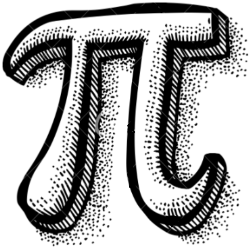
scidata



Gradiente

Conjuntos de nivel

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Definición (Derivada de un campo escalar)

Sea $f : \mathbb{R}^n \to \mathbb{R}$ un campo escalar. Definimos la derivada de f a partir del punto $a \in \mathbb{R}^n$ en la dirección $y \in \mathbb{R}^n$, como

$$f'(a; y) = \lim_{h \to 0} \frac{f(a + hy) - f(a)}{h}, \qquad (1)$$

siempre que dicho límite exista.

$\mathbb{R}^n$

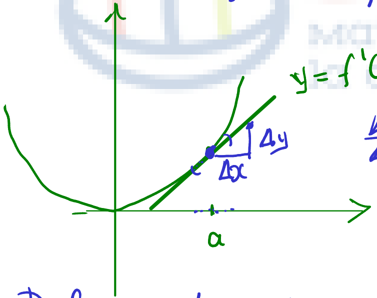
$a = (a_1, \dots, a_n)$

$y = (y_1, \dots, y_n)$

Nota: Si el vector dirección y es de norma 1, la derivada anterior se llama **derivada direccional, en la dirección de** y y el cálculo se simplifica con el gradiente.

$f(x) = x^2$

$f : \mathbb{R} \to \mathbb{R}$



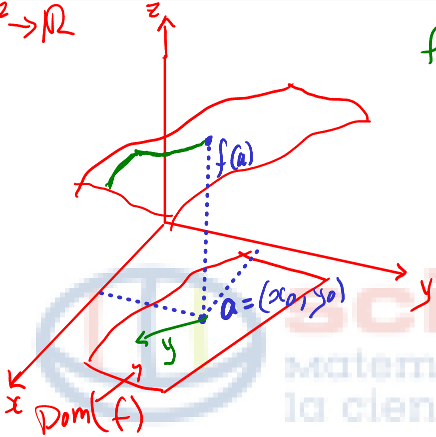
$y = f'(a)(x-a) + f(a)$ Ecuación de la recta tangenta a f en $x = a$.

$\dfrac{\Delta y}{\Delta x}$

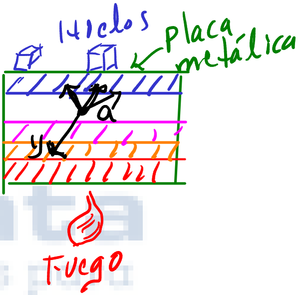
$f'(a) \longmapsto$ Razón de cambio o pendiente de la recta tangente.

Diferencial. "Aproximación lineal"

$f: \mathbb{R}^2 \to \mathbb{R}$



$$f'(a;y) = \lim_{h \to 0} \frac{f(a+hy) - f(a)}{h}$$



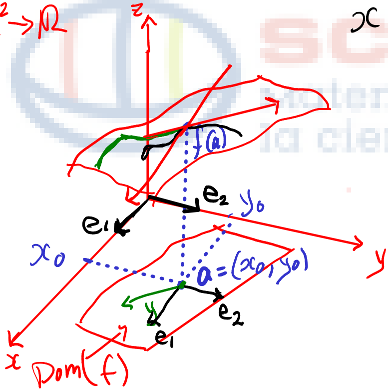
Definición (Derivadas parciales)

Sea $f : \mathbb{R}^n \to \mathbb{R}$ un campo escalar. Definimos la derivada parcial de f en x_i, $i = 1, 2, ..., n$, en punto $a \in \mathbb{R}^n$ como

$$\frac{\partial f}{\partial x_i} = \lim_{h \to 0} \frac{f(a + he_i) - f(a)}{h}, \tag{2}$$

siempre que dicho límite exista. El vector e_i es el vector coordenado donde todas las componentes son cero, excepto en la coordenada i que es 1.

