Lasers

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Contents

1   Introduction

In this chapter, we will explore how lasers work. However, before understanding the mechanism of lasers, we need to learn about the concept of energy levels, the interaction between radiation and matter, population distribution, Einstein coefficients, and more.

1.1   Energy Level

An energy level is a specific, quantized state that an e^- can occupy in an atom or molecule, each with a fixed energy value. We will be studying about these energy levels in depth when we will be learning Quantum Physics.

    <u>What does an atom getting excited mean?</u> An atom getting excited means that the atom has gained some energy (relative to its ground state). When an electron of the atom get excited, it gains energy and as an electron is part of an atom, so the atom too does gain energy. From whatever we know till now, when a photon is absorbed by an atom, it is more accurate to say that the electron has been excited. But, in this chapter, we will often keep atom as our point of reference. So, whenever we say that an atom has been excited, we refer to excitation of the outermost valence shell electron. Refer to Born-Oppenheimer approximation for a better idea on how nucleus of an atom is not affected when the photon is absorbed by the atom.

> If an electron has been excited, then the atom too gets excited but if an atom has been excited, there are many possibilities. But, in this chapter, we will be only considering the excitation of the outermost valence shell electron as the only possibility.

1.2 Energy Gap

Usually for an **isolated gaseous atom**, there are many energy levels in discrete which the electron can occupy. But, in the case of solids, due to tightly packed atoms and their orbital overlap, the discrete energy levels become very close and appear to be continuous. Such continuous energy levels are called as energy bands and the e^- can take any energy value between the lowest and highest value of energy band. The concept of energy bands will be discussed again in Band Theory of Solids in depth.

1.3 Classical Representation of Atom

In classical view (an extended version of Newtonian mechanics), these energy levels are related with the orbits of the e^- revolving around the nucleus (See figure 1).
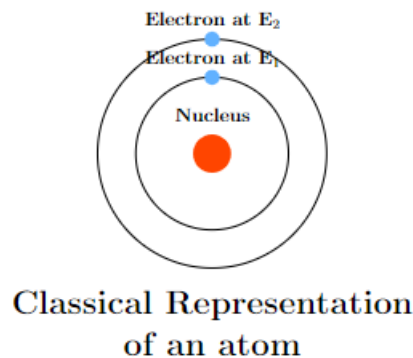
Figure 1: Classical Representation of an Atom

This idea is very outdated and is wrong as we move towards the concept of Quantum Mechanics. The atom is often represented like this in some books to keep things simple. In the quantum mechanical model of the atom, we consider electrons to be in the orbitals corresponding to their energy levels.

1.4 Energy Level Diagrams

To explain phenomena related to interaction between radiation and matter, we need a good diagrammatic representation of an atom. As stated earlier, we are not interested in outdated classical representation of atom anymore in favour of the quantum mechanical model.

Now, if we follow the quantum mechanical model, representing orbitals in an atom is highly complicated, making it even more challenging to use

those diagrams to explain certain phenomena. We have discussed earlier that in this chapter we are going to equate energy states of an atom with the energy state of its outermost valence electron, i.e. when an electron excites from E_1 to E_2, the atom too excites from E_1 to E_2. So, instead of using diagrams of electrons, orbitals or orbits, we will be using the diagram of energy levels (see figure 2) to explain several phenomena.
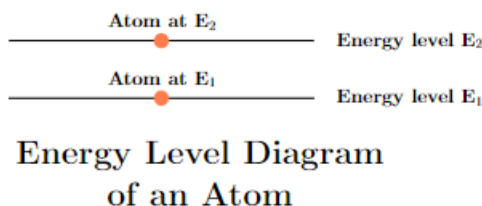

Figure 2: Energy level diagram of an Atom

This energy level diagram helps use to visualise in which energy state the atom (or its outermost valence shell e^-) is in without violating any idea of quantum mechanical model of an atom. Compare figure 2 with figure 1.

1.5   Wave nature of Photons

In this chapter, we will only deal with wave nature of Photons. Because in phenomena like lasers, it is observed that photons behave like waves. By waves, do not confuse it with a physical mechanical wave but they are just electromagnetic waves. For the sake of representation, we will represent the wave nature of Photons as wavy arrow (see in Figure (3)).

What is the difference between Photon and Radiation? Radiation is a broader term that encompasses the emission and transmission of energy, either as waves or as particles. Radiations can be of many types like electromagnetic, thermal, etc. While, Photons are specific quantum (individual element) of an electromagnetic radiation.

To put it simply, "electromagnetic radiation" is made up of discrete energy quantums called as "photons" just like how "matter" is made up of discrete units called as "atoms".

Using **Planck's formula**, we can determine the energy E of a photon from its frequency ν,

$$E = h\nu \tag{1}$$

The energy of an electromagnetic radiation will be the sum of the energies of all photons in the radiation. But, if we assume the electromagnetic radiation having n photons has an uniform isotropic frequency ν, i.e. all photons in the radiation has frequency ν independent of directions,

$$E_{\text{radiation}} = h\nu + h\nu + h\nu + ... \text{ (n times)}$$
$$\Rightarrow E_{\text{radiation}} = nh\nu \tag{2}$$

In reality, for any radiation, there will be slight variations in frequency among its photons, making this formula somewhat impractical. Therefore, for radiation with a frequency ν, the photons will have frequencies ranging from ν to $\nu + d\nu$ (where $d\nu$ represents the slight variation in frequency). By plotting this as a distribution, we may obtain a more accurate formula for the energy of the radiation (we will deal with this in later chapters).

Also, in this chapter, unless and until specified, by "radiation", we mean "electromagnetic radiation". We will be using the term "radiation" when we are talking about the system or the matter as whole while "photon" when we are talking about the "atom" alone.

1.6   References

- Read Basavaraju's book on Engineering Physics for fundamental ideas (Chapter 5 Lasers)

- Watch this YT video for a complete animated video of the whole mechanism of LASER. The only con of this video will be its attempt of using the classical representation of atom in it.

- See this Reddit post for question on energy levels.

This page has to be updated once we complete studying Quantum Physics and its mechanics.

2   Types of Interactions

Working of laser depends on the three different ways of interaction of radiation (photons) and matter (atoms). In this chapter, there will only be an introduction to the interactions and will study them later in depth in Quantum Mechanics. **Note** that these interactions happen only in isolated atoms with discrete E levels.

