Spectral Atmospheric General Circulation Model in Python

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Chapter 1

Introduction

Chapter 2

Dynamical Core

2.1 Vertical Hybrid Coordinate

In the hybrid coordinate, pressure is a function of surface pressure and hybrid coefficients:

$$p(\lambda, \mu, \eta) = A(\eta)p_0 + B(\eta)p_s(\lambda, \mu) \tag{2.1}$$

Where, λ is the longitude, $\mu = \sin\varphi$ (where φ is the latitude), η varies from 0 at the top of atmosphere (TOA) to 1 at the surface; p_0 is defined as 1×10^5 Pa.

From 2.1, we can also derive the following useful relation:

$$\frac{\partial p}{\partial p_s} = B(\eta) \tag{2.2}$$

2.2 Primitive Equations

The full prognostic equations over a spherical planet with topography written in the vorticity-divergence forms are:

$$\frac{\partial \zeta}{\partial t} = \frac{1}{a(1-\mu^2)}\frac{\partial n_V}{\partial \lambda} - \frac{1}{a}\frac{\partial n_U}{\partial \mu} + F_{\zeta H} \tag{2.3}$$

$$\frac{\partial \delta}{\partial t} = \frac{1}{a(1-\mu^2)}\frac{\partial n_U}{\partial \lambda} + \frac{1}{a}\frac{\partial n_V}{\partial \mu} - \nabla^2(E+\Phi) + F_{\delta H} \tag{2.4}$$

$$\frac{\partial T}{\partial t} = -\frac{1}{a(1-\mu^2)}\frac{\partial (TU)}{\partial \lambda} - \frac{1}{a}\frac{\partial (TV)}{\partial \mu} + T\delta - \dot{\eta}\frac{\partial p}{\partial \eta}\frac{\partial T}{\partial p} + \frac{RT_v}{c_p^*}\frac{\omega}{p} + Q + F_{TH} + F_{FH} \tag{2.5}$$

$$\frac{\partial q}{\partial t} = -\frac{1}{a(1-\mu^2)}\frac{\partial (qU)}{\partial \lambda} - \frac{1}{a}\frac{\partial (qV)}{\partial \mu} + q\delta - \dot{\eta}\frac{\partial p}{\partial \eta}\frac{\partial q}{\partial p} + S \tag{2.6}$$

$$\frac{\partial \ln p_s}{\partial t} = -\int_{(\eta_t)}^{(1)} \left(\vec{V}\cdot\nabla \ln p_s\right) d\left(\frac{\partial p}{\partial p_s}\right) - \frac{1}{p_s}\int_{p(\eta_t)}^{p(1)} \delta dp \tag{2.7}$$

Where, the prognostic variables:

ζ —— relative vorticity (s^{-1})
δ —— divergence (s^{-1})
T —— temperature (K)
q —— specific humidity ($kg\ kg^{-1}$)
$\ln p_s$ —— log(surface pressure) ($\ln(Pa)$)

Parameters and other variables:

a —— the planet radius;
R —— gas constant;
$f = 2\Omega \sin\varphi$ —— Coriolis parameter (where Ω is the angular velocity of the planet);
$(U, V) = (u\cos\varphi, v\cos\varphi)$ —— latitude-scaled zonal and meridional wind components;
$F_{\zeta H}$ —— vorticity tendency due to horizontal diffusion;
$F_{\delta H}$ —— divergence tendency due to horizontal diffusion;
F_{TH} —— temperature tendency due to horizontal diffusion;
F_{FH} —— frictional heating due to the horizontal diffusion on momentum;
Q —— temperature tendency due to physical parameterization (diabatic heating);
S —— moisture tendency due to physical parameterization (source);
$E = \dfrac{U^2+V^2}{2(1-\mu^2)}$ —— kinetic energy;
$\Phi = \Phi_s + R\displaystyle\int_{p(\eta)}^{p(1)} T_v d\ln p$ —— geopotential, where Φ_s is the surface geopotential;
$T_v = \left[1 + \left(\dfrac{R_v}{R} - 1\right)q\right]T$ —— virtual temperature;
$c_p^* = \left[1 + \left(\dfrac{c_{pv}}{c_p} - 1\right)q\right]c_p$ —— water vapor weighted specific heat;
$\vec{V}\cdot\nabla\ln p_s = \dfrac{U\partial\ln p_s}{a(1-\mu^2)\partial\lambda} + \dfrac{V\partial\ln p_s}{a\partial\mu}$ —— horizontal advection of log(surface pressure);
$n_U = (\zeta+f)V - \dot{\eta}\dfrac{\partial p}{\partial\eta}\dfrac{\partial U}{\partial p} - \dfrac{RT_v}{a}\dfrac{p_s}{p}\dfrac{\partial p}{\partial p_s}\dfrac{\partial\ln p_s}{\partial\lambda} + F_u\cos(\varphi)$
$n_V = -(\zeta+f)U - \dot{\eta}\dfrac{\partial p}{\partial\eta}\dfrac{\partial V}{\partial p} - \dfrac{RT_v}{a}\dfrac{p_s}{p}\dfrac{\partial p}{\partial p_s}(1-\mu^2)\dfrac{\partial\ln p_s}{\partial\mu} + F_v\cos(\varphi)$, where F_u, F_v are the tendency of zonal and meridional wind due to the physical parameterization.

