STA305/1004-Class 22

April 3, 2017

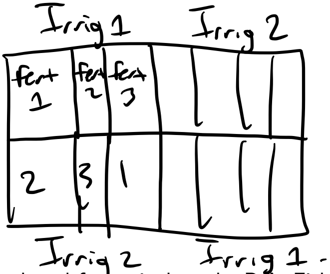
Today's Class

Exam review session on Wednesday during Class - NOT on Thurs.

- Split plot designs
  - Corrosion study example
  - Split plot versus factorial designs
  - Why choose a plit plot design?
  - ANOVA for split plot designs
  - How not to do it
  - How to do it
  - Randomizing a split plot design

Split plot designs

- Irrigation method
  watering.
- fertilizer



- These designs were originally developed for agriculture by R.A. Fisher and F. Yates.
- Due to their applicability outside agriculture they could be called split-unit designs.
- But we will use split-plot . . .

Split plot designs

- Some factors need to be applied to larger plots compared to other factors.
- For example, if type of irrigation method and the type of fertilizer used are the two factors then irrigation requires a larger plot.
- Apply a specific irrigation method to a large plot, and fertilizer to a smaller plot.

Split plot designs

Figure 1: Split plot design examples from agriculture.

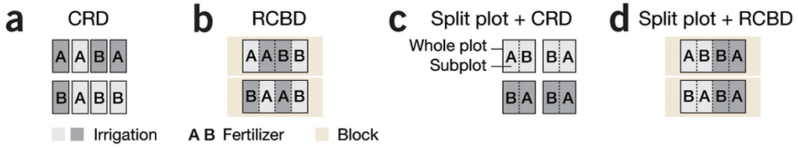
From

Points of Significance: Split plot design

Naomi Altman & **Martin Krzywinski**

Nature Methods **12**, 165–166 (2015) | doi:10.1038/nmeth.3293

Published online 26 February 2015

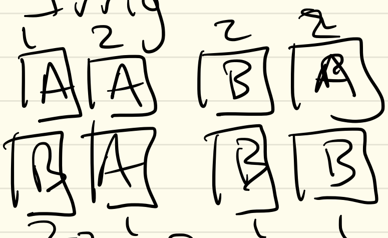


(**a**) In CRD, levels of irrigation and fertilizer are assigned to plots of land (experimental units) in a random and balanced fashion. (**b**) In RCBD, similar experimental units are grouped (for example, by field) into blocks and treatments are distributed in a CRD fashion within the block. (**c**) If irrigation is more difficult to vary on a small scale and fields are large enough to be split, a split plot design becomes appropriate. Irrigation levels are assigned to whole plots by CRD and fertilizer is assigned to subplots using RCBD (irrigation is the block). (**d**) If the fields are large enough, they can be used as blocks for two levels of irrigation. Each field is composed of two whole plots, each composed of two subplots. Irrigation is assigned to whole plots using RCBD (blocked by field) and fertilizer assigned to subplots using RCBD (blocked by irrigation).

Figure 1:

Points of Significance: Split plot design. Nature Methods 12, 165–166 (2015)

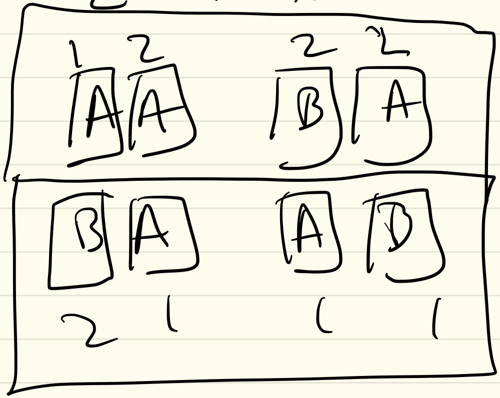
(A) Irrig



large areas are irrigated then fertizer is randomly assigned to plot.