2.1 Stimulated Absorption

This interaction is also called as **Induced Absorption** in some books (like in Basavaraju's) while in some others, it is called as **Stimulated Absorption**.
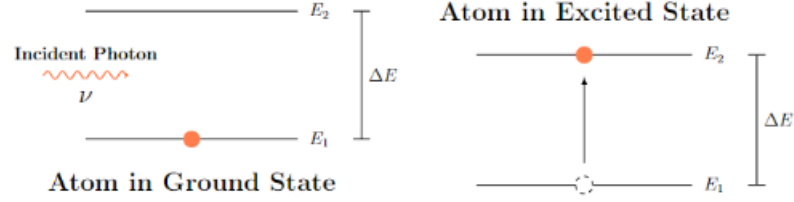
Figure 3: Stimulated Absorption

Here (and for the other 2 interactions too), assume E_1 is the ground state and E_2 is some excited state of that atom.

$$\Delta E = E_2 - E_1 \tag{3}$$

Note although E_1 will be greater than E_2 in magnitude but their values are in negative thus making the ΔE positive.

Here, the photon is absorbed by the atom to excite itself from the ground state to an excited state. This is only possible if the energy of photon is same as the energy different between two levels. So, from Planck's law in Equation (1), we get,

$$\Delta E = E_{\text{photon}} = h\nu \tag{4}$$

2.2 Spontaneous Emission

Figure 4: Spontaneous Emission

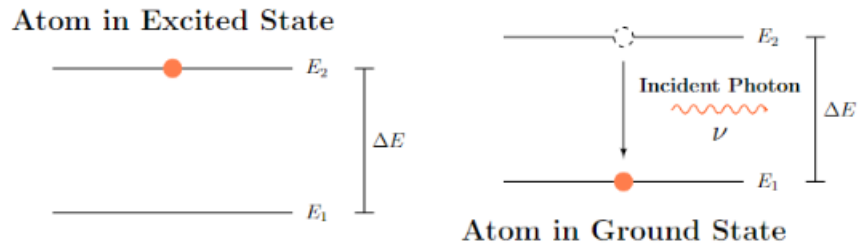Any atom in an excited state is unstable and tends to de-excite to the ground state without any external aid (thus spontaneous). During this process, it emits an photon of an energy equal to the energy difference. **Note** that, there is an universal tendency for any system to be in the lowest energy state to keep itself stable.

Photons emitted by two atoms excited under (and are in) identical conditions undergoing spontaneous emission, **will not have any phase relationship** (i.e. in any direction and can have phase differences). So, photons emitted under this process are incoherent (i.e. no phase relationship). Such incoherent emissions can be seen in Thermoionic emission in bulb, candle flame, etc.

2.3 Stimulated Emission
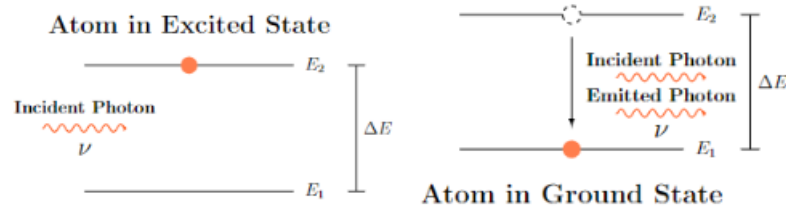
Figure 5: Stimulated Emission

Although, atoms can get de-excited without any external aid as seen in Spontaneous emission, they can also get de-excited by an incident photon of an energy equal to the energy difference between the excited levels. When such de-excitation happens, the atom emits another photon which will be in phase (i.e. coherent) with the incident photon. This process is also referred to as "negative absorption".

3 Population distribution of Atoms in Excited states

In a system, let n_i be the number density (i.e. number of units per unit volume) of atoms existing in the energy state i. To keep things more simple in understanding the lasing process, we will use the energy level diagrams to show the distribution of atoms of system in the excited states (see figure 6). Also, by "population", we mean the "number" of atoms.

Figure 6: Population distribution of atoms in the system

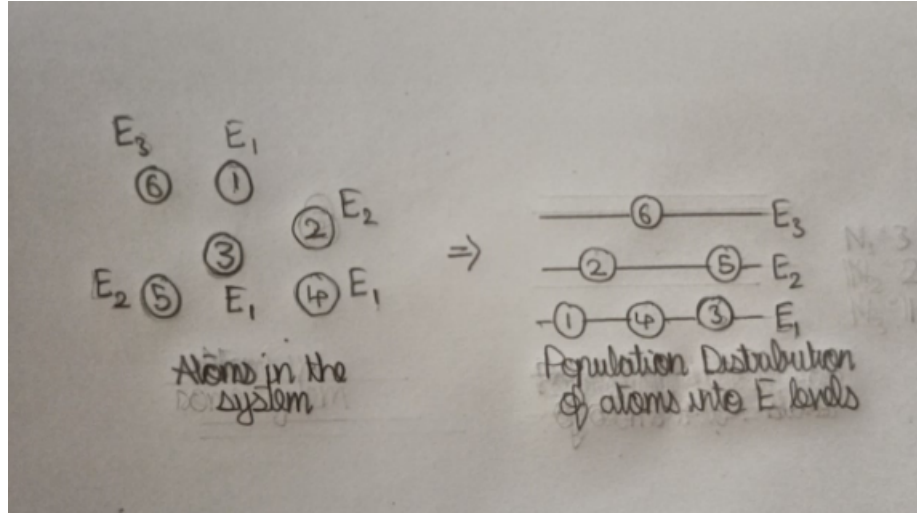**Note** that population distribution is only for identical atoms. In real systems, the number of atoms exceed the number of energy levels possible in that system. (??)

3.1 Rate of change of Population Density

Then, dn_i/dt will be the rate at which the number density of atoms exciting or de-exciting (depending on the sign of the derivative) from/to an energy state i.

Let's consider a system having energy states E_1, E_2, ..., E_i. Let it have a total population density (of atoms) n,

$$n_1 + n_2 + ... + n_i = n \tag{5}$$

We can differentiate this Equation (w.r.t t) to get a relation between the rate of atoms exciting and de-exciting in each energy state,

$$\frac{dn_1}{dt} + \frac{dn_2}{dt} + ... + \frac{dn_i}{dt} = \frac{dn}{dt}$$

In a fixed system, we can assume n to be constant, which makes dn/dt to be 0,

$$\frac{dn_1}{dt} + \frac{dn_2}{dt} + ... + \frac{dn_i}{dt} = 0$$

In transition of atoms, involving only two states, the rates of other states

will become zero,

$$\frac{dn_1}{dt} + \frac{dn_2}{dt} = 0$$

$$\Rightarrow \frac{dn_2}{dt} = -\frac{dn_1}{dt} \tag{6}$$

3.2 Thermodynamic Equilibrium

At Thermodynamic Equilibrium, the net energy exchanged will be 0 (energy exchange will happen but the sum will be 0). Excitation and de-excitation of atoms are energy exchanges too as they involve absorbing and emitting radiation (energy), so at Thermodynamic Equilibrium, the net change in the number of excited atoms will be 0 too.

