Partial Derivatives

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1 Introduction

1.1 Partial Derivative

Partial Derivative represents how a multivariate function changes with respect to one variable while keeping all other variables constant. It is represented by $\frac{\partial}{\partial x}$

Now, let's consider a multivariate function $f(x_1, x_2, ..., x_n)$ which depends on n independent variables. We can express the partial derivation of the function with respect to the variable x_i as a limit,

$$\frac{\partial}{\partial x_i} f(x_1, x_2, ..., x_i, ..., x_n) = \lim_{h \to 0} \frac{f(x_1, x_2, ..., x_i + h, ..., x_n) - f(x_1, x_2, ..., x_i, ..., x_n)}{h} \tag{1}$$

This partial derivative can also be represented as $\frac{\partial f}{\partial x_i}$, $\partial_{x_i} f$ or f'_{x_i}.

1.2 Total Derivative

Total derivatives represents how a multivariate function changes with respect to one variable considering the changes in all variables that the function depends on. It is represented by $\frac{d}{dx}$

We can express the total derivative of f with respect to a parameter x using the partial derivatives with respect to each of its variables,

$$\frac{d}{dx}f(x_1, x_2, ..., x_n) = \frac{\partial f}{\partial x_1}\frac{dx_1}{dx} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dx} + ... + \frac{\partial f}{\partial x_n}\frac{dx_n}{dx} \tag{2}$$

One can prove this by simply reverse engineering the chain rule of derivatives.

1.2.1 Total Derivative w.r.t an Independent variable

If the n variables of the function f are independent of each other, then the total derivative of the function with respect to any of it's variable will be same as it's partial derivative.

Proof:

Let's consider a function $f(x_1, x_2, ..., x_n)$ depending on n independent variables,

$$\frac{dx_i}{dx_j} = 0, \forall\ 0 < i, j \leq n,\ i \neq j$$

Now applying the total derivative formula, to find the total derivative of f with respect to x_i $(0 < i \leq n)$,

$$\begin{aligned} \frac{df}{dx_i} &= \frac{\partial f}{\partial x_1}\frac{dx_1}{dx_i} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dx_i} + ... + \frac{\partial f}{\partial x_i}\frac{dx_i}{dx_i} + ... + \frac{\partial f}{\partial x_n}\frac{dx_n}{dx_i} \\ \Rightarrow \frac{df}{dx_i} &= \frac{\partial f}{\partial x_1}\cdot 0 + \frac{\partial f}{\partial x_2}\cdot 0 + ... + \frac{\partial f}{\partial x_i}\cdot 1 + ... + \frac{\partial f}{\partial x_n}\cdot 0 \\ \Rightarrow \frac{df}{dx_i} &= \frac{\partial f}{\partial x_i} \end{aligned}$$

1.3 Derivative of Univariate Functions

So far, we have only been dealing with univariate functions (functions which depend only on one variable) that depend on a single variable, so there was no need for 'partial derivatives' and 'total derivatives,' as both of them are the same and are simply referred to as 'derivatives'.

Proof:

Applying the total derivative formula for an univariate function $p(x)$,

$$\frac{dp}{dx} = \frac{\partial p}{\partial x}\frac{dx}{dx}$$
$$\Rightarrow \frac{dp}{dx} = \frac{\partial p}{\partial x}$$

We can think of the 'derivatives' of the univariate functions we've been dealing with so far as the partial derivatives with respect to their only variable.

2 Equality of Mixed Partials

Exchanging the order of partial derivatives of a multivariate function does not change the result if certain conditions are satisfied. See Wikipedia.

$$\frac{\partial}{\partial x_i}(\frac{\partial f}{\partial x_j}) = \frac{\partial}{\partial x_j}(\frac{\partial f}{\partial x_i}) \tag{3}$$

The mixed partial $\frac{\partial}{\partial x_i}(\frac{\partial f}{\partial x_j})$ can also be represented as,

$$\frac{\partial^2 f}{\partial x_i \partial x_j},\ \partial_{x_i}\partial x_j f \text{ or } f'_{x_i x_j}$$

3 Integrating Partial Derivatives

We have learnt that derivative and integral cancels out for a univariate function. Take $p(x)$,

$$\int \frac{dp}{dx}\, dx = p(x) + c$$

where, c is some constant (i.e. independent of x). This is called as **First Fundamental Theorem of Calculus**.

When taking about multivariate functions, it is better to say that partial derivative and integral cancels out. Take $f(x_1, ..., x_i, ..., x_n)$,

$$\int \frac{\partial}{\partial x_i} f(x_1, ..., x_n)\ dx_i = f(x_1, ..., x_n) + C(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$$

where $C(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$ is a function which depends on all variables on f except for x_i (i.e. independent of x_i or constant with respect to x_i).

The above equation can be taken as a generalised form of **First Fundamental Theorem of Calculus** for a multivariate function.