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An Exploration of Lissajous Figures

A sine wave or a sinusoid is characterized by being the graphical representation of smooth repetitive oscillation. When one sine wave is paired with another sine wave, they both still retain their individual shapes. Lissajous figures or Lissajous curves appear as a result this relationship. When two sine waves have a relationship where the frequencies are certain ratios, the resultant image is a Lissajous figure. There are many ways to create Lissajous figures. In this paper I talk about three of the ways to make Lissajous figures. The first is by graphing two parametric equations in a Java program. The next is by viewing the two waves on an oscilloscope wherein I set up an oscilloscope with two function generators and played around with different settings. I compare those observations to the figures I've created using Java. The third way to create Lissajous figures is by using an oscillator that physically draws them. I talk about the different mechanisms that can be used to draw the figures, how the parameters can be changed, and cite some images from others who have experimented with oscillators in the past and have drawn interesting figures.

Lissajous figures are created when using a series of parametric equations specifically those shown below. These equations are a description of complex harmonic oscillation.

$$x(t)=A\ cos(a\bullet t+\phi)$$

$$y(t)=B\ cos(b\bullet t)$$

The paramedic equations are functions of time and represent the vertical and horizontal components of the sine waves. In the equation, $x(t)$ and $y(t)$ are obviously the vertical and horizontal components as a function of time, where t is time. A and B are the amplitudes and a and b determine the frequencies of the waves. The phase is represented by ϕ, where $\pi/2$ is ninety degrees out of phase, π is one hundred and eighty degrees out of phase, and so on. The figures appear when the frequencies of the two waves are at a certain ratio to each other. The ratio determines the number of major lobes in the figure. As shown in Figure 1, the ratio of the frequencies is 1:4, and there are four major lobes that can be counted in the figure. The waves have to be out of phase for the Lissajous figure to appear. If the sine waves are in phase, then what shows up just looks like a single simple sine wave. The figure also depends on the vertical and horizontal component, which are the values of a and b in the equations above. The phase also makes a difference. Figure 2 shows the

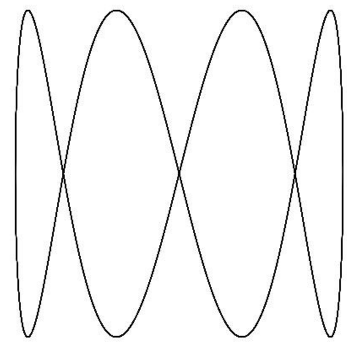

Figure 1: Frequencies with 1:4 ratio and the waves are 90 degrees out of phase.

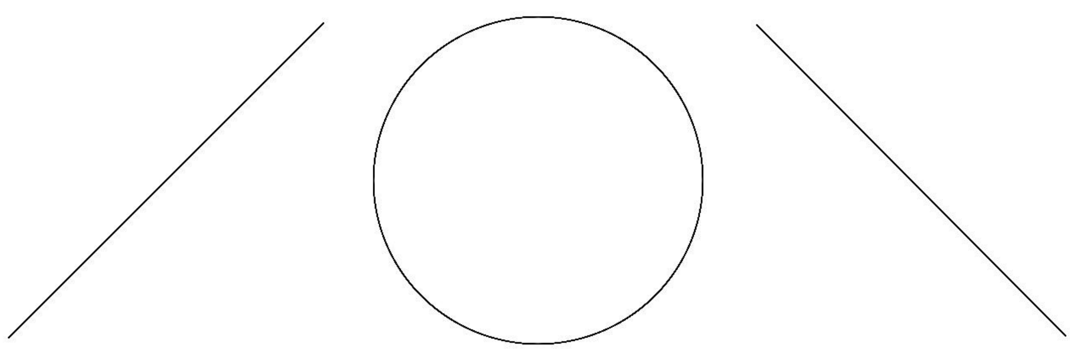

Figure 2: Where the waves are in phase, 90 degrees out of phase, and 180 degrees out of phase, but the frequencies are equal.

frequencies with a 1:1 ratio, but the waves are in phase, ninety degrees out of phase, and one hundred and eighty degrees out phase. Figure 3 then shows figures where the frequencies have 1:3 ration, but the first figure shows them in phase and the second shows them ninety degrees out of phase.