$f: \mathbb{R}^2 \to \mathbb{R}$



$x \to e_1 = (1,0)$, $y \mapsto e_2 = (0,1)$

$\frac{\partial f}{\partial x} \Rightarrow$ f es una función que depende de al. menos 2 variables

$(e_1 = (1,0,0,0))$

$(f(x) \rightsquigarrow f'(x))$

Ej.: $f(x,y) = x^2 + 5y^3$

$$\frac{\partial f}{\partial x} = 2x \quad , \quad \frac{\partial f}{\partial y} = 15y^2 \qquad \frac{\partial f(x,y)}{\partial x}$$

partial derivative x^2+5y^3

NATURAL LANGUAGE | MATH INPUT

Input interpretation

partial derivatives	$x^2 + 5\,y^3$

Results

$$\frac{\partial}{\partial x}(x^2 + 5\,y^3) = 2\,x$$

$$\frac{\partial}{\partial y}(x^2 + 5\,y^3) = 15\,y^2$$

3D plots

Enlarge

$$\frac{\partial f}{\partial x}(1,0) = 2(1) = 2 .$$

$$\frac{\partial f}{\partial y}(1,0) = 0$$

diff

$f(x,y,z) = e^{-x^2 y} + sen(yz)$

partial derivative e^(-x^2y)+sin(yz)

NATURAL LANGUAGE MATH INPUT EXTE

Input interpretation

partial derivatives	$e^{-x^2 y} + \sin(y\,z)$

Results Approximate forms

$$\frac{\partial}{\partial x}\left(e^{-x^2 y} + \sin(y\,z)\right) = -2\,x\,y\,e^{-x^2 y} = \frac{\partial f}{\partial x}$$

$$\frac{\partial}{\partial y}\left(e^{-x^2 y} + \sin(y\,z)\right) = z\cos(y\,z) - x^2 e^{-x^2 y} = \frac{\partial f}{\partial y}$$

$$\frac{\partial}{\partial z}\left(e^{-x^2 y} + \sin(y\,z)\right) = y\cos(y\,z) = \frac{\partial f}{\partial z}$$

Differential

$\frac{\partial f}{\partial x}(1,-1,0) = 2e$

$\frac{\partial f}{\partial y}(1,-1,0) = -e$

$\frac{\partial f}{\partial z}(1,-1,0) = -1$

Definición (Gradiente)

Sea $f : \mathbb{R}^n \to \mathbb{R}$ un campo escalar. Definimos el gradiente de f en el punto a, denotado por $\nabla f(a)$, como

$$\nabla f(a) = \left(\frac{\partial f(a)}{\partial x_1}, \frac{\partial f(a)}{\partial x_2}, \dots, \frac{\partial f(a)}{\partial x_n} \right). \tag{3}$$

$a \in \mathbb{R}^n$

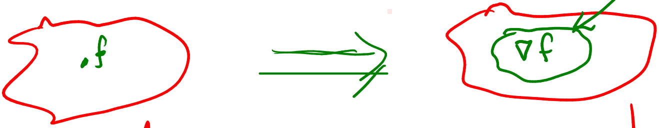
nabla, pero se lee gradiente de.

$f(x_1, x_2, \dots, x_n)$

$$\nabla f(x) = \left(\frac{\partial f(x)}{\partial x_1}, \dots, \frac{\partial f(x)}{\partial x_n} \right)$$ campo vectorial

$x \in \mathbb{R}^n$

campos conservativos

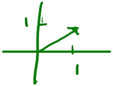


Campos escalares

Campos vectoriales

$$f(x,y) = x + y + 5$$

$$\frac{\partial f}{\partial x} = 1 \ , \ \frac{\partial f}{\partial y} = 1 \Rightarrow \nabla f(x,y) = (1,1)$$



gradient of x+y+5

NATURAL LANGUAGE | MATH INPUT

Input interpretation

$\operatorname{grad}(x + y + 5)$

Del operator form

$\nabla(x + y + 5)$

Result in 2D Cartesian coordi | Enlarge

$\operatorname{grad}(x + y + 5) = (1, 1)$

(x: first Cartesian coordinate | y: second Cartesian coordinate)

