt{(a^2-x^2)^3}} = \frac{x}{a^2\cdot\sqrt{a^2-x^2}} + C$$

61. $\int \frac{x}{\sqrt{a^2-x^2}}\,dx = -\sqrt{a^2-x^2} + \mathrm{C} \qquad (a>0)$

证明:$\int \frac{x}{\sqrt{a^2-x^2}}\,dx = \frac{1}{2}\int (a^2-x^2)^{-\frac{1}{2}}\,dx^2$

$$= -\frac{1}{2}\int (a^2-x^2)^{-\frac{1}{2}}\,d\,(a^2-x^2)$$

$$= -\frac{1}{2}\times\frac{1}{1-\frac{1}{2}}\cdot(a^2-x)^{1-\frac{1}{2}} + C$$

$$= -\sqrt{a^2-x^2} + \mathrm{C}$$

62. $\int \frac{x}{\sqrt{(a^2-x^2)^3}}\ dx = \frac{1}{\sqrt{a^2-x^2}}\ +\text{C} \qquad (a>0)$

证明：$\int \frac{x}{\sqrt{(a^2-x^2)^3}}\ dx = \frac{1}{2}\int (a^2-x^2)^{-\frac{3}{2}}\ d(x^2)$

$$= -\frac{1}{2}\int (a^2-x^2)^{-\frac{3}{2}}\ d\ (a^2-x^2)$$

$$= -\frac{1}{2}\times\frac{1}{1-\frac{3}{2}}\cdot(a^2-x^2)^{1-\frac{3}{2}}+C$$

$$= \frac{1}{\sqrt{a^2-x^2}}\ +\text{C}$$

63. $\int \frac{x^2}{\sqrt{a^2-x^2}}dx = -\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\cdot arcsin\frac{x}{a}+C \qquad (a>0)$

证明：被积函数 $f(x)=\frac{x^2}{\sqrt{a^2-x^2}}$ 的定义域为 $\{x\,|-a<x<a\}$

$\therefore$ 可设$x=a\cdot sint \quad (-\frac{\pi}{2}<t<\frac{\pi}{2})$，则$dx=a\cdot\cos t\ dt$ ，$\frac{x^2}{\sqrt{a^2-x^2}}=\frac{a^2\cdot sin^2t}{|\ a\cdot cost\ |}$

$\because\ -\frac{\pi}{2}<t<\frac{\pi}{2}$ ，$\cos t>0 \quad \therefore \quad \frac{x^2}{\sqrt{a^2-x^2}}=\frac{a\cdot sin^2t}{cost}$

$$\therefore\ \int \frac{x^2}{\sqrt{a^2-x^2}}dx = \int \frac{a\cdot sin^2t}{cost}\cdot a\cdot\cos t\ dt$$

$$= a^2\int sin^2t\ dt$$

$$= a^2\int \frac{1-\cos 2t}{2}\ dt$$

$$= \frac{a^2}{2}\int dt-\frac{a^2}{4}\int \cos 2t\ d(2t)$$

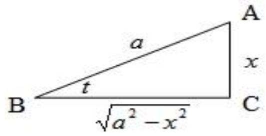
$$= \frac{a^2}{2}\cdot t-\frac{a^2}{4}\cdot sin2t+C$$

$$= \frac{a^2}{2}\cdot t-\frac{a^2}{2}\cdot sint\cdot cost+C$$

> 提示：$cos2t=cos^2t-sin^2t$
> $=1-2sin^2t$
> $sin2t=2\cdot sint\cdot cost$

在Rt $\varDelta ABC$中，设 $\angle B=t$，$|\ AB\ |=a$，则$|\ AC\ |=x$，$|\ BC\ |=\sqrt{a^2-x^2}$

$\therefore\ sint=\frac{x}{a}$ ，$cost=\frac{\sqrt{a^2-x^2}}{a}$



$$\therefore\ \int \frac{x^2}{\sqrt{a^2-x^2}}dx = -\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\cdot arcsin\frac{x}{a}+C$$

64. $\int \frac{x^2}{\sqrt{(a^2-x^2)^3}}\,dx = \frac{x}{\sqrt{a^2-x^2}} - arcsin\frac{x}{a} + C \qquad (a>0)$

证明:被积函数 $f(x)=\frac{x^2}{\sqrt{(a^2-x^2)^3}}$ 的定义域为 $\{x \mid -a<x<a\}$

$\therefore$ 可设$x=a\cdot sint \quad (-\frac{\pi}{2}<t<\frac{\pi}{2})$，则$dx=a\cdot\cos t\,dt$ ，$\frac{x^2}{\sqrt{(a^2-x^2)^3}}=\frac{a^2\cdot sin^2t}{|\,a^3\cdot cos^3t\,|}$

$\because -\frac{\pi}{2}<t<\frac{\pi}{2}$ ，$\cos t>0 \quad \therefore \quad \frac{x^2}{\sqrt{(a^2-x^2)^3}}=\frac{sin^2t}{a\cdot cos^3t}$

$$\therefore \int\frac{x^2}{\sqrt{a^2-x^2}}\,dx=\int\frac{sin^2t}{a\cdot cos^3t}\cdot a\cdot\cos t\,dt$$

$$=\int\frac{sin^2t}{cos^2t}\,dt$$

$$=\int\frac{1-cos^2t}{cos^2t}\,dt$$

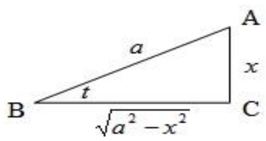
$$=\int\frac{1}{cos^2t}\,dt-\int dt$$

$$=\int d\,tant-\int dt$$

$$=tant-t+C$$

在Rt ΔABC中，设 $\angle B=t$,$|AB|=a$,则$|AC|=x$,$|BC|=\sqrt{a^2-x^2}$

$$\therefore \quad tant=\frac{x}{\sqrt{a^2-x^2}}$$



$$\therefore \int\frac{x^2}{\sqrt{(a^2-x^2)^3}}\,dx=\frac{x}{\sqrt{a^2-x^2}}-arcsin\frac{x}{a}+C$$

65. $\int \frac{dx}{x\sqrt{a^2-x^2}}=\frac{1}{a}\cdot ln\frac{a-\sqrt{a^2-x^2}}{|x|}+C \qquad (a>0)$

证明：被积函数$f(x)=\frac{1}{x\sqrt{a^2-x^2}}$的定义域为$\{x|-a<x<a且x\neq 0\}$

1.当$-a<x<0$时，可设$x=a\cdot sint \quad (-\frac{\pi}{2}<t<0)$，则$dx=a\cdot cos\,t\,dt$

$x\sqrt{a^2-x^2}=a\cdot sint\cdot|a\cdot cos\,t| \quad \because -\frac{\pi}{2}<t<0$，$cos\,t>0 \quad \therefore \quad x\sqrt{a^2-x^2}=a^2\cdot sint\cdot cos\,t$

$$\therefore \int \frac{dx}{x\sqrt{a^2-x^2}}=\int \frac{1}{a^2\cdot sint\cdot cos\,t}\cdot a\cdot cos\,t\,dt$$

$$=\frac{1}{a}\int \frac{1}{sint}dt$$

$$=\frac{1}{a}\int \frac{sint}{sin^2t}dt$$

$$=-\frac{1}{a}\int \frac{1}{1-cos^2t}d\,cos\,t$$

$$=-\frac{1}{2a}\int (\frac{1}{1+cost}+\frac{1}{1-cost})d\,cos\,t$$

$$=-\frac{1}{2a}\int \frac{1}{1+cost}d(cos\,t+1)+\frac{1}{2a}\int \frac{1}{1-cost}d(1-cos\,t)$$

$$=-\frac{1}{2a}\cdot ln|1+cost|+\frac{1}{2a}\cdot ln|cost-1|+C_1$$

$$=\frac{1}{2a}\cdot ln\left|\frac{cost-1}{1+cost}\right|+C_1$$

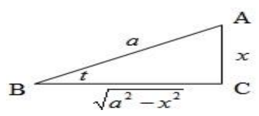
$$=\frac{1}{2a}\cdot ln\left|\frac{(cost-1)^2}{1-cos^2t}\cdot(-1)\right|+C_1$$

$$=\frac{1}{2a}\cdot ln\left|\frac{(cost-1)^2}{sin^2t}\right|+C_2$$

$$=\frac{1}{a}\cdot ln\left|\frac{cost-1}{sint}\right|+C_2$$

$$=\frac{1}{a}\cdot ln|cott-csct|+C_2$$

在Rt ΔABC中，设 $\angle B=t$，$|AB|=a$，则$|AC|=x$，$|BC|=\sqrt{a^2-x^2}$



$$\therefore \quad cott=\frac{\sqrt{a^2-x^2}}{x}，csct=\frac{1}{sint}=\frac{a}{x}$$

$$\therefore \int \frac{dx}{x\sqrt{a^2-x^2}}=\frac{1}{a}\cdot ln\left|\frac{\sqrt{a^2-x^2}-a}{x}\right|+C_2=\frac{1}{a}\cdot ln\left|\frac{a-\sqrt{a^2-x^2}}{x}\cdot(-1)\right|+C_2$$

$$=\frac{1}{a}\cdot ln\left|\frac{a-\sqrt{a^2-x^2}}{x}\right|+C_3$$

$$\because \ a-\sqrt{a^2-x^2}>0$$

$$\therefore \int \frac{dx}{x\sqrt{a^2-x^2}}=\frac{1}{a}\cdot ln\frac{a-\sqrt{a^2-x^2}}{|x|}+C$$

2.当$0<x<a$时，可设$x=a\cdot sint \quad (0<t<\frac{\pi}{2})$，同理可证

综合讨论1，2得：$\int \frac{dx}{x\sqrt{a^2-x^2}}=\frac{1}{a}\cdot ln\frac{a-\sqrt{a^2-x^2}}{|x|}+C$

66. $\int \frac{dx}{x^2\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{a^2 x} + C \qquad (a>0)$

证明:被积函数 $f(x)=\frac{1}{x^2\sqrt{a^2-x^2}}$ 的定义域为 $\{x \mid -a<x<a \text{且} x\neq 0\}$

1.当 $-a<x<0$ 时,可设 $x=a\cdot sint \quad (-\frac{\pi}{2}<t<0)$,则 $dx = a\cdot\cos t\, dt$,

$$\frac{1}{x^2\sqrt{a^2-x^2}} = \frac{1}{a^2\cdot sin^2 t}\cdot\frac{1}{|\,a\cdot\cos t\,|}$$

$\because -\frac{\pi}{2}<t<\frac{\pi}{2}$, $\cos t>0$ $\therefore$ $\frac{1}{x^2\sqrt{a^2-x^2}} = \frac{1}{a^3\cdot sin^2 t\cdot cost}$

$$\therefore \int\frac{dx}{x^2\sqrt{a^2-x^2}} = \int\frac{1}{a^3\cdot sin^2 t\cdot cost}\cdot a\cdot\cos t\, dt$$

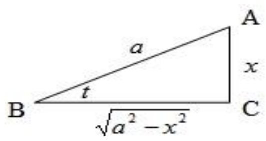
$$= \frac{1}{a^2}\int\frac{1}{sin^2 t}dt$$

$$= -\frac{1}{a^2}\int -csc^2 t\, dt$$

$$= -\frac{1}{a^2}\cdot cott + C$$

在Rt ΔABC 中, 设 $\angle B = t$,$|AB|=a$,则 $|AC|=x$,$|BC|=\sqrt{a^2-x^2}$

$$\therefore \ cott = \frac{\sqrt{a^2-x^2}}{x}$$



$$\therefore \int\frac{dx}{x^2\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{a^2 x} + C$$

2.当 $0<x<a$ 时,可设 $x=a\cdot sint \quad (0<t<\frac{\pi}{2})$,同理可证

综合讨论1,2得: $\int\frac{dx}{x^2\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{a^2 x} + C$

67. $\int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\cdot arcsin\frac{x}{a}+C \qquad (a>0)$

证明:被积函数 $f(x)=\sqrt{a^2-x^2}$ 的定义域为 $\{x\,|-a<x<a\}$

$\therefore$ 可设$x=a\cdot sint \quad (-\frac{\pi}{2}<t<\frac{\pi}{2})$，则$dx=a\cdot cos\,t\,dt$ ，$\sqrt{a^2-x^2}=|\,a\cdot cos\,t\,|$

$\because -\frac{\pi}{2}<t<\frac{\pi}{2}$ ，$cos\,t>0$ $\therefore$ $\sqrt{a^2-x^2}=a\cdot cos\,t$

$$\begin{aligned}\therefore \int\sqrt{a^2-x^2}\,dx&=\int a\cdot cos\,t\cdot a\cdot cos\,t\,dt\\&=a^2\int cos^2t\,dt\\&=a^2\int(1-sin^2t)\,dt\\&=a^2\int dt-a^2\int sin^2t\,dt \qquad ①\end{aligned}$$

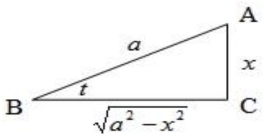
$$\begin{aligned}又\int\sqrt{a^2-x^2}\,dx&=a^2\int cos^2t\,dt\\&=a^2\int cost\,d\,sint\\&=a^2\cdot sint\cdot cost-a^2\int sint\,d\,cost\\&=a^2\cdot sint\cdot cost+a^2\int sin^2t\,dt \qquad ②\end{aligned}$$

由①+②得：$2\int\sqrt{a^2-x^2}\,dx=a^2\int dt+a^2\cdot sint\cdot cost=a^2t+a^2\cdot sint\cdot cost$

$\therefore \int\sqrt{a^2-x^2}\,dx=\frac{a^2}{2}t+\frac{a^2}{2}\cdot sint\cdot cost+C$

在Rt $\varDelta ABC$中，设 $\angle B=t$,$|AB|=a$, 则$|AC|=x$,$|BC|=\sqrt{a^2-x^2}$

$\therefore \; sint=\frac{x}{a}$ ，$cost=\frac{\sqrt{a^2-x^2}}{a}$



$$\begin{aligned}\therefore \int\sqrt{a^2-x^2}\,dx&=\frac{a^2}{2}\cdot arcsin\frac{x}{a}+\frac{a^2}{2}\cdot\frac{\sqrt{a^2-x^2}}{a}\cdot\frac{x}{a}+C\\&=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\cdot arcsin\frac{x}{a}+C\end{aligned}$$

68. $\int\sqrt{(a^2-x^2)^3}\,dx=\frac{x}{8}\cdot(5a^2-2x^2)\sqrt{a^2-x^2}+\frac{3}{8}\cdot a^4\cdot arcsin\frac{x}{a}+C \qquad (a>0)$

证明：$\int\sqrt{(a^2-x^2)^3}\,dx=x\cdot(a^2-x^2)^{\frac{3}{2}}-\int xd\,(a^2-x^2)^{\frac{3}{2}}$

$$=x\cdot(a^2-x^2)^{\frac{3}{2}}-\int x\cdot\frac{3}{2}\cdot(-2x)\cdot(a^2-x^2)^{\frac{1}{2}}d\,x$$

$$=x\cdot(a^2-x^2)^{\frac{3}{2}}+3\int x^2(a^2-x^2)^{\frac{1}{2}}d\,x$$

$$=x\cdot(a^2-x^2)^{\frac{3}{2}}+3\int(x^2-a^2+a^2)(a^2-x^2)^{\frac{1}{2}}d\,x$$

$$=x\cdot(a^2-x^2)^{\frac{3}{2}}-3\int(a^2-x^2)^{\frac{3}{2}}d\,x+3a^2\int(a^2-x^2)^{\frac{1}{2}}d\,x$$

移项并整理得：$\int\sqrt{(a^2-x^2)^3}\,dx=\frac{x}{4}\cdot(a^2-x^2)^{\frac{3}{2}}+\frac{3a^2}{4}\int(a^2-x^2)^{\frac{1}{2}}d\,x \qquad$ ①

又$\int(a^2-x^2)^{\frac{1}{2}}d\,x=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\cdot arcsin\frac{x}{a}+C \qquad$ (公式67) $\qquad$ ②

联立①②得：

$$\int\sqrt{(a^2-x^2)^3}\,dx=\frac{x}{4}(a^2-x^2)^{\frac{3}{2}}+\frac{3x}{8}\cdot a^2\cdot\sqrt{a^2-x^2}+\frac{3}{8}\cdot a^4\cdot arcsin\frac{x}{a}+C$$

$$=(\frac{a^2x}{4}-\frac{x^3}{4})\sqrt{a^2-x^2}+\frac{3x}{8}\cdot a^2\cdot\sqrt{a^2-x^2}+\frac{3}{8}\cdot a^4\cdot arcsin\frac{x}{a}+C$$

$$=\frac{x}{8}\cdot(5a^2-2x^2)\sqrt{a^2-x^2}+\frac{3}{8}\cdot a^4\cdot arcsin\frac{x}{a}+C$$

69. $\int x\sqrt{a^2-x^2}dx=-\frac{1}{3}\sqrt{(a^2-x^2)^3}+C \qquad (a>0)$

证明：被积函数 $f(x)=x\sqrt{a^2-x^2}$ 的定义域为 $\{x\,|-a<x<a\}$

$\therefore$ 可设$x=a\cdot sint \quad (-\frac{\pi}{2}<t<\frac{\pi}{2})$，则$dx=a\cdot\cos t\,dt$ ，$x\sqrt{a^2-x^2}=a\cdot\sin t\cdot|\,a\cdot\cos t\,|$

$\because -\frac{\pi}{2}<t<\frac{\pi}{2}$ ，$\cos t>0 \quad \therefore \quad x\sqrt{a^2-x^2}=a^2\cdot sint\cdot cost$

$\therefore \quad \int x\sqrt{a^2-x^2}dx=\int a^2\cdot sint\cdot cost\cdot a\cdot\cos t\,dt=a^3\int\cos^2t\cdot sint\,dt$

$$=-a^3\int cos^2t\,dcost=-\frac{a^3}{3}cos^3t+C$$

$$=-\frac{a^3}{3}(1-sin^2t)^{\frac{3}{2}}+C$$

$\because \quad x=a\cdot sint \quad (-\frac{\pi}{2}<t<\frac{\pi}{2})$ ，$\therefore \quad sint=\frac{x}{a}$

$\therefore \quad (1-sin^2t)^{\frac{3}{2}}=(\frac{a^2-x^2}{a^2})^{\frac{3}{2}}=\frac{\sqrt{(a^2-x^2)^3}}{a^3}$

$\therefore \quad \int x\sqrt{a^2-x^2}dx=-\frac{a^3}{3}(1-sin^2t)^{\frac{3}{2}}+C$

$$=-\frac{1}{3}\sqrt{(a^2-x^2)^3}+C$$

70. $\int x^2\sqrt{a^2-x^2}\,dx=\frac{x}{8}\cdot(2x^2-a^2)\sqrt{a^2-x^2}+\frac{a^4}{8}\cdot arcsin\frac{x}{a}+C \qquad (a>0)$

证明：被积函数 $f(x)=x^2\sqrt{a^2-x^2}$ 的定义域为 $\{x\,|\,-a<x<a\}$

$\therefore$ 可令 $x=a\cdot sint \quad (-\frac{\pi}{2}<t<\frac{\pi}{2})$, 则 $x^2\sqrt{a^2-x^2}=a^2\cdot sin^2t\,|a\cdot cost\,|$