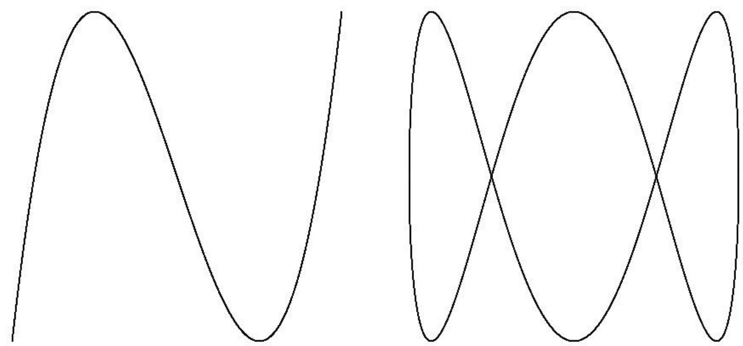

Figure 3: The frequencies have a 1:3 ratio and are first, in phase, then 90 degrees out of phase.

There are many kinds of harmonic oscillators. A simple harmonic oscillator which has a restoring force proportional to its displacement from equilibrium, and a damped oscillator where a damping force acts on the system and a viscous damping coefficient are two examples of harmonic oscillators[1]. A third type is a parametric oscillator that is also a type of driven oscillator. The drive energy is provided by varying the parameters of the oscillator, where the force oscillates in a sinusoidal manner[2]. A few parameters that can be changed are the resonance frequency of the oscillator and the damping of the system.

[1] Russell, Daniel A. "The Simple Harmonic Oscillator." *The Simple Harmonic Oscillator*. The Pennsylvania State University, 2011. Web. 12 May 2016.

[2] Elert, Glenn. "4.1 Harmonic Oscillator." *Hypertextbook*. The Chaos Hypertextbook, 2007. Web. 12 May 2016.

Dr. Java is program associated with JavaScript that allows mainly students to write Java programs using an easy-to-use intuitive interface[3]. Using Dr. Java, I used code to write a program where by inputting the parametric sine equations, establishing a time variable, and setting up the constants, was able to use the draw feature and create the resulting figure from the two equations. The values of A and B were both 1. The value for t was given a big enough number to be able to watch the figures get drawn. The values for a and b are both changeable as they represent the frequency of the wave, and the phase ϕ was also changed at times throughout testing. After compiling and running the software, the program would draw the image. It was very useful to watch the program draw the figure. because of some of the figures result in squares with a lot of crossing lines in the middle, so it is hard to determine how the figure actually came to be. Because of this, I videoed one of the figures being drawn in Dr. Java. While you can save the image file as a .jpg, it does not allow screenshots from the in progress image, so I had to take a video of the figure being drawn. The captured images are shown in Figure 4 below.

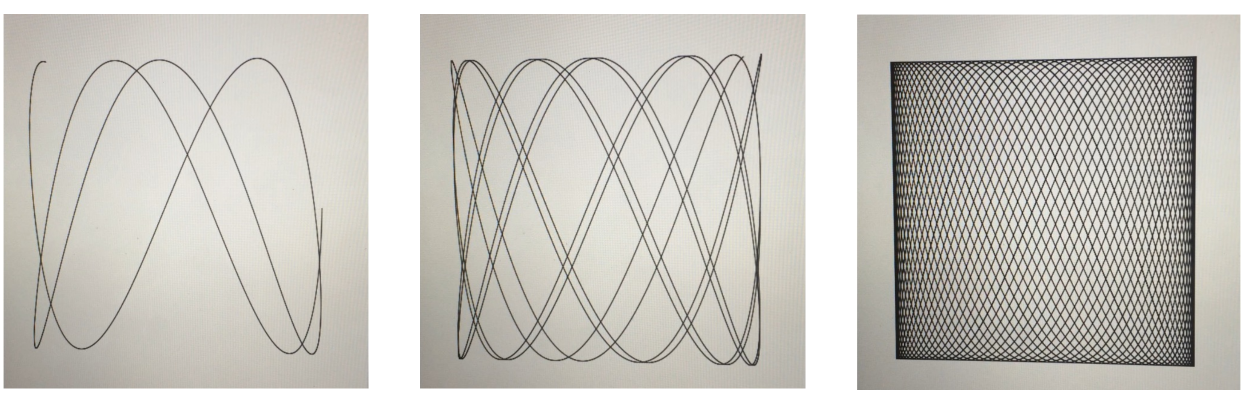

Figure 4: Images captured from video taken of Java drawing of a Lissajous figure.

