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$$\int_{\mathbb{R}} f \, d\mu$$

Integración por Cambio de Variable

Sustitución

Dr. Juan Luis Palacios Soto

palacios.s.j.l@gmail.com

¿Cuándo aplicar este teorema?

Este teorema se aplica cuando el integrando es complicado, ya sea por su naturaleza misma o porque el dominio de integración lo vuelve complicado.

Este teorema cobra mayor relevancia a más dimensiones (estadística multivariante, por mencionar alguna) donde se aplican conceptos del coordenadas polares, coordenas cilíndricas o coordenas esféricas.

$$\int_a^b \underline{f(x)}\, dx$$

Teorema (Cambio de Variable)

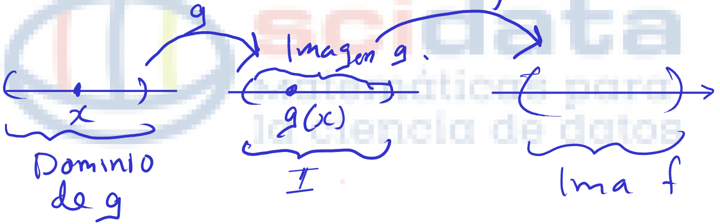
Si $u = g(x)$ es una función derivable cuya imagen es I y f es continua sobre I, entonces

$du = g'(x)dx$

$$\int f(g(x))\overbrace{g'(x)dx}^{du} = \int f(u)du$$

Ejemplo:

$$\int \frac{e^{4x}+3}{e^{3x}}dx$$



Teorema (Cambio de Variable)

Si $u = g(x)$ es una función derivable cuya imagen es I y f es continua sobre I, entonces

$$\int f(g(x))g'(x)dx = \int f(u)du$$

Ejemplo:

$$\int \frac{e^{4x}+3}{e^{3x}}dx \leftarrow$$

$\cancel{u = e^{4x}+3}$

$\cancel{du = 4e^{4x}dx}$

$\cancel{u = e^{3x}}$

$\cancel{du = 3e^{3x}dx}$

$\boxed{u = e^{x}} \Rightarrow du = e^{x}dx$

$u^4 = (e^x)^4 = e^{4x}$

$u^3 = (e^x)^3 = e^{3x}$

$du = u\,dx$

$\frac{du}{u} = dx$

$$= \int \frac{(u^4+3)}{u^3}\frac{du}{u}$$

$$= \int \frac{u^4+3}{u^4}du = \int \left(\frac{u^4}{u^4} + \frac{3}{u^4}\right)du = \int \left(1 + \frac{3}{u^4}\right)du$$

$$= \int 1du + \int 3u^{-4}du = u - u^{-3} + C$$

$$= e^x - (e^x)^{-3} + C = e^x - e^{-3x} + C.$$

Ejemplo:

$$\int \frac{1}{x\ln(x)}dx = \int \frac{1}{\ln(x)} \frac{dx}{x}$$

(1) $u = \ln(x)$

(2) $\cancel{u = x\ln(x)}$

$du = \ ? \ dx$

Si $u = x\ln(x) \Rightarrow du = (\ln(x) + 1)dx \Rightarrow \cancel{\int \frac{dx}{u}}$

$= \int \frac{du}{(\ln(x)+1)u}$?

Si $u = \ln(x) \Rightarrow du = \frac{dx}{x} \Rightarrow \int \frac{1}{u} du = \ln|u| + C.$

$= \ln|\ln(x)| + C.$

$$\log_a(b) = \frac{\ln(b)}{\ln(a)}. \qquad \log_3(2) = \frac{\ln(2)}{\ln(3)}$$

Teorema (Método de sustitución para integrales definidas)

Si $g'(x)$ es continua sobre $[a,b]$ y f es continua sobre el rango de $u = g(x)$, entonces

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Ejemplo:

$$(a) \int_0^1 \frac{x^2}{\sqrt[3]{1+2x}} 2dx \quad du$$

$u = 1+2x$

$du = 2dx$

$u - 1 = 2x$

$\left(\frac{u-1}{2}\right)^2 = x^2$

$\frac{u^2-2u+1}{4} = x^2$

$= \frac{1}{2}\int_1^3 \frac{x^2\, du}{\sqrt[3]{u}}$

$u(x) = 1+2x$

$u(0) = 1+2(0) = 1$

$u(1) = 1+2(1) = 3$

$= \frac{1}{2}\int_1^3 \frac{\frac{u^2-2u+1}{4}}{\frac{u^{1/3}}{1}} du$

$= \frac{1}{2}\int_1^3 \frac{(u^2-2u+1)du}{4u^{1/3}}$

$\cancel{u = x^2 \Rightarrow \sqrt{u} = x}$

$\cancel{du = 2x\,dx}$

$\cancel{\frac{du}{2\sqrt{u}} = dx}$

$$= \frac{1}{8}\int_1^3 \left(\frac{u^2}{u^{1/3}} - \frac{2u}{u^{1/3}} + \frac{1}{u^{1/3}}\right) du$$

$$= \frac{1}{8}\int_1^3 \left(u^{5/3} - 2u^{2/3} + u^{-\frac{1}{3}}\right) du \qquad \int u^n du = \frac{u^{n+1}}{n+1}$$

$$= \frac{1}{8}\left(\frac{u^{8/3}}{\frac{8}{3}} - 2\,\frac{u^{5/3}}{\frac{5}{3}} + \frac{u^{2/3}}{2/3}\right)\Bigg|_1^3$$