This means, for any state i at Thermodynamic Equilibrium,

$$\frac{dn_i}{dt} = 0 \tag{7}$$

3.3 Boltzmann Factor

At Thermodynamic Equilibrium, energy states having large population densities can be related with each other by the Boltzmann factor,

$$\frac{n_2}{n_1} = e^{-\frac{E_2 - E_1}{kT}} \tag{8}$$

This formula is built on approximation of the theoretical data and works only for large n values.

The derivation of this equation is given in this paper.

3.4 Non Thermodynamic Equilibrium Conditions

We know in energy level scheme, $E_2 > E_1$. By substituting this inequality in Boltzmann's equation in (8), we get $n_2 < n_1$. If we place the system at a very huge temperature (i.e. $T \rightarrow \infty$ but in equilibrium), then from Equation (8), we get $n_2 \rightarrow n_1$.

As Boltzmann's equation is only valid at Thermodynamic Equilibrium, we can say n_2 is always lesser than or equal to n_1 (i.e $n_2 \leq n_1$) under Thermodynamic Equilibrium. In short, n_2 can never be greater than n_1 under Thermodynamic Equilibrium conditions.

$$n_2 \leq n_1 \text{ (under Thermodynamic Equilibrium)}$$

So, to make $n_2 > n_1$, we have to feed energy in other ways to the system resulting in non equilibrium conditions which is essential for Lasing process (will be discussed later).

4   Einstein's Coefficients

The formation of atomic spectral lines is due to the three different interactions of radiation and matter which was proposed by Einstein in 1916. In his studies, he assumed matter is in Thermodynamic Equilibrium (see 3.2) with Black Body radiation.

Each of these processes (interactions) proposed by Einstein are associated with an Einstein coefficient. Einstein considered the case of isotropic radiation of frequency ν (i.e. having same frequency in all directions the radiation is propagating at) and spectral energy density $\rho(\nu)$.

Read more about this in Wikipedia.

4.1   Spectral Energy Density

For a radiation of continous spectrum of frequencies ranging from v_{min} to v_{max}, let $\rho(\nu)d\nu$ be the energy incident per unit volume (i.e. energy density) of the radiations having frequencies between ν and $\nu + d\nu$ in the spectrum.

Here, $\rho(\nu)$ is called as spectral energy density (having dimensions of $E/A\nu$) which is the energy density per frequency of radiations having frequencies between ν and $\nu + d\nu$. In some books (like Basavaraju's), this has been represented by U_γ where the frequency is represented by γ.

We already discussed about how the frequencies of photons vary with small values $d\nu$ in a radiation of frequency ν in Subsection (1.5). With this parameter, we can take account for the variation and find the total energy of the radiation more accurately than in Equation (2),

$$E_{\text{radiation}} = \text{Volume} \ \times \ \int_{\nu_{min}}^{\nu_{max}} \rho(\nu)d\nu \tag{9}$$

According to **Planck's law**, the spectral energy density is given as,

$$\rho(\nu) = \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{\frac{h\nu}{kT}} - 1} \tag{10}$$

We can substitute this into Equation (10) and get a complete equation for the $E_{\text{radiation}}$ but, we will deal with that later in Quantum Physics.

4.2 Description of Einstein's coefficients

Let's consider two energy levels E_1 (ground state) and E_2 (some excited state) where n_1 and n_2 are the respective population densities of atoms in those energy states. Now, let's see how the three kinds of interactions of radiation and matter are described by the Einstein's coefficients,

4.2.1 Stimulated Absorption

In this process, the atoms excite from E_1 to E_2 due to the incident photon of frequency ν. To measure the rate of this process (and for the others too), we will consider E_1 (ground state) and its population density n_1 as reference, i.e.,

Rate of stimulated = Rate at which atoms are undergoing stimulated

absorption   absorption from E_1 to E_2

$$= (\frac{dn_1}{dt})_{st.a.}$$

And, from experiments, we know,

$$\text{Rate of stimulated absorption} \propto -n_1\rho(\nu)$$

$$(\frac{dn_1}{dt})_{st.a.} \propto -n_1\rho(\nu)$$

As atoms excite from E_1 to E_2, it results in decrease in n_1 (and increase in n_2) which explains the negative sign. The rate of stimulated absorption will be more if there are more atoms in the ground state E_1 to get excited to E_2 so the rate depends on n_1. This points out the fact that the rate of stimulated absorption will decrease as more and more atoms in E_1 get excited. Also, as this process is initiated by the energy of the incident photon, it also depends on the spectral energy density of it.

The proportional constant for the equation is the Einstein coefficient B_{12} $[m^3 J^{-1} s^{-2}]$ where 12 indicates the direction of process (i.e. E_1 to E_2) and B indicates the process is stimulated (induced). After plugging the Einstein coefficient in the proportional equation, we get,

$$(\frac{dn_1}{dt})_{st.a.} = -B_{12}n_1\rho(\nu) \tag{11}$$

To find the rate at which the atoms are getting excited to E_2 by stimulated absorption, we can substitute the above rate in the Equation (6),

$$(\frac{dn_2}{dt})_{st.a.} = -(\frac{dn_1}{dt})_{st.a.} = B_{12}n_1\rho(\nu)$$

4.2.2 Spontaneous Emission

In this process, the atoms de-excite from E_2 to E_1 spontaneously (i.e. without any external aid). Like we did for stimulated absorption, we are going to consider E_1 (ground state) as our reference to find rate of this emission.