Diagnostic equations:

$$\begin{aligned}\dot{\eta}\frac{\partial p}{\partial\eta} = \frac{\partial p}{\partial p_s}&\left[p_s\int_{(\eta_t)}^{(1)}\left(\vec{V}\cdot\nabla\ln p_s\right)d\left(\frac{\partial p}{\partial p_s}\right) + \int_{p(\eta_t)}^{p(1)}\delta dp\right]\\ &-\left[p_s\int_{(\eta_t)}^{(\eta)}\left(\vec{V}\cdot\nabla\ln p_s\right)d\left(\frac{\partial p}{\partial p_s}\right) + \int_{p(\eta_t)}^{p(\eta)}\delta dp\right]\end{aligned} \tag{2.8}$$

$$\omega = \frac{\partial p}{\partial p_s}p_s\left(\vec{V}\cdot\nabla\ln p_s\right) - p_s\int_{(\eta_t)}^{(\eta)}\left(\vec{V}\cdot\nabla\ln p_s\right)d\left(\frac{\partial p}{\partial p_s}\right) - \int_{p(\eta_t)}^{p(\eta)}\delta dp \tag{2.9}$$

2.3 Vertical Difference Method

2.3.1 Hybridstatic Equation

Rewriting the multiple variables as column-vectors and the vertical integration as multiplication between a matrix and a column-vector, the hydrostatic equation becomes:

$$\underline{\Phi} = \Phi_s + R\boldsymbol{H}\underline{T_v} \tag{2.10}$$

where, the underlines represent column-vectors; $\boldsymbol{H}$ is an upper triangular matrix:

$$H_{kl} = \begin{cases} \Delta p_l/p_l, & (k<l) \\ \Delta p_l/(2p_l), & (k=l) \end{cases} \tag{2.11}$$

2.3.2 Vertical Advection Equation

To conserve the kinetic energy during vertical transport, the discrete form of the vertical advection equation of variable X should be:

$$\left(\dot{\eta}\frac{\partial p}{\partial \eta}\frac{\partial X}{\partial p}\right)_k = \frac{1}{2\Delta p_k}\left[\left(\eta\frac{\dot{\partial p}}{\partial \eta}\right)_{k+\frac{1}{2}}(X_{k+1}-X_k) + \left(\eta\frac{\dot{\partial p}}{\partial \eta}\right)_{k-\frac{1}{2}}(X_k - X_{k-1})\right] \tag{2.12}$$

Where, the η-coordinate vertical velocity is :

$$\left(\dot{\eta}\frac{\partial p}{\partial \eta}\right)_{k+\frac{1}{2}} = B_{k+\frac{1}{2}}\sum_{l=1}^{K}[\delta_l \Delta p_l + p_s(\vec{V_l}\cdot\nabla \ln p_s)\Delta B_l] - \sum_{l=1}^{k}[\delta_l \Delta p_l + p_s(\vec{V_l}\cdot\nabla \ln p_s)\Delta B_l] \tag{2.13}$$

Note: $\left(\dot{\eta}\frac{\partial p}{\partial \eta}\right)_{\frac{1}{2}} = \left(\dot{\eta}\frac{\partial p}{\partial \eta}\right)_{K+\frac{1}{2}} = 0.$