$$= \frac{1}{8}\left(\frac{3}{8}u^{8/3} - \frac{6}{5}u^{5/3} + \frac{3}{2}u^{2/3}\right)\Bigg|_1^3$$

$$= \frac{1}{8}\left(\left[\frac{3}{8}(3)^{8/3} - \frac{6}{5}\,3^{5/3} + \frac{3}{2}\,3^{2/3}\right] - \left[\frac{3}{8}\,1^{8/3} - \frac{6}{3}\,1^{5/3} + \frac{3}{2}\,1^{2/3}\right]\right)$$

$$= \longrightarrow$$

Ejemplo:

$$\int_0^2 x\sqrt{x+1}\,dx \longleftarrow = \underbrace{\int_0^2 u\sqrt{u+1}\,du}$$

$\cancel{u = x \,,\; du = dx}$

$\cancel{u=0}$

$\cancel{u=2}$

$u = x+1 \leftarrow u-1 = x$

$du = dx$

$u = 0+1 = 1$

$u = 2+1 = 3$

$\therefore = \int_1^3 (u-1)\sqrt{u}\,du$

$\left(\int u^n du = \frac{u^{n+1}}{n+1}\right)$

$$= \int_1^3 (u-1)\,u^{\frac{1}{2}}\,du = \int_1^3 \left(u^{\frac{3}{2}} - u^{1/2}\right) du =$$

$$= \int_1^3 u^{3/2}\,du - \int_1^3 u^{1/2}\,du = \left(\frac{u^{\frac{5}{2}}}{\frac{5}{2}} - \frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right)\Bigg|_1^3$$

$$= \frac{2}{5}u^{5/2}\Big|_1^3 - \frac{2}{3}u^{\frac{3}{2}}\Big|_1^3 = \frac{2}{5}\left(3^{\frac{5}{2}} - 1^{5/2}\right) - \frac{2}{3}\left(3^{\frac{3}{2}} - 1^{3/2}\right)$$

$=$

Ejemplo:

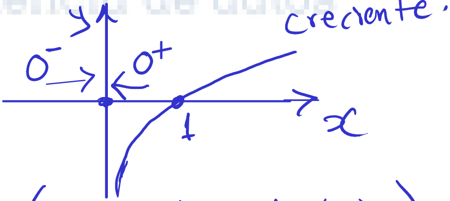
$$\cancel{\int_0^2 x\sqrt{x+1}dx}$$

$$\int_1^e \frac{\ln(x)}{x}\,\underbrace{dx}_{du} = \int_0^1 u\,du = \frac{u^2}{2}\Big|_0^1 = \frac{1^2}{2} - \frac{0^2}{2} = \frac{1}{2}$$

$u = \ln(x)$ $\quad$ $u = \ln(1) =$

$du = \frac{1}{x}dx$ $\quad$ $u = \ln(e^1) = 1$

$$\int_0^1 \ln(x)\,dx$$



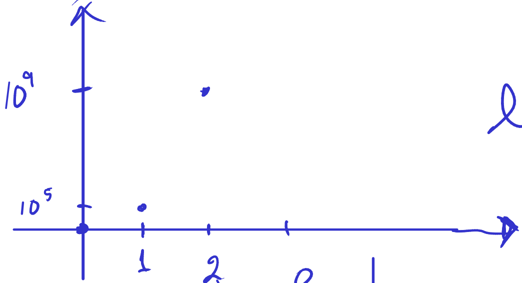
$\ln(10) \simeq 2.03$

$\ln(1000) = 6.09$

$\ln(x^n) = n\ln(x)$

$\ln(1000) = \ln(10^3) = 3\ln(10) = 6.09$

$\ln(10{,}000{,}000{,}000) = \ln(10^{10}) = 10\ln(10) = \underline{20.3}$



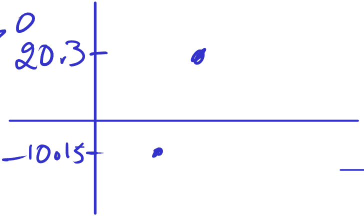
$\ln(\underline{0.00001})$

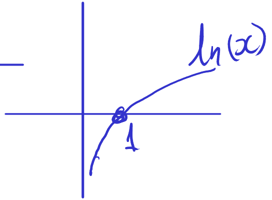
$= \ln(10^{-5}) = -5\ln(10)$

$= -10.15$

argumento $\to 0$

$\ln(|x|)$



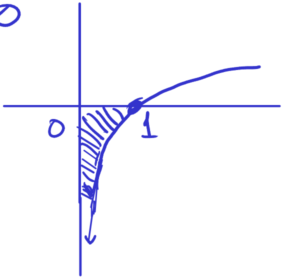


$$\ln(10^{-10}) = -10\ln(10) = -20.3$$

$$10^{-10} = 0.0000000001$$

$$\ln(10^{-100}) = -100\ln(10) = -203$$

$$\lim_{x\to 0^+} \ln(x) = -\infty$$



$$\int \ln(x)\,dx = \int u\, e^{u}\, du \;?$$

Integración por partes

$$\cancel{u = x}$$

$$u = \ln(x) \iff e^{u} = e^{\ln(x)} = x$$

$$du = \frac{dx}{x}$$

$$du = \frac{dx}{e^{u}} \Rightarrow \underline{e^{u}\,du} = \underline{dx}$$