$\because -\frac{\pi}{2}<t<\frac{\pi}{2}$, $\cos t>0$, $\therefore x^2\sqrt{a^2-x^2}=a^3\cdot sin^2t\cdot cost$

$$\therefore \int x^2\sqrt{a^2-x^2}\,dx=\int a^3sin^2t\cdot cost\,d\,(a\cdot sint)=a^4\int sin^2t\cdot cos^2t\,dt$$

$$=\frac{a^4}{3}\int 3\cdot sin^2t\cdot cost\cdot cost\,dt$$

$$=\frac{a^4}{3}\int cost\,d\,sin^3t$$

$$=\frac{a^4}{3}\cdot\cos t\cdot sin^3t-\frac{a^4}{3}\int sin^3t\,d\,cost$$

$$=\frac{a^4}{3}\cdot\cos t\cdot sin^3t-\frac{a^4}{3}\int sint\cdot(1-cos^2t)\,d\,cost$$

$$=\frac{a^4}{3}\cdot\cos t\cdot sin^3t-\frac{a^4}{3}\int sint\,d\,cost+\frac{a^4}{3}\int sint\cdot cos^2t\,d\,cost$$

$$=\frac{a^4}{3}\cdot\cos t\cdot sin^3t-\frac{a^4}{3}\int sint\,d\,cost+\frac{a^4}{3}\int sin^2t\cdot cos^2t\,dt$$

移项并整理得：$\int x^2\sqrt{a^2-x^2}\,dx=a^4\int sin^2t\cdot cos^2t\,dt$

$$=\frac{a^4}{4}\cdot\cos t\cdot sin^3t-\frac{a^4}{4}\int sint\,d\,cost \qquad ①$$

$$\because \int sint\,d\,cost=sint\cdot cost-\int cost\,d\,sint$$

$$=sint\cdot cost-\int\cos^2t\,dt$$

$$=sint\cdot cost-\int(1-sin^2t)dt$$

$$=sint\cdot cost-\int dt+\int sin^2t\,dt \qquad ②$$

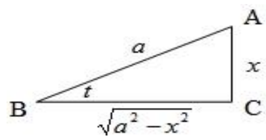
又 $\because \int sint\,d\,cost=-\int sin^2t\,dt \qquad ③$

联立②③得：$\int sint\,d\,cost=\frac{1}{2}\cdot sint\cdot cost-\frac{1}{2}\int dt$

$$=\frac{1}{2}\cdot sint\cdot cost-\frac{1}{2}\cdot t+C_1 \qquad ④$$

联立①④得：$\int x^2\sqrt{a^2-x^2}\,dx=\frac{a^4}{4}\cdot\cos t\cdot sin^3t-\frac{a^4}{8}\cdot sint\cdot cost-\frac{a^4}{8}\cdot t+C$

在 $Rt\triangle ABC$ 中，设 $\angle B=t$, $|AB|=a$，则 $|AC|=x$, $|BC|=\sqrt{a^2-x^2}$



$\therefore cost=\frac{\sqrt{a^2-x^2}}{a}$, $sint=\frac{x}{a}$

$$\therefore \int x^2\sqrt{a^2-x^2}\,dx=\frac{a^4}{4}\cdot\frac{\sqrt{a^2-x^2}}{a}\cdot\frac{x^3}{a^3}-\frac{a^4}{8}\cdot\frac{\sqrt{a^2-x^2}}{a}\cdot\frac{x}{a}+\frac{a^4}{8}\cdot arcsin\frac{x}{a}+C$$

$$=\frac{x}{8}\cdot(2x^2-a^2)\sqrt{a^2-x^2}+\frac{a^4}{8}\cdot arcsin\frac{x}{a}+C$$

71. $\int \frac{\sqrt{a^2-x^2}}{x}dx = \sqrt{a^2-x^2} + a\cdot ln\frac{a-\sqrt{a^2-x^2}}{|x|} + C \qquad (a>0)$

证明:被积函数 $f(x)=\frac{1}{x\sqrt{a^2-x^2}}$ 的定义域为 $\{x|-a<x<a且x\neq 0\}$

1.当$-a<x<0$时,可设$x=a\cdot sint \quad (-\frac{\pi}{2}<t<0)$,则$dx=a\cdot\cos t\,dt$

$$\frac{\sqrt{a^2-x^2}}{x}=\frac{|a\cdot\cos t|}{a\cdot sint} \quad \because -\frac{\pi}{2}<t<0 \text{ , } \cos t>0 \quad \therefore \frac{\sqrt{a^2-x^2}}{x}=\frac{\cos t}{sint}$$

$$\therefore \int\frac{\sqrt{a^2-x^2}}{x}dx=\int\frac{\cos t}{sint}\cdot a\cdot\cos t\,dt=a\int\frac{\cos^2 t}{sint}dt$$

$$=a\int\frac{1-sin^2t}{sint}dt=a\int\frac{1}{sint}dt-a\int sint\,dt$$

$$=a\int\frac{sint}{sin^2t}dt-a\int sint\,dt=-a\int\frac{1}{1-cos^2t}dcost-a\int sint\,dt$$

$$=-\frac{a}{2}\int(\frac{1}{1+cost}+\frac{1}{1-cost})d\cos t-a\int sint\,dt$$

$$=-\frac{a}{2}\int\frac{1}{1+cost}d(\cos t+1)+\frac{a}{2}\int\frac{1}{cost-1}d(cost-1)-a\int sint\,dt$$

$$=-\frac{a}{2}\cdot ln|1+cost|+\frac{a}{2}\cdot ln|cost-1|+a\cdot\cos t+C_1$$

$$=\frac{a}{2}\cdot ln\left|\frac{cost-1}{1+cost}\right|+a\cdot\cos t+C_1$$

$$=\frac{a}{2}\cdot ln\left|\frac{(cost-1)^2}{1-cos^2t}\cdot(-1)\right|+a\cdot\cos t+C_1$$

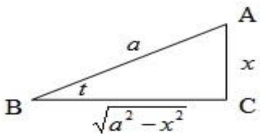
$$=\frac{a}{2}\cdot ln\left|\frac{(cost-1)^2}{sin^2t}\right|+a\cdot\cos t+C_2$$

$$=a\cdot ln\left|\frac{cost-1}{sint}\right|+a\cdot\cos t+C_2$$

$$=a\cdot ln|cott-csct|+a\cdot\cos t+C_2$$

在Rt $\varDelta ABC$中, 设 $\angle B=t,|AB|=a$,则$|AC|=x,|BC|=\sqrt{a^2-x^2}$

$$\therefore cott=\frac{\sqrt{a^2-x^2}}{x} \text{ , } csct=\frac{1}{sint}=\frac{a}{x} \text{ , } cost=\frac{\sqrt{a^2-x^2}}{a}$$



$$\therefore \int\frac{\sqrt{a^2-x^2}}{x}dx=a\cdot ln\left|\frac{\sqrt{a^2-x^2}-a}{x}\right|+a\cdot\frac{\sqrt{a^2-x^2}}{a}+C_2$$

$$=a\cdot ln\left|\frac{a-\sqrt{a^2-x^2}}{x}\cdot(-1)\right|+a\cdot\frac{\sqrt{a^2-x^2}}{a}+C_2$$

$$=\frac{1}{a}\cdot ln\left|\frac{a-\sqrt{a^2-x^2}}{x}\right|+a\cdot\frac{\sqrt{a^2-x^2}}{a}+C_3$$

$$\because a-\sqrt{a^2-x^2}>0$$

$$\therefore \int\frac{\sqrt{a^2-x^2}}{x}dx=\sqrt{a^2-x^2}+a\cdot ln\frac{a-\sqrt{a^2-x^2}}{|x|}+C$$

2.当$0<x<a$时,可设$x=a\cdot sint \quad (0<t<\frac{\pi}{2})$,同理可证

综合讨论1,2得:$\int\frac{\sqrt{a^2-x^2}}{x}dx=\sqrt{a^2-x^2}+a\cdot ln\frac{a-\sqrt{a^2-x^2}}{|x|}+C$

72. $\int \frac{\sqrt{a^2-x^2}}{x^2}dx = -\frac{\sqrt{a^2-x^2}}{x} - arcsin\frac{x}{a} + C \qquad (a>0)$

证明:被积函数 $f(x)=\frac{\sqrt{a^2-x^2}}{x^2}$ 的定义域为 $\{x \mid -a<x<a且x\neq 0\}$

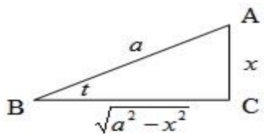
1.当 $-a<x<0$ 时,可设 $x=a\cdot sint \quad (-\frac{\pi}{2}<t<0)$，则 $dx=a\cdot\cos t\,dt$ ，$\frac{\sqrt{a^2-x^2}}{x^2}=\frac{|a\cdot\cos t|}{a^2\cdot sin^2t}$

$\because -\frac{\pi}{2}<t<0$ ，$\cos t>0$ $\therefore$ $\frac{\sqrt{a^2-x^2}}{x^2}=\frac{\cos t}{a\cdot sin^2t}$

$$\begin{aligned}
\therefore \int \frac{\sqrt{a^2-x^2}}{x^2}dx &= \int \frac{\cos t}{a\cdot sin^2t}\cdot a\cdot\cos t\,dt \\
&= \int \frac{cos^2t}{sin^2t}dt \\
&= \int \frac{1-sin^2t}{sin^2t}dt \\
&= \int csc^2tdt - \int dt \\
&= -cott - t + C
\end{aligned}$$

在Rt ΔABC 中，设 $\angle B=t$,$|AB|=a$,则 $|AC|=x$,$|BC|=\sqrt{a^2-x^2}$

$\therefore \ cott=\frac{\sqrt{a^2-x^2}}{x}$



$\therefore \ \int \frac{\sqrt{a^2-x^2}}{x^2}dx = -\frac{\sqrt{a^2-x^2}}{x} - arcsin\frac{x}{a} + C$

2.当 $0<x<a$ 时,可设 $x=a\cdot sint \quad (0<t<\frac{\pi}{2})$,同理可证

综合讨论1,2得: $\int \frac{\sqrt{a^2-x^2}}{x^2}dx = -\frac{\sqrt{a^2-x^2}}{x} - arcsin\frac{x}{a} + C$

（九）含有 $\sqrt{\pm a^2+bx+c}$ $(a>0)$ 的积分（73~78）

73. $\displaystyle\int\frac{dx}{\sqrt{ax^2+bx+c}}=\frac{1}{\sqrt{a}}\cdot ln\left|2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}\right|+C \qquad (a>0)$

证明：若被积函数 $f(x)=\dfrac{1}{\sqrt{ax^2+bx+c}}$ 成立，则 $ax^2+bx+c>0$ 恒成立

$$\because\ a>0 \quad \therefore\ \Delta=b^2-4ac>0$$

$$\because ax^2+bx+c=\frac{1}{4a}[(2ax+b)^2+4ac-b^2]$$

$$=\frac{1}{4a}[(2ax+b)^2-(\sqrt{b^2-4ac})^2]$$

$$\therefore\ \int\frac{dx}{\sqrt{ax^2+bx+c}}=2\sqrt{a}\int\frac{1}{\sqrt{(2ax+b)^2-(\sqrt{b^2-4ac})^2}}dx$$

$$=\frac{2\sqrt{a}}{2a}\int\frac{1}{\sqrt{(2ax+b)^2-(\sqrt{b^2-4ac})^2}}d(2ax+b)$$

$$=\frac{1}{\sqrt{a}}\int\frac{1}{\sqrt{(2ax+b)^2-(\sqrt{b^2-4ac})^2}}d(2ax+b)$$

公式45: $\displaystyle\int\frac{dx}{\sqrt{x^2-a^2}}=ln\left|x+\sqrt{x^2-a^2}\right|+C$

$$=\frac{1}{\sqrt{a}}\cdot ln\left|2ax+b+\sqrt{(2ax+b)^2-(\sqrt{b^2-4ac})^2}\right|+C$$

$$=\frac{1}{\sqrt{a}}\cdot ln\left|2ax+b+\sqrt{4a\cdot(ax^2+bx+c)}\right|+C$$

$$=\frac{1}{\sqrt{a}}\cdot ln\left|2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}\right|+C$$

74. $\displaystyle\int\sqrt{ax^2+bx+c}\,dx=\frac{2ax+b}{4a}\sqrt{ax^2+bx+c}+\frac{4ac-b^2}{8\sqrt{a^3}}\cdot ln\left|2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}\right|+C \qquad (a>0)$

证明：若被积函数 $f(x)=\sqrt{ax^2+bx+c}$ 成立，则 $ax^2+bx+c>0$ 恒成立

$$\because\ a>0 \quad \therefore\ \Delta=b^2-4ac>0$$

$$\because ax^2+bx+c=\frac{1}{4a}[(2ax+b)^2+4ac-b^2]$$

$$=\frac{1}{4a}[(2ax+b)^2-(\sqrt{b^2-4ac})^2]$$

公式53: $\displaystyle\int\sqrt{x^2-a^2}\,dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\cdot ln\left|x+\sqrt{x^2-a^2}\right|+C$

$$\therefore\ \int\sqrt{ax^2+bx+c}\,dx=\frac{1}{2\sqrt{a}}\int\sqrt{(2ax+b)^2-(\sqrt{b^2-4ac})^2}\,dx$$

$$=\frac{1}{2a\cdot 2\sqrt{a}}\int\sqrt{(2ax+b)^2-(\sqrt{b^2-4ac})^2}\,d(2ax+b)$$

$$=\frac{1}{4a\cdot\sqrt{a}}\cdot\left[\frac{2ax+b}{2}\sqrt{(2ax+b)^2-(\sqrt{b^2-4ac})^2}-\frac{b^2-4ac}{2}\cdot ln\left|2ax+b+\sqrt{(2ax+b)^2-(\sqrt{b^2-4ac})^2}\right|\right]$$

$$=\frac{1}{4\sqrt{a^3}}\cdot\frac{2ax+b}{2}\cdot 2\sqrt{a}\sqrt{ax^2+bx+c}+\frac{4ac-b^2}{8\sqrt{a^3}}\cdot ln\left|2ax+b+\sqrt{(2ax+b)^2-(\sqrt{b^2-4ac})^2}\right|+C$$

$$=\frac{1}{4\sqrt{a^3}}\cdot\frac{2ax+b}{2}\cdot 2\sqrt{a}\sqrt{ax^2+bx+c}+\frac{4ac-b^2}{8\sqrt{a^3}}\cdot ln\left|2ax+b+\sqrt{4a\cdot(ax^2+bx+c)}\right|+C$$

$$=\frac{2ax+b}{4a}\cdot\sqrt{ax^2+bx+c}+\frac{4ac-b^2}{8\sqrt{a^3}}\cdot ln\left|2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}\right|+C$$

75. $\int \frac{x}{\sqrt{ax^2+bx+c}}dx=\frac{1}{a}\sqrt{ax^2+bx+c}-\frac{b}{2\sqrt{a^3}}\cdot ln\left|2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}\right|+C \qquad (a>0)$

证明：$\because d(ax^2+bx+c)=(2ax+b)dx$

$\therefore$ 可将$\int \frac{x}{\sqrt{ax^2+bx+c}}dx$ 变换成$\int\left[\frac{1}{\sqrt{ax^2+bx+c}}\cdot\left(\frac{2ax+b}{2a}-\frac{b}{2a}\right)\right]dx$

$\therefore$ 上式$=\frac{1}{2a}\int\frac{1}{\sqrt{ax^2+bx+c}}\cdot(2ax+b)dx-\frac{b}{2a}\int\frac{1}{\sqrt{ax^2+bx+c}}dx$

$$=\frac{1}{2a}\int(ax^2+bx+c)^{-\frac{1}{2}}d(ax^2+bx+c)-\frac{b}{2a}\int\frac{1}{\sqrt{ax^2+bx+c}}dx$$

$$=\frac{1}{a}\sqrt{ax^2+bx+c}-\frac{b}{2a}\int\frac{1}{\sqrt{ax^2+bx+c}}dx$$

又$\frac{b}{2a}\int\frac{1}{\sqrt{ax^2+bx+c}}dx=\frac{b}{2a}\cdot\frac{1}{\sqrt{a}}\cdot ln\left|2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}\right|+C_1 \qquad$ (公式73)

$$=\frac{b}{2\sqrt{a^3}}\cdot ln\left|2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}\right|+C_1$$

$\therefore \int\frac{x}{\sqrt{ax^2+bx+c}}dx=\frac{1}{a}\sqrt{ax^2+bx+c}-\frac{b}{2\sqrt{a^3}}\cdot ln\left|2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}\right|+C$

76. $\int\frac{dx}{\sqrt{c+bx-ax^2}}=\frac{1}{\sqrt{a}}\cdot arcsin\frac{2ax-b}{\sqrt{b^2+4ac}}+C \qquad (a>0)$

证明：若被积函数 $f(x)=\frac{1}{\sqrt{c+bx-ax^2}}$成立，则$c+bx-ax^2>0$有解

$\because a>0 \quad \therefore \Delta=b^2+4ac>0$

$\because c+bx-ax^2=\frac{1}{4a}[b^2-(2ax-b)^2]+c$

$$=\frac{b^2+4ac}{4a}-\frac{(2ax-b)^2}{4a}$$

$\therefore \int\frac{dx}{\sqrt{c+bx-ax^2}}=2\sqrt{a}\int\frac{1}{\sqrt{(b^2+4ac)^2-(2ax-b)^2}}dx$

$$=\frac{1}{\sqrt{a}}\cdot arcsin\frac{2ax-b}{\sqrt{b^2+4ac}}+C$$

原题：$\int\frac{dx}{\sqrt{c+bx-ax^2}}=-\frac{1}{\sqrt{a}}\cdot arcisn\frac{2ax-b}{\sqrt{b^2+4ac}}+C$有误

77. $\int\sqrt{c+bx-ax^2}dx=\frac{2ax-b}{8a}\sqrt{c+bx-ax^2}+\frac{b^2+4ac}{8\sqrt{a^3}}\cdot arcsin\frac{2ax-b}{\sqrt{b^2+4ac}}+C \qquad (a>0)$

证明：若被积函数 $f(x)=\sqrt{c+bx-ax^2}$ 成立，则 $c+bx-ax^2\geq 0$ 有解

$$\because\ a>0 \quad \therefore\ \Delta=b^2+4ac\geq 0$$

$$\because c+bx-ax^2=\frac{1}{4a}[b^2-(2ax-b)^2]+c$$

$$=\frac{b^2+4ac}{4a}-\frac{(2ax-b)^2}{4a}$$

公式67：$\int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\cdot arcsin\frac{x}{a}+C$

$$\therefore\ \int\sqrt{c+bx-ax^2}\,dx=\frac{1}{2\sqrt{a}}\int\sqrt{(b^2+4ac)^2-(2ax-b)^2}\,dx$$

$$=\frac{1}{2\sqrt{a}\cdot 2a}\int\sqrt{(\sqrt{b^2+4ac})^2-(2ax-b)^2}\,d(2ax-b)$$

$$=\frac{1}{4\sqrt{a^3}}\left[\frac{2ax-b}{2}\sqrt{(\sqrt{b^2+4ac})^2-(2ax-b)^2}+\frac{b^2+4ac}{2}\cdot arcsin\frac{2ax-b}{\sqrt{b^2+4ac}}\right]+C$$

$$=\frac{2ax-b}{8\sqrt{a^3}}\sqrt{4a\cdot(c+bx-ax^2)}+\frac{b^2+4ac}{8\sqrt{a^3}}\cdot arcsin\frac{2ax-b}{\sqrt{b^2+4ac}}+C$$

$$=\frac{2ax-b}{8a}\sqrt{c+bx-ax^2}+\frac{b^2+4ac}{8\sqrt{a^3}}\cdot arcsin\frac{2ax-b}{\sqrt{b^2+4ac}}+C$$

78. $\int\frac{x}{\sqrt{c+bx-ax^2}}dx=-\frac{1}{a}\sqrt{c+bx-ax^2}+\frac{b}{2\sqrt{a^3}}\cdot arcsin\frac{2ax-b}{\sqrt{b^2+4ac}}+C \qquad (a>0)$