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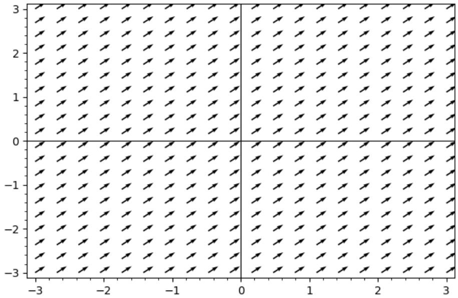
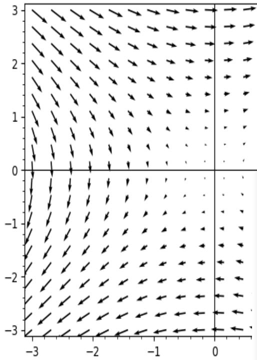
```
plot_vector_field((1,1),(x,-3,3),(y,-3,3))
```



$f(x,y) = xy \qquad \frac{\partial f}{\partial x} = y\,,\ \frac{\partial f}{\partial y} = x$

$\Rightarrow \nabla f(x,y) = (y, x)$

```
plot_vector_field((y,x),(x,-3,3),(y,-3,3))
```

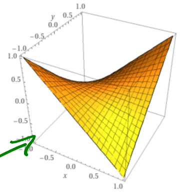


$f(x, y) = xy$

Geometric figure

hyperbolic paraboloid

3D plot



Gráfica de f

gradient of xy

NATURAL LANGUAGE

Input interpretation

grad(x y)

Del operator form

∇(x y)

Coordinate-free result

grad(x y) = y (grad x) + x

assuming x is a scalar-v
is a scalar-valued funct

Result in 2D Cartesian coordi

grad(x y) = (y, x)

(x: first Cartesian coordinate | y:

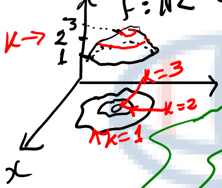
Definición (Conjuntos de nivel)

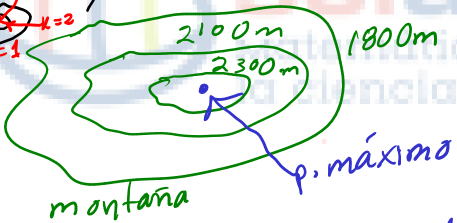
Dado un campo escalar $f : \mathbb{R}^n \to \mathbb{R}$*, definimos el conjunto de nivel, al nivel* $k \in \mathbb{R}$*, como el conjunto siguiente:*

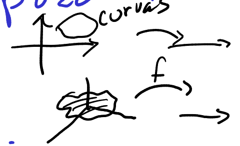
$$CN_k = \{x \in \mathbb{R}^n : f(x) = k\}.$$

Si $n = 2$*, dichos conjuntos de nivel se conocen como* **curvas de nivel** *(y* $CN_2 = CN$*),* $CN \subseteq \mathbb{R}^2$*, y si* $n = 3$ *se conocen como* **superficies de nivel** *(*$CN_3 = SN$*),* $SN \subseteq \mathbb{R}^3$*..*

$f : \mathbb{R}^2 \to \mathbb{R}$







$f : \mathbb{R}^2 \to \mathbb{R}$ curvas de nivel

$f : \mathbb{R}^3 \to \mathbb{R}$ superficies de nivel.