[3] "DrJava." *DrJava*. Rice University, n.d. Web. 12 May 2016.

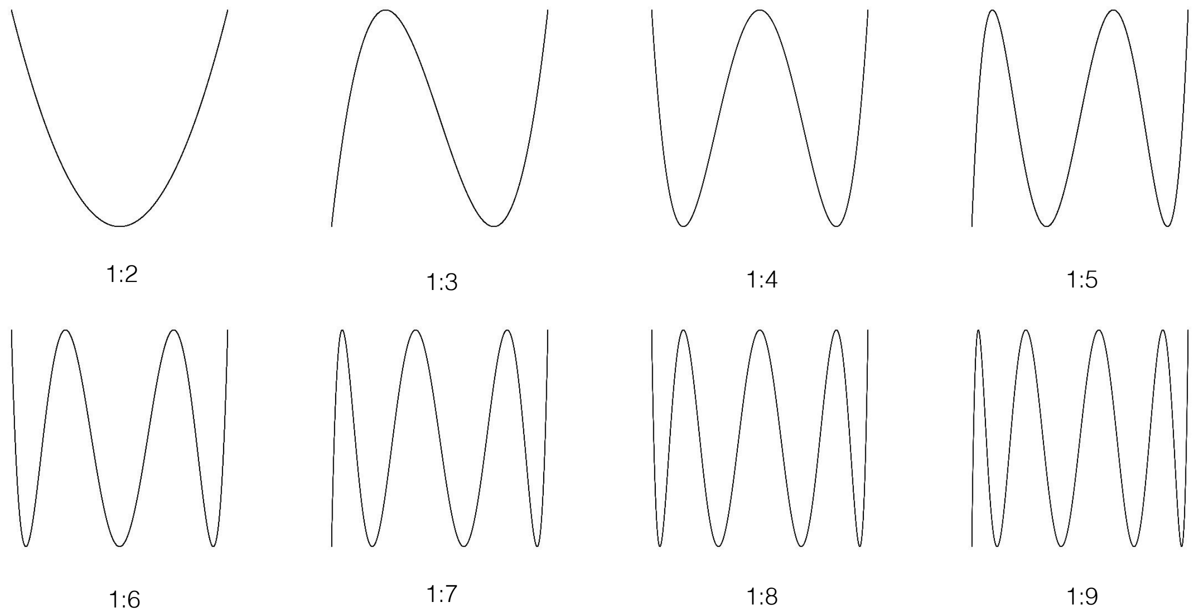

Figure 5: Figures created in Java using a series of ratios for the frequency values where all the waves are in phase.

Figure 5 shows a series of frequency ratios where the waves are in phase. Each wave has properties of a simple sinusoidal wave. As the ratio gets larger, the frequency of the resulting wave gets higher and the wavelength gets lower, as shown in Figure 5 by the wave becoming skinnier and skinner. Two sinusoidal wave functions that are in phase with matching frequencies create a diagonal line to the right. When the two wave functions are in phase but with frequencies that are ratios to each other or a certain degree out of phase, the resulting figures get more complex and interesting.

A large part of this project was attempting to make the same Lissajous figures I had seen on Java on an oscilloscope. An oscilloscope is a technological tool used to observe the change in electrical signals over time. Most modern oscilloscopes have the capability of plotting in X-Y mode which is very useful for graphing current versus voltage as well as Lissajous patterns. An oscilloscope can track the phase differences between multiple input signals, and when given the correct phase and frequencies, will create Lissajous figures. The benefit of viewing a Lissajous figure on the oscilloscope rather than digitally on the Java program is that the lines will fade out and gives a clearer image of the figure. On the Java program, the images created are still fascinating and telling about the inputs, but it is hard to discern it as anything other than a two-dimensional mess of lines.