证明：若被积函数 $f(x)=\frac{x}{\sqrt{c+bx-ax^2}}$ 成立，则 $c+bx-ax^2>0$ 有解

$$\because\ a>0 \quad \therefore\ \Delta=b^2+4ac>0$$

$$\because c+bx-ax^2=\frac{1}{4a}[b^2-(2ax-b)^2]+c$$

$$=\frac{1}{4a}\left[b^2+4ac-(2ax-b)^2\right]$$

$$\therefore\ \int\frac{x}{\sqrt{c+bx-ax^2}}dx=2\sqrt{a}\int\frac{x}{\sqrt{(\sqrt{b^2+4ac})^2-(2ax-b)^2}}\,dx$$

公式61：$\int\frac{x}{\sqrt{a^2-x^2}}\,dx=-\sqrt{a^2-x^2}+C$

$$=2\sqrt{a}\cdot\frac{1}{2a}\cdot\frac{1}{2a}\int\frac{2ax-b+b}{\sqrt{(\sqrt{b^2+4ac})^2-(2ax-b)^2}}\,d(2ax-b)$$

$$=\frac{1}{2\sqrt{a^3}}\int\frac{2ax-b}{\sqrt{(\sqrt{b^2+4ac})^2-(2ax-b)^2}}\,d(2ax-b)+\frac{b}{2\sqrt{a^3}}\int\frac{1}{\sqrt{(\sqrt{b^2+4ac})^2-(2ax-b)^2}}\,d(2ax-b)$$

$$=-\frac{1}{2\sqrt{a^3}}\sqrt{(\sqrt{b^2+4ac})^2-(2ax-b)^2}+\frac{b}{2\sqrt{a^3}}\cdot arcsin\frac{2ax-b}{\sqrt{b^2+4ac}}+C$$

公式59：$\int\frac{dx}{\sqrt{a^2-x^2}}=arcsin\frac{x}{a}+C$

$$=-\frac{1}{2\sqrt{a^3}}\sqrt{4a\cdot(c+bx-ax^2)}+\frac{b}{2\sqrt{a^3}}\cdot arcsin\frac{2ax-b}{\sqrt{b^2+4ac}}+C$$

$$=-\frac{1}{a}\sqrt{c+bx-ax^2}+\frac{b}{2\sqrt{a^3}}\cdot arcsin\frac{2ax-b}{\sqrt{b^2+4ac}}+C$$

（十）含有 $\sqrt{\pm\frac{x-a}{x-b}}$ 或 $\sqrt{(x-a)(b-x)}$ 的积分（79~82）

79. $\int\sqrt{\frac{x-a}{x-b}}dx=(x-b)\sqrt{\frac{x-a}{x-b}}+(b-a)\cdot ln\,(\sqrt{|x-a|}+\sqrt{|x-b|})+C$

证明：$\because\sqrt{\frac{x-a}{x-b}}>0$ 可令 $t=\sqrt{\frac{x-a}{x-b}}\quad(t>0)$，则 $x=\frac{a-bt^2}{1-t^2}$ ，$dx=\frac{2t\cdot(a-b)}{(1-t^2)^2}dt$

$$\therefore\int\sqrt{\frac{x-a}{x-b}}dx=\int t\cdot\frac{2t\cdot(a-b)}{(1-t^2)^2}dt=2(a-b)\int\frac{t^2}{(1-t^2)^2}dt$$

$$=2(b-a)\int\frac{1-t^2+1}{(1-t^2)^2}dt=2(b-a)\int[\frac{1}{1-t^2}-\frac{1}{(1-t^2)^2}]dt$$

$$=2(b-a)\int\frac{1}{1-t^2}dt-2(b-a)\int\frac{1}{(1-t^2)^2}dt\ =2(a-b)\int\frac{1}{t^2-1}dt+2(a-b)\int\frac{1}{(1-t^2)^2}dt$$

$$=2(a-b)\cdot\frac{1}{2}\cdot ln\left|\frac{t-1}{t+1}\right|+2(a-b)\int\frac{1}{(1-t^2)^2}dt=(a-b)\cdot ln\left|\frac{t-1}{t+1}\right|+2(a-b)\int\frac{1}{(1-t^2)^2}dt$$

对于 $\int\frac{1}{(1-t^2)^2}dt=\int\frac{1}{(t^2-1)^2}dt\qquad(t>0)$

$\therefore$ 可令 $t=sec\,k\qquad(0<k<\frac{\pi}{2})$，则 $(t^2-1)^2=tan^4 k$，$d\,sec\,k=sec\,k\cdot tan\,kdk$

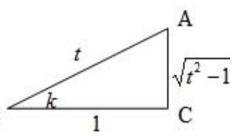
$$\therefore\int\frac{1}{(t^2-1)^2}dt=\int\frac{1}{tan^4 k}\cdot sec\,k\cdot tan\,kdk=\int\frac{sec\,k}{tan^4 k}dk=\int\frac{cos^2 k}{sin^3 k}dk$$

$$=\int\frac{1-sin^2 k}{sin^3 k}dk=\int\frac{1}{sin^3 k}dk-\int\frac{1}{sin\,k}dk=-\frac{1}{2}\cdot\frac{cos\,k}{sin^2 k}+\frac{1}{2}\int\frac{1}{sin\,k}dk-\int\frac{1}{sin\,k}dk$$

$$=-\frac{1}{2}\cdot\frac{cos\,k}{sin^2 k}-\frac{1}{2}\int\frac{1}{sin\,k}dk=-\frac{1}{2}\cdot ln\,|\,csck-cotk\,|-\frac{1}{2}\cdot\frac{cos\,k}{sin^2 k}$$

在 RtΔABC 中，$\angle B=k$ ，$|BC|=1$ 则 $|AC|=\sqrt{t^2-1}$ ，$|AB|=t$

$\therefore\ csc\,k=\frac{1}{sin\,k}=\frac{t}{\sqrt{t^2-1}}$ ，$cot\,k=\frac{1}{\sqrt{t^2-1}}$ ，$cos\,k=\frac{1}{t}$ ，$sin\,k=\frac{\sqrt{t^2-1}}{t}$



$$\therefore\ \int\sqrt{\frac{x-a}{x-b}}dx=(a-b)\cdot ln\left|\frac{t-1}{t+1}\right|+2(a-b)[-\frac{1}{2}\cdot ln\left|\frac{t-1}{\sqrt{t^2-1}}\right|-\frac{t}{2(t^2-1)}]+C_1$$

$$=(a-b)\cdot ln\left|\frac{t-1}{t+1}\right|-(a-b)\cdot ln\left|\frac{t-1}{\sqrt{t^2-1}}\right|-\frac{(a-b)\cdot t}{t^2-1}+C_1$$

$$=(a-b)\cdot ln\left|\frac{\sqrt{t^2-1}}{t+1}\right|-\frac{(a-b)\cdot t}{(t^2-1)}+C_1$$

将 $t=\sqrt{\frac{x-a}{x-b}}$ 代入上式得：$\therefore\ \int\sqrt{\frac{x-a}{x-b}}dx=(a-b)\cdot ln\left|\frac{\sqrt{\frac{b-a}{|x-b|}}}{\frac{\sqrt{|x-a|}+\sqrt{|x-b|}}{\sqrt{|x-b|}}}\right|-(a-b)\sqrt{\frac{x-a}{x-b}}\cdot\frac{x-b}{b-a}+C_1$

$$=(x-b)\sqrt{\frac{x-a}{x-b}}+(a-b)\,ln\left|\frac{\sqrt{b-a}}{\sqrt{|x-a|}+\sqrt{|x-b|}}\right|+C_1$$

$$=(x-b)\sqrt{\frac{x-a}{x-b}}+(a-b)\,ln\left|\sqrt{b-a}\right|+(b-a)\,ln\left|\sqrt{|x-a|}+\sqrt{|x-b|}\right|+C_1$$

$$=(x-b)\sqrt{\frac{x-a}{x-b}}+(b-a)\cdot ln\,(\sqrt{|x-a|}+\sqrt{|x-b|})+C$$

80. $\int\sqrt{\frac{x-a}{b-x}}dx=(x-b)\sqrt{\frac{x-a}{b-x}}+(b-a)\cdot arcsin\sqrt{\frac{x-a}{b-a}}+\mathrm{C}$

证明：$\because\sqrt{\frac{x-a}{b-x}}>0$ 可令 $t=\sqrt{\frac{x-a}{b-x}}\quad(t>0)$，则$x=\frac{a+bt^2}{1+t^2}$ ，$dx=\frac{2t\cdot(b-a)}{(1+t^2)^2}dt$

$$\therefore\int\sqrt{\frac{x-a}{b-x}}dx=\int t\cdot\frac{2t\cdot(b-a)}{(1+t^2)^2}dt=2(b-a)\int\frac{t^2}{(1+t^2)^2}dt$$

$$=2(b-a)\int\frac{1+t^2-1}{(1+t^2)^2}dt=2(b-a)\int[\frac{1}{1+t^2}-\frac{1}{(1+t^2)^2}]dt$$

$$=2(b-a)\int\frac{1}{1+t^2}dt-2(b-a)\int\frac{1}{(1+t^2)^2}dt=2(b-a)arcsint-2(a-b)\int\frac{1}{(1+t^2)^2}dt$$

$$=2(a-b)\cdot\frac{1}{2}\cdot ln\left|\frac{t-1}{t+1}\right|+2(a-b)\int\frac{1}{(1-t^2)^2}dt=(a-b)\cdot ln\left|\frac{t-1}{t+1}\right|+2(a-b)\int\frac{1}{(1-t^2)^2}dt$$

对于$\int\frac{1}{(1+t^2)^2}dt\quad(t>0)$

$\therefore$ 可令 $t=tan\,k\quad(0<k<\frac{\pi}{2})$，则$(t^2+1)^2=sec^4k$，$dt=sec^2kdk$

$$\therefore\int\frac{1}{(1+t^2)^2}dt=\int\frac{1}{sec^4k}\cdot sec^2kdk=\int\frac{1}{sec^2k}dk=\int cos^2kdk$$

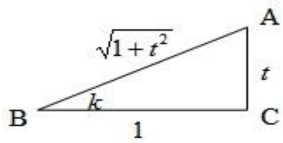
$$=\frac{1}{2}\int(1+cos\,2k)dk=\frac{1}{2}\int dk+\frac{1}{2}\int cos\,2kdk$$

$$=\frac{k}{2}+\frac{1}{4}\cdot sin\,2k+C_1$$

$$\therefore\int\sqrt{\frac{x-a}{b-x}}dx=2(b-a)k-2(b-a)[\frac{k}{2}+\frac{1}{4}\cdot sin\,2k]+C_1$$

$$=(b-a)k-(b-a)\sin k\cdot cos\,k+C_1$$

在$\mathrm{Rt}\varDelta ABC$中，$\angle\mathrm{B}=k$ ，$|\mathrm{BC}|=1$ 则$|\mathrm{AC}|=1$ ，$|\mathrm{AB}|=\sqrt{t^2+1}$



$\therefore\quad cos\,k=\frac{1}{\sqrt{t^2+1}}$ ，$sin\,k=\frac{t}{\sqrt{t^2+1}}$

$$\therefore\int\sqrt{\frac{x-a}{b-x}}dx=(b-a)\,arcsin\frac{t}{\sqrt{t^2+1}}-(b-a)\cdot\frac{1}{\sqrt{t^2+1}}\cdot\frac{t}{\sqrt{t^2+1}}+\mathrm{C}_1$$

$$=(b-a)\,arcsin\frac{t}{\sqrt{t^2+1}}-(b-a)\cdot\frac{t}{t^2+1}+\mathrm{C}_1$$

将$t=\sqrt{\frac{x-a}{b-x}}$代入上式得：$\therefore\int\sqrt{\frac{x-a}{b-x}}dx=(b-a)\,arcsin\left(\sqrt{\frac{b-x}{b-a}}\sqrt{\frac{x-a}{b-x}}\right)-(b-a)\cdot\sqrt{\frac{x-a}{b-x}}\sqrt{\frac{b-x}{b-a}}+\mathrm{C}_1$

$$=(b-a)\,arcsin\sqrt{\frac{x-a}{b-a}}-(b-x)\cdot\sqrt{\frac{x-a}{b-x}}+\mathrm{C}_1$$

$$=(b-a)\,arcsin\sqrt{\frac{x-a}{b-a}}+(x-b)\cdot\sqrt{\frac{x-a}{b-x}}+\mathrm{C}$$

81. $\int \frac{dx}{\sqrt{(x-a)(b-x)}} = 2\arcsin\sqrt{\frac{x-a}{b-a}} + C \qquad (a<b)$

证明：$\int \frac{dx}{\sqrt{(x-a)(b-x)}} = \int \frac{1}{|x-a|}\cdot\sqrt{\frac{x-a}{b-x}}\ dx$

令 $t=\sqrt{\frac{x-a}{b-x}}$，则$x=\frac{a+bt^2}{1+t^2}$，$|x-a|=\left|\frac{(b-a)t^2}{1+t^2}\right|$，$dx=\frac{2t(b-a)}{(1+t^2)^2}dt$

$\because b>a$，$\therefore$ $|x-a|=(b-a)\cdot\frac{t^2}{1+t^2}$

于是 $\int \frac{1}{|x-a|}\cdot\sqrt{\frac{x-a}{b-x}}\ dx = \int \frac{1}{b-a}\cdot\frac{1+t^2}{t^2}\cdot t\cdot\frac{2t\cdot(b-a)}{(1+t^2)^2}dt$

$$= 2\int\frac{1}{1+t^2}dt = 2\arctan t + C \qquad \text{(公式19)}$$

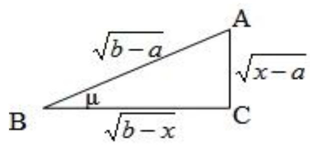
$$= 2\arctan\sqrt{\frac{x-a}{b-x}} + C$$

令 $\tan\mu=\sqrt{\frac{x-a}{b-x}}$，则 $\mu=\arctan\sqrt{\frac{x-a}{b-x}}$

在 Rt$\triangle ABC$中，$\angle \mathrm{B}=\mu$，$|\mathrm{AC}|=\sqrt{x-a}$

$\therefore$ $|\mathrm{BC}|=\sqrt{b-x}$，$|\mathrm{AB}|=\sqrt{|\mathrm{AC}|^2+|\mathrm{BC}|^2}=\sqrt{b-a}$

$\therefore$ $\sin\mu=\sqrt{\frac{x-a}{b-a}}$，$\therefore$ $\mu=\arcsin\sqrt{\frac{x-a}{b-a}}$



$\therefore$ $\int \frac{dx}{\sqrt{(x-a)(b-x)}} = 2\arcsin\sqrt{\frac{x-a}{b-a}} + C$

82. $\int\sqrt{(x-a)(b-x)}dx=\frac{2x-a-b}{4}\sqrt{(x-a)(b-x)}+\frac{(b-a)^2}{4}\cdot arcsin\sqrt{\frac{x-a}{b-x}}+\mathrm{C}\quad(a<b)$

证明: $\int\sqrt{(x-a)(b-x)}dx=\int|x-a|\sqrt{\frac{b-x}{x-a}}dx$

$\because\sqrt{\frac{b-x}{x-a}}>0$ 可令 $t=\sqrt{\frac{b-x}{x-a}}\quad(t>0)$，则$x=\frac{b+at^2}{1+t^2}$ ，$dx=\frac{2at\cdot(1+t^2)-2t(at^2+b)}{(1+t^2)^2}dt=\frac{2t(a-b)}{(1+t^2)^2}dt$

$$|x-a|=\left|\frac{at^2+b-a-at^2}{1+t^2}\right|=\left|\frac{b-a}{1+t^2}\right|$$

$\because\ a<b\qquad\therefore\ |x-a|=\frac{b-a}{1+t^2}$

$$\therefore\ \int\sqrt{(x-a)(b-x)}dx=\int\frac{b-a}{1+t^2}\cdot t\cdot\frac{2t(a-b)}{(1+t^2)^2}dt$$

$$=-2(a-b)^2\int\frac{t^2}{(1+t^2)^3}dt$$

对于$\int\frac{t^2}{(1+t^2)^3}dt\qquad(t>0)$ $\therefore$ 可令 $t=tan\,k\qquad(0<k<\frac{\pi}{2})$, 则$(t^2+1)^3=sec^6k$, $dt=sec^2kdk$

$$\therefore\ \int\frac{t^2}{(1+t^2)^3}dt=\int\frac{tan^2k}{sec^6k}\cdot sec^2kdk=\int\frac{tan^2k}{sec^4k}dk=\int sin^2k\cdot cos^2kdk$$

$$=\frac{1}{4}\int(2\sin k\cdot cos\,k)^2dk=\frac{1}{4}\int sin^2 2k\cdot dk$$

$$=\frac{1}{8}\left[\frac{2k}{2}-\frac{1}{4}\cdot sin\,4k\right]+C$$

$$=\frac{k}{8}-\frac{1}{32}\cdot sin\,4k+C$$

$$=\frac{k}{8}-\frac{1}{32}\cdot(4\sin k\cdot cos^3k-4\sin^3k\cdot cos\,k)+C$$

$$=\frac{k}{8}-\frac{1}{8}\cdot sin\,k\cdot cos^3k+\frac{1}{8}sin^3k\cdot cos\,k+C$$

联立以上两式得:$\int\sqrt{(x-a)(b-x)}dx=-2(b-a)^2\cdot\frac{1}{8}\cdot(k-sin\,k\cdot cos^3k+sin^3k\cdot cos\,k)+\mathrm{C}$

$$=-\frac{(b-a)^2}{4}\cdot(k-sin\,k\cdot cos^3k+sin^3k\cdot cos\,k)+\mathrm{C}$$

在$\mathrm{Rt}\varDelta ABC$中，$\angle\mathrm{B}=k$ ，$|\mathrm{BC}|=1$ 则$|\mathrm{AC}|=t$ ，$|\mathrm{AB}|=\sqrt{t^2+1}$

$\therefore\quad cos\,k=\frac{t}{\sqrt{t^2+1}}$ ，$sin\,k=\frac{1}{\sqrt{t^2+1}}$

$$\therefore\ \int\sqrt{(x-a)(b-x)}dx=-\frac{(b-a)^2}{4}\cdot\left(arcsin\frac{t}{\sqrt{t^2+1}}-\frac{1}{\sqrt{t^2+1}}\cdot\frac{t^2}{t^2+1}\cdot\frac{t}{\sqrt{t^2+1}}\right.$$

$$\left.+\frac{1}{t^2+1}\frac{1}{\sqrt{t^2+1}}\cdot\frac{t}{\sqrt{t^2+1}}\right)$$

$$=-\frac{(b-a)^2}{4}\cdot\left(arcsin\frac{t}{\sqrt{t^2+1}}-\frac{t^3}{(t^2+1)^2}+\frac{t}{(t^2+1)^2}\right)+\mathrm{C}$$

$$=-\frac{(b-a)^2}{4}\cdot\left(arcsin\frac{t}{\sqrt{t^2+1}}-\frac{t(t^2-1)}{(t^2+1)^2}\right)+\mathrm{C}$$

将$t=\sqrt{\frac{b-x}{x-a}}$代入上式得:$\int\sqrt{(x-a)(b-x)}dx=\frac{2x-a-b}{4}\sqrt{(x-a)(b-x)}+\frac{(b-a)^2}{4}\cdot arcsin\sqrt{\frac{x-a}{b-x}}+\mathrm{C}$

