Addition Formulae

These equations are very useful for expressing the sine, cosine and tangent of multiple angles in a different format. The ≡ symbol means "identical to" (i.e. sine squared theta plus cosine squared theta is identical to 1). This symbols means the relationship is always true, regardless of the values of α and β.

$$\sin(\alpha+\beta) \equiv \sin\alpha\cos\beta + \cos\alpha\,\sin\beta$$

$$\cos(\alpha+\beta) \equiv \cos\alpha\cos\beta - \sin\alpha\,\sin\beta$$

For example, how can we find $\sin 75$?
We can split 75 into two more familiar expressions

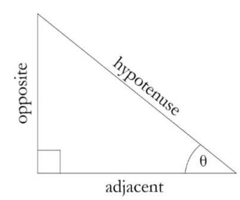
$$\sin 75 = \sin(30+45) = \sin 30\cos 45 + \cos 30 \sin 45$$

We should know what cos and sin of 30 and 45 are

$$\sin 75 = \frac{1}{2}\times\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\times\frac{\sqrt{2}}{2}$$

Therefore

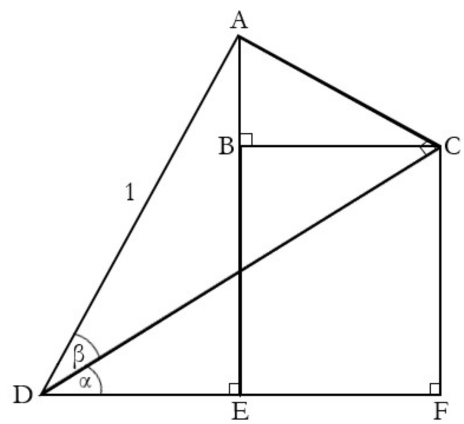
$$\sin 75 = \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4} = \frac{\sqrt{2}+\sqrt{6}}{4}$$



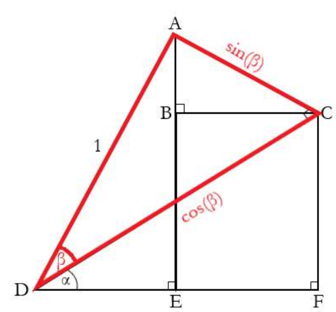
Proofs

SOH CAH TOA is used heavily in this proof. Remember that

$\sin\theta = \dfrac{opposite}{hypotenuse}$ and $\cos\theta = \dfrac{adjacent}{hypotenuse}$



Use the diagram on the right for the following proof. It is essentially two connected right-angled triangles so that the angle β adds to the angle α.



In triangle ACD

$$\cos\beta = \frac{DC}{1}$$

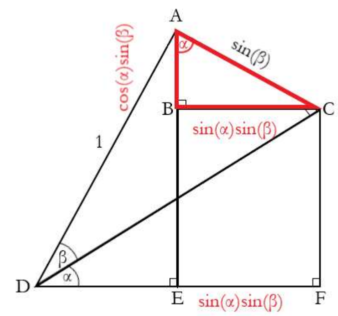
$$DC = \cos\beta$$

$$\sin\beta = \frac{AC}{1}$$

$$AC = \sin\beta$$

The angle CDF = α, and using z-angles we know that angle BCD = α and so angle ACB = 90 – α, so angle BAC = α

In triangle ABC, using the fact that $AC = \sin \beta$



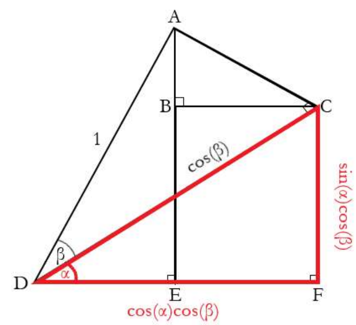
$$\cos \alpha = \frac{AB}{AC} = \frac{AB}{\sin \beta}$$

$$AB = \cos \alpha \sin \beta$$

$$\sin \alpha = \frac{BC}{AC} = \frac{BC}{\sin \beta}$$

$BC = \sin \alpha \sin \beta = EF$ (see diagram to see that $BC = EF$)

In triangle CDF, using the fact that $DC = \cos \beta$



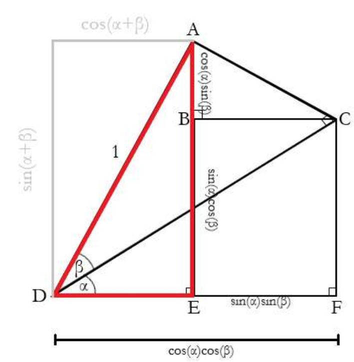
$$\cos \alpha = \frac{DF}{DC} = \frac{DF}{\cos \beta}$$

$$DF = \cos \alpha \cos \beta$$

$$\sin \alpha = \frac{CF}{\cos \beta}$$

$CF = \sin \alpha \cos \beta = BE$ (see diagram to confirm CF=BE)

In triangle ADE, using all of the information gathered



$$\cos(\alpha + \beta) = \frac{DE}{DA} = \frac{DE}{1} = DE = DF - EF$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\sin(\alpha + \beta) = \frac{AE}{DA} = \frac{AE}{1} = AE = BE + AB$$

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

There are other formulae of a similar nature. The last three are (evidently) used for subtracting angles from one another, but there is nothing fundamentally different about them.