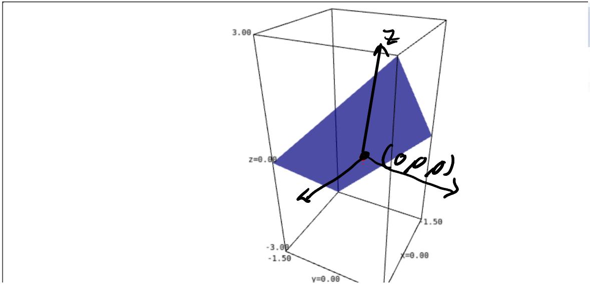
$f(x,y) = x + y \qquad z = f(x,y)$

$z = x + y \Rightarrow \underline{x + y - z = 0}$ E. lineal.

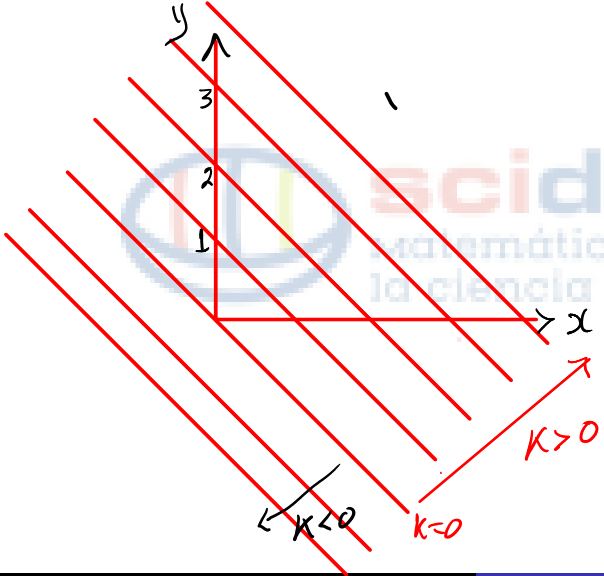
plano

1. Grafica del plano $f(x,y) = x + y$.

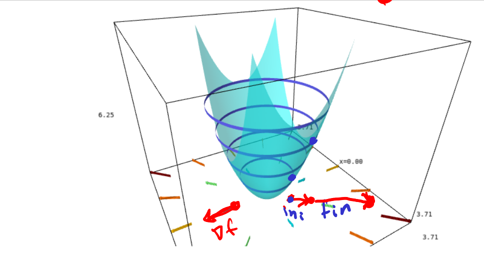
```
var('x,y,z')
f(x,y)=x+y
p=plot3d(f(x,y),(x,-1.5,1.5),(y,-1.5,1.5),color='blue',opacity=0.7)
show(p)
```



$$z=f(x,y)=\underbrace{x+y}_{\text{plano}}=k \qquad y=-x+k \qquad \underline{k\in\mathbb{R}}$$

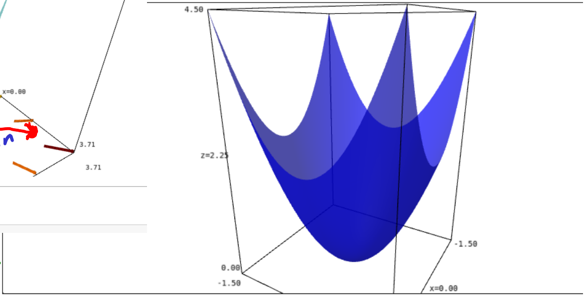


$$f(x,y) = \underbrace{x^2+y^2 = K}_{\text{cónica.}} \qquad C=(0,0) \quad r=\sqrt{K}, \quad \underline{K \geq 0}$$



áfica de $f(x,y) = -x^2 - y^2$.

