# The “Riemann Hypothesis” is true for period polynomials of almost all newforms

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## Abstract

The period polynomial for a weight and level *N* newform is the generating function for special values of *L(s, f)*. The functional equation for *L(s, f)* induces a functional equation on . Jin, Ma, Ono, and Soundararajan proved that for all newforms *f* of even weight and trivial nebentypus, the “Riemann Hypothesis” holds for : that is, all roots of lie on the circle of symmetry . We generalize their methods to prove that this phenomenon holds for all but possibly finitely many newforms f of weight with any nebentypus. We also show that the roots of are equidistributed if *N* or *k* is sufficiently large.

## Introduction and statement of results

Let be a newform of weight *k*, level *N*, and nebentypus . Associated with *f* is an *L*-function *L(s, f)*, which can be normalized so that the completed *L*-function

Equation 1

satisfies the functional equation

Equation 2

for some with .

The *period polynomial* associated with *f* is the degree polynomial defined by

Equation 3

By the binomial theorem, we have

Thus, is the generating function for the special values of the *L*-function associated with *f*. For background on period polynomials, we refer the reader to (Choie, Park and Zagier in press, Knopp 1974, Kohen and Zagier 1984, Paşol and Popa 2013, Zagier 1991).

When , the period polynomial is nonconstant, so one can consider where the roots of are located. To this end, we use the functional Equation 2 to observe that

Thus, if *ρ* is a root of , then is also a root. Much like the behavior of the nontrivial zeroes of *L(s, f)* predicted by the generalized Riemann Hypothesis, one can consider whether all the roots of lie on the curve of symmetry of the roots: in this case, the circle |. It is natural to expect the following conjecture, which is supported by extensive numerical evidence.

Conjecture  
(“Riemann Hypothesis” for period polynomials) Let be a newform. Then, the roots of all lie on the circle .

El-Guindy and Raji (2014) proved this for Hecke eigenforms on with full level (, for which the circle of symmetry is ). They were inspired by the work of Conrey et al. (2013), who showed an analogous result for the odd parts of these period polynomials, again with full level.

Recent work by Jin et al. (2016) proved the conjecture for all newforms of even weight and trivial nebentypus. They also showed that the roots of are equidistributed on the circle of symmetry for sufficiently large *N* or *k*. Using similar methods, Löbrich et al. (2016) proved an analogous result for polynomials generating special values of for a sufficiently well-behaved class of motives with odd weight and even rank.

In this paper, we generalize the methods of (Jin, et al. 2016) to prove the conjecture for all but possibly finitely many newforms.

### Theorem 1.1

The “Riemann Hypothesis” for period polynomials holds for all but possibly finitely many newforms with weight and nontrivial nebentypus.

### Remark

Note that for , the period polynomial is a constant. Therefore, Theorem 1.1 is essentially the best result for which one could hope, since an effective computation can check that Theorem 1.1 also holds for the finitely many possible exceptions. We denote the set of these finitely many newforms as , which consists of the following:

(1)

For , all newforms with level .

(2)

For , all newforms with level , where is a constant given by tables at the end of Sects. 4 and 5.

We know of no counterexamples to Theorem 1.1.

We also show that the roots of are equidistributed on the circle of symmetry for sufficiently large *N* or *k*.

### Theorem 1.2

Let be a newform of weight , level *N*, and nebentypus such that . Then, the following are true:

1. Suppose that , and let denote the roots of . Then for any real , where the implied constant depends only on and is effectively computable.
2. Suppose that . There exists such that the arguments of the roots of can be written as , where denotes the unique solution of , and the implied constant depends only on and is effectively computable.
3. Suppose that . There exists such that the arguments of the roots of can be written as , Here, is the unique solution to the equation , and the implied constant is absolute and effectively computable.

In [Sect. 2](#_Preliminaries), we introduce notation and lemmas that we will be using in our proof. In Sect. 3, we will prove our main results for using ad hoc arguments. For larger *k*, we prove Theorem 1.1 in Sect. 4 (the case of k even) and Sect. 5 (the case of *k* odd), and we prove Theorem 1.2 in Sect. 6. Finally, in Sect. 7, we detail our Sage computations suggesting that the roots of the period polynomial of the newform

are all on the circle . This newform *f* is in our finite set of possible exceptions, which suggests that Theorem 1.1 should be true even for newforms in .

## Preliminaries

Throughout this section, we assume that is a newform of weight , level *N*, and arbitrary nebentypus . We note that the nebentypus character will be essentially invisible throughout our proof, other than the fact that it determines the level of *f*. We now define some notation related to and prove lemmas about the values of and along the real line. The lemmas will be very similar in spirit to those proven in (Jin, et al. 2016).

Define to satisfy . Now, define

Equation 4

where denotes for and . Using Equation 4, one can compute

.

Therefore, if , then . Additionally, for , we also have , so is real for . Note that if and only if .. Therefore, to prove Theorem 1.1 and Theorem 1.2, it suffices to show that all roots of lie on the circle and are equidistributed.