Rate of spontaneous $=$ Rate at which atoms are undergoing spontaneous

emission   emission from E_2 to E_1

$$= (\frac{dn_1}{dt})_{sp.e.}$$

And, from experiments, we know,

$$\text{Rate of spontaneous emission} \propto n_2$$

$$(\frac{dn_1}{dt})_{sp.e.} \propto n_2$$

As atoms de-excite from E_2 to E_1, there will be an increase in n_1 (and decrease in n_2) which explains why it has no negative sign. The rate of spontaneous emission will be more if there are more excited atoms in E_2 (i.e. n_2) to get de-excited spontaneously, so the rate depends on n_2. As this process is independent of any external aid (i.e. no incident photon like in stimulated absorption), the rate is independent of spectral energy density.

The proportional constant for the equation is the Einstein coefficient A_{21} $[s^{-1}]$ where 21 indicates the direction of process (i.e. E_2 to E_1) and A indicates the process is spontaneous. After plugging the Einstein coefficient in the proportional equation, we get,

$$(\frac{dn_1}{dt})_{sp.e.} = A_{21}n_2 \tag{12}$$

We can get the rate of this emission keeping E_2 as our reference by applying Equation (6) like we did earlier in stimulated absorption.

4.2.3 Stimulated Emission

In this process, the atoms de-excite from E_2 to E_1 due to incident of photon of frequency ν. Taking E_1 (ground state) as our reference like we did earlier.

$$\text{Rate of stimulated} = \text{Rate at which atoms are undergoing stimulated}$$
$$\text{emission} \quad \text{emission from } E_2 \text{ to } E_1$$
$$= (\frac{dn_1}{dt})_{st.e.}$$

And, from experiments, we know,

$$\text{Rate of stimulated emission} \propto n_2\rho(\nu)$$
$$(\frac{dn_1}{dt})_{st.e.} \propto n_2\rho(\nu)$$

Like in spontaneous emission, the atoms get de-excited from E_2 to E_1 making the rate dependent on the population density in excited state (i.e. n_2) and has a positive sign (due to the increase in n_1). Unlike in spontaneous emission, here the emission takes place because of the incident radiation, so the rate here depends on the spectral energy density of the incident radiation (like in stimulated absorption).

The proportional constant for the equation is the Einstein coefficient B_{21} $[m^3 J^{-1} s^{-2}]$ where 21 indicates the direction of process (i.e. E_2 to E_1) and B indicates the process is stimulated. After plugging the Einstein coefficient in the proportional equation, we get,

$$(\frac{dn_1}{dt})_{sp.e.} = B_{21}n_2\rho(\nu) \tag{13}$$

Again, we can get the rate of this emission keeping E_2 as our reference by applying Equation (6) like we did earlier in stimulated absorption.

4.3 Probability aspect of Einstein Coefficients

4.3.1 Stimulated absorption-only system

Let's consider a system where atoms only undergo stimulated absorption (no stimulated emission or spontaneous emission) from its E_1 state to E_2 state when a photon of required energy is incident on it.

In that system, let $P(t)$ be the probability of an atom to undergo stimulated absorption from E_1 to E_2 at time t and let $P^c(t)$ be the probability of an atom to remain in the same E_1 state at time t (i.e. not undergoing any transition),

$$P(t) + P^c(t) = 1 \tag{14}$$

As more and more atoms in E_1 undergo stimulated absorption to E_2, the population density n_1 decreases which means an atom in E_1 state has more chances (i.e. probability) to absorb the incident photon and get excited to E_2. One can imagine it like this: if the supplies are fixed and the number of people in the crowd is decreasing, then it becomes more probable for each person in the crowd to obtain a supply. This explains why is the probability of an atom getting excited or not is a function of time t (i.e. dependent on time t).

We can define the probability of an atom to remain in the same state (i.e. $P^c(t)$) as,

$$
\begin{aligned}
\text{Probability of an atom to} &= \text{Ratio of population density of atoms} \\
\text{remain in the state} &\quad \text{in that state at t=t to the population} \\
&\quad \text{population density at t=0}
\end{aligned}
$$

$$\Rightarrow P^c(t) = \frac{n_1(t)}{n_1(0)} \tag{15}$$

Here, $n_1(t)$ is the population density at E_1 state at time t.

Since we assumed there is only stimulated absorption taking place in the system, the change in population density of atoms in E_1 is only due to stimulated absorption,

$$\frac{dn_1}{dt} = \left(\frac{dn_1}{dt}\right)_{st.a.}$$

Now, we can plug Equation (11) into it,

$$\frac{dn_1}{dt} = -B_{12} n_1 \rho(\nu)$$

After rearranging the terms, we can integrate the equation on the LHS from $n_1(0)$ to $n_1(t)$ and on the RHS from 0 to t, where $n_1(t)$ is the popu-

lation density n_1 at time t,

$$\Rightarrow \frac{dn_1}{n_1} = -B_{12}\rho(\nu)dt$$

$$\Rightarrow \int_{n_1(0)}^{n_1(t)} \frac{dn_1}{n_1} = \int_0^t -B_{12}\rho(\nu)dt$$

$$\Rightarrow ln(\frac{n_1(t)}{n_0(0)}) = -B_{12}\rho(\nu)(t-0)$$

$$\Rightarrow \frac{n_1(t)}{n_1(0)} = e^{-B_{12}\rho(\nu)t} \qquad (16)$$

Now, we can substitute the Equation (15) into it,

$$P^c(t) = e^{-B_{12}\rho(\nu)t}$$

Using Equation (14), we can find the value of $P(t)$,

$$P(t) = 1 - P^c(t)$$

$$\Rightarrow P(t) = 1 - e^{-B_{12}\rho(\nu)t} \qquad (17)$$

Thus, we have arrived at the equation for the probability of an atom undergoing stimulated absorption . **Note** that this has been derived under the assumption that no other interactions (such as stimulated emission or spontaneous emission) are taking place. In reality (e.g., in the lasing process), all three interactions (i.e. stimulated absorption, stimulated emission, and spontaneous emission) take place when radiation having required frequency is incident on the system, so in such cases, we cannot apply this equation.

Interactions like stimulated absorption usually occur almost instantly, so we can consider $t \to 0$. To simplify our equation further, we can use the limit $t \to 0$, for which we will divide our equation with t first,

$$\Rightarrow \frac{P(t)}{t} = \frac{1 - e^{-B_{12}\rho(\nu)t}}{t}$$

Now, apply limits only on the RHS of the equation,

$$\Rightarrow \frac{P(t)}{t} = \lim_{t \to 0} \frac{1 - e^{-B_{12}\rho(\nu)t}}{t}$$

$$\Rightarrow \frac{P(t)}{t} = B_{12}\rho(\nu)$$

$$\Rightarrow P(t) = B_{12}\rho(\nu)t \qquad (18)$$

Usually, when applying limits, we have to apply it on both sides not just on one side of the equation. However, in this case, we were trying to simplify by approximation. **Note** that if t is significant enough, this equation fails because of the approximation, so in such cases, we have to use Equation (17).