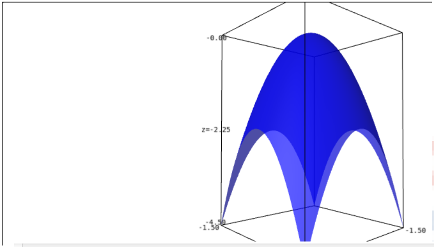
1.5),(y,-1.5,1.5),color='blue',opacity=0.7)



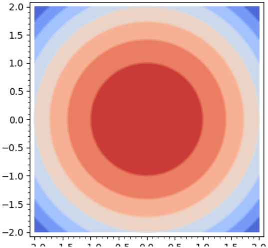
$f(x,y) = -x^2 - y^2 = k \Rightarrow x^2 + y^2 = -k$

$C = (0,0) \quad r = \sqrt{-k} \quad k \leq 0$

```
var('x,y,z')
f(x,y)=-x^2-y^2
p=plot3d(f(x,y),(x,-1.5,1.5),(y,-1.5,1.5),color='blue',opacity=0.7)
show(p)
```




```
4]: var("x,y")
p = contour_plot(-x^2-y^2, (x,-2,2), (y,-2,2),cmap="coolwarm")
show(p)
```



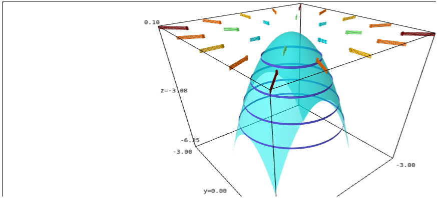
máx

mín

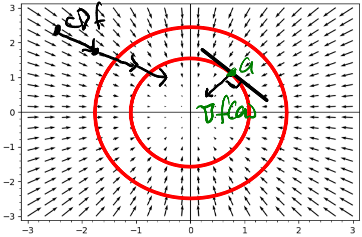
u

$-u$

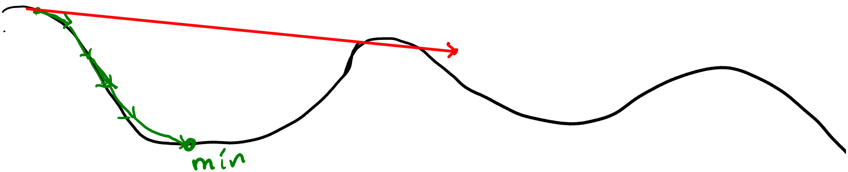
```
var('x,y,z')
f(x,y)=-x^2-y^2
g(x,y,z)=(-2*x,-2*y,0)
levels=[-1,-2,-3,-4]
epsilon=0.1
p=plot3d(f(x,y),(x,-2,2),(y,-1.5,1.5),color='cyan',opacity=0.5)
for h in levels:
    p+=implicit_plot3d(f(x,y)==h,(x,-2,2),(y,-2,2),(z,h,h+epsilon))
q=p+plot_vector_field3d(g,(x,-3,3),(y,-3,3),(z,0,0.1))
show(q)
```




```
5]: plot_vector_field((-2*x,-2*y),(x,-3,3),(y,-3,3))
5]:
```



Todos los elementos juntos de $f(x,y) = -x^2 - y^2$.


mín

Definición (Campo escalar diferenciable)

Sea $f : \mathbb{R}^n \to \mathbb{R}$ un campo escalar. Decimos que f es diferenciable en $a \in \mathbb{R}^n$, si existe una transformación lineal $T_a : \mathbb{R}^n \to \mathbb{R}$, y una función escalar E_a tal que

$$f(a+v) = f(a) + T_a(v) + E_a$$

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

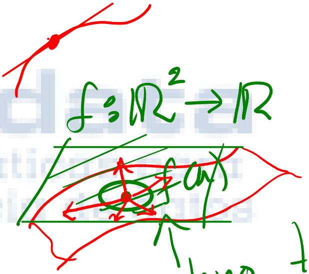
$$f'(a) \approx \frac{f(a+h) - f(a)}{h}$$

$$f'(a) \cdot h \approx f(a+h) - f(a)$$

$$f(a+h) = f(a) + f'(a) \cdot h + e.$$

Fórmula de Taylor de 1er orden

$f : \mathbb{R}^2 \to \mathbb{R}$



plano tangente

$$\underbrace{\text{Diferenciable}}_{\text{Transformación lineal}} \Rightarrow \underbrace{\text{derivable}}_{\text{número}} \Rightarrow \text{continua}$$