2.3.3 Omega Equation

$$\omega_k = B_k p_s \vec{V_k}\cdot\nabla \ln p_s - \sum_{l=1}^{k} C_{kl}[\delta_l \Delta p_l + p_s(\vec{V_l}\cdot\nabla \ln p_s)\Delta B_l] \tag{2.14}$$

Where, $\boldsymbol{C}$ is a lower triangular matrix:

$$C_{kl} = \begin{cases} 1, & (k > l) \\ 0.5, & (k = l) \end{cases} \tag{2.15}$$

Comparing matrix $\boldsymbol{C}$ with matrix $\boldsymbol{H}$, we can easily find their relationship:

$$H_{kl} = C_{lk}\frac{\Delta p_l}{p_l} \tag{2.16}$$

2.4 Semi-Implicit Time Integration

Basically, the idea of semi-implicit time integration is using central difference scheme for nonlinear terms (i.e. slow geostrophic evolution) and backwards difference scheme for linear terms (i.e. fast geostrophic adjusment). Mathematically, the semi-implicit time discrete equation for a variable X can be written as:

$$\begin{aligned}\frac{X^{t+\Delta t} - X^{t-\Delta t}}{2\Delta t} &= \boldsymbol{N}(X^t) + \boldsymbol{L}(\frac{X^{t-\Delta t} + X^{t+\Delta t}}{2}) \\ &= \boldsymbol{RHS}(X^t) + \boldsymbol{L}(\frac{X^{t-\Delta t} + X^{t+\Delta t}}{2} - X^t)\end{aligned} \tag{2.17}$$

where, $\boldsymbol{RHS}(X)$ represents the right-hand-side of the prognostic equation of X; $\boldsymbol{N}(X)$ represents the nonlinear term, and $\boldsymbol{L}(X)$ represents the linear term; the superscript $t, t-\Delta t, t+\Delta t$ represent the current, backwards, and forwards time step, respectively.

Define the reference temperature profile and reference pressure for linearization:

$$\begin{aligned}&T(\lambda,\mu,\eta,t) = T^r(\eta) + T'(\lambda,\mu,\eta,t) \\ &p(\lambda,\mu,\eta,t) = p^r(\eta) + p'(\lambda,\mu,\eta,t) \\ &p^r(\eta) = A(\eta)p_0 + B(\eta)p_s^r\end{aligned} \tag{2.18}$$

Where, the reference surface pressure p_s^r is defined as 1×10^5 Pa.

2.4.1 Linear Terms in Divergence Equation

$$\begin{aligned}
&-\nabla\cdot\left(\nabla\Phi + RT_v\frac{p_s}{p}\frac{\partial p}{\partial p_s}\nabla\ln p_s\right) + ... \\
&\Rightarrow -R\nabla^2\int_{p(\eta)}^{p(1)} T_v d\ln p - RT_v\frac{p_s}{p}\frac{\partial p}{\partial p_s}\nabla^2\ln p_s + ... \\
&\Rightarrow -R\int_{p^r}^{p_s^r}\nabla^2 T d\ln p^r - R\int_{p}^{p_s} T^r\nabla^2 d\ln p - RT^r\frac{p_s^r}{p^r}\frac{\partial p}{\partial p_s}\nabla^2\ln p_s + ... \\
&\Rightarrow -R\int_{p^r}^{p_s^r}\nabla^2 T d\ln p^r - R\int_{(\eta)}^{(1)} T^r\frac{p_s^r}{p^r} d\left(\frac{\partial p}{\partial p_s}\right)\nabla^2\ln p_s - RT^r\frac{p_s^r}{p^r}\frac{\partial p}{\partial p_s}\nabla^2\ln p_s + ... \\
&\Rightarrow -R\boldsymbol{H^r}\nabla^2\underline{T} - R\left(\underline{b^r} + \underline{h^r}\right)\nabla^2\ln p_s + ...
\end{aligned} \tag{2.19}$$

Where, $\boldsymbol{H^r}$ is an upper triangular matrix:

$$H_{kl}^r = \begin{cases} \Delta p_l^r/p_l^r, & (k<l) \\ \Delta p_l^r/(2p_l^r), & (k=l) \end{cases} \tag{2.20}$$