Time Saving Result

Exponential functions of form like in Equation (17) are very common in Physics. They can be simplified by approximation if $x \to 0$ by applying limits on the RHS.

$$y = 1 - e^{kx}$$
$$\Rightarrow y \approx -kx \quad \because x \to 0$$

By rearranging the terms in Equation (18), we can arrive at,

$$B_{12} = \frac{P(t)}{\rho(\nu)t} \tag{19}$$

Thus, we can define B_{12} as the probability per unit time per unit energy density of the radiation field per unit frequency that an atom in E_1 absorb the incident photon (of required energy) and get excited to E_2 in a system where no other interactions (processes) are taking place in a small interval of time.

A lot of books or articles (like Wikipedia), refers Einstein coefficients as "probability" and by "probability", this is what they mean. Truly speaking, Einstein coefficients are more of rate constants (which we can see even in the differential equations) than probability. As the value of einstein coefficient increases, the rate increases and as the rate increases, more atoms excite from E_1 to E_2 so there are more chances for an atom in E_1 to undergo stimulated absorption thus affecting the value of probability. So, even in a system where all kind of interactions takes place, the einstein coefficient is related to probability but not linearly proportional as derived above under several assumptions.

4.3.2 Stimulated emission-only system

Let's consider a system where only atoms undergo stimulated emission from E_2 to E_1 when a radiation having required frequency is incident on it.

In this system, let $P(t)$ be the probability of an atom to undergo stimulated emission from E_2 to E_1 at time t and let $P^c(t)$ be the probability

of an atom to remain in the same E_2 state at time t (i.e. not undergoing any transition).

Unlike in absorption, where atoms move from E_1 to E_2, in emission, atoms move from E_2 to E_1, so we will be using dn_2/dt and n_2 to obtain probability equation,

$$P^c(t) = \frac{n_2(t)}{n_2(0)}$$

$P^c(t)$ represents the probability of atoms that did not undergo the process. In the absorption process, $P^c(t)$ refers to the atoms in the state E_1 that did not undergo absorption. Conversely, in the emission process, $P^c(t)$ refers to the atoms in the state E_2 that did not undergo emission. Therefore, we use n_2 instead of n_1 to determine $P^c(t)$ in the emission process.

Since we assumed there is only stimulated emission taking place in the system, the change in population density of atoms in E_2 is only due to stimulated emission,

$$\frac{dn_2}{dt} = (\frac{dn_2}{dt})_{st.e.}$$

Here, we are finding the value of dn_2/dt instead of dn_1/dt because we are interested in the value of the fraction $n_2(t)/n_2(0)$.

As we are familiar with the math and the procedure, we can skip steps,

$$\frac{dn_2}{dt} = -B_{21}n_2\rho(\nu)$$

$$\Rightarrow \frac{n_2(t)}{n_2(0)} = e^{-B_{21}\rho(\nu)t} \quad \because \text{Integrating both sides}$$

$$\Rightarrow P^c(t) = e^{-B_{21}\rho(\nu)t}$$

$$\Rightarrow P(t) = 1 - e^{-B_{21}\rho(\nu)t} \quad \because P(t) = 1 - P^c(t)$$

$$\Rightarrow P(t) \approx B_{21}\rho(\nu)t \quad \because t \to 0$$

Now, by rearranging the terms, we arrive at,

$$B_{21} = \frac{P(t)}{\rho(\nu)t} \tag{20}$$

Thus, we can define B_{21} as the probability per unit time per unit energy density of the radiation field per unit frequency that an atom in E_2 gets stimulated by the incident photon (of required energy) and get de-excited to E_1 emitting a photon of an equal energy in a system where no other interactions (processes) are taking place in a small interval of time.

4.3.3 Spontaneous emission-only system

Let's consider a system where only atoms undergo spontaneous emission from E_2 to E_1. In this system, let $P(t)$ be the probability of an atom to undergo spontaneous emission from E_2 to E_1 at time t and let $P^c(t)$ be the probability of an atom to remain in the same E_2 state at time t (i.e. not undergoing any transition).

Since spontaneous emission is a type of emission process similar to stimulated emission, with the key difference being that spontaneous emission does not require incident photon to stimulate the atom and is thus independent of the spectral energy density of the incident radiation, we can modify the equation for the probability of stimulated emission by removing $\rho(\nu)$ and changing the Einstein coefficient B_{21} to A_{21}. This modification yields the probability of spontaneous emission.

$$\frac{dn_2}{dt} = -A_{21}n_2)$$

$$\Rightarrow \frac{n_2(t)}{n_2(0)} = e^{-A_{21}t} \quad \because \text{Integrating both sides}$$

$$\Rightarrow P^c(t) = e^{-A_{21}t}$$

$$\Rightarrow P(t) = 1 - e^{-A_{21}t} \quad \because P(t) = 1 - P^c(t)$$

$$\Rightarrow P(t) \approx A_{21}t \quad \because t \to 0$$

Now, by rearranging the terms, we arrive at,

$$A_{21} = \frac{P(t)}{t} \tag{21}$$

Thus, we can define A_{21} as the probability per unit time per unit energy density of the radiation field per unit frequency that an atom in E_2 gets de-excited to E_1 emitting a photon (of a required energy) in a system where no other interactions (processes) are taking place in a small interval of time.