We will require the following monotonicity result.

### Lemma 2.1

We have

Also, for all ,

.

Proof

As is entire of order 1, we apply the Hadamard factorization theorem to write

,

where the sum is taken over all roots of . By [4, Proposition 5.7(3)], we have that

.

Note that . This implies that is increasing for and , from which the lemma follows. ◻

We also prove a useful inequality on ratios of *L*-function values.

### Lemma 2.2

For all , we have

.

We have that

.

If we express

,

then Deligne’s bound on the eigenvalues of the Hecke operators on states that

Equation 5

where denotes the von Mangoldt function; for a reference, see [ (Ono 2004), Theorem 2.32]. Therefore, we have that

Equation 6

Now the lemma follows from the inequality . ◻

Finally, we show a lemma that serve as our main means of proving Theorem 1.1 for period polynomials.

### Lemma 2.3

Let equal for negative real numbers *r*, *1* for positive real numbers *r*, and *0* for . If there exist real numbers such that either

,

or

,

then all solutions to satisfy .

### Proof

First, is real for , so is well defined. Now, by the intermediate value theorem, there exist such that for all . This gives us roots of that lie on . When k is even, we also get a root in the range by the intermediate value theorem. When *k* is odd, we may redefine the square root in order to move the discontinuity into an interval outside of . This would only affect the sign of . By the intermediate value theorem, this shows the existence of a zero with argument in the range as desired. As for at most *k* values of *z*, the above argument shows that we have found all of them. ◻

## Proof for weights 3, 4, and 5

Here we prove Theorem 1.1 and Theorem 1.2 for .

### The weight 3 case

For , Equation 4 gives that

.

By Equation 2, we know that

,

so the root of lies on the unit circle.

### The weight 4 case

For , we have

.

Now, note that for it follows that

.

By Lemma 2.1, we have that

,

so there exist 2 values of z with such that

,

as desired.

In order to prove Theorem 1.2(iii), we need to bound . First, note that

.

In order to bound , we appeal to the Phragmén–Lindelöf principle; specifically, see [ (Iwaniec and Kowalski 2004), Lemma 5.2, Theorem 5.53] and apply Equation 5. This allows to obtain for any

,

Thus, we have that

and the values of *z* satisfying satisfy

### The weight 5 case

For , we have

.

Once again, for , we have

.

There exist three reals such that for , and alternates in sign. Thus, by Lemma 2.3, we are done if we are able to show that

,

which is equivalent to proving

.

Let . By Lemma 2.1 and Lemma 2.2, it follows that

.

Choosing , the last expression is less than for , which completes the proof for .

To show the desired equidistribution property, define and as above. Now let , for to be chosen later. Then, we see that

with the sign of being different for and . If we can show that

then Lemma 2.3 will show that the root has an argument lying between and . By the bounding above, we only require

.

Choosing suffices.

## Proof for remaining even weights

In this section, we will show Theorem 1.1 for all even weights . Throughout the section, we will restrict our attention to those *z* such that . For simplicity, let , and define

Equation 7

This satisfies

.

Next, define

.

Note that . As in (Jin, et al. 2016), rewrite

,

where we define

.

For , note that

Equation 8

,

where C is a fixed constant depending on and . Therefore, we can pick *k* values of z on the circle such that the previous expression has argument for integers ℓ.ℓ. The value of at these points has alternating positive and negative real part with magnitude at least . By Lemma 2.3, it suffices to show that

.

To bound , we use Lemma 2.2 in the form . This gives

.

For the term in the above expression, we use the bound . For , note that is decreasing for . Therefore, for , we find that

.

Now, we combine the above estimates with to obtain

Equation 9

To finish, we estimate || using Lemma 2.1 and then Lemma 2.2.

Equation 10

By using Equation 9 and Equation 10, it suffices to verify

Equation 11

For each value of m in the first row on the following table, the value *N(m)* is such that inequality Equation 11 holds for all . Note that the case was done in [Sect. 3](#_Proof_for_weights).

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| *m* | 29 | 21 | 18 | 16 | 14 | 13 | 12 | 11 | 10 | 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 |
| *N(m)* | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 9 | 11 | 14 | 19 | 27 | 41 | 69 | 142 | 433 | 5875 |

Therefore, for all , . This completes our proof of Theorem 1.1 for *k* even.

## Proof for remaining odd weights

In this section, we will show Theorem 1.1 for all odd weights . As in the above section, we will restrict our attention to those *z* such that . For simplicity, let , and define

Equation 12

,

so . As in the above section, define

,

where , and are defined as follows.

.

As in [Sect. 4](#_Proof_for_remaining), it suffices to show that

.

The proof of this will proceed in a very similar way to that of the above section. Note that the function is decreasing for . By Lemma 2.2, for , we can bound

.

By Lemma 2.2, we have

Equation 13

.

Now we use Lemma 2.1 to bound .

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