（十一）含有三角函数的积分（83~112）

83. $\int sinx\, dx = -cosx + C$

证明: $\int sinx\, dx = -\int (-sinx)\, dx$

$\because\ (cosx)' = -sinx$ 即 $cosx$ 为 $-sinx$ 的原函数

$\therefore\ \int sinx\, dx = -\int dcosx$

$= -cosx + C$

84. $\int cos\,x\, dx = sin\,x + C$

证明: $\because\ (sin\,x)' = cos\,x$ 即 $sin\,x$ 为 $cos\,x$ 的原函数

$\therefore\ \int cos\,x\, dx = \int d\,sin\,x$

$= sin\,x + C$

85. $\int tan\,x\, dx = -ln\,|\,cosx\,| + C$

证明: $\int tan\,x\, dx = \int \dfrac{sinx}{cos\,x}\, dx$

$= -\int \dfrac{1}{cos\,x}\, d\,cos\,x$

$= -ln\,|\,cosx\,| + C$

86. $\int cot\,x\, dx = ln\,|\,sin\,x\,| + C$

证明: $\int cot\,x\, dx = \int \dfrac{cos\,x}{sin\,x}\, dx$

$= \int \dfrac{1}{sin\,x}\, d\,sin\,x$

$= ln\,|\,sin\,x\,| + C$

87. $\int sec\,xdx = ln\,|\,tan\,(\dfrac{\pi}{4} + \dfrac{x}{2})\,| + \mathrm{C} = ln|sec\,x + tan\,x| + \mathrm{C}$

证明: $\int sec\,xdx = \int \dfrac{1}{cos\,x}dx = \int \dfrac{cos\,x}{cos^2\,x}dx$

$= \int \dfrac{1}{1-sin^2\,x}d\,sin\,x = \dfrac{1}{2}\int \dfrac{1}{1+sin\,x}d\,sin\,x + \dfrac{1}{2}\int \dfrac{1}{1-sin\,x}d\,sin\,x$

$= \dfrac{1}{2}\cdot ln\,|1+sin\,x| - \dfrac{1}{2}\cdot ln\,|1-sin\,x| + \mathrm{C}$

$= \dfrac{1}{2}\cdot ln\,|\dfrac{1+sin\,x}{1-sin\,x}| + \mathrm{C} = \dfrac{1}{2}\cdot ln\,\left|\dfrac{(1+sin\,x)^2}{1-sin^2\,x}\right| + C$

$= \dfrac{1}{2}\cdot ln\left|\dfrac{(1+sin\,x)^2}{cos^2\,x}\right| + C = ln\,|\dfrac{1+sin\,x}{cos\,x}| + \mathrm{C}$

$= ln\,\left|\dfrac{1}{cos\,x} - \dfrac{sin\,x}{cox}\right| + C$

$= ln\,|\,sec\,x + tan\,x\,| + \mathrm{C}$

88. $\int csc\,x\,dx = ln\left|\,tan\frac{x}{2}\right| + C = ln\,|\,csc\,x - cot\,x\,| + C$

证法1：$\because csc\,x = \frac{1}{sin\,x} = \frac{1}{2\cdot sin\frac{x}{2}\cdot cos\frac{x}{2}} = \frac{sin^2\frac{x}{2} + cos^2\frac{x}{2}}{2\cdot sin\frac{x}{2}\cdot cos\frac{x}{2}} = \frac{1 + tan^2\frac{x}{2}}{2\,tan\frac{x}{2}}$

又 $\because\ d\,tan\frac{x}{2} = \frac{1}{2}\cdot\frac{1}{cos^2\frac{x}{2}}dx$

$\therefore\ dx = 2\cdot cos^2\frac{x}{2}\,d\,tan\frac{x}{2}$

$\therefore\ \int csc\,x\,dx = \int\frac{1}{2\cdot sin\frac{x}{2}\cdot cos\frac{x}{2}}\cdot 2\cdot cos^2\frac{x}{2}\,d\,tan\frac{x}{2}$

$= \int\frac{1}{tan\frac{x}{2}}\,d\,tan\frac{x}{2}$

$= ln\left|\,tan\frac{x}{2}\right| + C$

$\because\ tan\frac{x}{2} = \frac{sin\frac{x}{2}}{cos\frac{x}{2}} = \frac{sin^2\frac{x}{2}}{sin\frac{x}{2}\cdot cos\frac{x}{2}} = \frac{2\,sin^2\frac{x}{2}}{2\,sin\frac{x}{2}\cdot cos\frac{x}{2}} = \frac{1 - cos\,x}{sin\,x} = csc\,x - cot\,x$

$\therefore\ \int csc\,x\,dx = ln\left|\,tan\frac{x}{2}\right| + C = ln\,|\,csc\,x - cot\,x\,| + C$

证法2：$\int csc\,x\,dx = \int\frac{1}{sint}\,dt$

$= \int\frac{sint}{sin^2 t}\,dt$

$= -\int\frac{1}{1 - cos^2 t}\,d\,cos\,t$

$= -\frac{1}{2}\int(\frac{1}{1 + cost} + \frac{1}{1 - cost})d\,cos\,t$

$= -\frac{1}{2}\int\frac{1}{1 + cost}\,d(cos\,t + 1) + \frac{1}{2}\int\frac{1}{1 - cost}\,d(1 - cos\,t)$

$= -\frac{1}{2}\cdot ln\,|\,1 + cost\,| + \frac{1}{2}\cdot ln\,|\,cost - 1\,| + C_1$

$= \frac{1}{2}\cdot ln\left|\frac{cost - 1}{1 + cost}\right| + C_1$

$= \frac{1}{2}\cdot ln\left|\frac{(1 - cos\,t)^2}{1 - cos^2 t}\cdot(-1)\right| + C_1$

$= \frac{1}{2}\cdot ln\left|\frac{(1 - cos\,t)^2}{sin^2 t}\right| + C_2$

$= ln\left|\frac{1 - cost}{sint}\right| + C_2$

$= ln\,|\,csc\,x - cot\,x\,| + C$

89. $\int sec^2 x\,dx = tan\,x + C$

证明：$\because\ (tan\,x)' = sec^2 x$即 $tan\,x$为$sec^2 x$的原函数

$$\therefore \int sec^2 x\,dx = \int d\,tan\,t$$

$$= tan\,x + C$$

90. $\int csc^2 x\,dx = -cotx + C$

证明：$\int csc^2 x\,dx = -\int (-csc^2 x)\,dx$

$\because\ (cotx)' = -csc^2 x$即 $cot\,x$为$-csc^2 x$的原函数

$$\therefore \int csc^2 x\,dx = -\int dcotx$$

$$= -cotx + C$$

91. $\int sec\,x \cdot tan\,x\,dx = sec\,x + C$

证明：$\because\ (sec\,x)' = sec\,x \cdot tan\,x$即 $sec\,x$为$sec\,x \cdot tan\,x$的原函数

$$\therefore \int sec\,x \cdot tan\,x\,dx = \int d\,sec\,x$$

$$= sec\,x + C$$

92. $\int cscx \cdot cot\,x\,dx = -csc\,x + C$

证明：$\int cscx \cdot cot\,x\,dx = -\int (-cscx \cdot cot\,x)\,dx$

$\because\ (csc\,x)' = -cscx \cdot cot\,x$即 $csc\,x$为$-cscx \cdot cot\,x$的原函数

$$\therefore \int cscx \cdot cot\,x\,dx = -\int d\,csc\,x$$

$$= -csc\,x + C$$

93. $\int sin^2 x\,dx = \frac{x}{2} - \frac{1}{4} \cdot sin\,2x + C$

证明：$\int sin^2 x\,dx = \int (\frac{1}{2} - \frac{1}{2} \cdot cos2x)\,dx$

$$= \frac{1}{2}\int dx - \frac{1}{4}\int cos2x\,d\,2x$$

$$= \frac{x}{2} - \frac{1}{4} sin\,2x + C$$

$\boxed{\text{提示}: sin^2 x = \frac{1 - cos\,2x}{2}}$

94. $\int cos^2 x\,dx = \frac{x}{2} + \frac{1}{4} \cdot sin\,2x + C$

证明：$\int cos^2 x\,dx = \int (\frac{1}{2} + \frac{1}{2} \cdot cos2x)\,dx$

$$= \frac{1}{2}\int dx + \frac{1}{4}\int cos2x\,d\,2x$$

$$= \frac{x}{2} + \frac{1}{4} sin\,2x + C$$

$\boxed{\text{提示}: cos^2 x = \frac{1 + cos\,2x}{2}}$

95. $\int sin^n x\,dx = -\frac{1}{n}\cdot sin^{n-1} x\cdot cos\,x + \frac{n-1}{n}\int sin^{n-2} x\,dx$

证明：$\int sin^n x\,dx = \int sin^{n-1} x\cdot sin\,x\,dx$

$= -\int sin^{n-1} x\,d\,cos\,x$

$= -cos\,x\cdot sin^{n-1} x + \int cos\,x\,d\,sin^{n-1} x$

$= -cos\,x\cdot sin^{n-1} x + \int cos\,x\cdot(n-1)\cdot sin^{n-2} x\cdot cos\,x\,dx$

$= -cos\,x\cdot sin^{n-1} x + (n-1)\int cos^2 x\cdot sin^{n-2} x\,dx$

$= -cos\,x\cdot sin^{n-1} x + (n-1)\int (1-sin^2 x)\cdot sin^{n-2} x\,dx$

$= -cos\,x\cdot sin^{n-1} x + (n-1)\int sin^{n-2} x\,dx - (n-1)\int sin^n x\,dx$

移项并整理得：$n\int sin^n x\,dx = -cos\,x\cdot sin^{n-1} x + (n-1)\int sin^{n-2} x\,dx$

$\therefore \int sin^n x\,dx = -\frac{1}{n}\cdot sin^{n-1} x\cdot cos\,x + \frac{n-1}{n}\int sin^{n-2} x\,dx$

96. $\int cos^n x\,dx = \frac{1}{n}\cdot cos^{n-1} x\cdot sin\,x + \frac{n-1}{n}\int cos^{n-2} x\,dx$

证明：$\int cos^n x\,dx = \int cos^{n-1} x\cdot cos\,x\,dx$

$= \int cos^{n-1} x\,d\,sin\,x$

$= sin\,x\cdot cos^{n-1} x - \int sin\,x\,d\,cos^{n-1} x$

$= sin\,x\cdot cos^{n-1} x + \int sin\,x\cdot(n-1)\cdot cos^{n-2} x\cdot sin\,x\,dx$

$= sin\,x\cdot cos^{n-1} x + (n-1)\int sin^2 x\cdot s\,cos^{n-2} x\,dx$

$= sin\,x\cdot cos^{n-1} x + (n-1)\int (1-cos^2 x)\cdot cos^{n-2} x\,dx$

$= sin\,x\cdot cos^{n-1} x + (n-1)\int cos^{n-2} x\,dx - (n-1)\int cos^n x\,dx$

移项并整理得：$n\int cos^n x\,dx = sin\,x\cdot cos^{n-1} x + (n-1)\int cos^{n-2} x\,dx$

$\therefore \int sin^n x\,dx = \frac{1}{n}\cdot sin\,x\cdot cos^{n-1} x + \frac{n-1}{n}\int cos^{n-2} x\,dx$

97. $\int \frac{dx}{\sin^n x} dx = -\frac{1}{n-1} \cdot \frac{\cos x}{\sin^{n-1} x} + \frac{n-2}{n-1} \int \frac{dx}{\sin^{n-2} x}$

证明：$\int \frac{dx}{\sin^n x} dx = -\int \frac{1}{\sin^{n-2} x} \cdot \frac{1}{-\sin^2 x} dx$

$$= -\int \frac{1}{\sin^{n-2} x} d \cot x$$

$$= -\frac{\cot x}{\sin^{n-2} x} + \int \cot x \, d \frac{1}{\sin^{n-2} x}$$

$$= -\frac{\cot x}{\sin^{n-2} x} + \int \cot x \cdot (2-n) \cdot \sin^{1-n} x \cdot \cos x \, dx$$

$$= -\frac{\cot x}{\sin^{n-2} x} + (2-n) \int \frac{\cos^2 x}{\sin^n x} dx$$

$$= -\frac{\cot x}{\sin^{n-2} x} + (2-n) \int \frac{1-\sin^2 x}{\sin^n x} dx$$

$$= -\frac{\cot x}{\sin^{n-2} x} + (2-n) \int \frac{dx}{\sin^n x} dx - (2-n) \int \frac{1}{\sin^{n-2} x} dx$$

移项并整理得：$(n-1) \int \frac{dx}{\sin^n x} dx = -\frac{\cot x}{\sin^{n-2} x} - (2-n) \int \frac{1}{\sin^{n-2} x} dx$

$$= -\frac{\cos x}{\sin^{n-1} x} + (n-2) \int \frac{1}{\sin^{n-2} x} dx$$

$\therefore \int \frac{dx}{\sin^n x} dx = -\frac{1}{n-1} \cdot \frac{\cos x}{\sin^{n-1} x} + \frac{n-2}{n-1} \int \frac{dx}{\sin^{n-2} x}$

98. $\int \frac{dx}{\cos^n x} = -\frac{1}{n-1} \cdot \frac{\sin x}{\cos^{n-1} x} + \frac{n-2}{n-1} \int \frac{dx}{\cos^{n-2} x}$

证明：$\int \frac{dx}{\cos^n x} = \int \frac{1}{\cos^{n-2} x} \cdot \frac{1}{\cos^2 x} dx$

$$= \int \frac{1}{\cos^{n-2} x} d \tan x$$

$$= \frac{\tan x}{\cos^{n-2} x} + \int \tan x \, d \frac{1}{\cos^{n-2} x}$$

$$= \frac{\tan x}{\cos^{n-2} x} + \int \tan x \cdot (2-n) \cdot \cos^{1-n} x \cdot \sin x \, dx$$

$$= \frac{\tan x}{\cos^{n-2} x} - (n-2) \int \frac{\sin^2 x}{\cos^n x} dx$$

$$= \frac{\tan x}{\cos^{n-2} x} - (n-2) \int \frac{1-\cos^2 x}{\cos^n x} dx$$

$$= \frac{\sin x}{\cos^{n-1} x} - (n-2) \int \frac{dx}{\cos^n x} dx + (n-2) \int \frac{1}{\cos^{n-2} x} dx$$

移项并整理得：$(n-1) \int \frac{dx}{\cos^n x} = \frac{\sin x}{\cos^{n-1} x} + (n-2) \int \frac{1}{\cos^{n-2} x} dx$

$$= \frac{\sin x}{\cos^{n-1} x} + (n-2) \int \frac{1}{\cos^{n-2} x} dx$$

$\therefore \int \frac{dx}{\cos^n x} = -\frac{1}{n-1} \cdot \frac{\sin x}{\cos^{n-1} x} + \frac{n-2}{n-1} \int \frac{dx}{\cos^{n-2} x}$

99. $\int cos^m x\cdot sin^n xdx=\frac{1}{m+n}\cdot cos^{m-1}x\cdot sin^{n+1}x+\frac{m-1}{m+n}\int cos^{m-2}x\cdot sin^n xdx$ ①

$=-\frac{1}{m+n}\cdot cos^{m+1}x\cdot sin^{n-1}x+\frac{n-1}{m+n}\int cos^m x\cdot sin^{n-2}xdx$ ②

证明①：$\because d\,sin^{m+n}xdx=(m+n)\cdot sin^{m+n-1}x\cdot cos\,xdx$

$$\therefore \int cos^m x\cdot sin^n xdx=\frac{1}{m+n}\int cos^{m-1}x\cdot sin^{1-m}xd\,sin^{m+n}x$$

$$=\frac{1}{m+n}\cdot cos^{m-1}x\cdot sin^{n+1}x-\frac{1}{m+n}\int sin^{m+n}xd(cos^{m-1}x\cdot sin^{1-m}x)$$

$\because d(cos^{m-1}x\cdot sin^{1-m}x)=[-(m-1)\cdot cos^{m-2}x\cdot sin\,x\cdot sin^{1-m}x+(1-m)\cdot sin^{1-m-1}x\cdot cos\,x\cdot cos^{m-1}x]dx$

$$=[(1-m)\cdot sin^{-m}x\cdot cos^m x\cdot(sin^2 x\cdot cos^{-2}x+1)]dx$$

$$=[(1-m)\cdot sin^{-m}x\cdot cos^m x\cdot(\frac{sin^2 x+cos^2 x}{cos^2 x})]dx$$

$$=[(1-m)\cdot sin^{-m}x\cdot cos^{m-2}x]dx$$

$$\therefore -\frac{1}{m+n}\int sin^{m+n}xd(cos^{m-1}x\cdot sin^{1-m}x)=\frac{m-1}{m+n}\int cos^{m-2}x\cdot sin^n xdx$$

$$\therefore \int cos^m x\cdot sin^n xdx=\frac{1}{m+n}\cdot cos^{m-1}x\cdot sin^{n+1}x+\frac{m-1}{m+n}\int cos^{m-2}x\cdot sin^n xdx$$

证明②：$\because d\,cos^{m+n}x=-(m+n)\cdot cos^{m+n-1}x\cdot sin\,xdx$

$$\therefore \int cos^m x\cdot sin^n xdx=\frac{-1}{m+n}\int cos^{1-n}x\cdot sin^{n-1}xd\,cos^{m+n}x$$

$$=\frac{-1}{m+n}\cdot sin^{n-1}x\cdot cos^{m+1}x+\frac{1}{m+n}\int cos^{m+n}xd(sin^{n-1}x\cdot cos^{1-n}x)$$

$\because d(sin^{n-1}x\cdot cos^{1-n}x)=[(n-1)\cdot sin^{n-2}x\cdot cos\,x\cdot cos^{1-n}x-(1-n)\cdot cos^{1-n-1}x\cdot sin\,x\cdot sin^{n-1}x]dx$

$$=[(n-1)\cdot cos^{-n}x\cdot sin^n x\cdot(sin^{-2}x\cdot cos^2 x+1)]dx$$

$$=[(n-1)\cdot cos^{-n}x\cdot sin^n x\cdot(\frac{sin^2 x+cos^2 x}{sin^2 x})]dx$$

$$=[(n-1)\cdot cos^{-n}x\cdot sin^{n-2}x]dx$$

$$\therefore \frac{1}{m+n}\int cos^{m+n}xd(sin^{n-1}x\cdot cos^{1-n}x)=\frac{n-1}{m+n}\int cos^m x\cdot sin^{n-2}xdx$$

$$\therefore \int cos^m x\cdot sin^n xdx=-\frac{1}{m+n}\cdot cos^{m+1}x\cdot sin^{n-1}x+\frac{n-1}{m+n}\int cos^m x\cdot sin^{n-2}xdx$$

100. $\int \sin ax \cdot \cos bx\, dx = -\frac{1}{2(a+b)} \cdot \cos(a+b)x - \frac{1}{2(a-b)} \cdot \cos(a-b)x + C$

证明：$\int \sin ax \cdot \cos bx\, dx = \int \frac{1}{2}[\sin(a+b)x + \sin(a-b)x]dx$ $\boxed{\text{提示}: \sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]}$

$$= \frac{1}{2}\int \sin(a+b)x\, dx \frac{1}{2}\int \sin(a-b)x dx$$

$$= \frac{1}{2(a+b)}\int \sin(a+b)x\, d(a+b)x + \frac{1}{2(a-b)}\int \sin(a-b)x\, d(a-b)x$$

$$= -\frac{1}{2(a+b)} \cdot \cos(a+b)x - \frac{1}{2(a-b)} \cdot \cos(a-b)x$$

101. $\int \sin ax \cdot \sin bx\, dx = -\frac{1}{2(a+b)} \cdot \sin(a+b)x + \frac{1}{2(a-b)} \cdot \sin(a-b)x + C$

证明：$\int \sin ax \cdot \sin bx\, dx = \int \frac{1}{2}[\cos(a-b)x - \cos(a+b)x]dx$ $\boxed{\text{提示}: \sin\alpha\sin\beta = -\frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)]}$

$$= \frac{1}{2}\int \cos(a-b)x\, dx - \frac{1}{2}\int \cos(a+b)x\, dx$$

$$= \frac{1}{2(a-b)}\int \cos(a-b)x\, d(a-b)x - \frac{1}{2(a+b)}\int \cos(a+b)x\, d(a+b)x$$

$$= \frac{1}{2(a-b)} \cdot \sin(a-b)x - \frac{1}{2(a+b)} \cdot \sin(a+b)x + C$$

102. $\int \cos ax \cdot \cos bx\, dx = \frac{1}{2(a+b)} \cdot \sin(a+b)x + \frac{1}{2(a-b)} \cdot \sin(a-b)x + C$