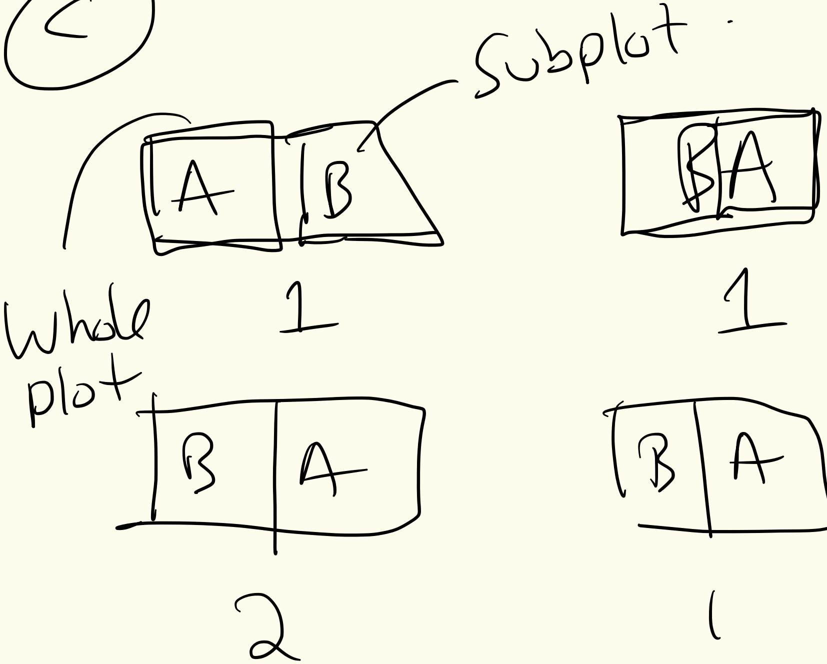
2^2 factorial design. – fert. and irrig.

(B)



If land is divided into two large areas that differ in some way then block by area → randomize w/in each block.

(c)



- Assign irrigation to wholeplots using randomization.
- Assign fertilizer to Subplots using randomization

Split plot designs - corrosion study

- An experiment of corrosion resistance of steel bars treated with four different coatings C_1, C_2, C_3, C_4 was conducted.
- Three furnace temperatures were investigated and four differently coated bars randomly arranged in the furnace within each heat.
- Positions of the coated steel bars in the furnace randomized within each heat.
- Furnace temperature was difficult to change so heats were run in systematic order shown.

hard to change factor.

Temperature	Position 1	Position 2	Position 3	Position 4
360°				
370°				
380°				
380°				
370°				
360°				

Split plot designs - corrosion study

The split-plot experiment of corrosion resistance is shown for the first replicate at 360.

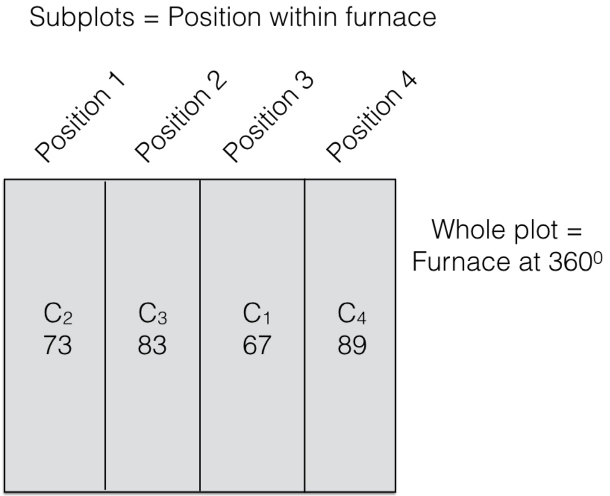


Figure 2:

Split plot designs - corrosion study

- There are $I = 3$ furnace temperatures arranged in $n = 2$ replications.
- Each furnace temperature is called a **whole plot**.
- The whole plot treatments are $T_1 = 360^\circ$, $T_2 = 370^\circ$, $T_3 = 380^\circ$.
- Within each furnace temperature there are $J = 4$ subplots.
- The four sub plot treatments C_1, C_2, C_3, C_4 are randomly applied to the sub plots within each whole plot.

Temperature	Position 1	Position 2	Position 3	Position 4
360°	C_2	C_3	C_1	C_4
370°	C_1	C_3	C_4	C_2
380°	C_3	C_1	C_2	C_4

Temperature	Position 1	Position 2	Position 3	Position 4
380°	C_4	C_3	C_2	C_1
370°	C_4	C_1	C_3	C_2
360°	C_1	C_4	C_2	C_3

Split plot designs - corrosion study

Analysis of the whole plots.

