The Reissner–Nordström Black hole

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Introduction :
A Reissner-Nordström black hole is a theoretical type of black hole characterized by having mass and electric charge, but no spin . It is a static, spherically symmetric solution to the Einstein-Maxwell field equations, describing the gravitational field of a charged, non-rotating body . Discovered between 1916 and 1921, this model differs from a standard Schwarzschild black hole by possessing two horizons when its mass is greater than its charge: an outer event horizon and an inner Cauchy horizon . If the mass and charge are equal, the horizons merge to form an "extremal" black hole . However, if the charge were to exceed the mass, no event horizon could form, which would result in a naked singularity

Electrodynamics in Special Relativity

Maxwell's equations describe the generation and interaction of electric and magnetic fields with each other and by charges and currents. With the electric field $\mathbf{E}$, magnetic field $\mathbf{B}$, charge density ρ and current density $\mathbf{J}$, Maxwell's equations are

$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$
$\nabla \cdot \mathbf{B} = 0$
$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$
$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$

The Maxwell's equations in covariant form are written as;

$\mathrm{F}^{\alpha\beta}_{\ ,\beta} = -\mu_0 j^{\alpha}$
$F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0$

The em stress-energy tensor is given by

$$T^{\alpha\beta} = \frac{1}{\mu_0}\left(\frac{1}{4}\eta^{\alpha\beta} F^{\mu\nu} F_{\mu\nu} - F^{\alpha\mu} F^{\beta}_{\ \mu}\right)$$

The em field tensor is given in the matrix form as:

$\mathrm{F}^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$

The em field stress-energy tensor is given by

$\mathrm{T}^{\alpha\beta} = \frac{1}{\mu_0}\left(\frac{1}{4}\eta^{\alpha\beta} F^{\mu\nu} F_{\mu\nu} - F^{\alpha\mu} F^{\beta\mu}\right)$

Einstein' Field Equation

$\mathrm{R}_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta}$

We can write this as also: $\mathrm{R}_{\alpha\beta} = \frac{8\pi G}{c^4}(T_{\alpha\beta} - \frac{1}{2}T g_{\alpha\beta})$

Derivation of Reissner-Nordstrom Metric :

In order to derive it,we all use the following assumptions:

1. Static and spherically symmetric body of mass m.
2. The body is charged with charge of Q.
3. There is only an em field only and vacuum outside this mass. 4. The spacetime is asymptotically flat, which means $\lim_{r\to\infty} g_{\mu\nu}(r) = \eta_{\mu\nu}$.

5.$AsQ \to 0$, this metric reduces to the Schwarzschild metric.

Let's assume the form of the spacetime metric for the charged blackhole is as follow:

$$ds^2 = A(t,r)\, c^2\, dt^2 - B(t,r)\, dr^2 - r^2 \sin^2\theta\, d\phi^2$$

The corresponding metric tensor $g_{\alpha\beta}$ in coordinates (t, r, θ, ϕ) is: $\mathrm{g}_{\alpha\beta} = \begin{pmatrix} A(t,r)\, c^2 & 0 & 0 & 0 \\ 0 & -B(t,r) & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2\theta \end{pmatrix}$

Calculatiopn of Christoffel Symbols Components

In order to calculate the Christoffel symbolls ,we use the following equation:

$$\Gamma^{\lambda}_{\alpha\beta} = \frac{1}{2} g^{\lambda\mu} \left(\partial_\alpha g_{\beta\mu} + \partial_\beta g_{\mu\alpha} - \partial_\mu g_{\alpha\beta} \right)$$

To calculate Γ^0_{00}, we use the general formula for Christoffel symbols:

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} \left(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} \right)$$

Substituting $\lambda = 0$, $\mu = 0$, $\nu = 0$, we get:

$$\Gamma^0_{00} = \frac{1}{2} g^{0\rho} \left(\partial_0 g_{\rho 0} + \partial_0 g_{\rho 0} - \partial_\rho g_{00} \right)$$

$$= \frac{1}{2} g^{0\rho} \left(2\partial_0 g_{\rho 0} - \partial_\rho g_{00} \right)$$

Assuming a diagonal metric such that $g_{\rho 0} = 0$ for $\rho \neq 0$, and only $g_{00} \neq 0$, we consider only $\rho = 0$ term:

$$\Gamma^0_{00} = \frac{1}{2} g^{00} \left(2\partial_0 g_{00} - \partial_0 g_{00} \right) = \frac{1}{2} g^{00} \partial_0 g_{00}$$

$$\Gamma^0_{00} = \frac{1}{2} g^{00} \partial_0 g_{00} = \frac{1}{2} \frac{1}{A} \frac{1}{c} \partial_t A = \frac{\dot{A}}{2AC}$$

Similarly all other non-zero Christoffel symbolls can be calculated and they are written below:

1. $\Gamma^0_{00} = \frac{\dot{A}}{2AC}$

2. $\Gamma^1_{01} = \Gamma^1_{10} = \frac{\dot{B}}{2Bc}$

3. $\Gamma^0_{01} = \Gamma^0_{10} = \frac{A'}{2A}$

4. $\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}$

5. $\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}$

6. $\Gamma^3_{23} = \Gamma^3_{32} = cot\theta$

7. $\Gamma^2_{33} = -sin\theta cos\theta$

8. $\Gamma^0_{11} = \frac{\dot{B}}{2Ac}$

9. $\Gamma^1_{00} = \frac{A'}{2B}$

10. $\Gamma^1_{11} = \frac{B'}{2B}$

11. $\Gamma^1_{22} = \frac{-r}{B}$

12. $\Gamma^1_{33} = \frac{-rsin^2\theta}{B}$

Please note that prime(') represents derivative w.r.t. r and dot(.) represents derivative w.r.t. time t.