证明：$\int \cos ax \cdot \cos bx\, dx = \int \frac{1}{2}[\cos(a+b)x + \cos(a-b)x]dx$ $\boxed{\text{提示}: \cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)]}$

$$= \frac{1}{2}\int \cos(a+b)x\, dx + \frac{1}{2}\int \cos(a-b)x\, dx$$

$$= \frac{1}{2(a+b)}\int \cos(a+b)x\, d(a+b)x + \frac{1}{2(a-b)}\int \cos(a-b)x\, d(a-b)x$$

$$= \frac{1}{2(a+b)} \cdot \sin(a+b)x + \frac{1}{2(a-b)} \cdot \sin(a-b)x + C$$

103. $\int\frac{dx}{a+b\cdot\sin x}=\frac{2}{\sqrt{a^2-b^2}}\cdot\arctan\frac{a\cdot\tan\frac{x}{2}+b}{\sqrt{a^2-b^2}}+C \qquad (a^2>b^2)$

证明：令$t=\tan\frac{x}{2}$，则 $\sin x=2\cdot\sin\frac{x}{2}\cdot\cos\frac{x}{2}=\frac{2\cdot\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}=\frac{2t}{1+t^2}$

$$dt=(\tan\frac{x}{2})dx=\frac{1}{2}\cdot\sec^2\frac{x}{2}dx=\frac{1}{2}(1+\tan^2\frac{x}{2})\,dx=\frac{1}{2}(1+t^2)\,dx$$

$$\therefore\ dx=\frac{2}{1+t^2}dt\ ,\ a+b\cdot\sin x=a+\frac{2bt}{1+t^2}=\frac{a\,(1+t^2)+2bt}{1+t^2}$$

$$\therefore\ \int\frac{dx}{a+b\cdot\sin x}=\int\frac{1+t^2}{a\,(1+t^2)+2bt}\cdot\frac{2}{1+t^2}dt$$

$$=2\int\frac{1}{a\,t^2+2bt+a}dt$$

$$=2\int\frac{1}{a\,(t+\frac{b}{a})^2-\frac{b^2}{a}+a}dt$$

$$=2a\int\frac{1}{(at+b)^2+(a^2-b^2)}dt$$

$$=2\int\frac{1}{(at+b)^2+(a^2-b^2)}d(at+b)$$

当$a^2>b^2$,即$a^2-b^2>0$时

$$2\int\frac{1}{(at+b)^2+(a^2-b^2)}d(at+b)=2\int\frac{1}{(at+b)^2+(\sqrt{a^2-b^2})^2}d(at+b)$$

$\boxed{公式19：\int\frac{dx}{x^2+a^2}=\frac{1}{a}\cdot\arctan\frac{x}{a}+C}$ $\qquad =\frac{2}{\sqrt{a^2-b^2}}\cdot\arctan\frac{at+b}{\sqrt{a^2-b^2}}+C$

将$t=\tan\frac{x}{2}$代入上式得：$\int\frac{dx}{a+b\sin x}=\frac{2}{\sqrt{a^2-b^2}}\cdot\arctan\frac{a\cdot\tan\frac{x}{2}+b}{\sqrt{a^2-b^2}}+C$

104. $\int \frac{dx}{a+b\sin x} = \frac{1}{\sqrt{b^2-a^2}} \cdot ln\left| \frac{a \cdot tan\frac{x}{2}+b-\sqrt{b^2-a^2}}{a \cdot tan\frac{x}{2}+b+\sqrt{b^2-a^2}} \right| + C \qquad (a^2<b^2)$

证明：令$t = tan\frac{x}{2}$，则 $\sin x = 2 \cdot sin\frac{x}{2} \cdot cos\frac{x}{2} = \frac{2 \cdot tan\frac{x}{2}}{1+tan^2\frac{x}{2}} = \frac{2t}{1+t^2}$

$$dt = (tan\frac{x}{2})dx = \frac{1}{2} \cdot sec^2\frac{x}{2}dx = \frac{1}{2}(1+tan^2\frac{x}{2})\,dx = \frac{1}{2}(1+t^2)\,dx$$

$$\therefore\ dx = \frac{2}{1+t^2}dt\ ,\ a+b\sin x = a+\frac{2bt}{1+t^2} = \frac{a\,(1+t^2)+2bt}{1+t^2}$$

$$\therefore\ \int \frac{dx}{a+b\sin x} = \int \frac{1+t^2}{a\,(1+t^2)+2bt} \cdot \frac{2}{1+t^2}dt$$

$$= 2\int \frac{1}{a\,t^2+2bt+a}dt$$

$$= 2\int \frac{1}{a\,(t+\frac{b}{a})^2 - \frac{b^2}{a} + a}dt$$

$$= 2a\int \frac{1}{(at+b)^2+(a^2-b^2)}dt$$

$$= 2\int \frac{1}{(at+b)^2+(a^2-b^2)}d(at+b)$$

当$a^2<b^2$,即$a^2-b^2<0$时

$$2\int \frac{1}{(at+b)^2+(a^2-b^2)}d(at+b) = 2\int \frac{1}{(at+b)^2-(b^2-a^2)}d(at+b)$$

$$= 2\int \frac{1}{(at+b)^2-(\sqrt{b^2-a^2})^2}d(at+b)$$

$\boxed{公式21:\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \cdot ln\left|\frac{x-a}{x+a}\right| + C}$

$$= 2\times\frac{1}{2\sqrt{b^2-a^2}} \cdot ln\left| \frac{at+b-\sqrt{b^2-a^2}}{at+b+\sqrt{b^2-a^2}} \right| + C$$

将$t = tan\frac{x}{2}$代入上式得：$\int \frac{dx}{a+b\sin x} = \frac{1}{\sqrt{b^2-a^2}} \cdot ln\left| \frac{a \cdot tan\frac{x}{2}+b-\sqrt{b^2-a^2}}{a \cdot tan\frac{x}{2}+b+\sqrt{b^2-a^2}} \right| + C$

105. $\int\frac{dx}{a+b\cdot cos\,x}=\frac{2}{a+b}\cdot\sqrt{\frac{a+b}{a-b}}\,arctan\left(\sqrt{\frac{a-b}{a+b}}\cdot tan\frac{x}{2}\right)+C \qquad (a^2>\mathrm{b}^2)$

证明：令$t=tan\frac{x}{2}$，则 $cos\,x=\frac{1-tan^2\frac{x}{2}}{1+tan^2\frac{x}{2}}=\frac{1-t^2}{1+t^2}$

$$\therefore\ a+b\cdot cos\,x=a+b\cdot\frac{1-t^2}{1+t^2}=\frac{(a+b)+t^2(a-b)}{1+t^2}$$

$$\because\ dt=d\,tan\frac{x}{2}=\frac{1}{2}\cdot sec^2\frac{x}{2}dx=\frac{1}{2cos^2\frac{x}{2}}dx=\frac{1}{1+cos\,x}dx=\frac{1+t^2}{2}dx$$

$$\therefore\ dx=\frac{2}{1+t^2}dt$$

提示：$cos^2\theta=\frac{1+cos\,2\theta}{2}$

$$\therefore\ \int\frac{dx}{a+b\cdot cos\,x}=\int\frac{2}{(a+b)+t^2(a-b)}dt$$

当$|a|>|b|$，即$a^2>b^2$时

$$\int\frac{2}{(a+b)+t^2(a-b)}dt=\frac{2}{a-b}\int\frac{1}{\left(\sqrt{\frac{a+b}{a-b}}\right)^2+t^2}dt$$

公式19：$\int\frac{dx}{x^2+a^2}=\frac{1}{a}\cdot arctan\frac{x}{a}+C$

$$=\frac{2}{a-b}\cdot\sqrt{\frac{a-b}{a+b}}\cdot arctan\left(\sqrt{\frac{a-b}{a+b}}\cdot t\right)+C$$

$$=\frac{2}{a-b}\cdot\sqrt{\frac{a-b}{a+b}}\cdot arctan\left(\sqrt{\frac{a-b}{a+b}}\cdot t\right)+C$$

$$=2\sqrt{\frac{1}{(a+b)\cdot(a-b)}}\cdot arctan\left(\sqrt{\frac{a-b}{a+b}}\cdot t\right)+C$$

$$=\frac{2}{a+b}\cdot\sqrt{\frac{a+b}{a-b}}\cdot arctan\left(\sqrt{\frac{a-b}{a+b}}\cdot t\right)+C$$

将$t=tan\frac{x}{2}$代入上式得：$\int\frac{dx}{a+b\cdot cos\,x}=\frac{2}{a+b}\cdot\sqrt{\frac{a+b}{a-b}}\,arctan\left(\sqrt{\frac{a-b}{a+b}}\cdot tan\frac{x}{2}\right)+C$

106. $\int \frac{dx}{a+b\cdot \cos x}=\frac{1}{a+b}\cdot \sqrt{\frac{a+b}{b-a}}\cdot \ln\left|\frac{\tan\frac{x}{2}+\sqrt{\frac{a+b}{b-a}}}{\tan\frac{x}{2}-\sqrt{\frac{a+b}{b-a}}}\right|+C \qquad (a^2<b^2)$

证明：令$t=\tan\frac{x}{2}$，则 $\cos x=\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}=\frac{1-t^2}{1+t^2}$

$$\therefore\ a+b\cdot\cos x=a+b\cdot\frac{1-t^2}{1+t^2}=\frac{(a+b)+t^2(a-b)}{1+t^2}$$

$$\because\ dt=d\tan\frac{x}{2}=\frac{1}{2}\cdot\sec^2\frac{x}{2}dx=\frac{1}{2\cos^2\frac{x}{2}}dx=\frac{1}{1+\cos x}dx=\frac{1+t^2}{2}dx$$

$$\therefore\ dx=\frac{2}{1+t^2}dt$$

提示：$\cos^2\theta=\frac{1+\cos 2\theta}{2}$

$$\therefore\ \int\frac{dx}{a+b\cdot\cos x}=\int\frac{2}{(a+b)+t^2(a-b)}dt$$

当 $a^2<b^2$，即$|a|<|b|$，$\therefore\ b-a>0$

$$\int\frac{2}{(a+b)+t^2(a-b)}dt=\int\frac{2}{(a+b)-t^2(b-a)}dt$$

$$=\frac{2}{b-a}\int\frac{1}{\left(\sqrt{\frac{a+b}{b-a}}\right)^2-t^2}dt=\frac{2}{a-b}\int\frac{1}{t^2-\left(\sqrt{\frac{a+b}{b-a}}\right)^2}dt$$

$$=\frac{2}{a-b}\cdot\frac{1}{2}\cdot\sqrt{\frac{b-a}{a+b}}\cdot\ln\left|\frac{t-\sqrt{\frac{a+b}{b-a}}}{t+\sqrt{\frac{a+b}{b-a}}}\right|+C=\frac{1}{a-b}\cdot\sqrt{\frac{b-a}{a+b}}\cdot\ln\left|\frac{t-\sqrt{\frac{a+b}{b-a}}}{t+\sqrt{\frac{a+b}{b-a}}}\right|+C$$

公式21：$\int\frac{dx}{x^2-a^2}=\frac{1}{2a}\cdot\ln\left|\frac{x-a}{x+a}\right|+C$

$$=(-1)\sqrt{\frac{1}{(a+b)\cdot(b-a)}}\cdot\ln\left|\frac{t-\sqrt{\frac{a+b}{b-a}}}{t+\sqrt{\frac{a+b}{b-a}}}\right|+C\ =-\frac{1}{a+b}\cdot\sqrt{\frac{a+b}{b-a}}\cdot\ln\left|\frac{t-\sqrt{\frac{a+b}{b-a}}}{t+\sqrt{\frac{a+b}{b-a}}}\right|+C$$

$$=\frac{1}{a+b}\cdot\sqrt{\frac{a+b}{b-a}}\cdot\ln\left|\frac{t+\sqrt{\frac{a+b}{b-a}}}{t-\sqrt{\frac{a+b}{b-a}}}\right|+C$$

将$t=\tan\frac{x}{2}$代入上式得：$\int\frac{dx}{a+b\cdot\cos x}=\frac{1}{a+b}\cdot\sqrt{\frac{a+b}{b-a}}\cdot\ln\left|\frac{\tan\frac{x}{2}+\sqrt{\frac{a+b}{b-a}}}{\tan\frac{x}{2}-\sqrt{\frac{a+b}{b-a}}}\right|+C$

107. $\int \frac{dx}{a^2\cos^2 x + b^2\sin^2 x} = \frac{1}{ab}\cdot arctan\left(\frac{b}{a}\cdot \tan x\right) + C$

证明：
$$\int \frac{dx}{a^2\cos^2 x + b^2\sin^2 x} = \int \frac{1}{\cos^2 x}\cdot\frac{1}{a^2 + b^2\tan^2 x}dx$$
$$= \int \frac{1}{a^2 + b^2\tan^2 x}\,d\tan x$$
$$= \frac{1}{b^2}\int \frac{1}{(\frac{a^2}{b^2} + \tan^2 x)}\,d\tan x$$
$$= \frac{1}{b^2}\int \frac{1}{((\frac{a}{b})^2 + \tan^2 x)}\,d\tan x$$

$\boxed{公式19：\int \frac{dx}{x^2+a^2} = \frac{1}{a}\cdot arctan\ \frac{x}{a} + C}$

$$= \frac{1}{b^2}\cdot\frac{b}{a}\cdot arctan\left(\frac{b}{a}\cdot\tan x\right) + C$$
$$= \frac{1}{ab}\cdot arctan\left(\frac{b}{a}\cdot\tan x\right) + C$$

108. $\int \frac{dx}{a^2\cos^2 x - b^2\sin^2 x} = \frac{1}{2ab}\cdot ln\left|\frac{b\cdot\tan x + a}{b\cdot\tan x - a}\right| + C$

证明：
$$\int \frac{dx}{a^2\cos^2 x - b^2\sin^2 x} = \int \frac{1}{\cos^2 x}\cdot\frac{1}{a^2 - b^2\tan^2 x}dx$$
$$= \int \frac{1}{a^2 - b^2\tan^2 x}\,d\tan x$$
$$= \frac{1}{b}\int \frac{1}{a^2 - (b\cdot\tan x)^2}\,d\,(b\cdot\tan x)$$
$$= -\frac{1}{b}\int \frac{1}{(b\cdot\tan x)^2 - a^2}\,d\,(b\cdot\tan x)$$

$\boxed{公式21：\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\cdot ln\left|\frac{x-a}{x+a}\right| + C}$

$$= -\frac{1}{b}\cdot\frac{1}{2a}\cdot ln\left|\frac{b\cdot\tan x - a}{b\cdot\tan x + a}\right| + C$$
$$= -\frac{1}{2ab}\cdot ln\left|\frac{b\cdot\tan x - a}{b\cdot\tan x + a}\right| + C$$

$\boxed{提示：log_a b^{-1} = -log_a b}$

$$= \frac{1}{2ab}\cdot ln\left|\frac{b\cdot\tan x + a}{b\cdot\tan x - a}\right| + C$$

109. $\int x\cdot \sin ax\,dx=\frac{1}{a^2}\cdot \sin ax-\frac{1}{a}\cdot x\cdot \cos ax+C$

证明：$\int x\cdot \sin ax\,dx=-\frac{1}{a}\int x\,d\cos ax$

$$=-\frac{1}{a}\cdot x\cdot \cos ax+\frac{1}{a}\int \cos ax\,dx$$

$$=-\frac{1}{a}\cdot x\cdot \cos ax+\frac{1}{a^2}\int \cos ax\,dax$$

$$=-\frac{1}{a}\cdot x\cdot \cos ax+\frac{1}{a^2}\cdot \sin ax+C$$

110. $\int x^2\cdot \sin ax\,dx=-\frac{1}{a}\cdot x^2\cdot \cos ax+\frac{2}{a^2}\cdot x\cdot \sin ax+\frac{2}{a^3}\cdot \cos ax+C$

证明：$\int x^2\cdot \sin ax\,dx=-\frac{1}{a}\int x^2\,d\cos ax$

$$=-\frac{1}{a}\cdot x^2\cdot \cos ax+\frac{1}{a}\int \cos ax\,dx^2$$

$$=-\frac{1}{a}\cdot x^2\cdot \cos ax+\frac{2}{a}\int x\cdot \cos ax\,dx$$

$$=-\frac{1}{a}\cdot x^2\cdot \cos ax+\frac{2}{a^2}\cdot \int x\,d\sin ax$$

$$=-\frac{1}{a}\cdot x^2\cdot \cos ax+\frac{2}{a^2}\cdot x\cdot \sin ax-\frac{2}{a^3}\cdot \int \sin ax\,dax$$

$$=-\frac{1}{a}\cdot x^2\cdot \cos ax+\frac{2}{a^2}\cdot x\cdot \sin ax+\frac{2}{a^3}\cdot \cos ax$$

111. $\int x\cdot \cos ax\,dx=\frac{1}{a^2}\cdot \cos ax-\frac{1}{a}\cdot x\cdot \sin ax+C$

证明：$\int x\cdot \cos ax\,dx=\frac{1}{a}\int x\,d\sin ax$

$$=\frac{1}{a}\cdot x\cdot \sin ax-\frac{1}{a}\int \sin ax\,dx$$

$$=\frac{1}{a}\cdot x\cdot \sin ax-\frac{1}{a^2}\int \sin ax\,dax$$

$$=\frac{1}{a}\cdot x\cdot \sin ax+\frac{1}{a^2}\cdot \cos ax+C$$

112. $\int x^2\cdot \cos ax\,dx=\frac{1}{a}\cdot x^2\cdot \sin ax+\frac{2}{a^2}\cdot x\cdot \cos ax-\frac{2}{a^3}\cdot \sin ax+C$

证明：$\int x^2\cdot \cos ax\,dx=\frac{1}{a}\int x^2\,d\sin ax$

$$=\frac{1}{a}\cdot x^2\cdot \sin ax-\frac{1}{a}\int \sin ax\,d(x^2)$$

$$=\frac{1}{a}\cdot x^2\cdot \sin ax+\frac{2}{a}\int x\cdot \sin ax\,dx$$

$$=\frac{1}{a}\cdot x^2\cdot \sin ax-\frac{2}{a^2}\cdot \int x\,d\cos ax$$

$$=\frac{1}{a}\cdot x^2\cdot \sin ax+\frac{2}{a^2}\cdot x\cdot \cos ax-\frac{2}{a^3}\cdot \int \cos ax\,dax$$

$$=\frac{1}{a}\cdot x^2\cdot \sin ax+\frac{2}{a^2}\cdot x\cdot \cos ax-\frac{2}{a^3}\cdot \sin ax+C$$

（十二）含有反三角函数的积分（其中$a>0$）（113~121）

113. $\int arcsin\ \frac{x}{a}dx = x\cdot arcsin\ \frac{x}{a}+\sqrt{a^2-x^2}+C \qquad (a>0)$