Source	df
Replications	$2-1$
Temperature	$3-1$
Whole plot Error	$(3-1)(2-1)$
Between whole plots	$3 \cdot 2-1$

Split plot designs - corrosion study

Analysis of the sub plots.

Source	df
Between whole plots	$3 \cdot 2 - 1$
Coatings	$4 - 1$
Coatings $\times$ Temperature	$(4-1)(3-1)$
Sub plot Error	$3(2-1)(4-1)$
Total	$3 \cdot 2 \cdot 4 - 1$

Split plot designs - corrosion study

The primary interest were the comparison of coatings and how they interacted with temperature.

Some of the data from the experiment is shown below.

run	heats	coating	position	replication	resistance
r1	T360	C2	1	1	73
r1	T360	C3	2	1	83
r1	T360	C1	3	1	67
r1	T360	C4	4	1	89
r2	T370	C1	1	1	65
r2	T370	C3	2	1	87

Split-plot designs versus Factorial Designs

- How does the split-plot design compare with a 3x4 factorial design of coating and temperature?
- In the factorial design an oven temperature-coating combination would be randomly selected then we would obtain a corrosion resistance measure.
- Then randomly select another oven temperature-coating combination and obtain another corrosion resistance measure until we have a resistance measure for all 12 oven temperature-coating combinations.
- To run each combination in random order would require adjusting the furnace temperature up to 24 times (since there were two replicates) and would have resulted in a much larger variance.
- The split plot is like a randomized block design (with replications as blocks) in which the opportunity is taken to introduce additional factors between blocks.
- In this design there is only one source of error influencing the resistance.

Factorial design ...

	360°	370°	380°
C_1	C_1 360°	C_1 370°	C_1 380°
C_2	C_2 360°	..	..
C_3	.	.	.
C_4	.	.	C_4 380°

How many treatments? 12 treatments.

Require Changing oven temp. many times.

Split-plot designs versus Factorial Designs

There are two different experimental units:

- The six different furnace heats, called whole plots.
- The four positions within each furnace heat, called subplots, where the differently coated bars could be placed in the furnace.
- Misleading to treat as if only one error source and one variance.
- Two different experimental units: six furnace heats (whole plots); and four positions (subplots) where differently coated bars placed in furnace.
- Two different variances: σ_W^2 for whole plots and σ_S^2 for subplots.
- It would be misleading to treat as if only one error source and one variance.

Split plot designs versus Factorial Designs

- Achieving and maintaining a given temperature in this furnace was very imprecise.
- The whole plot variance, measuring variation from one heat to another, was expected to be large.
- The subplot variance measuring variation from position to position, within a given heat, was expected to be small.
- The subplot effects and subplot-main plot interaction are estimated using with the same subplot error.

Why choose a split plot design?

- Two considerations important in choosing an experimental design are feasibility and efficiency.
- In industrial experimentation a split-plot design is often convenient and the only practical possibility.
- This is the case whenever there are certain factors that are *difficult to change* and others that are *easy to change*.
- In this example changing the *furnace temperature was difficult to change*; rearranging the *positions of the coated bars in the furnace was easy to change*.

Question

A large-scale bakery is designing a new brownie recipe. They are experimenting with two levels of chocolate and sugar using two different baking temperatures. In the experiment four trays of brownies will baked, and each tray will have one of the chocolate/sugar combinations. However, to save time they decide to bake four trays of brownies at the same time instead of baking each tray individually. A statistician recommends a split plot design.