Column vectors:

$$\begin{aligned}
b_k^r &= p_s^r\sum_{l=k+1}^{K} T_l^r\left[\frac{B_{l+\frac{1}{2}}}{p_{l+\frac{1}{2}}^r} - \frac{B_{l-\frac{1}{2}}}{p_{l-\frac{1}{2}}^r}\right] \\
h_k^r &= T_k^r\frac{p_s^r}{p_k^r}B_k
\end{aligned} \tag{2.21}$$

2.4.2 Linear Terms in Temperature Equation

$$\begin{aligned}
&-\dot\eta\frac{\partial p}{\partial\eta}\frac{\partial T}{\partial p} + \frac{RT_v}{c_p^*}\frac{\omega}{p} + ... \\
&\Rightarrow -\left(\frac{\partial p}{\partial p_s}\int_{p(\eta_t)}^{p(1)}\delta dp - \int_{p(\eta_t)}^{p(\eta)}\delta dp\right)\frac{\partial T}{\partial p} - \frac{RT_v}{c_p^* p}\int_{p(\eta_t)}^{p(\eta)}\delta dp + ... \\
&\Rightarrow -\left(\frac{\partial p}{\partial p_s}\int_{p^r(\eta_t)}^{p_s^r}\delta dp^r - \int_{p^r(\eta_t)}^{p^r}\delta dp^r\right)\frac{\partial T^r}{\partial p} - \frac{RT^r}{c_p p^r}\int_{p^r(\eta_t)}^{p^r}\delta dp^r + ... \\
&\Rightarrow -\boldsymbol{D^r}\underline{\delta} + ...
\end{aligned} \tag{2.22}$$

Where, $\boldsymbol{D^r}$ is a full matrix:

$$\begin{aligned}
&D_{kl}^r = \frac{\Delta p_l^r}{2\Delta p_k^r}\left[(T_k^r - T_{k-1}^r)(B_{k-\frac{1}{2}} - \epsilon_{kl}) + (T_{k+1}^r - T_k^r)(B_{k+\frac{1}{2}} - \sigma_{kl})\right] + \frac{RT_k^r}{c_p}\frac{\Delta p_l^r}{p_k^r}C_{kl} \\
&\epsilon_{kl} = \begin{cases} 1, & (k>l) \\ 0, & (k\le l) \end{cases}, \qquad \sigma_{kl} = \begin{cases} 1, & (k\ge l) \\ 0, & (k<l) \end{cases}
\end{aligned} \tag{2.23}$$

Note: if $\frac{\partial T^r}{\partial p} = 0$, the vertical advection contribution vanishes and $\boldsymbol{D^r}$ becomes a lower triangular matrix, determined by the lower triangular matrix $\boldsymbol{C}$ in previous section.

2.4.3 Linear Terms in log(surface pressure) Equation

$$\begin{aligned}
&-\frac{1}{p_s}\int_{p(\eta_t)}^{p(1)}\delta dp + ... \\
&\Rightarrow -\frac{1}{p_s^r}(\underline{\Delta p^r})^T\underline{\delta} + ...
\end{aligned} \tag{2.24}$$

Where, the superscript T means transposing the column vector to a row vector.

2.4.4 Discrete Equations in Grid Point Space

$$(\underline{\zeta})^{t+\Delta t} = (\underline{\zeta})^{t-\Delta t} + \frac{2\Delta t}{a(1-\mu^2)}\left[\frac{\partial}{\partial\lambda}(\underline{n_V})^t - (1-\mu^2)\frac{\partial}{\partial\mu}(\underline{n_U})^t\right] + 2\Delta t\underline{F_{\zeta H}} \tag{2.25}$$

$$\begin{aligned}(\underline{\delta})^{t+\Delta t} = (\underline{\delta})^{t-\Delta t} &+ \frac{2\Delta t}{a(1-\mu^2)}\left[\frac{\partial}{\partial\lambda}(\underline{n_U})^t + (1-\mu^2)\frac{\partial}{\partial\mu}(\underline{n_V})^t\right]\\ &-2\Delta t\nabla^2[(\underline{E})^t + (\underline{\Phi})^t]\\ &-2\Delta tR\boldsymbol{H^r}\nabla^2\left[\frac{(\underline{T})^{t-\Delta t}+(\underline{T})^{t+\Delta t}}{2} - (\underline{T})^t\right]\\ &-2\Delta tR(\underline{b^r}+\underline{h^r})\nabla^2\left[\frac{(\ln p_s)^{t-\Delta t}+(\ln p_s)^{t+\Delta t}}{2} - (\ln p_s)^t\right]\\ &+2\Delta t\underline{F_{\delta H}}\end{aligned} \tag{2.26}$$