4.4 Relation between the Einstein Coefficients

Let's consider a system having two energy levels i.e., E_1 (ground state) and E_2 (excited state). If radiation having a frequency equal to $\nu = \Delta E/h$ between the energy states is incident on the system, then, all the three kind of interactions between radiation and matter will happen. So, at any

point of time t,

Rate of change in = Rate at which atoms − Rate at which atoms
no. of atoms in E_i are excited from E_i are de-excited to E_i

$$\Rightarrow \frac{dn_i}{dt} = (\frac{dn_i}{dt})_{st.a.} - ((\frac{dn_i}{dt})_{sp.e.} + (\frac{dn_i}{dt})_{st.e.})$$

As we have considered, E_1 (ground state) to be our reference for rate equations, we will take $i = 1$ and plug-in the equations (11), (12), (13) in the above one,

$$\Rightarrow \frac{dn_1}{dt} = (\frac{dn_1}{dt})_{st.a.} - ((\frac{dn_1}{dt})_{sp.e.} + (\frac{dn_1}{dt})_{st.e.})$$

$$\Rightarrow \frac{dn_1}{dt} = B_{12}n_1\rho(\nu) - (A_{21}n_2 + B_{21}n_2\rho(\nu)) \tag{22}$$

Now, we can consider Thermodynamic Equilibrium which will make $dn_1/dt = 0$ by using the Equation (7),

$$\Rightarrow 0 = B_{12}n_1\rho(\nu) - (A_{21}n_2 + B_{21}n_2\rho(\nu))$$

$$\Rightarrow B_{12}n_1\rho(\nu) = A_{21}n_2 + B_{21}n_2\rho(\nu) \tag{23}$$

Thus, we have arrived at an relationship between the Einstein's coefficients at Thermodynamic Equilibrium. Considering $dn_1/dt = 0$ under equilibrium conditions is called as **Detailed Balancing**.

Also, we can further manipulate the equation to get the spectral energy density of the radiation,

$$\Rightarrow \rho(\nu) = \frac{A_{21}n_2}{B_{12}n_1 - B_{21}n_2}$$

$$\Rightarrow \rho(\nu) = \frac{A_{21}}{B_{21}} \frac{1}{\frac{B_{12}n_1}{B_{21}n_2} - 1}$$

As we have already assumed Thermodynamic Equilibrium, we can substitute the Boltzmann factor of Equation (8) here,

$$\Rightarrow \rho(\nu) = \frac{A_{21}}{B_{21}} \frac{1}{\frac{B_{12}}{B_{21}}e^{-\frac{E_1-E_2}{kT}} - 1}$$

We are more interested with the frequency of incident photon involved in the interaction (also spectral energy density is a function of that frequency), so we can substitute Equation (4) here,

$$\Rightarrow \rho(\nu) = \frac{A_{21}}{B_{21}} \frac{1}{\frac{B_{12}}{B_{21}}e^{\frac{hf}{kT}} - 1} \tag{24}$$

Now, compare this equation with Planck's law in Equation (9), we get,

$$\frac{A_{21}}{B_{21}} = \frac{8\pi h \nu^3}{c^3} \tag{25}$$

$$B_{12} = B_{21} \tag{26}$$

Note that these relations are valid only under Thermodynamic Equilibrium conditions.

Now, B_{12} and B_{21} are same (independent of directions), we can drop the subscript notation for the coefficients when discussing equilibrium conditions, i.e., A is the Einstein coefficient for spontaneous process ($A = A_{21}$) and B is the Einstein coefficient for stimulated process ($B = B_{12} = B_{21}$). With this, we can make Equation (24) look more simple,

$$\rho(\nu) = \frac{A}{B} \frac{1}{e^{\frac{hf}{kT}} - 1}$$

From (25), we can see that the rate constant of both stimulated emission and stimulated absorption becomes equal under equilibrium conditions.

From (26), we can get this proportional equation,

$$\frac{A_{21}}{B_{21}} \propto \nu^3$$

Now, substituting (3) in that proportional equation gives us,

$$\frac{A_{21}}{B_{21}} \propto (\Delta E)^3 \quad \because \nu = \frac{\Delta E}{h} \tag{27}$$

Here, we can deduce the following,

1) If ΔE is very large,

Then, $A_{21} >> B_{21}$, this means that fewer atoms are de-excited by stimulated emission compared to spontaneous emission, as Einstein coefficients are rate constants.

Therefore, it is more probable for atoms in E_2 to de-excite spontaneously to E_1 rather than by stimulated emission.

As $B_{12} = B_{21}$, we can also say, $A_{21} >> B_{12}$, this means there will be more emissions (spontaneous) than absorptions.

2) If ΔE is very small,

Then, $A_{21} << B_{21}$, this means that fewer atoms are de-excited by spontaneous emission compared to stimulated emission, as Einstein coefficients are rate constants.

Therefore, it is more probable for atoms in E_2 to de-excite by stimulated emission to E_1 rather than by spontaneous emission.

As $B_{12} = B_{21}$, we can also say, $A_{21} << B_{12}$, this means there will be more absorptions than emissions.

This makes sense because, the instability of an excited atom increases with energy (i.e. ΔE) absorbed by it to get excited. The instability of an atom makes the atom to de-excite faster without any external aid to make itself stable which makes it probable for spontaneous emission to occur when ΔE is higher.

To understand how high is considered high and how low is considered low, we can rearrange the terms in Equation (24),

$$\rho(\nu) = \frac{A_{21}}{B_{21}} \frac{1}{e^{\frac{h\nu}{kT}} - 1} \quad \because \frac{B_{12}}{B_{21}} = 1$$

$$\Rightarrow \rho(\nu)\frac{B_{21}}{A_{21}} = \frac{1}{e^{\frac{h\nu}{kT}} - 1}$$

$$\Rightarrow \frac{1}{\rho(\nu)}\frac{A_{21}}{B_{21}} = e^{\frac{h\nu}{kT}} - 1$$

$$\Rightarrow \frac{1}{\rho(\nu)}\frac{A_{21}}{B_{21}} = e^{\frac{\Delta E}{kT}} - 1 \quad \because \Delta E = h\nu \tag{28}$$

From this, we can get the idea of how high is considered as high and how low is considered as low,

1) If $\Delta E >> kT$,

$$e^{\frac{h\nu}{kT}} >> 1$$
$$\Rightarrow e^{\frac{h\nu}{kT}} - 1 >> 1$$
$$\Rightarrow \frac{1}{\rho(\nu)}\frac{A_{21}}{B_{21}} >> 1$$
$$\Rightarrow A_{21} >> B_{21}$$

Spontaneous emission becomes more probable than stimulated emission. Thus, when we state ΔE is very large, it follows that $\Delta E >> kT$.

2) If $\Delta E << kT$,

$$e^{\frac{h\nu}{kT}} << 1$$
$$\Rightarrow e^{\frac{h\nu}{kT}} - 1 << 1$$
$$\Rightarrow A_{21} << B_{21}$$

Stimulated emission becomes more probable than spontaneous emission. Thus, when we state ΔE is very small, it follows that $\Delta E << kT$.

5 Production of Lasers

From here on, we will understand how a laser works and how it is constructed, using the concepts we have studied so far in the chapter.