The whole plots and sub plots, respectively, are:

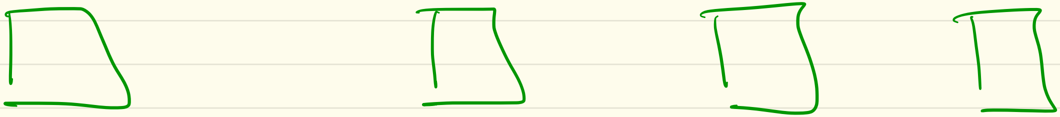
Respond at **PollEv.com/nathantaback**

Text **NATHANTABACK** to **37607** once to join, then **A, B, C, or D**

Whole plot = tray; sub plot = amount of sugar and chocolate	~~A~~	4
6 Whole plot = oven temperature; sub plot = amount of sugar and chocolate	B	15
6 Whole plot = oven temperature; sub plot = tray	C	8
Whole plot = tray; sub plot = oven temperature	~~D~~	1

Temp 1

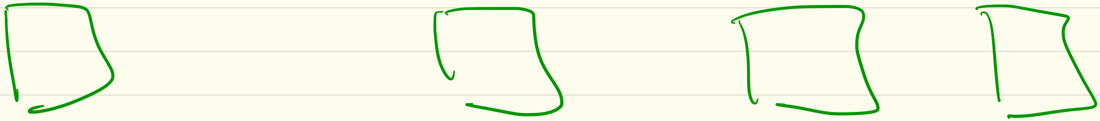
Choc/Sugar



+ −   − +   − −   + +

Temp 2

Choc/Sugar



whole plot is temp
Subplot is tray

ANOVA table for split plot experiment

- The numerical calculations for the ANOVA of a split-plot design are the same as for other balanced designs (designs where all treatment combinations have the same number of observations) and can be performed in R or with other statistical software.
- Experimenters sometimes have difficulty identifying appropriate error terms.

```
spfurcoat1 <- aov(resistance~ replication*heats*coating,data=tab0901)
summary(spfurcoat1)
```

```
                          Df Sum Sq Mean Sq
replication                1    782     782
heats                      2  26519   13260
coating                    3   4289    1430
replication:heats          2  13658    6829
replication:coating        3    254      85
heats:coating              6   3270     545
replication:heats:coating  6    867     144
```

ANOVA table for split plot experiment - whole plot

```
spfurcoat1 <- aov(resistance~ replication*heats*coating,data=tab0901)
summary(spfurcoat1)
```

```
                          Df Sum Sq Mean Sq
replication                1    782     782
heats                      2  26519   13260
coating                    3   4289    1430
replication:heats          2  13658    6829
replication:coating        3    254      85
heats:coating              6   3270     545
replication:heats:coating  6    867     144
```

The ANOVA table for the whole plots is:

Source	DF	SS	MS
replication	1	782	782
heats	2	26519	13260
replication:heats (whole plot error)	2	13658	6829

The whole plot mean square error is 6829. This measures the differences between the replicated heats at the three different temperatures.

ANOVA table for split plot experiment - sub plot

```
spfurcoat1 <- aov(resistance~ replication*heats*coating,data=tab0901)
summary(spfurcoat1)
```

```
                        Df Sum Sq Mean Sq
replication              1    782     782
heats                    2  26519   13260
coating                  3   4289    1430
replication:heats        2  13658    6829
replication:coating      3    254      85
heats:coating            6   3270     545
replication:heats:coating 6   867     144
```

The subplot effects are:

Source	DF	SS	MS
`coating`	3	4289	1430
`heats:coating`	6	3270	545
Sub plot error	9	1121	124.6

- The subplot mean square error is $(254 + 867)/(3 + 6) = 124.6$.
- The subplot error measures to what extent the coatings give dissimilar results within each of the replicated temperatures.

ANOVA table for split plot experiment - using aov() with Error()

In R the ANOVA table for whole plot and sub plot effects can obtained by specifying the subplot error structure explicit using Error().

```
spfurcoat <- aov(resistance~ replication + heats + replication:heats
                 + coating + heats:coating
                 +Error(heats/replication),data=tab0901)
summary(spfurcoat)
```

```
Error: heats
      Df Sum Sq Mean Sq
heats  2  26519   13260

Error: heats:replication
                  Df Sum Sq Mean Sq
replication        1    782     782
replication:heats  2  13658    6829

Error: Within
              Df Sum Sq Mean Sq F value  Pr(>F)
coating        3   4289  1429.7  11.480 0.00198 **
heats:coating  6   3270   545.0   4.376 0.02407 *
Residuals      9   1121   124.5
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Whole plot

Subplot.

ANOVA for split plot experiment - using aov() with Error()- whole plot

The headings Error: heats and Error: heats:replication contain the ANOVA for a randomized block design with replicates as blocks.