$$\begin{aligned}(\underline{T})^{t+\Delta t} = (\underline{T})^{t-\Delta t} &- \frac{2\Delta t}{a(1-\mu^2)}\left[\frac{\partial}{\partial\lambda}(\underline{TU})^t + (1-\mu^2)\frac{\partial}{\partial\mu}(\underline{TV})^t\right]\\ &+2\Delta t\left[(\underline{T})^t(\underline{\delta})^t + \left(\frac{RT_v}{c_p^*}\frac{\omega}{p}\right)^t + (\underline{Q})^t - \underline{(\dot\eta\frac{\partial p}{\partial\eta}\frac{\partial T}{\partial p})^t}\right]\\ &-2\Delta t\boldsymbol{D^r}\left[\frac{(\underline{\delta})^{t-\Delta t}+(\underline{\delta})^{t+\Delta t}}{2} - (\underline{\delta})^t\right]\\ &+2\Delta t(\underline{F_{TH}}+\underline{F_{FH}})\end{aligned} \tag{2.27}$$

$$\begin{aligned}(\ln p_s)^{t+\Delta t} = (\ln p_s)^{t-\Delta t} &- 2\Delta t\left[\underline{(\vec V\cdot\nabla\ln p_s)^{tT}}\underline{\Delta B} + \frac{1}{(\underline{p_s})^t}(\underline{\delta})^{tT}(\underline{\Delta p})^t\right]\\ &-\frac{2\Delta t}{p_s^r}(\underline{\Delta p^r})^T\left[\frac{(\underline{\delta})^{t-\Delta t}+(\underline{\delta})^{t+\Delta t}}{2} - (\underline{\delta})^t\right]\end{aligned} \tag{2.28}$$

2.4.5 Discrete Equations in Spectral Space

$$(\underline{\zeta_n^m})^{t+\Delta t} = \underline{VS_n^m} \tag{2.29}$$

$$\begin{aligned}(\underline{\delta_n^m})^{t+\Delta t} = \underline{DS_n^m} &+ \Delta tR\boldsymbol{H^r}\frac{n(n+1)}{a^2}(\underline{T_n^m})^{t+\Delta t}\\ &+\Delta tR(\underline{b^r}+\underline{h^r})\frac{n(n+1)}{a^2}[(\ln p_s)_n^m]^{t+\Delta t}\end{aligned} \tag{2.30}$$

$$(\underline{T_n^m})^{t+\Delta t} = \underline{TS_n^m} - \Delta t\boldsymbol{D^r}(\underline{\delta_n^m})^{t+\Delta t} \tag{2.31}$$

$$[(\ln p_s)_n^m]^{t+\Delta t} = PS_n^m - \frac{\Delta t}{p_s^r}(\underline{\Delta p^r})^T(\underline{\delta_n^m})^{t+\Delta t} \tag{2.32}$$

Where, the first terms on the right-hand-side of the equations are:

$$\begin{aligned}\underline{VS_n^m} &= \frac{1}{2\pi}\int_{-1}^{1}\int_0^{2\pi}\left\{(\underline{\zeta})^{t-\Delta t} + \frac{2\Delta t}{a(1-\mu^2)}\left[\frac{\partial}{\partial\lambda}(\underline{n_V})^t - (1-\mu^2)\frac{\partial}{\partial\mu}(\underline{n_U})^t\right]\right\}P_n^m(\mu)e^{-im\lambda}d\lambda d\mu\\ &= (\underline{\zeta_n^m})^{t-\Delta t} + \sum_{j=1}^{J}\left[im\underline{n_V^m(\mu_j)^t}P_n^m(\mu_j) + \underline{n_U^m(\mu_j)^t}H_n^m(\mu_j)\right]\frac{2\Delta tw_j}{a(1-\mu_j^2)}\end{aligned} \tag{2.33}$$