Teorema (Condición suficiente de diferenciabilidad)

Sea $f : \mathbb{R}^n \to \mathbb{R}$ *un campo escalar. Decimos que* f *es diferenciable en* a *con diferencial* T_a*, si* $\dfrac{\partial f(a)}{\partial x_i}$ *es continua para toda* $i = 1, 2, ..., n$ *y* $T_a(v) = \nabla f(a) \cdot v$.

Extra. $f : \mathbb{R}^n \to \mathbb{R}^m$

$f(x_1, \dots, x_n) = (f_1, f_2, \dots, f_m)$

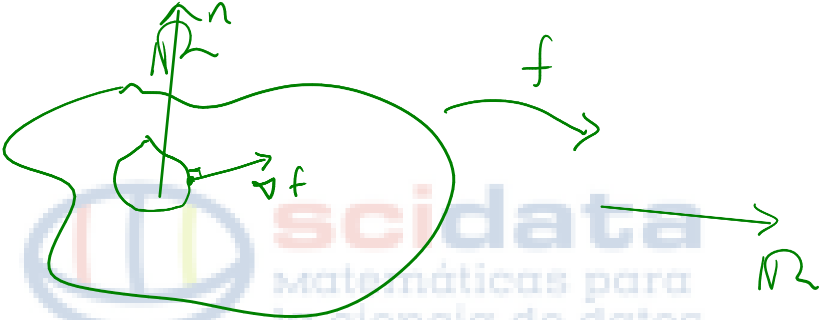
$A_T \in \mathcal{M}_{m \times n}$

$$A = J = \begin{bmatrix} \nabla f_1 \\ \nabla f_2 \\ \vdots \\ \nabla f_m \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Matriz Jacobiana

$|J| \to$ Jacobiano. (determinante de J)

$f: \mathbb{R}^n \to \mathbb{R}$

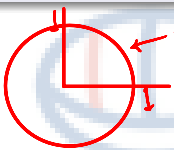


Teorema (Plano tangente)

Sea $f : \mathbb{R}^n \to \mathbb{R}$ *un campo escalar. Si* f *es diferenciable en* $a = (a_1, a_2, ..., a_n)$*, entonces existe un plano tangente en el punto* $f(a)$*, el cual está dado por*

$$\nabla F(a) \cdot x = 0,$$

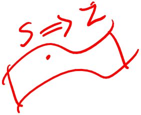
donde $F : \mathbb{R}^{n+1} \to \mathbb{R}$ *esta dada por* $F(a) = (a_1, a_2, ..., a_n, f(a_1, a_2, ..., a_n))$, $x_{n+1} = f(x_1, x_2, ..., n_n)$.



$x^2+y^2=1$ cónica = curva de nivel "0"