```
summary( aov(resistance~heats*replication,tab0901))
```

```
                  Df Sum Sq Mean Sq F value   Pr(>F)
heats              2  26519   13260  27.498 3.37e-06 ***
replication        1    782     782   1.622 0.219043
heats:replication  2  13658    6829  14.161 0.000202 ***
Residuals         18   8680     482
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

for correct p-value see next slide

Caution: The p-value for heats is incorrect in this output!

ANOVA for split plot experiment - using `aov()` with `Error()`- whole plot

- The ratio of mean square errors follows an $F_{2,2}$.
- The F statistic for whole plots is 13260/6829$=$ 1.94.
- The p-value of the test

$$H_0 : \mu_{360} = \mu_{370} = \mu_{380}$$

```
1-pf(q = 13260/6829,df1 = 2,df2 = 2)
```

```
[1] 0.3399373
```

This is the correct p-value for `heats`.

ANOVA for split plot experiment - using `aov()` with `Error()`- sub plot

- The subplot effects of coating and the interaction of temperature and coating can be tested by forming F statistics using the subplot mean square error.
- These tests are given in the ANOVA table under the heading `Error: Within`.
- There are statistically significant differences between coatings and the interaction between temperature and coating.

ANOVA for split plot experiment

The values for the split plot experiment can be put into one ANOVA table.

Source	DF	SS	MS	F	P
Whole plot:					
replication	1	782	782	782/6829=0.12	0.77
heats	2	26519	13260	13260/6829=1.9	0.34
replication $\times$ heats	2	13658	6829		
(whole plot error)					
Subplot:					
coating	3	4289	1430	11.48	0.002
coating $\times$ heats	6	3270	545	4.376	0.02
Subplot error	9	1121	124.5		

Estimating whole plot and sub plot variances

- The whole plot error mean square 6829 is an estimate of

$$4\sigma_W^2 + \sigma_S^2$$ (this is also called the expected mean square)

σ_w^2 is multiplied by 4 $\because$ there are four coatings in each heat.

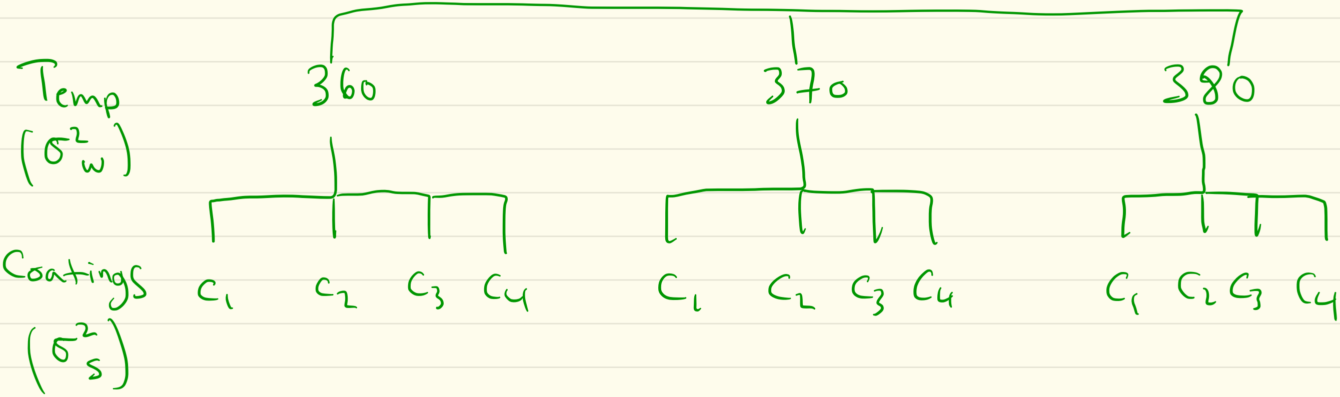
$$6829 = 4\hat{\sigma}_W^2 + \hat{\sigma}_S^2.$$

- The subplot mean square error is 125 so $\hat{\sigma}_S^2 = 125$. Estimates of the whole plot and sub plot standard deviations are,

$$\hat{\sigma}_W = \sqrt{\left(\frac{6829 - 125}{4}\right)} = 40.9, \qquad \hat{\sigma}_S = \sqrt{125} = 11.1.$$

- The estimated standard deviation of furnace heats is approximately four times as large as the standard deviation for coatings.