5.1 What is LASER?

Laser (or LASER) is an abbreviation for **Light Amplification by Stimulated Emission of Radiation**. As the name suggests, laser is achieved by amplification of light by stimulated emission.

How is amplification of light achieved by stimulated emission? When we studied about Stimulated Emission, we know than an excited atom gets stimulated by the incident photon (of required energy) and de excites to the ground state by emitting another photon which travels in the same direction and is in the same phase with the incident photon. But, what if there are many atoms in an excited state?
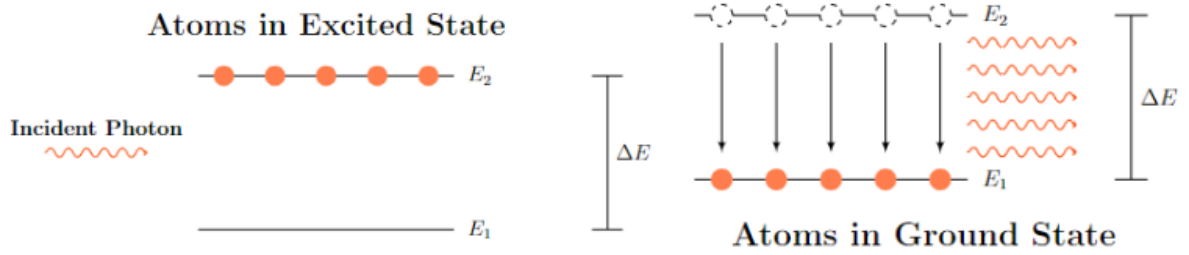
Figure 7: Stimulated emission from multiple excited atoms in an energy level diagram.

The incident photon stimulates the first excited atom, resulting in two in-phase photons traveling in the same direction. These two photons then stimulate two other excited atoms, resulting in four in-phase photons travelling in the same direction. These four photons stimulate four more excited atoms, resulting in eight in-phase photons travelling in the same direction, and so on, creating a chain reaction and a bunch of photons traveling in the same direction resulting in the amplification of light.

Although this may sound simple, there is a lot going on in reality that we need to take into account. Starting with the fact that the diagrammatic representation of this process in Figure (7) is only from the perspective of energy levels (i.e., energy level diagrams), as we have already discussed, these atoms are actually scattered throughout the system. Therefore, achieving unidirectional, highly collimated, monochromatic, coherent light (i.e., laser light) in reality requires a great deal of work in the construction of a laser.

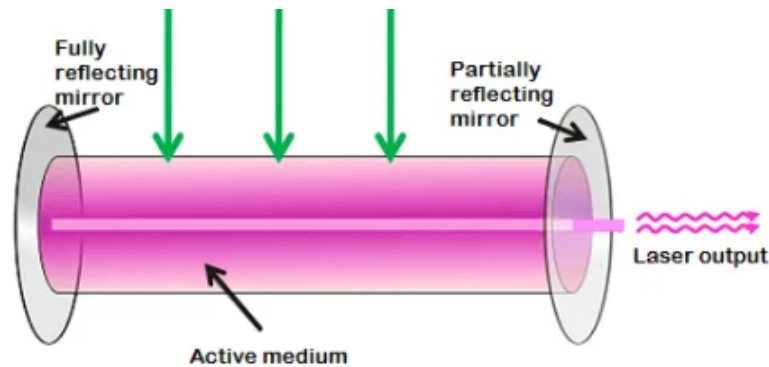## 5.2   Construction of a Laser

Figure 8: Simple diagrammatic representation of Laser

This diagram shows a very simplistic model of a LASER based on the very first lasers humans built. In this diagram, the **Active medium** (or **Gain medium**) is where the lasing process takes place.

Anyway, at present, the modern lasers available are very compact and use several methods to improve efficiency but the fundamental principles of lasing process used in the very first lasers are still used.

5.2.1 Population Inversion

We learnt that in a system under Thermodynamic equilibrium, there are more atoms in the ground state than in the excited state (according to Boltzmann's law). But to amplify light through stimulated emission, we need more atoms in the excited state to make it more probable for photons to find its way to an excited atom and amplify the radiation.

Why do we have to make it more probable for photons to stimulate the atoms? Because, if not, photons will take a lot of time to find excited atoms and stimulate it resulting in a very low intensity laser beam (which we are not interested in).

So, we have to make the system (i.e. active medium in the laser) to undergo non equilibrium conditions by supplying energy (represented by green arrows in Figure (8)) to the system from an external source. This process is called as **Pumping**.

The supplied energy excites alot of atoms in the ground state to the excited state thus making more atoms to be in excited state than in the ground state resulting in a **Population Inversion**. This makes it more probable for the photons to find the excited atoms and amplify the radiation more.
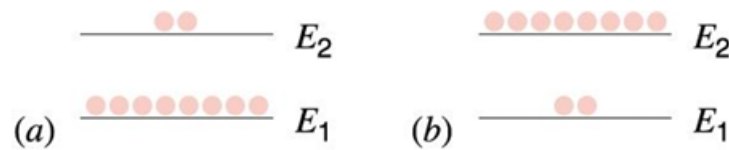

Figure 9: System under equilibrium (a) and System after population inversion (b)

5.2.2 Metastable states

When an atom excites to an excited state, it becomes very unstable and tends to de-excite spontaneously (almost instantly). The average time for which an atom remains in an excited state is 10^{-8} s, after which it de-excites spontaneously. This is a problem because the atom gets de-excited

spontaneously before even the photon reaches the atom to de-excite it through stimulation and amplify the radiation.

There are certain exceptional energy states in certain elements where the average time taken for an atom to remain in those energy states are as high as $10^{-3}s$ which is large enough (compared to $10^{-8}s$) for photons to excite it through stimulation. Theses states are called as **Metastable states**.

This is why active medium in the lasers consists of atoms or ions with such exceptional **Metastable states** to absorb energy and undergo stimulated emission which increases the output of the laser beam by being excited for long time and making it more probable for a photon to stimulate an excited atom.

5.2.3   Formation of Standing Waves

One of the most important aspect of laser is that how it's beam is highly collimated and coherent. This is achieved by the way how lasers are constructed.

When pumping takes place, alot of atoms in the active medium gets excited. Initially, there will not be many photons in the active medium to start the chain of stimulated emission so few excited atoms in the active medium have to undergo spontaneous emission releasing photons. We know, in spontaneous emissions, there is no phase relationship between the absorbed and emitted radiation so the photons are emitted in various directions. But, we need our laser beam to be highly collimated so we have to filter out photons which are not travelling parallel the axis of the active medium.