Setting up in the lab required the following materials: two function generators, an oscilloscope, gator clips, and wire. The lab setup in eluding the two function generators and the oscilloscope are shown in Figure 6. After hooking up the function generators and fine-tuning the many settings, it came time to experiment with the frequencies. You first had to find the starting frequency. To do this, you had to adjust the dial of one of the

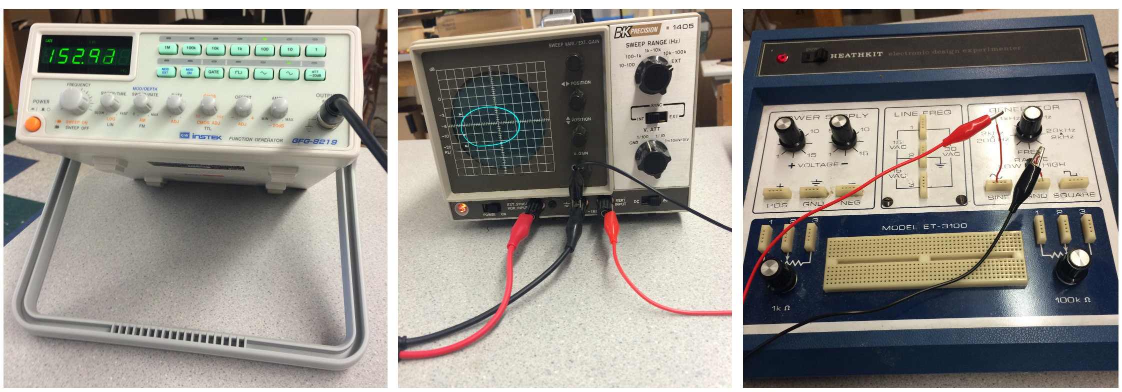

Figure 6: The lab setup with the two function generators on either side of the oscilloscope.

function generators to the point where a figure showed up on the oscilloscope. At this point you fine-tune it, to get it as exact as possible. The figure that shows up is almost a circle, that appears to be rotating. The two frequencies fluctuate, so they can not really ever be the same value. On the oscilloscope, it is very difficult to find the frequency that would allow the two waves to be in phase and then the correct degree of phase difference necessary to create a Lissajous figure. The next issue is that it was virtually impossible to make the frequency match the exact number wanted due to the sensitivity of the dial.

The first goal was to match frequencies with the waves ninety degrees out of phase which should create a circle like the one earlier in Figure 2. However, as said previously, it is very difficult to match frequencies, so there is a little difference in the frequencies. On the oscilloscope, this can be seen by what appears to be a rotating oval like in Figure 6 on the oscilloscope monitor. It is visually similar to what happens as a coin spinning on a table decelerates. This was then recreated using the Java program and is shown in Figure 7, where the waves are in phase but the frequencies are slightly different. The overlapping of lines gives slight indication that the object has some movement, but it is hard to tell exactly what the shape is doing. This is why it is helpful to view on the oscilloscope. Another ratio that was viewed using the oscilloscope was 3:5. The figure that resulted appeared to be rotating due to the

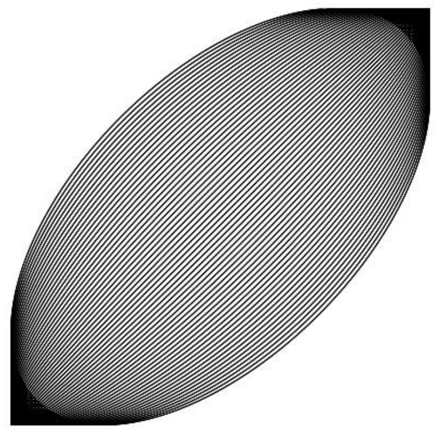

Figure 7: Figure drawn in Java where the waves are in phase but the frequencies slightly differ.

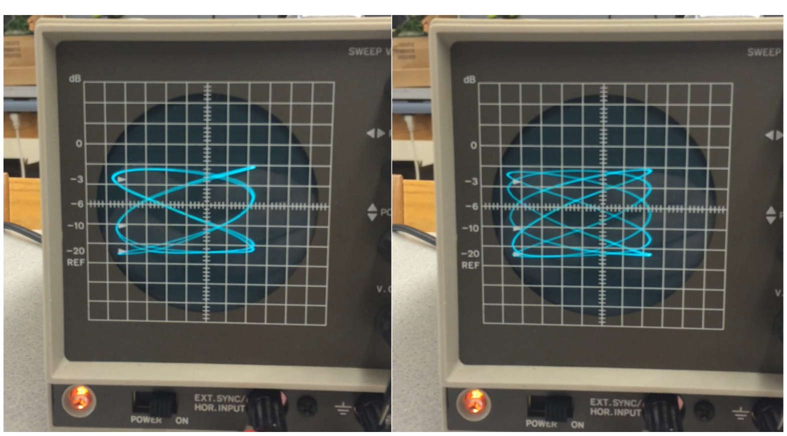

Figure 8: Images from the oscilloscope where the frequencies have a ratio of 3:5.