Additional Notes on previous Slide (added after Class)



- each temp average is based on four coatings.

$\therefore$ 6829 is an estimate of $4\sigma_w^2 + \sigma_s^2$

The main point of the previous Slide is to show that the estimated Standard deviation of furnace heats is approx. four times as large Standard deviation for Coatings.

ANOVA for split plot experiment

- Suppose that a split plot experiment is conducted with whole factor plot A with I levels and sub plot factor B with J levels.
- The experiment is replicated n times.

Source	DF	SS
Whole plot:		
replication	$n-1$	SS_{Rep}
A	$I-1$	SS_A
replication $\times$ A	$(n-1)(I-1)$	SS_W
(whole plot error)		
Subplot:		
B	$J-1$	SS_B
$A \times B$	$(I-1)(J-1)$	$SS_{A\times B}$
Subplot error	$I(J-1)(n-1)$	SS_S

Split plot ANOVA - how not to do it

- Suppose that you didn't know about the split-plot structure.
- So the experimenter analyzes the data as a two-way ANOVA.
- Would you reach the same conclusions?

Split plot ANOVA - how not to do it

```
furcoatanova <- aov(resistance~heats*coating,data=tab0901)
summary(furcoatanova)
```

```
              Df Sum Sq Mean Sq F value  Pr(>F)
heats          2  26519   13260  10.226 0.00256 **
coating        3   4289    1430   1.103 0.38602
heats:coating  6   3270     545   0.420 0.85180
Residuals     12  15560    1297
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

The two-way ANOVA shows that there is no evidence of a difference in the four coatings, evidence of a difference between temperatures, and no evidence of an interaction between temperature and coating.

Split plot ANOVA - how not to do it

What happened?

- The two factors temperature and coating use different randomization schemes and the number of replicates is different for each factor.
- The subplot factor, coatings, restricted randomization to the four positions within a given temperature (whole plot).
- For the whole plot factor, complete randomization can usually be applied in assigning the levels of A to the whole plots (although this was not the case for the corrosion study).
- Therefore, the error should consist of two parts: whole plot error and subplot error.
- In order to test the significance of the whole plot factor and the subplot factor we need respective mean squares with the respective whole plot error component and subplot error component respectively.

Split plot ANOVA - how not to do it

The (incorrect) two-way ANOVA model is

$$y_{ijk} = \eta + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \epsilon_{ijk},\, \epsilon_{ijk} \sim N(0, \sigma^2)$$

y_{ijk} is the observation for the kth replicate of the ith level of factor A and the jth level of factor B. (adapted from Wu and Hamada)

Split plot ANOVA - how to do it

The correct model is

