8.1 Coherence Functional (Concrete Choice)

8.1.1 Minimal Coherence Functional and Selection Rule

Formal layer — Choose one coherence functional

Define the projected state at stroboscopic step n as $\rho_n^{\text{loc}} = \Lambda \rho_n \Lambda^\dagger$. Use a single coherence functional in a fixed reference basis $\{|i\rangle\}$:

$$\mathcal{C}(\rho_n^{\text{loc}}) \equiv \sum_{i \neq j} \left| \rho_{ij}^{\text{loc}}(n) \right| .$$

This is the ℓ_1 coherence measure (minimal, basis-explicit, and computable).

Formal layer — Selection functional (free-energy style)

Define a stability-selection functional over one recurrence window:

$$\mathcal{F}_n = E_n - \kappa\, \mathcal{C}(\rho_n^{\text{loc}}), \qquad \kappa > 0,$$

where $E_n = \mathrm{Tr}\big[H_{\text{eff}} \rho_n^{\text{loc}}\big]$ is effective projected energy. A sector is preferred if it minimizes time-averaged selection cost:

$$\overline{\mathcal{F}}_N = \frac{1}{N} \sum_{n=1}^{N} \mathcal{F}_n.$$

Interpretation — Interpretation

The model now has one explicit knob: coherence-weight κ. Larger κ rewards sectors that maintain off-diagonal structure under recurrence; smaller κ reduces to energy-dominant selection.

Test hook — Operational use

Scan candidate sectors and estimate $(E_n, \mathcal{C}_n)$ from projected trajectories. Stable classical-looking sectors should occupy low-variance bands of $\overline{\mathcal{F}}_N$ over long windows.

8.2 Compatibility Condition (Precise Threshold)

8.2.1 Precise Compatibility & Stability Thresholds

Formal layer — The Commutativity Constraint

Let $\mathcal{B}(S)$ be the algebra of bounded operators on the substrate. Define the stability error as:

$$\epsilon := \left\| \Lambda \mathcal{F} - \mathcal{F}_{\text{loc}} \Lambda \right\|_{\text{op}} .$$

The emergent sector S_{loc} is stably coherent if there exists a local propagator $\mathcal{F}_{\text{loc}}$ such that:

$$\epsilon < \delta_{\text{coh}},$$

where δ_{coh} is a coherence margin induced by Δ (and practically tied to a spectral separation scale of H_{tot}).

Interpretation — Dynamical drift criterion

If $\epsilon \approx 0$, projected dynamics faithfully track the global update. Once ϵ exceeds δ_{coh}, representation drift appears: Ω can no longer sustain a stable local narrative, signaling decoherence of the projected sector.

Test hook — Empirical hook

Tune control parameters in a driven coherent system and monitor recurrence observables. Crossing the inferred threshold ($\epsilon \sim \delta_{\text{coh}}$) should produce a sharp loss of periodic structure (effective time-crystal melting / thermalization onset).

8.3 Geometry Extraction (Operational Pipeline)

8.3.1 Emergent Geometry from Distinguishability

Formal layer — Metric choice on projected states

For neighboring projected states, define infinitesimal distance with quantum Fisher metric:

$$\mathrm{d}s^2 = \frac{1}{4}\,\mathcal{I}_Q\big(\rho_\theta^{\text{loc}}\big)\,\mathrm{d}\theta^2,$$

where θ indexes effective coordinates in the projected sector. For discrete samples $(\rho_a^{\text{loc}}, \rho_b^{\text{loc}})$ use Bures proxy:

$$d_{ab}^2 = 2\Big(1 - \sqrt{F(\rho_a^{\text{loc}}, \rho_b^{\text{loc}})}\Big),$$

with fidelity $F(\rho_a, \rho_b) = \big(\mathrm{Tr}\,\sqrt{\sqrt{\rho_a}\rho_b\sqrt{\rho_a}}\big)^2$.

Formal layer — Graph-to-geometry protocol

Build a weighted graph over sampled projected states with edge weights

$$w_{ab} = \exp\!\big(-d_{ab}^2/\sigma^2\big)\,.$$

Estimate effective local metric tensor $g_{\mu\nu}$ via neighborhood regression (diffusion-map/Laplacian embedding), then extract curvature scalars from $g_{\mu\nu}$ on the learned manifold.

Interpretation — Interpretation

Geometry is reconstructed from state distinguishability, not assumed a priori. Distances, neighborhoods, and curvature emerge from transform difficulty under coherence constraints.

Test hook — Falsifiable signature

Persistent coherence bottlenecks should map to positive effective-curvature zones and correlate with stronger recurrence stability under Floquet updates.