$$
\begin{aligned}
\underline{DS_n^m} = \frac{1}{2\pi}\int_{-1}^{1}\int_{0}^{2\pi} & \left\{(\underline{\delta})^{t-\Delta t} + \frac{2\Delta t}{a(1-\mu^2)}\left[\frac{\partial}{\partial\lambda}(\underline{n_U})^t + (1-\mu^2)\frac{\partial}{\partial\mu}(\underline{n_V})^t\right]\right. \\
& -2\Delta t\nabla^2[(\underline{E})^t + (\underline{\Phi})^t] \\
& -\Delta t R\boldsymbol{H^r}\nabla^2\left[(\underline{T})^{t-\Delta t} - 2(\underline{T})^t\right] \\
& \left.-\Delta t R(\underline{b^r}+\underline{h^r})\nabla^2\left[(\ln p_s)^{t-\Delta t} - 2(\ln p_s)^t\right]\right\} P_n^m(\mu)e^{-im\lambda}d\lambda d\mu \\
= (\underline{\delta_n^m})^{t-\Delta t} + \sum_{j=1}^{J} & \left[im\underline{n_U^m(\mu_j)}^t P_n^m(\mu_j) - \underline{n_V^m(\mu_j)}^t H_n^m(\mu_j)\right]\frac{2\Delta t w_j}{a(1-\mu_j^2)} \\
+\frac{n(n+1)\Delta t}{a^2} & \left\{2[(\underline{E})^t + (\underline{\Phi})^t] + R\boldsymbol{H^r}\left[(\underline{T})^{t-\Delta t} - 2(\underline{T})^t\right] + R(\underline{b^r}+\underline{h^r})\left[(\ln p_s)^{t-\Delta t} - 2(\ln p_s)^t\right]\right\}_n^m
\end{aligned}
\tag{2.34}
$$

$$
\begin{aligned}
\underline{TS_n^m} = \frac{1}{2\pi}\int_{-1}^{1}\int_{0}^{2\pi} & \left\{(\underline{T})^{t-\Delta t} - \frac{2\Delta t}{a(1-\mu^2)}\left[\frac{\partial}{\partial\lambda}(\underline{TU})^t + (1-\mu^2)\frac{\partial}{\partial\mu}(\underline{TV})^t\right]\right. \\
& +2\Delta t\left[(\underline{T})^t(\underline{\delta})^t + \left(\underline{\frac{RT_v}{c_p^*}\frac{\omega}{p}}\right)^t + (\underline{Q})^t - (\underline{\dot{\eta}\frac{\partial p}{\partial\eta}\frac{\partial T}{\partial p}})^t\right] \\
& \left.-\Delta t\boldsymbol{D^r}\left[(\underline{\delta})^{t-\Delta t} - 2(\underline{\delta})^t\right]\right\} P_n^m(\mu)e^{-im\lambda}d\lambda d\mu \\
= (\underline{T_n^m})^{t-\Delta t} - \sum_{j=1}^{J} & \left[im\underline{(TU)^m(\mu_j)}^t P_n^m(\mu_j) - \underline{(TV)^m(\mu_j)}^t H_n^m(\mu_j)\right]\frac{2\Delta t w_j}{a(1-\mu_j^2)} \\
+ & \left\{2\Delta t\left[(\underline{T})^t(\underline{\delta})^t + \left(\underline{\frac{RT_v}{c_p^*}\frac{\omega}{p}}\right)^t + (\underline{Q})^t - (\underline{\dot{\eta}\frac{\partial p}{\partial\eta}\frac{\partial T}{\partial p}})^t\right] - \Delta t\boldsymbol{D^r}\left[(\underline{\delta})^{t-\Delta t} - 2(\underline{\delta})^t\right]\right\}_n^m
\end{aligned}
\tag{2.35}
$$

$$
\begin{aligned}
PS_n^m = & \left\{(\ln p_s)^{t-\Delta t} - 2\Delta t\left[(\underline{\vec{V}\cdot\nabla\ln p_s})^{tT}\underline{\Delta B} + \frac{1}{(\underline{p_s})^t}(\underline{\delta})^{tT}(\underline{\Delta p})^t\right]\right. \\
& \left.-\frac{\Delta t}{p_s^r}(\underline{\Delta p^r})^T\left[(\underline{\delta})^{t-\Delta t} - 2(\underline{\delta})^t\right]\right\}_n^m
\end{aligned}
\tag{2.36}
$$

