Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

This rule is used when one needs to differentiate a function within a function. u is an intermediate which expresses the inner function.

For example, say we want to differentiate $y = (x^2 + 2)^5$
We could expand the bracket and differentiate term-by-term, but this would take a great deal of time and effort. Instead, we can implement the chain rule.

If we let $u = x^2 + 5$
This means

$$y = u^5$$

Differentiating this

$$\frac{dy}{du} = 5u^4$$

To find $\frac{dy}{dx}$ we need to know $\frac{du}{dx}$

$$u = x^2 + 5$$

So

$$\frac{du}{dx} = 2x + 0 = 2x$$

Using the chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 2x \times 5u^4 = 10xu^4$$

Using the fact that $u = x^2 + 5$

$$\frac{dy}{dx} = 10x(x^2 + 5)^4$$

Proof

Let there be a function $y = f\big(g(x)\big)$
We can attempt to differentiate this from first principles

$$\frac{dy}{dx} = \lim_{h \to 0} \left(\frac{f(g(x+h) - f(g(x))}{h} \right)$$

An expression multiplied by one is equal to that expression, so

$$\frac{dy}{dx} = \lim_{h \to 0} \left(\frac{f(g(x+h) - f(g(x))}{h} \times \frac{g(x+h) - g(x)}{g(x+h) - g(x)} \right)$$

We can re-arrange this to give

$$\frac{dy}{dx} = \lim_{h \to 0} \left(\frac{f(g(x+h) - f(g(x))}{g(x+h) - g(x)} \times \frac{g(x+h) - g(x)}{h} \right)$$

We can use a property of limits to split this into two expressions

$$\frac{dy}{dx} = \lim_{h \to 0} \left(\frac{f(g(x+h) - f(g(x))}{g(x+h) - g(x)} \right) \times \lim_{h \to 0} \left(\frac{g(x+h) - g(x)}{h} \right)$$

The above factor on the right is clearly equivalent to the derivative of $g(x)$ with respect to x

$$\frac{dy}{dx} = \lim_{h \to 0} \left(\frac{f(g(x+h) - f(g(x))}{g(x+h) - g(x)} \right) \times \frac{dg(x)}{dx}$$

We can now carry out the substitution $k = g(x+h) - g(x)$ to attempt to find expression for the above left hand product. If $h \to 0$, $k = g(x) - g(x)$ so $k \to 0$. Also note that $g(x+h) = g(x) + k$

$$\frac{dy}{dx} = \lim_{k \to 0}\left(\frac{f(g(x)+k) - f\big(g(x)\big)}{k}\right) \times \frac{dg(x)}{dx}$$

This above left hand factor is clearly the derivative of $f\big(g(x)\big)$ with respect to $g(x)$, as the $g(x)$ is equivalent to x in a more typical differentiation from first principles

$$\frac{dy}{dx} = \frac{df\big(g(x)\big)}{dg(x)} \times \frac{dg(x)}{dx}$$

Let $g(x) = u$, and remember that $y = f(g(x))$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Note

In step 2 the expression was multiplied by $\frac{g(x+h)-g(x)}{g(x+h)-g(x)}$, which had no reason other than it works to produce the answer we want.

See also

- Differentiation from First Principles
- Integration

References

Hrabovsky, G. and Susskind, L. (2014). *Classical Mechanics - The Theoretical Minimum*. London: Penguin. pp.35-37.