$f : \mathbb{R}^2 \to \mathbb{R}$

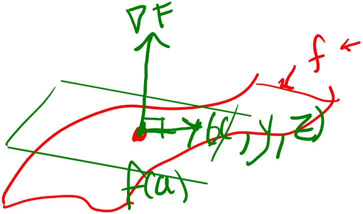
$z = f(x,y)$



$F = f(x,y) - z = 0$

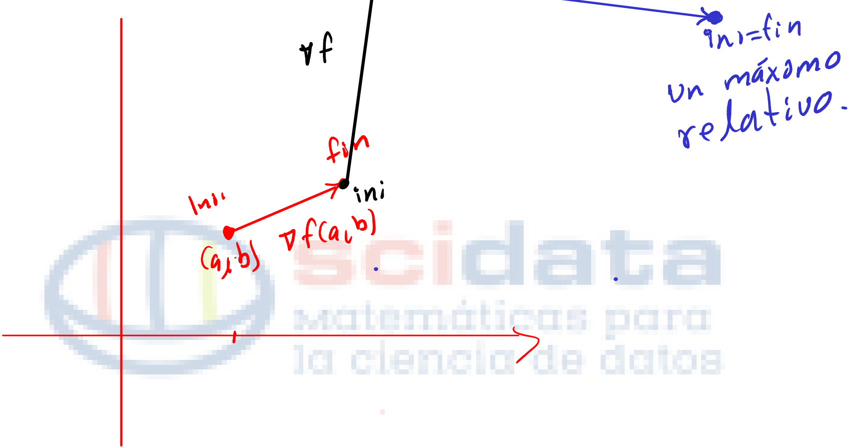
$F(x,y,z) = f(x,y) - z$

$\nabla F(x,y,z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1\right)$

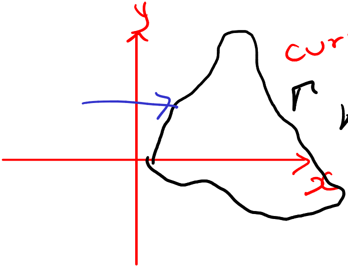


$$\nabla F \cdot (x, y, z) = 0$$

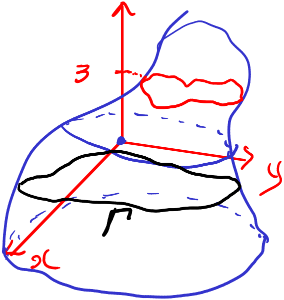
$f: \mathbb{R}^2 \to \mathbb{R}$

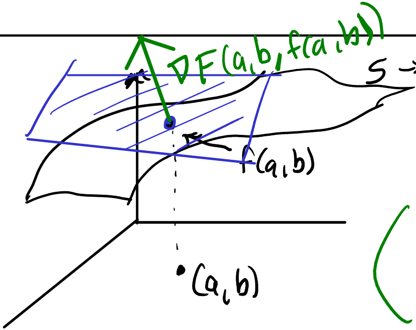


curva cerrada. $f: \mathbb{R}^2 \to \mathbb{R}$



$\underline{K=0}$

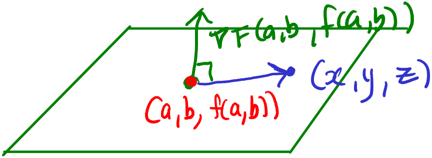




$S \to z = f(x,y)$ $\qquad f: \mathbb{R}^2 \to \mathbb{R}$

$F: \mathbb{R}^3 \to \mathbb{R}$ $\qquad K=0$

$$\left(\begin{array}{l} f(x,y) = x+y \quad (z = f(x,y)) \\ x+y=K \Rightarrow x+y=0 \end{array} \right)$$



$$\underline{\nabla F(a,b,f(a,b))\cdot\big((x,y,z)-(a,b,f(a,b))\big)=0}$$

E. plano tangente a f en (a,b).

Ej. $\underline{f(x,y)=x^2+y^2}$ en $(1,2)$

$F(x,y,z)=x^2+y^2-z$

$z=f(x,y)$

$z=x^2+y^2$

$x^2+y^2-z=0 \leftarrow K$

$\nabla F(x,y,z)=(2x,2y,-1)$

$z=(1)^2+2^2=5$

$\Rightarrow \nabla F(1,2,5)=(2,4,-1)$ $\quad\therefore\ (1,2,5)$

$\Rightarrow \nabla F(1,2,5)\cdot\big((x,y,z)-(1,2,5)\big)=0$

$$(2,4,-1)\cdot(x-1,\ y-2,\ z-5)=0$$

$$2(x-1)+4(y-2)-(z-5)=0$$

$$2x-2+4y-8-z+5=0$$

$$2x+4y-z-5=0$$

$$z=2x+4y-5$$

$$g(x,y)=2x+4y-5$$