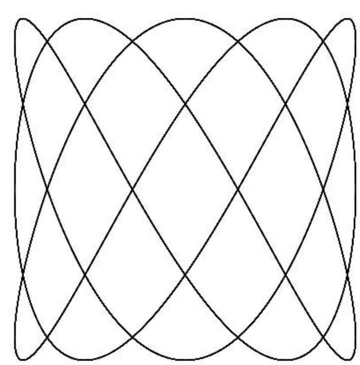

Figure 9: The frequencies have a 3:5 ratio and the waves are 90 degrees out of phase.

slight difference in the frequencies. You could count the number of lobes by watching it oscillate. Two images from the oscilloscope of the rotating object are shown in Figure 8. The ratio was then recreated in the Java program, which made the image in Figure 9, and the two look very similar.

One of the most well known harmonic oscillators is a pendulum. Pendulums have a mass at the bottom of a string or rod and the have a resonant frequency. When describing a pendulum, one has to take into account the mass, length of the string or rod, gravity, the period of oscillation, and the amplitude for systems with larger amplitudes[4]. Multiple pendulums can be used to make Lissajous figures like those shown in the figures on the previous or more complex drawings. A system of pendulums that employs harmonic motion to draw figures is called a harmonograph. In a lateral harmonograph, there a two pendulums: one controlling the pencil and the other controlling the paper. An example of this is shown in Figure 10. A slightly more complex harmonograph is a rotary harmonograph that uses three pendulums shown in Figure 11 on the next page. The difference between the two is that the more complex harmonograph has more drawing capabilities and can draw at a higher level of complexity; however, a simpler

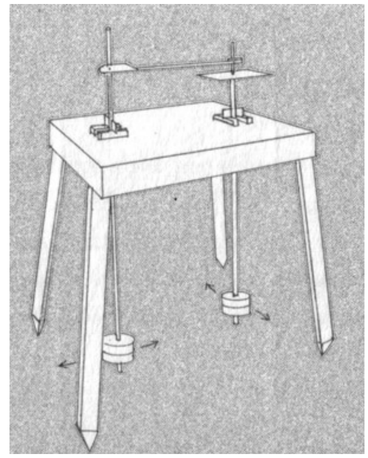

Figure 10: Lateral harmonograph using two pendulums.

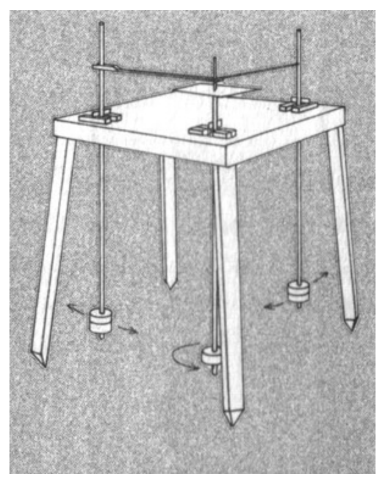

Figure 11: Rotary harmonograph using three pendulums.

[4] "Simple Pendulum." *HyperPhysics*. N.p., n.d. Web. 12 May 2016.

Figure 10 and 11 taken from http://www.birdandbee.co.uk/index.php/harmony/what-is-a-harmonograph/

harmonograph can still draw ellipses, figure eights, or spirals. The following figures were taken from various third party sources to show the capabilities and complexity of harmonographs and their creations.