If you observe Figure (8), the active medium is enclosed by two mirrors on either side, with its length exposed. Photons that do not travel parallel to the axis of the active medium undergo several reflections between the mirrors and eventually escape the medium. In contrast, photons traveling parallel to the axis remain perpendicular to the mirrors. Even after multiple reflections, they stay in the same line, allowing only the photons traveling parallel to the axis to remain within the active medium.
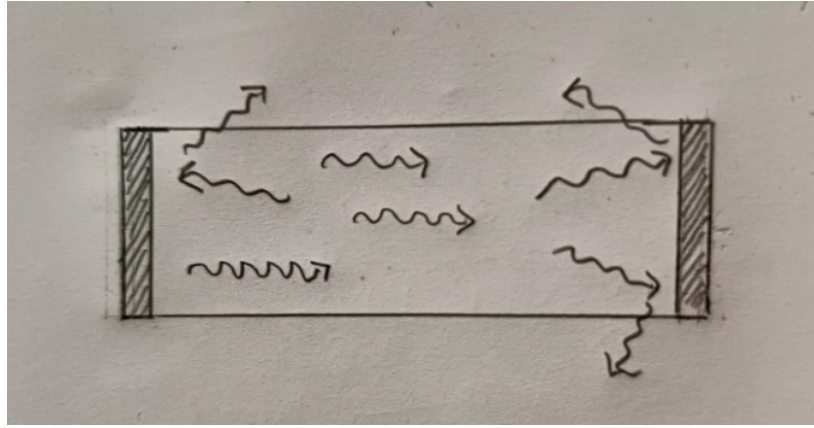
Figure 10: Filtration of photons which are not travelling parallel to the axis.

Once the filtration is done, photons which are travelling along the direction of the axis will be only left. From there onwards, those photons start the chain process of stimulated emissions. We know, in stimulated emission, there is a phase relationship with the stimulated and emitted photon so the photons which are emitted from the stimulated emissions therafter will also be travelling parallel to the principal axis thus making the beam highly collimated.

Now, with the filtration being done, the photons only travel parallel to the axis but they still have the freedom to move either in left to right or right to left. So, it is very much probable that a photon meets another photon in the same path but from the opposite direction resulting in an interference of waves.
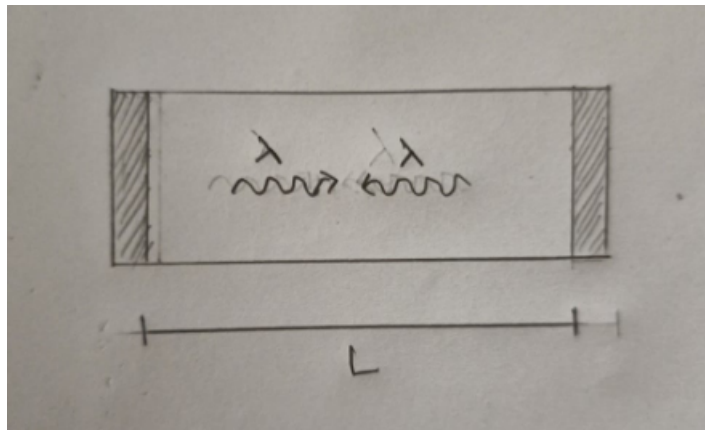

Figure 11: Two photons meeting along the same line

This interference can be either constructive or destructive. We are interested in a laser beam with maximum intensity, so we will be only considering the case of constructive interference. We know, when waves

from two opposite directions constructively interfere, it results in **Standing Waves**. Similarly, these photons constructively interfere and result in standing waves. Here, the active medium having mirrors on both of its sides acts like a closed organ pipe. So, we can simply apply the formula of closed organ pipe,

$$L = n\frac{\lambda}{2}, \quad n \in \mathbb{N} \tag{29}$$
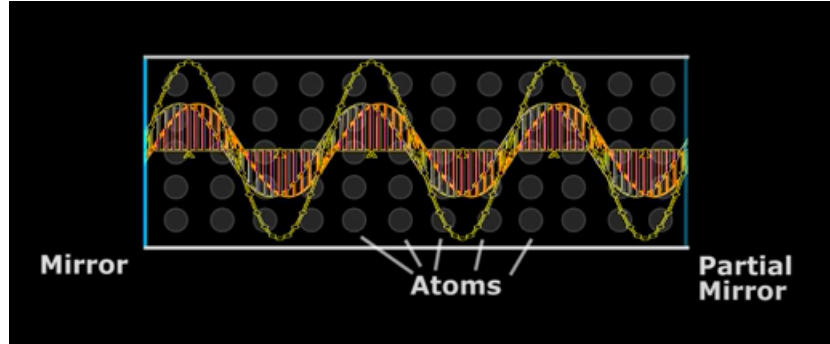

Figure 12: Standing waves in the active medium

Note that the diagrammatic representation of waves in this figure (or any figure in this chapter) is for convenience in illustrating the concept. They don't actually span across the active medium or represent physical waves but are instead a mathematical tool (wave equation) used to understand their behavior.

Are photons particles or waves? In this chapter, we focus solely on the wave nature of photons (probabilistic), rather than their particle nature. Why? In phenomena such as lasers, photons exhibit wave-like behavior, making it convenient to model them as waves for mathematical analysis. For example, if we keep the wavelength of photons in the active medium, λ, constant and vary the length of the active medium, L, we observe changes in intensity due to interference effects (see Equation (29)). Thus, for these cases, it is simpler to perform calculations by assuming photons as waves.

With more and more interference with photons, the standing wave amplifies (i.e. amplitude increases). Given, there is a finite amount of excited atoms, it will reach a saturation point at some point of time.

How do we emit our standing wave as laser beam? Although, we did say there are mirrors on both the side of the active medium. Only one of them is 100% reflecting (left one in Figure (8)) while the other one is 99% reflexting (right one in Figure (8)), i.e. only 1% of light incident on that

mirror is actually emitted as laser beam while the remaining 99% is used to maintain standing wave and amplify itself.

Also, note that there will be no one big standing wave in the as illustrated in Figure (12), there will be multiple standing wave across each possible path parallel to the axis which are emitted together from the partially reflecting mirror. Unlike in Figure (8), the laser beam output will be all area of the cross section of the active medium. From this, we can understand that, to keep the laser beam more thin, we have to reduce the radius of the active medium.