$$y_{ijk} = \eta + \tau_k + \alpha_i + (\tau\alpha)_{ki} + \beta_j + (\alpha\beta)_{ij} + (\tau\beta)_{kj} + (\tau\alpha\beta)_{kij} + \epsilon'_{ijk},\ \epsilon'_{ijk} \sim N(0, \sigma^2)$$

$i = 1, ..., I;\ j = 1, ..., J;\ k = 1, ..., n.$

y_{ijk} is the observation for the kth replicate of the ith level of factor A and the jth level of factor B.

Whole plot effects

- τ_k is the effect of the kth replicate.
- α_i is the ith main effect for A
- $(\tau\alpha)_{ki}$ is the (k, i)th interaction effect between replicate and A. This is the whole plot error term.

Subplot effects

- β_j is the jth main effect of B
- $(\alpha\beta)_{ij}$ is the (i, j)th interaction between A and B.
- $(\tau\beta)_{kj}$ is the (k, j)th interaction between the replicate and B.
- $(\tau\alpha\beta)_{kij}$ is the (k, i, j)th interaction between the replicate, A, and B.
- ϵ'_{ijk} is the error term.

The term $\epsilon_{kij} = (\tau\beta)_{kj} + (\tau\alpha\beta)_{kij} + \epsilon'_{ijk}$ is the subplot error term.

What is a split plot?

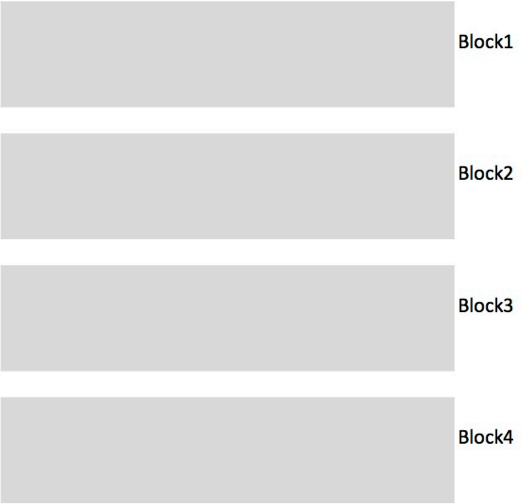
- A split-plot can be thought of as a blocked experiment where the blocks themselves serve as experimental units for a subset of the factors.
- Corresponding to two levels of experimental units are two levels of randomization.
- One randomization to to determine assignment to whole plots.
- A randomization of treatments to split-plot experimental units occurs within each plot.

Randomizing a split plot experiment

The three steps in randomizing a basic split-plot experiment consisting of 5 blocks (replicates), 4 levels of whole plot factor A, and 8 levels of split-plot factor B are:

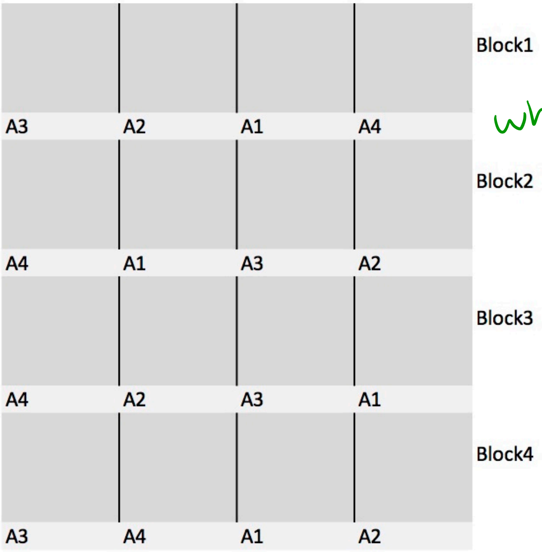
1. Division of experimental area or material into five blocks

Replicates



Randomizing a split plot experiment

2. Randomization of four levels of whole plot factor A to each of the five blocks.



whole plots
are
randomized

Figure 4:

Randomizing a split plot experiment

3. Randomization of eight levels of split plot factor B within each level of whole plot factor A.

Two Randomizations.

B3	B6	B8	B8	
B7	B7	B7	B7	Block1
B6	B5	B1	B4	
B2	B3	B6	B3	
B4	B1	B3	B6	
B1	B2	B4	B5	
B5	B4	B2	B1	
B8	B8	B5	B2	
A3	A2	A1	A4	
B7	B7	B5	B4	
B8	B5	B2	B7	Block2
B3	B6	B1	B8	
B1	B1	B7	B1	
B2	B4	B4	B5	
B5	B8	B6	B2	
B6	B2	B8	B6	
B4	B3	B3	B3	
A4	A1	A3	A2	
B2	B4	B6	B8	
B5	B2	B3	B3	Block3
B3	B3	B5	B4	
B6	B1	B7	B2	
B8	B8	B4	B1	
B7	B5	B2	B5	
B1	B6	B8	B7	
B4	B7	B1	B6	
A4	A2	A3	A1	
B2	B3	B5	B1	
B3	B2	B3	B5	Block4
B6	B5	B7	B2	
B7	B1	B4	B6	
B5	B6	B8	B4	
B8	B7	B2	B8	
B4	B4	B6	B3	
B1	B8	B1	B7	
A3	A4	A1	A2	

randomize the split-plots.

THE LAST SLIDE OF THE COURSE (well almost . . .)

Thanks for being a great class!

p.s - don't forget about the exam review session
Wednesday, April 5, 11:00-1:00