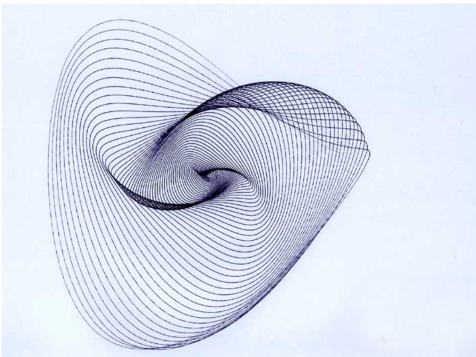

Figure 12: Taken from https://en.wikipedia.org/wiki/Harmonograph

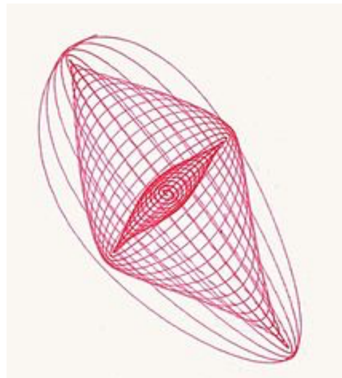

Figure 13: Taken from https://en.wikipedia.org/wiki/Harmonograph

Figure 14: Taken from http://www.lukewallin.co.uk/misc/harmonograph

Not only are Lissajous figures important in mathematics for mathematical reasons, but they are also important in art as well. There is not thought to be much perfection in art; it is expected to be freeform and an expression of human life which is inherently not perfect. Lissajous figures are mathematical perfection. Whether generated on an oscilloscope, using a software that draws it, a pendulum system, or digitally, they show how amazing math can be. The thing that surprised me the most while doing this was how aesthetically pleasing the figures are while drawing. They are absolutely amazing. After learning about the math involved and the specific ratios of frequencies or the degree of phase difference. Next, I will talk about the most amazing figures that I came across while playing around with the software.

First, a series of figures that I found very interesting was when keeping the phase $\pi/2$, but having the frequencies be slightly different. When you keep the frequency ratio the same but increase the overall frequency, so the figures appear to be darker in Figure 15 due to the lines being closer together as the frequency increases.

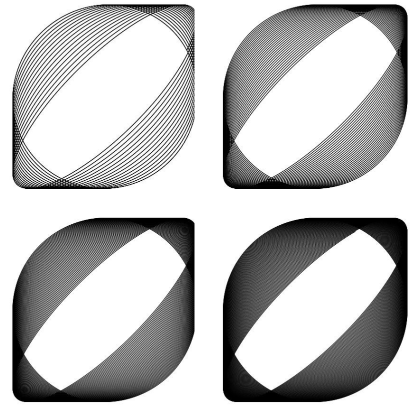

Figure 15: Same ratio of increasing frequencies 90 degrees out of phase.

Next, are a few figures where a small change in frequency made the ratios slightly off and resulted in these Lissajous figures. The phase remained at $\pi/2$. These figures are very interesting because you can see the overlap of the lines. The first looks like a coil, that if the figure were to continue upward, then the shape would be a visible coil. The second also appears to be coil-like, just thicker. The third is a lot more confusing to

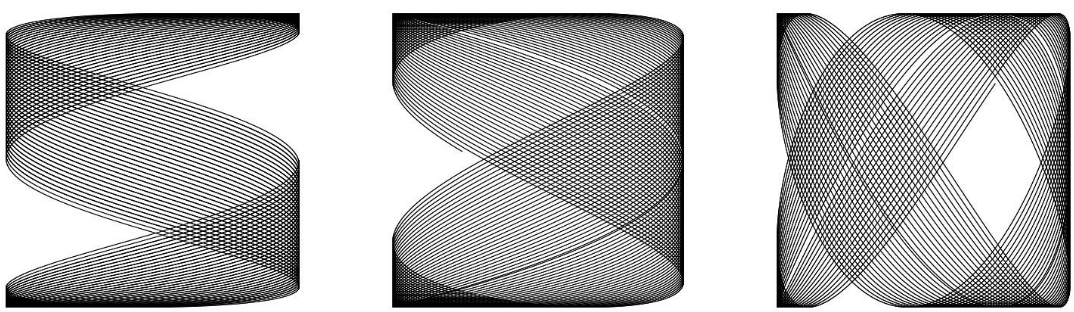

Figure 16: Figures made in the Dr. Java program with similar frequency ratios and the same phase difference.

discern. It's obvious that there is a lot of overlap, but it's just a very intriguing figure to try to figure out.

Throughout the project of learning about Lissajous figures and trying the different ways of creating them through computer software, oscilloscopes, watching a harmonograph at work, and doing the math, I have come to a better understanding of harmonic oscillators and their motion. I have written out the equations and changed the parameters probably a hundred times. I have tried many ratios and phase differences to find absolutely everything interesting that I can. The accumulation of images and figures that I now have is massive compared the little knowledge I had on Lissajous curves at the beginning of this exploration. Overall, this project was successful in comparing different methods of making Lissajous figures and discovering the limits that the figures will reach. There is a lot more that can be done in the future relating to Lissajous figures and the many ways to make them. Pieces of mathematical perfection is an amazing way to foster an appreciation of numbers and mathematical relations, and Lissajous figures couldn't be any more perfect way of doing that.