证明：$\int arcsin\ \frac{x}{a}dx = x\cdot arcsin\ \frac{x}{a}-\int x\ d\ \ arcsin\ \frac{x}{a}$

$$= x\cdot arcsin\ \frac{x}{a}-\int x\cdot\frac{1}{\sqrt{1-(\frac{x}{a})^2}}\cdot\frac{1}{a}dx$$

$$= x\cdot arcsin\ \frac{x}{a}-\int\frac{x}{\sqrt{a^2-x^2}}dx$$

$$= x\cdot arcsin\ \frac{x}{a}-\frac{1}{2}\int\frac{1}{\sqrt{a^2-x^2}}d(x^2)$$

$$= x\cdot arcsin\ \frac{x}{a}+\frac{1}{2}\int(a^2-x^2)^{-\frac{1}{2}}d(a^2-x^2)$$

$$= x\cdot arcsin\ \frac{x}{a}+\frac{1}{2}\cdot\frac{1}{1-\frac{1}{2}}\cdot(a^2-x^2)^{1-\frac{1}{2}}+C$$

$$= x\cdot arcsin\ \frac{x}{a}+\sqrt{a^2-x^2}+C$$

114. $\int x\cdot arcsin\frac{x}{a}dx = (\frac{x^2}{2}-\frac{a^2}{4})\cdot arcsin\frac{x}{a}+\frac{x}{4}\sqrt{a^2-x^2}+C \qquad (a>0)$

证明：令$t = arcsin\frac{x}{a}$，则 $x = a\cdot sin\,t$

$$\therefore\ \int x\cdot arcsin\frac{x}{a}dx = \int a\cdot sin\,t\cdot t\ d(a\cdot sin\,t) = a^2\int t\cdot sin\,t\cdot cos\,t\ dt$$

$$= \frac{a^2}{2}\int t\cdot sin\,2t\ dt = -\frac{a^2}{4}\int t\ d\ cos\,2t$$

$$= -\frac{a^2}{4}\cdot t\cdot cos\,2t+\frac{a^2}{4}\int cos\,2t\ dt$$

$$= -\frac{a^2}{4}\cdot t\cdot cos\,2t+\frac{a^2}{8}\int cos\,2t\ d2t$$

$$= -\frac{a^2}{4}\cdot t\cdot cos\,2t+\frac{a^2}{8}\cdot sin\,2t+C$$

$$= -\frac{a^2}{4}\cdot t\cdot(2cos^2 t-1)+\frac{a^2}{4}\cdot sin\,t\cdot cos\,t+C$$

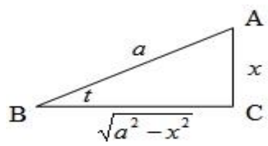
$$= -\frac{a^2}{2}\cdot t\cdot cos^2 t+\frac{a^2}{4}\cdot t+\frac{a^2}{4}\cdot sin\,t\cdot cos\,t+C$$

提示：$sin\,2x = 2\cdot sin\,x\cdot cos\,x$

$cos\,2x = cos^2 x-sin^2 x$

$= 2cos^2 x-1$

在Rt $\varDelta ABC$中，可设$\angle B = t$，$|AB| = a$，则$|AC| = x$，$|BC| = \sqrt{a^2-x^2}$



$$\therefore\ cos\,t = \frac{\sqrt{a^2-x^2}}{a}\ ,\ \ sin\,t = \frac{x}{a}$$

$$\therefore\ \int x\cdot arcsin\frac{x}{a}dx = -\frac{a^2}{2}\cdot arcsin\frac{x}{a}\cdot\frac{a^2-x^2}{a^2}+\frac{a^2}{4}\cdot arcsin\frac{x}{a}+\frac{a^2}{4}\cdot\frac{x}{a}\cdot\frac{\sqrt{a^2-x^2}}{a}+C$$

$$= \frac{x^2-a^2}{2}\cdot arcsin\frac{x}{a}+\frac{a^2}{4}\cdot arcsin\frac{x}{a}+\frac{x}{4}\cdot\sqrt{a^2-x^2}+C$$

$$= (\frac{x^2}{2}-\frac{a^2}{4})\cdot arcsin\frac{x}{a}+\frac{x}{4}\sqrt{a^2-x^2}+C$$

115. $\int x^2 \cdot arcsin\frac{x}{a}dx = \frac{x^3}{3}\cdot arcsin\frac{x}{a} + \frac{1}{9}(x^2+2a^2)\sqrt{a^2-x^2}+C \qquad (a>0)$

证明：令$t = arcsin\frac{x}{a}$，则 $x = a\cdot sin t$

$$\therefore \int x^2 \cdot arcsin\frac{x}{a}dx = \int a^2\cdot sin^2 t\cdot t\, d(a\cdot sin t) = a^3\int t\cdot sin^2 t\cdot cos t\, dt$$

$$= \frac{a^3}{3}\int t\, d\, sin^3 t$$

$$= \frac{a^3}{3}\cdot t\cdot sin^3 t - \frac{a^3}{3}\int sin^3 t\, dt$$

$$= \frac{a^3}{3}\cdot t\cdot sin^3 t - \frac{a^3}{3}\int sin t\,(1-cos^2 t)\, dt$$

$$= \frac{a^3}{3}\cdot t\cdot sin^3 t - \frac{a^3}{3}\int sin t\, dt + \frac{a^3}{3}\int sin t\cdot cos^2 t\, dt$$

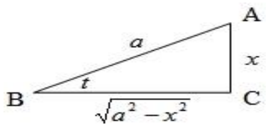
$$= \frac{a^3}{3}\cdot t\cdot sin^3 t + \frac{a^3}{3}\cdot cos t - \frac{a^3}{3}\int cos^2 t\, d\, cos t$$

$$= \frac{a^3}{3}\cdot t\cdot sin^3 t + \frac{a^3}{3}\cdot cos t - \frac{a^3}{3}\cdot\frac{1}{1+2}\cdot cos^3 t + C$$

$$= \frac{a^3}{3}\cdot t\cdot sin^3 t + \frac{a^3}{3}\cdot cos t - \frac{a^3}{9}\cdot cos^3 t + C$$

在Rt ΔABC中，可设$\angle B = t$,$|AB| = a$,则$|AC| = x$,$|BC| = \sqrt{a^2-x^2}$

$\therefore\ cos t = \frac{\sqrt{a^2-x^2}}{a}$ ， $sin t = \frac{x}{a}$



$$\therefore \int x^2\cdot arcsin\frac{x}{a}dx = \frac{a^3}{3}\cdot arcsin\frac{x}{a}\cdot\frac{x^3}{a^3} + \frac{a^3}{3}\cdot\frac{\sqrt{a^2-x^2}}{a} - \frac{a^3}{9}\cdot\frac{a^2-x^2}{a^3}\cdot\sqrt{a^2-x^2}+C$$

$$= \frac{x^3}{3}\cdot arcsin\frac{x}{a} + \frac{a^2}{3}\cdot\sqrt{a^2-x^2} - \frac{a^2-x^2}{9}\cdot\sqrt{a^2-x^2}+C$$

$$= \frac{x^3}{3}\cdot arcsin\frac{x}{a} + \frac{1}{9}(x^2+2a^2)\sqrt{a^2-x^2}+C$$

116. $\int arccos\ \frac{x}{a}dx = x \cdot arccos\ \frac{x}{a} - \sqrt{a^2 - x^2} + C \qquad (a > 0)$

证明: $\int arccos\ \frac{x}{a}dx = x \cdot arccos\ \frac{x}{a} - \int x\ d\ \ arccos\ \frac{x}{a}$

$$= x \cdot arccos\ \frac{x}{a} + \int x \cdot \frac{1}{\sqrt{1 - (\frac{x}{a})^2}} \cdot \frac{1}{a}dx$$

$$= x \cdot arccos\ \frac{x}{a} + \int \frac{x}{\sqrt{a^2 - x^2}}dx$$

$$= x \cdot arccos\ \frac{x}{a} + \frac{1}{2}\int \frac{1}{\sqrt{a^2 - x^2}}d(x^2)$$

$$= x \cdot arccos\ \frac{x}{a} - \frac{1}{2}\int (a^2 - x^2)^{-\frac{1}{2}}d(a^2 - x^2)$$

$$= x \cdot arccos\ \frac{x}{a} - \frac{1}{2} \cdot \frac{1}{1 - \frac{1}{2}} \cdot (a^2 - x^2)^{1-\frac{1}{2}} + C$$

$$= x \cdot arccos\ \frac{x}{a} - \sqrt{a^2 - x^2} + C$$

117. $\int x \cdot arccos\frac{x}{a}dx = (\frac{x^2}{2} - \frac{a^2}{4}) \cdot arccos\frac{x}{a} - \frac{x}{4}\sqrt{a^2 - x^2} + C \qquad (a > 0)$

证明: 令$t = arccos\frac{x}{a}$ ，则 $x = a \cdot cos\,t$

$$\therefore \int x \cdot arccos\frac{x}{a}dx = \int a \cdot cos\,t \cdot t\ d(a \cdot cos\,t) = -a^2\int t \cdot cos\,t \cdot\ sin\,t\ dt$$

$$= -\frac{a^2}{2}\int t \cdot sin\,2t\ dt = \frac{a^2}{4}\int t\ d\,cos\,2t$$

$$= \frac{a^2}{4} \cdot t \cdot cos\,2t - \frac{a^2}{4}\int cos\,2t\ dt$$

$$= \frac{a^2}{4} \cdot t \cdot cos\,2t - \frac{a^2}{8}\int cos\,2t\ d\,2t$$

提示: $sin\,2x = 2 \cdot sin\,x \cdot cos\,x$

$cos\,2x = cos^2 x - sin^2 x$

$= 2cos^2 x - 1$

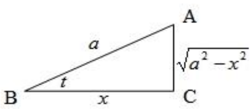
$$= \frac{a^2}{4} \cdot t \cdot cos\,2t - \frac{a^2}{8} \cdot sin\,2t + C$$

$$= \frac{a^2}{4} \cdot t \cdot (2cos^2 t - 1) - \frac{a^2}{4} \cdot sin\,t \cdot cos\,t + C$$

$$= \frac{a^2}{2} \cdot t \cdot cos^2 t - \frac{a^2}{4} \cdot t - \frac{a^2}{4} \cdot sin\,t \cdot cos\,t + C$$

在Rt ΔABC中， 可设 $\angle B = t$,$|AB| = a$,则 $|BC| = x$,$|AC| = \sqrt{a^2 - x^2}$

$$\therefore\ \ sin\,t = \frac{\sqrt{a^2 - x^2}}{a}\ ,\ \ cos\,t = \frac{x}{a}$$



$$\therefore \int x \cdot arccos\frac{x}{a}dx = \frac{a^2}{2} \cdot arcsin\frac{x}{a} \cdot \frac{x^2}{a^2} - \frac{a^2}{4} \cdot arcsin\frac{x}{a} - \frac{a^2}{4} \cdot \frac{x}{a} \cdot \frac{\sqrt{a^2 - x^2}}{a} + C$$

$$= \frac{x^2}{2} \cdot arcsin\frac{x}{a} - \frac{a^2}{4} \cdot arcsin\frac{x}{a} - \frac{x}{4} \cdot \sqrt{a^2 - x^2} + C$$

$$= (\frac{x^2}{2} - \frac{a^2}{4}) \cdot arcsin\frac{x}{a} + \frac{x}{4}\sqrt{a^2 - x^2} + C$$

118. $\int x^2 \cdot arccos\frac{x}{a}dx = \frac{x^3}{3}\cdot arccos\frac{x}{a} - \frac{1}{9}(x^2+2a^2)\sqrt{a^2-x^2}+C \qquad (a>0)$

证明：令$t = arccos\frac{x}{a}$，则 $x = a\cdot cos\,t$

$$\therefore \int x^2\cdot arccos\frac{x}{a}dx = \int a^2\cdot cos^2 t\cdot t\, d(a\cdot cos\,t) = -a^3\int t\cdot cos^2 t\cdot\ sin\,t\,dt$$

$$= \frac{a^3}{3}\int t\,d\,cos^3 t$$

$$= \frac{a^3}{3}\cdot t\cdot cos^3 t - \frac{a^3}{3}\int cos^3 t\,dt$$

$$= \frac{a^3}{3}\cdot t\cdot cos^3 t - \frac{a^3}{3}\int cos\,t\,(1-sin^2 t)\,dt$$

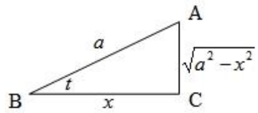
$$= \frac{a^3}{3}\cdot t\cdot cos^3 t - \frac{a^3}{3}\int cos\,t\,dt + \frac{a^3}{3}\int cos\,t\cdot sin^2 t\,dt$$

$$= \frac{a^3}{3}\cdot t\cdot cos^3 t - \frac{a^3}{3}\cdot sin\,t + \frac{a^3}{3}\int sin^2 t\,d\,sin\,t$$

$$= \frac{a^3}{3}\cdot t\cdot cos^3 t - \frac{a^3}{3}\cdot sin\,t + \frac{a^3}{3}\cdot\frac{1}{1+2}\cdot sin^3 t + C$$

$$= \frac{a^3}{3}\cdot t\cdot cos^3 t - \frac{a^3}{3}\cdot sin\,t + \frac{a^3}{9}\cdot sin^3 t + C$$

在Rt ΔABC中，可设$\angle B = t$，$|AB| = a$，则$|BC| = x$，$|AC| = \sqrt{a^2-x^2}$



$$\therefore\ \ sin\,t = \frac{\sqrt{a^2-x^2}}{a}\ ,\ \ cos\,t = \frac{x}{a}$$

$$\therefore \int x^2\cdot arccos\frac{x}{a}dx = \frac{a^3}{3}\cdot arcsin\frac{x}{a}\cdot\frac{x^3}{a^3} - \frac{a^3}{3}\cdot\frac{\sqrt{a^2-x^2}}{a} + \frac{a^3}{9}\cdot\frac{a^2-x^2}{a^3}\cdot\sqrt{a^2-x^2}+C$$

$$= \frac{x^3}{3}\cdot arcsin\frac{x}{a} - \frac{a^2}{3}\cdot\sqrt{a^2-x^2} + \frac{a^2-x^2}{9}\cdot\sqrt{a^2-x^2}+C$$

$$= \frac{x^3}{3}\cdot arcsin\frac{x}{a} - \frac{1}{9}(x^2+2a^2)\sqrt{a^2-x^2}+C$$

119. $\int arctan\ \frac{x}{a}dx = x\cdot arctan\ \frac{x}{a} - \frac{a}{2}\cdot ln\ (a^2+x^2)+C \qquad (a>0)$

证明：$\int arctan\ \frac{x}{a}dx = x\cdot arctan\ \frac{x}{a} - \int x\ d\ \ x\cdot arctan\ \frac{x}{a}$

$$= x\cdot arctan\ \frac{x}{a} - \int x\cdot\frac{1}{1+(\frac{x}{a})^2}\cdot\frac{1}{a}dx$$

$$= x\cdot arctan\ \frac{x}{a} - a\int\frac{x}{a^2+x^2}dx$$

$$= x\cdot arctan\ \frac{x}{a} - \frac{a}{2}\int\frac{1}{a^2+x^2}d(x^2)$$

$$= x\cdot arctan\ \frac{x}{a} - \frac{a}{2}\int\frac{1}{a^2+x^2}d(a^2+x^2)$$

$$= x\cdot arctan\ \frac{x}{a} - \frac{a}{2}\cdot ln\ \left|\ a^2+x^2\ \right| + C$$

$\because\ \ a^2+x^2>0$

$$\therefore\ \ \int arctan\ \frac{x}{a}dx = x\cdot arctan\ \frac{x}{a} - \frac{a}{2}\cdot ln\ (a^2+x^2)+C$$

120. $\int x\cdot arctan\frac{x}{a}dx=\frac{1}{2}(a^2+x^2)\cdot arctan\frac{x}{a}-\frac{a}{2}\cdot x+C \qquad (a>0)$

证明：令$t=arctan\frac{x}{a}$，则 $x=a\cdot tan\,t$

$$\therefore \int x\cdot arctan\frac{x}{a}dx=\int a\cdot tan\,t\cdot t\,d(a\cdot tan\,t)=a^2\int t\cdot sec^2 t\cdot tan\,t\,dt$$

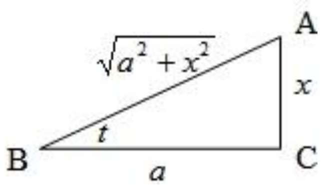
$$=\frac{a^2}{2}\int t\,d\,sec^2 t$$

$$=\frac{a^2}{2}\cdot t\cdot sec^2 t-\frac{a^2}{2}\int sec^2 t\,dt$$

$$=\frac{a^2}{2}\cdot t\cdot sec^2 t-\frac{a^2}{2}\cdot tan\,t+C$$

在Rt ΔABC中，可设$\angle B=t$，$|BC|=a$，则$|AC|=x$，$|AB|=\sqrt{a^2+x^2}$

$$\therefore\ sec\,t=\frac{1}{cos\,t}=\frac{\sqrt{a^2+x^2}}{a},\quad tan\,t=\frac{x}{a}$$



$$\therefore \int x\cdot arctan\frac{x}{a}dx=\frac{a^2}{2}\cdot arctan\frac{x}{a}\cdot\frac{a^2+x^2}{a^2}-\frac{a^2}{2}\cdot\frac{x}{a}+C$$

$$=\frac{1}{2}(a^2+x^2)\cdot arctan\frac{x}{a}-\frac{a}{2}\cdot x+C$$

121. $\int x^2\cdot arctan\ \frac{x}{a}dx=\frac{x^3}{3}\cdot arctan\ \frac{x}{a}-\frac{a}{6}\cdot x^2+\frac{a^3}{6}ln\,(a^2+x^2)+C \qquad (a>0)$

证明：$\therefore\ \int x^2\cdot arctan\ \frac{x}{a}dx=\frac{1}{3}\int arctan\ \frac{x}{a}dx^3$

$$=\frac{x^3}{3}\cdot arctan\ \frac{x}{a}-\frac{1}{3}\int x^3\cdot\frac{1}{1+(\frac{x}{a})^2}\cdot\frac{1}{a}dx$$

$$=\frac{x^3}{3}\cdot arctan\ \frac{x}{a}-\frac{a}{3}\int\frac{x^3}{a^2+x^2}dx$$

$$=\frac{x^3}{3}\cdot arctan\ \frac{x}{a}-\frac{a}{6}\int\frac{x^2}{a^2+x^2}d(x^2)$$

$$=\frac{x^3}{3}\cdot arctan\ \frac{x}{a}-\frac{a}{6}\int\frac{x^2+a^2-a^2}{a^2+x^2}d(x^2)$$

$$=\frac{x^3}{3}\cdot arctan\ \frac{x}{a}-\frac{a}{6}\int dx^2+\frac{a}{6}\int\frac{a^2}{a^2+x^2}d(x^2)$$

$$=\frac{x^3}{3}\cdot arctan\ \frac{x}{a}-\frac{a}{6}\int dx^2+\frac{a^3}{6}\int\frac{1}{a^2+x^2}d(x^2+a^2)$$

$$=\frac{x^3}{3}\cdot arctan\ \frac{x}{a}-\frac{a}{6}\cdot x^2+\frac{a^3}{6}ln\,\left|a^2+x^2\right|+C$$

$\because\quad a^2+x^2>0$

$$\therefore\ \int x^2\cdot arctan\ \frac{x}{a}dx=\frac{x^3}{3}\cdot arctan\ \frac{x}{a}-\frac{a}{6}\cdot x^2+\frac{a^3}{6}ln\,(a^2+x^2)+C$$

