To solve the compatibility condition for the **COSMOS** model, we define the precise relationship between global stroboscopic evolution and its local projection.

Here is the LaTeX subsection for **Section 8.2.1**:

"'latex

8.2.1 Precise Compatibility & Stability Thresholds

Formal layer — The Commutativity Constraint

Let $\mathcal{B}(S)$ be the algebra of bounded operators on the substrate. We define the **Stability Error** ϵ as the operator norm of the mismatch between the global update and the induced local dynamics:

$$\epsilon := \|\Lambda\mathcal{F} - \mathcal{F}_{\text{loc}}\Lambda\|_{\text{op}} = \sup_{|\psi\rangle \in S, \langle\psi|\psi\rangle=1} |(\Lambda\mathcal{F} - \mathcal{F}_{\text{loc}}\Lambda)\,|\psi\rangle|$$

The emergent sector S_{loc} is **stably coherent** if there exists a local propagator $\mathcal{F}_{\text{loc}}$ such that:

$$\epsilon < \delta_{\text{coh}}$$

where δ_{coh} is the *coherence margin* defined by the integration operator Δ (typically related to the spectral gap of the global generator H_{tot}).

Interpretation — Dynamical Drift and Decoherence

The condition $\epsilon \approx 0$ implies that the local "map" is a faithful representation of the global "land." If ϵ exceeds the threshold δ_{coh}, the local sector undergoes **representation drift**.
In this state, the effective operator Ω can no longer consistently organize histories; the projected sector "decouples" from the global rhythm. This is the origin of decoherence: not necessarily an interaction with an external environment (as there is no "outside" the substrate), but a failure of the local projection to track the global periodic update.

Child intuition — The Movie Projector

Imagine a movie projector ($\mathcal{F}$) spinning a film. The projection operator (Λ) is the lens that puts the light on the screen. If the lens and the film-strip are perfectly synchronized, the movie looks sharp ($\epsilon < \delta_{\text{coh}}$). But if the lens starts wobbling or the film slips, the picture gets blurry and eventually disappears into static. That "blurriness" is ϵ.

Test hook — Phase Transitions in Time Crystals

What would count as evidence? A sharp transition in the "classicality" of a system. If we tune a system's parameters such that ϵ crosses δ_{coh}, we should observe a sudden loss of periodic recurrence—a "melting" of the emergent time crystal into a state of maximum entropy (thermalization) where Ω no longer yields stable geometry.

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