Substituting the temperature and log(surface pressure) equation into divergence equation yields the Helmholtz equation:

$$
\boldsymbol{A}_n(\underline{\delta_n^m})^{t+\Delta t} = \underline{DS_n^m} + \Delta t\frac{n(n+1)}{a^2}\left[R\boldsymbol{H^r}\underline{TS_n^m} + R(\underline{b^r}+\underline{h^r})PS_n^m\right] \tag{2.37}
$$

Where,

$$
\boldsymbol{A}_n = \boldsymbol{I} + \Delta t^2\frac{n(n+1)}{a^2}\left[R\boldsymbol{H^r}\boldsymbol{D^r} + R(\underline{b^r}+\underline{h^r})(\underline{\Delta p^r})^T\frac{1}{p_s^r}\right] \tag{2.38}
$$

Note: $\underline{\delta_0^0} = 0$.

Chapter 3

Physical Parameterization

Chapter 4

Test Cases

Appendix A

Spectral Transform Method

A.1 Spherical harmonic transform (from physical space to spectral space)

The spherical harmonic transform contains two steps:

1) Fourier transform:

$$A^m(\mu) = \frac{1}{2\pi}\int_0^{2\pi} A(\lambda,\mu)e^{-im\lambda}d\lambda \tag{A.1}$$

2) Legendre transform:

$$A_n^m = \int_{-1}^{1} A^m(\mu)P_n^m(\mu)d\mu \tag{A.2}$$

where, $P_n^m(\mu)$ is the normalized associated Legendre polynomial.
Combining step 1) and 2), the spherical harmonic transform can be written as:

$$A_n^m = \frac{1}{2\pi}\int_{-1}^{1}\int_0^{2\pi} A(\lambda,\mu)Y_n^{m*}(\lambda,\mu)d\lambda d\mu \tag{A.3}$$

Where, $Y_n^{m*}(\lambda,\mu) = P_n^m(\mu)e^{-im\lambda}$ is the conjugate spherical harmonic function.
Practically, step 1) and 2) are performed discretely as:

$$A^m(\mu_j) = \frac{1}{2\pi}\sum_{i=1}^{I} A(\lambda_i,\mu_j)e^{-\mathbf{i}m\lambda_i} \tag{A.4}$$

$$A_n^m = \sum_{j=1}^{J} A^m(\mu_j)P_n^m(\mu_j)w(\mu_j) \tag{A.5}$$

Where, $\mathbf{i} = \sqrt{-1}$; i and j are the indexes of longitude and latitude; $w(\mu_j)$ are the Gaussian weights.

A.2 Inverse spherical harmonic transform (from spectral space to physical space)

The spherical harmonic transform also contains two steps, but in discrete form only:

1) Inverse Legendre transform:

$$A^m(\mu) = \sum_{n=|m|}^{N} A_n^m P_n^m(\mu) \tag{A.6}$$

Note: $A^{-m}(\mu)$ is conjugate to $A^m(\mu)$.
2) Inverse Fourier transform:

$$A(\lambda, \mu) = Re\left[\sum_{m=-M}^{M} A^m(\mu) e^{\mathbf{i}m\lambda}\right] \tag{A.7}$$

Where, Re means using the real part only.

Appendix B

Normalized Associated Legendre Polynomial Functions

The normalized associated Legendre polynomial functions are generated according the following steps:

$$P_0^0 = 1/\sqrt{2} \tag{B.1}$$

$$P_m^m(x) = -\sqrt{\frac{2m+1}{2m}}\sqrt{1-x^2}P_{m-1}^{m-1}(x) \tag{B.2}$$

$$P_m^{m-1}(x) = x\sqrt{2m+1}P_{m-1}^{m-1}(x) \tag{B.3}$$

$$\begin{aligned} \epsilon_{n+1}^m P_{n+1}^m(x) &= xP_n^m(x) - \epsilon_n^m P_{n-1}^m(x) \\ \epsilon_n^m &= \sqrt{\frac{n^2-m^2}{4n^2-1}} \end{aligned} \tag{B.4}$$

The first derivative of the normalized associated Legendre polynomial functions are generated according the following steps:

$$H_n^m = (1-x^2)\frac{dP_n^m(x)}{dx} = (2n+1)\epsilon_n^m P_{n-1}^m(x) - nxP_n^m(x) \tag{B.5}$$

$$H_m^m = (1-x^2)\frac{dP_m^m(x)}{dx} = -mxP_m^m(x) \tag{B.6}$$