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\sin(\alpha-\beta) \equiv \sin\alpha\cos\beta - \cos\alpha\ \sin\beta$$

$$\cos(\alpha-\beta) \equiv \cos\alpha\cos\beta + \sin\alpha\ \sin\beta$$

$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

Proofs

Considering the fact that $\tan\theta = \frac{\sin\theta}{\cos\theta}$

$$\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}$$

Using the addition formulae for sine and cosine

$$\tan(\alpha+\beta) = \frac{\sin\alpha\cos\beta + \cos\alpha\ \sin\beta}{\cos\alpha\cos\beta - \sin\alpha\ \sin\beta}$$

Dividing both the top and the bottom of the fraction by $\cos\alpha$

$$\tan(\alpha+\beta) = \frac{\left(\frac{\sin\alpha\cos\beta + \cos\alpha\ \sin\beta}{\cos\alpha}\right)}{\left(\frac{\cos\alpha\cos\beta - \sin\alpha\ \sin\beta}{\cos\alpha}\right)}$$

This can be written

$$\tan(\alpha+\beta) = \frac{\left(\frac{\sin\alpha}{\cos\alpha}\cos\beta + \frac{\cos\alpha}{\cos\alpha}\sin\beta\right)}{\left(\frac{\cos\alpha}{\cos\alpha}\cos\beta - \frac{\sin\alpha}{\cos\alpha}\sin\beta\right)}$$

Cancelling and again using $\tan\theta = \frac{\sin\theta}{\cos\theta}$

$$\tan(\alpha+\beta) = \frac{\tan\alpha\cos\beta + \sin\beta}{\cos\beta - \tan\alpha\sin\beta}$$

Dividing both the top and the bottom of the fraction by $\cos\beta$

$$\tan(\alpha+\beta) = \frac{\left(\tan\alpha\frac{\cos\beta}{\cos\beta} + \frac{\sin\beta}{\cos\beta}\right)}{\left(\frac{\cos\beta}{\cos\beta} - \tan\alpha\frac{\sin\beta}{\cos\beta}\right)}$$

So

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

For the next formulae, consider the positive version, inputting $-\beta$ instead of β

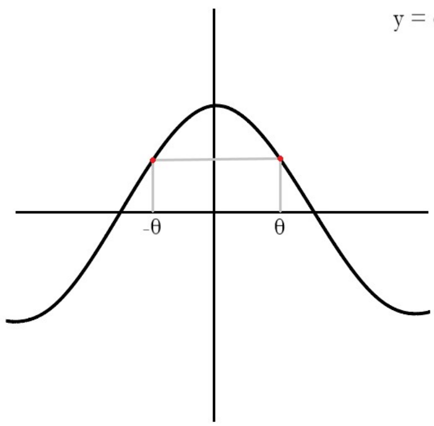
We know that $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$.

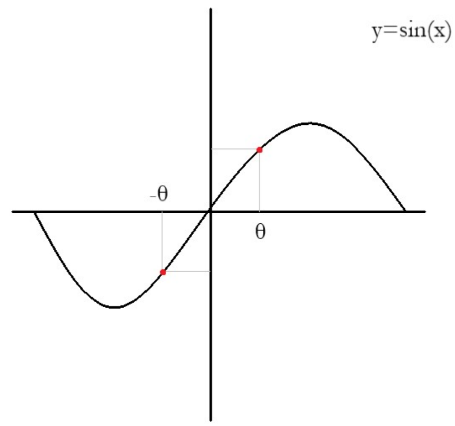
$$\sin(\alpha-\beta) = \sin(\alpha+(-\beta)) = \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta)$$

Consider the graphs of $y = \cos x$ and $y = \sin x$

If $x = \theta$ or $x = -\theta$ for $y = \cos x$ it is clear they give the same value. This means $\cos x = \cos(-x)$.

If $x = \theta$ or $x = -\theta$ it is clear the y-value for one is the negative of the y-value for the other. So $\sin x = -\sin(-x)$.





Using this, we now know that

$$\sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

So

$$\sin(\alpha-\beta) \equiv \sin\alpha\cos\beta - \cos\alpha\ \sin\beta$$

We can use a similar argument for $\cos(\alpha-\beta)$

We know that

$$\cos(\alpha+\beta) \equiv \cos\alpha\cos\beta - \sin\alpha\ \sin\beta$$

So

$$\cos(\alpha-\beta) = \cos\alpha\cos(-\beta) - \sin\alpha\ \sin(-\beta)$$

So

$$\cos(\alpha-\beta) \equiv \cos\alpha\cos\beta + \sin\alpha\ \sin\beta$$

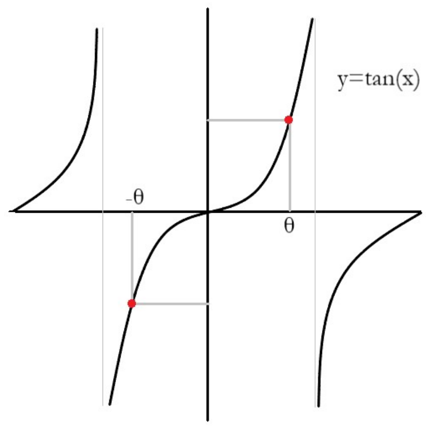
For $\tan(\alpha-\beta)$

$$\tan(\alpha-\beta) = \frac{\tan\alpha + \tan(-\beta)}{1 - \tan\alpha\tan(-\beta)}$$

From the graph it is clear that $\tan x = -\tan(-x)$

so

$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$



<u>See also</u>
- Sine, Cosine and Tangent (SOH CAH TOA)
- Double Angle Formulae

<u>References</u>
Attwood, G. et al. (2017). *Edexcel A level Mathematics - Pure Mathematics - Year 2*. London: Pearson Education. pp.167-168.