（十三）含有指数函数的积分（122~131）

122. $\int a^x dx = \frac{1}{\ln a} \cdot a^x + C$

证明：$\int a^x dx = \frac{1}{\ln a} \int \ln a \cdot a^x dx$

$\because \ (a^x)' = a^x \ln a$，即$a^x \ln a$的原函数为$a^x$

$\therefore \ \int a^x dx = \frac{1}{\ln a} \int d(a^x)$

$$= \frac{1}{\ln a} \cdot a^x + C$$

123. $\int e^{ax} dx = \frac{1}{a} \cdot e^{ax} + C$

证明：令 $ax = \mu$，则 $x = \frac{\mu}{a}$，$dx = \frac{1}{a} d\mu$

$\therefore \ \int e^{ax} dx = \frac{1}{a} \int e^{\mu} d\mu = \frac{1}{a} \cdot e^{\mu} + C$

$$= \frac{1}{a} \cdot e^{ax} + C$$

124. $\int x \cdot e^{ax} dx = \frac{1}{a^2}(ax-1)e^{ax} + C$

证明：$\int x \cdot e^{ax} dx = \frac{1}{a} \int x \ d(e^{ax})$

$$= \frac{1}{a} \cdot x \cdot \ e^{ax} - \frac{1}{a} \int e^{ax} dx$$

$$= \frac{1}{a} \cdot x \cdot \ e^{ax} - \frac{1}{a^2} \int e^{ax} d(ax)$$

$$= \frac{1}{a} \cdot x \cdot \ e^{ax} - \frac{1}{a^2} e^{ax} + C$$

$$= \frac{1}{a^2}(ax-1)e^{ax} + C$$

125. $\int x^n \cdot e^{ax} dx = \frac{1}{a} \cdot x^n \cdot e^{ax} - \frac{n}{a} \int x^{n-1} \cdot e^{ax} dx$

证明：$\int x^n \cdot e^{ax} dx = \frac{1}{a} \int x^n \ d(e^{ax})$

$$= \frac{1}{a} \cdot x^n \cdot \ e^{ax} - \frac{1}{a} \int e^{ax} d(x^n)$$

$$= \frac{1}{a} \cdot x^n \cdot \ e^{ax} - \frac{n}{a} \int x^{n-1} \cdot e^{ax} dx$$

126. $\int x\cdot a^x dx=\frac{x}{\ln a}\cdot a^x-\frac{1}{(\ln a)^2}\cdot a^x+C$

证明：$\int x\cdot a^x dx=\frac{1}{\ln a}\int x\, d(a^x)$

$$=\frac{1}{\ln a}\cdot x\cdot\ a^x-\frac{1}{\ln a}\int a^x dx \quad \boxed{\text{公式122：}\int a^x dx=\frac{1}{\ln a}\cdot a^x+C}$$

$$=\frac{1}{\ln a}\cdot x\cdot\ a^x-\frac{1}{(\ln a)^2}\cdot a^x+C$$

127. $\int x^n\cdot a^x dx=\frac{1}{\ln a}\cdot x^n\cdot a^x-\frac{n}{\ln a}\int x^{n-1}\cdot a^x dx$

证明：$\int x^n\cdot a^x dx=\frac{1}{\ln a}\int x^n\ d(a^x)$

$$=\frac{1}{\ln a}\cdot x^n\cdot\ a^x-\frac{1}{\ln a}\int a^x d(x^n)$$

$$=\frac{1}{\ln a}\cdot x^n\cdot a^x-\frac{n}{\ln a}\int x^{n-1}\cdot a^x dx$$

128. $\int e^{ax}\cdot\sin bx\ dx=\frac{1}{a^2+b^2}\cdot e^{ax}(a\cdot\sin bx-b\cdot\cos bx)+C$

证明：$\int e^{ax}\cdot\sin bx\ dx=-\frac{1}{b}\int e^{ax}d\cos bx$

$$=-\frac{1}{b}\cdot e^{ax}\cdot\cos bx+\frac{1}{b}\int\cos bxd(e^{ax})$$

$$=-\frac{1}{b}\cdot e^{ax}\cdot\cos bx+\frac{a}{b^2}\cdot e^{ax}\cdot\sin bx-\frac{a}{b^2}\int\sin bx\ d(e^{ax})$$

$$=-\frac{1}{b}\cdot e^{ax}\cdot\cos bx+\frac{a}{b^2}\cdot e^{ax}\cdot\sin bx-\frac{a}{b^2}\int\sin bx\ d(e^{ax})$$

移项并整理得：$\frac{a^2+b^2}{b^2}\int e^{ax}\cdot\sin bx\ dx=-\frac{1}{b}\cdot e^{ax}\cdot\cos bx+\frac{a}{b^2}\cdot e^{ax}\cdot\sin bx+C$

$\therefore\ \int e^{ax}\cdot\sin bx\ dx=-\frac{b}{a^2+b^2}\cdot e^{ax}\cdot\cos bx+\frac{a}{a^2+b^2}\cdot e^{ax}\cdot\sin bx+C$

$$=\frac{1}{a^2+b^2}\cdot e^{ax}(a\cdot\sin bx-b\cdot\cos bx)+C$$

129. $\int e^{ax}\cdot\cos bxdx=\frac{1}{a^2+b^2}\cdot e^{ax}(b\cdot\sin bx+a\cdot\cos bx)+C$

证明：$\int e^{ax}\cdot\cos bxdx=\frac{1}{b}\int e^{ax}d\sin bx$

$$
\begin{aligned}
&=\frac{1}{b}\cdot e^{ax}\cdot\sin bx-\frac{1}{b}\int\sin bxd(e^{ax})\\
&=\frac{1}{b}\cdot e^{ax}\cdot\sin bx-\frac{a}{b}\int\sin bx\cdot e^{ax}dx\\
&=\frac{1}{b}\cdot e^{ax}\cdot\sin bx+\frac{a}{b^2}\int e^{ax}d\cos bx\\
&=\frac{1}{b}\cdot e^{ax}\cdot\sin bx+\frac{a}{b^2}\cdot e^{ax}\cdot\cos bx-\frac{a}{b^2}\int\cos bxd(e^{ax})\\
&=\frac{1}{b}\cdot e^{ax}\cdot\sin bx+\frac{a}{b^2}\cdot e^{ax}\cdot\cos bx-\frac{a^2}{b^2}\int e^{ax}\cdot\cos bxdx
\end{aligned}
$$

$\therefore\quad(1+\frac{a^2}{b^2})\int e^{ax}\cdot\cos bxdx=\frac{a^2+b^2}{b^2}\int e^{ax}\cdot\cos bxdx=\frac{1}{b}\cdot e^{ax}\cdot\sin bx+\frac{a}{b^2}\cdot e^{ax}\cdot\cos bx$

$\therefore\quad\int e^{ax}\cdot\cos bxdx=\frac{1}{a^2+b^2}\cdot e^{ax}(b\cdot\sin bx+a\cdot\cos bx)+C$

130. $\int e^{ax}\cdot sin^n bx\,dx=\frac{1}{a^2+b^2n^2}e^{ax}\cdot sin^{n-1}bx(a\cdot sin bx-nb\cdot cos bx)$

$$+\frac{n\cdot(n-1)b^2}{a^2+b^2n^2}\int e^{ax}\cdot sin^{n-2}bx\,dx$$

证明：$\int e^{ax}\cdot sin^n bx\,dx=\int e^{ax}\cdot sin^{n-2}bx\cdot sin^2 bx\,dx=\int e^{ax}\cdot sin^{n-2}bx\cdot(1-cos^2 bx)\,dx$

$$=\int e^{ax}\cdot sin^{n-2}bx\,dx-\int e^{ax}\cdot sin^{n-2}bx\cdot cos^2 bx\,dx \quad ①$$

又 $\int e^{ax}\cdot sin^{n-2}bx\cdot cos^2 bx\,dx=\frac{1}{b\cdot(n-1)}\int e^{ax}\cdot cos bx\,d\,sin^{n-1}bx$

$$=\frac{1}{b\cdot(n-1)}\cdot e^{ax}\cdot cos bx\cdot sin^{n-1}bx-\frac{1}{b\cdot(n-1)}\int sin^{n-1}bx\,d(e^{ax}\cdot cos bx) \quad ②$$

又 $\int sin^{n-1}bx\,d(e^{ax}\cdot cos bx)=\int sin^{n-1}bx\,(a\cdot e^{ax}\cdot cos bx-b\cdot sin bx\cdot e^{ax})dx$

$$=a\int e^{ax}\cdot sin^{n-1}bx\cdot cos bx\,dx-b\int sin^n bx\cdot e^{ax}dx \quad ③$$

又 $\int e^{ax}\cdot sin^{n-1}bx\cdot cos bx\,dx=\frac{1}{b}\int e^{ax}\cdot sin^{n-1}bx\,d\,sin bx$

$$=\frac{1}{b}\cdot e^{ax}\cdot sin^n bx-\frac{1}{b}\int sin bx\,d(e^{ax}\cdot sin^{n-1}bx)$$

$$=\frac{1}{b}\cdot e^{ax}\cdot sin^n bx-\frac{1}{b}\int sin bx[a\cdot e^{ax}\cdot sin^{n-1}bx+b\cdot(n-1)sin^{n-2}bx\cdot cos bx\cdot e^{ax}]dx$$

$$=\frac{1}{b}\cdot e^{ax}\cdot sin^n bx-\frac{a}{b}\int sin^n bx\cdot e^{ax}dx-(n-1)\int e^{ax}\cdot sin^{n-1}bx\cdot cos bx\,dx$$

移项并整理得：$\int e^{ax}\cdot sin^{n-1}bx\cdot cos bx\,dx=\frac{1}{bn}\cdot e^{ax}\cdot sin^n bx-\frac{a}{bn}\int sin^n bx\cdot e^{ax}dx \quad ④$

将④式代入③式的得：$\int sin^{n-1}bx\,d(e^{ax}\cdot cos bx)$

$$=\frac{a}{bn}\cdot e^{ax}\cdot sin^n bx-\frac{a^2}{bn}\int sin^n bx\cdot e^{ax}dx-b\int sin^n bx\cdot e^{ax}dx$$

$$=\frac{a}{bn}\cdot e^{ax}\cdot sin^n bx-\frac{a^2+b^2n}{bn}\int sin^n bx\cdot e^{ax}dx \quad ⑤$$

将⑤式代入②式得：$\int e^{ax}\cdot sin^{n-2}bx\cdot cos^2 bx\,dx=\frac{1}{b\cdot(n-1)}\cdot e^{ax}\cdot cos bx\cdot sin^{n-1}bx$

$$-\frac{a}{b^2\cdot n\cdot(n-1)}\cdot e^{ax}\cdot sin^n bx+\frac{a^2+b^2n}{b^2\cdot n\cdot(n-1)}\int sin^n bx\cdot e^{ax}dx \quad ⑥$$

将⑥式代入①式得：$\int e^{ax}\cdot sin^n bx\,dx=\int e^{ax}\cdot sin^{n-2}bx\,dx-\frac{1}{b\cdot(n-1)}\cdot e^{ax}\cdot cos bx\cdot sin^{n-1}bx$

$$+\frac{a}{b^2\cdot n\cdot(n-1)}\cdot e^{ax}\cdot sin^n bx-\frac{a^2+b^2n}{b^2\cdot n\cdot(n-1)}\int sin^n bx\cdot e^{ax}dx$$

移项并整理得：$\int e^{ax}\cdot sin^n bx\,dx$

$$=\frac{n\cdot(n-1)b^2}{a^2+b^2n^2}\left[\int e^{ax}\cdot sin^{n-2}bx\,dx-\frac{1}{b\cdot(n-1)}\cdot e^{ax}\cdot cos bx\cdot sin^{n-1}bx+\frac{1}{n\cdot(n-1)b^2}\cdot e^{ax}\cdot sin^n bx\right]$$

$$=\frac{n\cdot(n-1)b^2}{a^2+b^2n^2}\cdot\int e^{ax}\cdot sin^{n-2}bx\,dx-\frac{bn}{a^2+b^2n^2}\cdot e^{ax}\cdot cos bx\cdot sin^{n-1}bx+\frac{a}{a^2+b^2n^2}\cdot e^{ax}\cdot sin^n bx$$

$$=\frac{1}{a^2+b^2n^2}\cdot e^{ax}\cdot sin^{n-1}bx(a\cdot sin bx-nb\cdot cos bx)$$

$$+\frac{n\cdot(n-1)b^2}{a^2+b^2n^2}\int e^{ax}\cdot sin^{n-2}bx\,dx$$

131. $$\int e^{ax}\cdot\cos^n bx\,dx=\frac{1}{a^2+b^2n^2}\cdot e^{ax}\cdot\cos^{n-1}bx(a\cdot\cos bx+nb\cdot\sin bx)+\frac{n\cdot(n-1)b^2}{a^2+b^2n^2}\int e^{ax}\cdot\cos^{n-2}bx\,dx$$

证明：$$\int e^{ax}\cdot\cos^n bx\,dx=\int e^{ax}\cdot\cos^{n-2}bx\cdot\cos^2 bx\,dx=\int e^{ax}\cdot\cos^{n-2}bx\cdot(1-\sin^2 bx)\,dx$$

$$=\int e^{ax}\cdot\cos^{n-2}bx\,dx-\int e^{ax}\cdot\cos^{n-2}bx\cdot\sin^2 bx\,dx \qquad ①$$

又 $$\int e^{ax}\cdot\cos^{n-2}bx\cdot\sin^2 bx\,dx=\frac{1}{b\cdot(1-n)}\int e^{ax}\cdot\sin bx\,d\cos^{n-1}bx$$

$$=\frac{1}{b\cdot(1-n)}\cdot e^{ax}\cdot\sin bx\cdot\cos^{n-1}bx-\frac{1}{b\cdot(1-n)}\int\cos^{n-1}bx\,d(e^{ax}\cdot\sin bx) \qquad ②$$

又 $$\int\cos^{n-1}bx\,d(e^{ax}\cdot\sin bx)=\int\cos^{n-1}bx\,(a\cdot e^{ax}\cdot\sin bx+b\cdot\cos bx\cdot e^{ax})dx$$

$$=a\int e^{ax}\cdot\cos^{n-1}bx\cdot\sin bx\,dx+b\int\cos^n bx\cdot e^{ax}dx \qquad ③$$

又 $$\int e^{ax}\cdot\cos^{n-1}bx\cdot\sin bx\,dx=-\frac{1}{b}\int e^{ax}\cdot\cos^{n-1}bx\,d\cos bx$$

$$=-\frac{1}{b}\cdot e^{ax}\cdot\cos^n bx+\frac{1}{b}\int\cos bx\,d(e^{ax}\cdot\cos^{n-1}bx)$$

$$=-\frac{1}{b}\cdot e^{ax}\cdot\cos^n bx+\frac{1}{b}\int\cos bx[a\cdot e^{ax}\cdot\cos^{n-1}bx-b\cdot(n-1)\cos^{n-2}bx\cdot\sin bx\cdot e^{ax}]dx$$

$$=-\frac{1}{b}\cdot e^{ax}\cdot\cos^n bx+\frac{a}{b}\int\cos^n bx\cdot e^{ax}dx-(n-1)\int e^{ax}\cdot\cos^{n-1}bx\cdot\sin bx\,dx$$

移项并整理得：$$\int e^{ax}\cdot\cos^{n-1}bx\cdot\sin bx\,dx=-\frac{1}{bn}\cdot e^{ax}\cdot\cos^n bx+\frac{a}{bn}\int\cos^n bx\cdot e^{ax}dx \qquad ④$$

将④式代入③式的得：$$\int\cos^{n-1}bx\,d(e^{ax}\cdot\sin bx)$$

$$=-\frac{a}{bn}\cdot e^{ax}\cdot\cos^n bx+\frac{a^2}{bn}\int\cos^n bx\cdot e^{ax}dx+b\int\cos^n bx\cdot e^{ax}dx$$

$$=-\frac{a}{bn}\cdot e^{ax}\cdot\cos^n bx+\frac{a^2+b^2n}{bn}\int\cos^n bx\cdot e^{ax}dx \qquad ⑤$$

将⑤式代入②式得：$$\int e^{ax}\cdot\cos^{n-2}bx\cdot\sin^2 bx\,dx=\frac{1}{b\cdot(1-n)}\cdot e^{ax}\cdot\sin bx\cdot\cos^{n-1}bx$$

$$+\frac{a}{b^2\cdot n\cdot(1-n)}\cdot e^{ax}\cdot\cos^n bx-\frac{a^2+b^2n}{b^2\cdot n\cdot(1-n)}\int\cos^n bx\cdot e^{ax}dx \qquad ⑥$$

将⑥式代入①式得：$$\int e^{ax}\cdot\cos^n bx\,dx=\int e^{ax}\cdot\cos^{n-2}bx\,dx-\frac{1}{b\cdot(1-n)}\cdot e^{ax}\cdot\sin bx\cdot\cos^{n-1}bx$$

$$+\frac{a}{b^2\cdot n\cdot(n-1)}\cdot e^{ax}\cdot\cos^n bx-\frac{a^2+b^2n}{b^2\cdot n\cdot(n-1)}\int\cos^n bx\cdot e^{ax}dx$$

移项并整理得：$\int e^{ax}\cdot\cos^n bx\,dx$

$$=\frac{n\cdot(1-n)b^2}{-a^2-b^2n^2}\left[\int e^{ax}\cdot\cos^{n-2}bx\,dx-\frac{1}{b\cdot(1-n)}\cdot e^{ax}\cdot\sin bx\cdot\cos^{n-1}bx-\frac{a}{n\cdot(1-n)b^2}\cdot e^{ax}\cdot\cos^n bx\right]$$

$$=\frac{n\cdot(n-1)b^2}{a^2+b^2n^2}\cdot\int e^{ax}\cdot\cos^{n-2}bx\,dx+\frac{bn}{a^2+b^2n^2}\cdot e^{ax}\cdot\sin bx\cdot\cos^{n-1}bx+\frac{a}{a^2+b^2n^2}\cdot e^{ax}\cdot\cos^n bx$$

$$=\frac{1}{a^2+b^2n^2}\cdot e^{ax}\cdot\cos^{n-1}bx(a\cdot\cos bx+nb\cdot\sin bx)+\frac{n\cdot(n-1)b^2}{a^2+b^2n^2}\int e^{ax}\cdot\cos^{n-2}bx\,dx$$

（十四）含有对数函数的积分（132~136）

132. $\int ln\, xdx = x \cdot ln\, x - x + C$

证明：$\int ln\, xdx = x \cdot ln\, x - \int x\, d\, ln\, x$

$$= x \cdot ln\, x - \int x \cdot \frac{1}{x}\, dx$$

$$= x \cdot ln\, x - \int dx$$

$$= x \cdot ln\, x - x + C$$

133. $\int \frac{dx}{x \cdot ln\, x} dx = ln \left| ln\, x \right| + C$

证明：$\int \frac{dx}{x \cdot ln\, x} dx = \int \frac{1}{ln\, x} d\, ln\, x$

$$= ln \left| ln\, x \right| + C$$

$\boxed{\text{提示：}(ln\, x)' = \frac{1}{x}}$

134. $\int x^n \cdot ln\, x\ dx = \frac{1}{n+1} \cdot x^{n+1}(ln\, x - \frac{1}{n+1}) + C$

证明：$\int x^n \cdot ln\, x\ dx = \int \frac{ln\, x}{n+1} \cdot (n+1) \cdot x^n dx$

$$= \int \frac{ln\, x}{n+1}\ dx^{n+!}$$

$$= \frac{ln\, x}{n+1} \cdot x^{n+1} - \frac{1}{n+1} \int x^{n+1} d(ln\, x)$$

$$= \frac{ln\, x}{n+1} \cdot x^{n+1} - \frac{1}{n+1} \int x^n dx$$

$$= \frac{ln\, x}{n+1} \cdot x^{n+1} - (\frac{1}{n+1})^2 \cdot x^{n+1} + C$$

$$= \frac{1}{n+1} \cdot x^{n+1}(ln\, x - \frac{1}{n+1}) + C$$

135. $\int (lnx)^n dx = x\cdot(lnx)^n - n\int (lnx)^{n-1}dx$

$$= x\sum_{k=0}^{n}(-1)^{n-k}\cdot\frac{n!}{k!}\cdot(lnx)^k$$

证明：$\int (lnx)^n dx = x\cdot(lnx)^n - \int xd(\ln x)^n$

$$= x\cdot(lnx)^n - \int x\cdot n\cdot(\ln x)^{n-1}\cdot\frac{1}{x}dx$$

$$= x\cdot(lnx)^n - n\int (lnx)^{n-1}dx$$

$$= x\cdot(lnx)^n - n\cdot x\cdot(lnx)^{n-1} + n\int xd(\ln x)^{n-1}$$

$$= x\cdot(lnx)^n - n\cdot x\cdot(lnx)^{n-1} + n\cdot(n-1)\int (lnx)^{n-2}dx$$

$$= x\cdot(lnx)^n - n\cdot x\cdot(lnx)^{n-1} + n\cdot(n-1)\cdot x\cdot(lnx)^{n-2} - n\cdot(n-1)\cdot(n-2)\int (lnx)^{n-3}dx$$

. . . .

$$= x\cdot(lnx)^n - n\cdot x\cdot(lnx)^{n-1} + n\cdot(n-1)\cdot x\cdot(lnx)^{n-2} - n\cdot(n-1)\cdot(n-2)(lnx)^{n-3}$$

$$+\cdots\cdots+(-1)^{n-k}\cdot n\cdot(n-1)\cdot(n-2)\cdots\cdots(n-k+1)\cdot(lnx)^{n-k}+\cdots\cdots$$

$$+(-1)^2\cdot n\cdot(n-1)\cdot(n-2)\cdots\cdots5\times4\times3\cdot(lnx)^{3-1}\cdot x$$

$$+(-1)^1\cdot n\cdot(n-1)\cdot(n-2)\cdots\cdots4\times3\times2\cdot(lnx)^{2-1}\cdot x$$

$$+(-1)^0\cdot n\cdot(n-1)\cdot(n-2)\cdots\cdots3\times2\times1\cdot(lnx)^{1-1}\cdot x$$

$$= x\sum_{k=0}^{n}(-1)^{n-k}\cdot\frac{n!}{k!}\cdot(lnx)^k$$

136. $\int x^m\cdot(\ln x)^{\mathrm{n}}\ dx = \frac{1}{m+1}\cdot x^{m+1}\cdot(\ln x)^{\mathrm{n}} - \frac{\mathrm{n}}{m+1}\int x^m\cdot(\ln x)^{\mathrm{n}-1}\ dx$

证明：$\int x^m\cdot(\ln x)^{\mathrm{n}}\ dx = \frac{1}{m+1}\int(\ln x)^{\mathrm{n}}d(x^{m+1})$

$$= \frac{1}{m+1}\cdot x^{m+1}\cdot(\ln x)^{\mathrm{n}} - \frac{1}{m+1}\int x^{m+1}\ d(\ln x)^{\mathrm{n}}$$

$$= \frac{1}{m+1}\cdot x^{m+1}\cdot(\ln x)^{\mathrm{n}} - \frac{\mathrm{n}}{m+1}\int x^{m+1}\cdot(\ln x)^{\mathrm{n}-1}\cdot\frac{1}{x}dx$$

$$= \frac{1}{m+1}\cdot x^{m+1}\cdot(\ln x)^{\mathrm{n}} - \frac{\mathrm{n}}{m+1}\int x^m\cdot(\ln x)^{\mathrm{n}-1}\ dx$$

（十五）含有双曲函数的积分（137~141）

137. $\int shx\,dx = chx + C$

证明：$\because\ (chx)' = shx$，即chx为shx的原函数

$$\therefore \int shx\,dx = \int d\,chx$$

$$= chx + C$$

138. $\int ch\,x\,dx = shx + C$

证明：$\because\ (shx)' = chx$，即shx为chx的原函数

$$\therefore \int ch\,x\,dx = \int d\,shx$$

$$= shx + C$$

139. $\int th\,x\,dx = ln\,chx + C$

证明：$\int th\,x\,dx = \int \frac{shx}{chx}dx$

$$= \int \frac{1}{chx}d\,chx$$

$$= ln\,chx + C$$

140. $\int sh^2x\,dx = -\frac{x}{2} + \frac{1}{4}sh\,2x + C$

证明：$\int sh^2x\,dx = \int\left(\frac{e^x - e^{-x}}{2}\right)^2 dx$

$$= \frac{1}{4}\int(e^{2x} + e^{-2x} - 2)dx$$

$$= \frac{e^{2x}}{8} - \frac{e^{-2x}}{8} - \frac{x}{2} + C$$

$$= -\frac{x}{2} + \frac{1}{4}\cdot\frac{e^{2x} - e^{-2x}}{2} + C$$

$$= -\frac{x}{2} + \frac{1}{4}\cdot sh\,2x + C$$

提示：$chx = \frac{e^x + e^{-x}}{2}$ (双曲余弦)

$shx = \frac{e^x - e^{-x}}{2}$ (双曲余弦)

141. $\int ch^2x\,dx = \frac{x}{2} + \frac{1}{4}\cdot sh\,2x + C$

证明：$\int ch^2x\,dx = \int\left(\frac{e^x + e^{-x}}{2}\right)^2 dx$

$$= \frac{1}{4}\int(e^{2x} + e^{-2x} + 2)dx$$

$$= \frac{e^{2x}}{8} - \frac{e^{-2x}}{8} + \frac{x}{2} + C$$

$$= \frac{x}{2} + \frac{1}{4}\cdot\frac{e^{2x} - e^{-2x}}{2} + C$$

$$= \frac{x}{2} + \frac{1}{4}\cdot sh\,2x + C$$

提示：$chx = \frac{e^x + e^{-x}}{2}$ (双曲余弦)

$shx = \frac{e^x - e^{-x}}{2}$ (双曲余弦)

（十六）定积分（142~147）

142. $\int_{-\pi}^{\pi} \cos nx\, dx = \int_{-\pi}^{\pi} \sin nx\, dx = 0$

证明①：$\int_{-\pi}^{\pi} \cos nx\, dx = \frac{1}{n}\int_{-\pi}^{\pi} \cos nx\, d(nx)$

$$= \frac{1}{n}\cdot(\sin nx|_{-\pi}^{\pi})$$
$$= \frac{1}{n}\cdot\sin\,(n\pi) - \frac{1}{n}\cdot\sin\,(-n\pi)$$
$$= \frac{2}{n}\cdot\sin\,(n\pi)$$
$$= 0$$

证明②：$\int_{-\pi}^{\pi} \sin nx\, dx = \frac{1}{n}\int_{-\pi}^{\pi} \sin nx\, d(nx)$

$$= -\frac{1}{n}\cdot(\cos nx|_{-\pi}^{\pi})$$
$$= -\frac{1}{n}\cdot\cos\,(n\pi) + \frac{1}{n}\cdot\cos\,(-n\pi)$$
$$= 0$$

综合证明①②得：$\int_{-\pi}^{\pi} \cos nx\, dx = \int_{-\pi}^{\pi} \sin nx\, dx = 0$

143. $\int_{-\pi}^{\pi} \cos mx\cdot\sin nx\, dx = 0$ $\boxed{\text{公式100：}\int \sin ax\cdot\cos bx\, dx = -\frac{1}{2(a+b)}\cdot\cos(a+b)x - \frac{1}{2(a-b)}\cdot\cos(a-b)x + C}$

证明：1. 当$m \neq n$时

$$\int_{-\pi}^{\pi} \cos mx\cdot\sin nx\, dx = -\frac{1}{2(m+n)}\cdot\cos(m+n)x\Big|_{-\pi}^{\pi} - \frac{1}{2(n-m)}\cos(n-m)x\Big|_{-\pi}^{\pi}$$
$$= -\frac{1}{2(m+n)}[\cos(m+n)\pi - \cos(m+n)\pi] - \frac{1}{2(n-m)}[\cos(n-m)\pi - \cos(n-m)(-\pi)]$$
$$= 0 + 0 = 0$$

2. 当$m = n$时

$$\int_{-\pi}^{\pi} \cos mx\cdot\sin nx\, dx = \int_{-\pi}^{\pi} \cos mx\cdot\sin mx\, dx$$ $\boxed{\text{提示：}\sin 2x = 2\cdot\sin x\cdot\cos x}$

$$= \frac{1}{2m}\int_{-\pi}^{\pi} \sin 2mx\, d(mx)$$
$$= \frac{1}{4m}\int_{-\pi}^{\pi} \sin 2mx\, d(2mx)$$
$$= -\frac{1}{4m}\cdot\cos 2mx\Big|_{-\pi}^{\pi}$$
$$= -\frac{1}{4m}\cdot[\cos 2m\pi - \cos(-2m\pi)]$$
$$= 0$$

综合讨论 1，2 得：$\int_{-\pi}^{\pi} \cos nx\, dx = \int_{-\pi}^{\pi} \cos mx\cdot\sin nx\, dx = 0$

144. $\int_{-\pi}^{\pi} \cos mx \cdot \cos nx \, dx = \begin{cases} 0\ , & m \neq n \\ \pi, & m = n \end{cases}$

证明：1. 当$m \neq n$时

$$\int_{-\pi}^{\pi} \cos mx \cdot \cos nx \, dx = \frac{1}{2(m+n)} \cdot \sin\ (m+n)x \Big|_{-\pi}^{\pi} - \frac{1}{2(m-n)} \sin\ (m-n)x \Big|_{-\pi}^{\pi}$$

$$= \frac{1}{2(m+n)}[\sin\ (m+n)\pi - \sin\ (m+n)(-\pi)] - \frac{1}{2(m-n)}[\sin\ (m-n)\pi + \sin\ (m-n)(-\pi)]$$

$$= 0 - 0 = 0$$

$\boxed{公式102: \int \cos ax \cdot \cos bx \, dx = \frac{1}{2(a+b)} \cdot \sin(a+b)x + \frac{1}{2(a-b)} \cdot \sin(a-b)x + C}$

2. 当$m = n$时

$$\int_{-\pi}^{\pi} \cos mx \cdot \cos nx \, dx = \int_{-\pi}^{\pi} \cos mx \cdot \cos mx \, dx$$

$$= \frac{1}{m}\int_{-\pi}^{\pi} \cos^2 mx \, d(mx)$$

$\boxed{公式94: \int \cos^2 x \, dx = \frac{x}{2} + \frac{1}{4} \cdot \sin 2x + C}$

$$= \frac{1}{4m} \cdot \sin 2mx \Big|_{-\pi}^{\pi} + \frac{1}{2m} \cdot mx \Big|_{-\pi}^{\pi}$$

$$= \frac{1}{4m} \cdot [\sin 2m\pi - \sin\ (-2m\pi)] + \frac{\pi}{2} + \frac{\pi}{2}$$

$$= \pi$$

综合讨论 1 , 2 得：$\int_{-\pi}^{\pi} \cos mx \cdot \cos nx \, dx = \begin{cases} 0\ , & m \neq n \\ \pi, & m = n \end{cases}$

145. $\int_{-\pi}^{\pi} \sin mx \cdot \sin nx \, dx = \begin{cases} 0\ , & m \neq n \\ \pi, & m = n \end{cases}$

证明：1. 当$m \neq n$时

$$\int_{-\pi}^{\pi} \sin mx \cdot \sin nx \, dx = -\frac{1}{2(m+n)} \cdot \sin\ (m+n)x \Big|_{-\pi}^{\pi} + \frac{1}{2(m-n)} \sin\ (m-n)x \Big|_{-\pi}^{\pi}$$

$$= -\frac{1}{2(m+n)}[\sin\ (m+n)\pi - \sin\ (m+n)(-\pi)] + \frac{1}{2(m-n)}[\sin\ (m-n)\pi - \sin\ (m-n)(-\pi)]$$

$$= 0 + 0 = 0$$

$\boxed{公式101: \int \sin ax \cdot \sin bx \, dx = -\frac{1}{2(a+b)} \cdot \sin(a+b)x + \frac{1}{2(a-b)} \cdot \sin(a-b)x + C}$

2. 当$m = n$时

$$\int_{-\pi}^{\pi} \sin mx \cdot \sin nx \, dx = \int_{-\pi}^{\pi} \sin^2 mx \, dx$$

$\boxed{公式93: \int \sin^2 x \, dx = \frac{x}{2} - \frac{1}{4} \cdot \sin 2x + C}$

$$= \frac{1}{m}\int_{-\pi}^{\pi} \sin^2 mx \, d(mx)$$

$$= \frac{1}{2m} \cdot mx \Big|_{-\pi}^{\pi} - \frac{1}{4m} \cdot \sin 2mx \Big|_{-\pi}^{\pi}$$

$$= -\frac{1}{4m} \cdot [\sin 2m\pi - \sin\ (-2m\pi)] + \frac{\pi}{2} + \frac{\pi}{2}$$

$$= \pi$$

综合讨论 1 , 2 得：$\int_{-\pi}^{\pi} \sin mx \cdot \sin nx \, dx = \begin{cases} 0\ , & m \neq n \\ \pi, & m = n \end{cases}$

146. $\int_0^{\pi} sin\,mx\cdot sin\,nx\;dx=\int_0^{\pi} cos\,mx\cdot cos\,nx\;dx\begin{cases}0\,,\;m\neq n\\ \dfrac{\pi}{2},\;m=n\end{cases}$

证明：1. 当$m\neq n$时

$$\int_0^{\pi} sin\,mx\cdot sin\,nx\;dx=-\frac{1}{2(m+n)}\cdot sin\;(m+n)x\Big|_0^{\pi}+\frac{1}{2(m-n)}sin\;(m-n)x\Big|_0^{\pi}$$
$$=-\frac{1}{2(m+n)}[sin\;(m+n)\pi-sin\;0]+\frac{1}{2(m-n)}[sin\;(m-n)\pi-sin\;0]$$
$$=0+0=0$$

$$\int_0^{\pi} cos\,mx\cdot cos\,nx\;dx=\frac{1}{2(m+n)}\cdot sin\;(m+n)x\Big|_0^{\pi}+\frac{1}{2(m-n)}sin\;(m-n)x\Big|_0^{\pi}$$
$$=\frac{1}{2(m+n)}[sin\;(m+n)\pi-sin\;0]+\frac{1}{2(m-n)}[sin\;(m-n)\pi+sin\;0]$$
$$=0+0=0$$

2. 当$m=n$时

$$\int_0^{\pi} sin\,mx\cdot sin\,nx\;dx\;=\int_0^{\pi} sin^2\,mx\;dx$$
$$=\frac{1}{m}\int_0^{\pi} sin^2\,mx\;d(mx)$$
$$=\frac{1}{2m}\cdot mx\Big|_0^{\pi}-\frac{1}{4m}\cdot sin\,2mx\Big|_0^{\pi}$$
$$=-\frac{1}{4m}\cdot[sin\,2m\pi-sin\;0]+\frac{\pi}{2}+0$$
$$=\frac{\pi}{2}$$

$$\int_0^{\pi} cos\,mx\cdot cos\,nx\;dx=\int_0^{\pi} cos\,mx\cdot cos\,mx\;dx$$
$$=\frac{1}{m}\int_0^{\pi} cos^2\,mx\;d(mx)$$
$$=\frac{1}{4m}\cdot sin\,2mx\Big|_0^{\pi}+\frac{1}{2m}\cdot mx\Big|_0^{\pi}$$
$$=\frac{1}{4m}\cdot[sin\,2m\pi-sin\;0]+\frac{\pi}{2}+0$$
$$=\frac{\pi}{2}$$

综合讨论 1，2 得：$\int_0^{\pi} sin\,mx\cdot sin\,nx\;dx=\int_0^{\pi} cos\,mx\cdot cos\,nx\;dx\begin{cases}0\,,\;m\neq n\\ \dfrac{\pi}{2},\;m=n\end{cases}$

以上所用公式：

公式101：$\int sin\,ax\cdot sin\,bx\;dx=-\frac{1}{2(a+b)}\cdot sin\,(a+b)x+\frac{1}{2(a-b)}\cdot sin\,(a-b)x+C$

公式102：$\int cos\,ax\cdot cos\,bx\;dx=\frac{1}{2(a+b)}\cdot sin\,(a+b)x+\frac{1}{2(a-b)}\cdot sin\,(a-b)x+C$

公式93：$\int sin^2 x\;dx=\frac{x}{2}-\frac{1}{4}\cdot sin\,2x+C$

公式94：$\int cos^2\;x\;dx=\frac{x}{2}+\frac{1}{4}\cdot sin\,2x+C$

147. $I_n=\int_0^{\frac{\pi}{2}}\sin^n x\,dx=\int_0^{\frac{\pi}{2}}\cos^n x\,dx$

$$I_n=\frac{n-1}{n}I_{n-2}$$

$$=\begin{cases}\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\cdots\frac{4}{5}\cdot\frac{2}{3}\ (n\text{为大于}1\text{的正奇数}),\ I_1=1\\ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\cdots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\ (n\text{为正偶数}),\ I_0=\frac{\pi}{2}\end{cases}$$

证明①：$I_n=\int_0^{\frac{\pi}{2}}\sin^n x\,dx=-\frac{1}{n}\cdot\sin^{n-1}x\cdot\cos x\Big|_0^{\frac{\pi}{2}}+\frac{n-1}{n}\int_0^{\frac{\pi}{2}}\sin^{n-2}x\,dx$

$$=-\frac{1}{n}(\sin^{n-1}\frac{\pi}{2}\cdot\cos\frac{\pi}{2}-\sin^{n-1}0\cdot\cos 0)+\frac{n-1}{n}\int_0^{\frac{\pi}{2}}\sin^{n-2}x\,dx$$

$$=\frac{n-1}{n}\int_0^{\frac{\pi}{2}}\sin^{n-2}x\,dx=\frac{n-1}{n}I_{n-2}$$

当n为正奇数时

$$I_n=\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\cdots\frac{4}{5}\cdot\frac{2}{3}\cdot\int_0^{\frac{\pi}{2}}\sin x\,dx$$

$$=\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\cdots\frac{4}{5}\cdot\frac{2}{3}\cdot(-\cos x)\Big|_0^{\frac{\pi}{2}}$$

$$=\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\cdots\frac{4}{5}\cdot\frac{2}{3}\cdot 1$$

特别的，当$n=1$时，$I_n=\int_0^{\frac{\pi}{2}}\sin x\,dx=(-\cos x)\Big|_0^{\frac{\pi}{2}}=1$

当n为正偶数时

$$I_n=\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\cdots\frac{3}{4}\cdot\frac{1}{2}\cdot\int_0^{\frac{\pi}{2}}\sin^0 x\,dx$$

$$=\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\cdots\frac{3}{4}\cdot\frac{1}{2}\cdot(x)\Big|_0^{\frac{\pi}{2}}$$

$$=\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\cdots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

特别的，当$n=0$时，$I_n=\int_0^{\frac{\pi}{2}}\sin^0 x\,dx=(x)\Big|_0^{\frac{\pi}{2}}=\frac{\pi}{2}$

证明②：$I_n=\int_0^{\frac{\pi}{2}}\cos^n x\,dx\cdots\cdots$亦同理可证

附录：常数和基本初等函数导数公式

1. $(C)' = 0$ $\quad(C为常数)$

2. $(x^{\mu})' = \mu \cdot x^{\mu-1}$ $\quad(x \neq 0)$

3. $(sinx)' = cosx$

4. $(cosx)' = -sinx$

5. $(tanx)' = sec^2 x$

6. $(cotx)' = -csc^2 x$

7. $(secx)' = secx \cdot tanx$

8. $(cscx)' = -cscx \cdot cotx$

9. $(a^x)' = a^x \cdot lna$ $\quad(a为常数)$

10. $(e^x)' = e^x$

11. $(log_a x)' = \dfrac{1}{x \cdot lna}$ $\quad(a > 0)$

12. $(lnx)' = \dfrac{1}{x}$

13. $(arcsinx)' = \dfrac{1}{\sqrt{1-x^2}}$

14. $(arccosx)' = \dfrac{1}{-\sqrt{1-x^2}}$

15. $(arctanx)' = \dfrac{1}{1+x^2}$

16. $(arccotx)' = -\dfrac{1}{1+x^2}$