1 Calculation of Ricci tensor components

The Ricci Tensor is given by:
$R_{\alpha\beta} = \partial_\mu \Gamma^\mu_{\alpha\beta} - \partial_\beta \Gamma^\mu_{\alpha\mu} + \Gamma^\mu_{\nu\beta}\Gamma^\nu_{\alpha\beta} - \Gamma^\mu_{\nu\beta}\Gamma^\nu_{\alpha\mu}$

To Calculate R_{01},

$$R_{01} = \partial_\mu \Gamma^\mu_{01} - \partial_1 \Gamma^\mu_{0\mu} + \Gamma^\mu_{\nu\mu}\Gamma^\nu_{01} - \Gamma^\mu_{\nu 1}\Gamma^\nu_{0\mu}$$

Using non-zero values of Christofell symbols, we get

$$R_{01} = \frac{\dot{B}}{Brc}$$

similarly we get all the non-zero components of the Ricci Tensor:

$$R_{00} = -\frac{A'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{A''}{2B} + \frac{A''}{Br} - \frac{\ddot{B}}{4Bc^2} + \frac{\dot{B}}{4Bc^2}\left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right)$$

$$R_{11} = \frac{A'}{4A}(\frac{A'}{a} + \frac{B'}{B}) - \frac{A''}{2A} + \frac{B'}{Br} - \frac{\ddot{B}}{2Ac^2} - \frac{\dot{B}}{4Ac^2}(\frac{\dot{A}}{A} - \frac{\dot{B}}{B})$$

$$R_{22} = -\frac{r}{2B}(\frac{A'}{A} - \frac{B'}{B}) - \frac{1}{B} + 1$$

$$R_{33} = R_{22} sin^2\theta$$

$$R_{01} = R_{10} = \frac{\dot{B}}{Brc}$$

Due to spherical symmetry,the electric field should have only radial component and so
$E_r = E_1 = E_1(t, r) = cF_{01} = -cF_{10}$

All other components are zero since there are no currents or magnetic monopoles. Therefore,we have

$$F_{\alpha\beta} = \begin{pmatrix} 0 & \frac{E_r}{c} & 0 & 0 \\ \frac{-E_r}{c} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We know that

$$T_{\alpha\beta} = \frac{1}{\mu_o}(\frac{1}{4} g_{\alpha\beta} F_{\mu\nu} F^{\mu\nu} - g_{\beta\nu} F_{\alpha\mu} F^{\nu\mu})$$

Now we evaluate the following:

$\frac{1}{4}g_{\alpha\beta}F_{\mu\nu}F^{\mu\nu}$
$= \frac{1}{4}g_{\alpha\beta}\left(F_{\mu 0}F^{\mu 0} + F_{\mu 1}F^{\mu 1}\right)$
$= \frac{1}{4}g_{\alpha\beta}(F_{10}F^{10}+F_{01}F^{01}) = \frac{1}{4}g_{\alpha\beta}(2F_{01}F^{01}) = \frac{1}{2}g_{\alpha\beta}F_{01}F^{01}$

and also
$g_{\beta\nu}F_{\alpha\mu}F^{\nu\mu} = g_{\beta}F_{\alpha}F^{10} + g_{\beta 0}F_{\alpha 1}F^{1}$

Now we evaluate the corresponding non-zero components of energy-momentum tensor:
$\mathrm{T}_{\alpha\beta} = \frac{1}{\mu o}(\frac{1}{2}g_{\alpha\beta}F_{01}F^{01} - g_{\beta 1}F_{\alpha 0}F^{10} - g_{\beta 0}F_{\alpha 1}F^{01})$
$\mathrm{T}_{00} = -\frac{1}{2\mu o}AF_{01}F^{01}$
$T_{11} = \frac{1}{2\mu o}BF_{01}F^{01}$
$T_{22} = -\frac{1}{2\mu o}r^2F_{01}F^{01}$
$T_{33} = T_{22}sin^2\theta$

Since $\mathrm{T}_{01} = 0, R_{01} = R_{10} = 0$

and using a value of $\mathrm{R}_{10}, we have \frac{\dot{B}}{Brc} = 0 => B = constant$
$Also, \frac{T_{00}}{A} + \frac{T_{11}}{B} = 0 => \frac{R_{00}}{A} + \frac{R_{11}}{B} = 0 = \frac{1}{rB}(\frac{A'}{A} + \frac{B'}{B}) => \frac{A'}{A} + \frac{B'}{B} = 0 => \frac{\partial}{\partial r}ln(AB) = 0 => AB = const. = f(t)$

We have that $\mathrm{A}{=}\mathrm{g}_{00} and B = -g_{00} => g_{00} = -\frac{f}{g_{11}}$
$Since F_{\mu\nu} = g_{\mu\alpha}g_{\nu\beta}F^{\alpha\beta}$

we have : $F_{01} = g_{00}g_{11}F^{01} = -f(t)F^{01}$

We know that : $F_{\alpha\beta};\gamma + F_{\beta\gamma};\alpha + F_{\gamma\alpha};\beta = 0 => F_{01};0 + F_{10};0 + F_{00};1 = 0 => F_{01};0 + F_{10};0 = 0$

We also know that $F^{\alpha\beta};\beta = 0 and T^{\alpha\beta};r = \partial_r T^{\alpha\beta} + \Gamma^{\alpha}_{\mu\gamma}T^{\mu\beta} + \Gamma^{\beta}_{\mu\beta}T^{\alpha\mu} for \alpha = 1 and \beta = 0, we have :$
$0 = \partial_0 F^{10} + CF^{\mu 0} + \Gamma^{0}_{\mu 0}F^{1\mu}$

the second and the third terms vanish so that

$\partial_0 F^{10} = 0 \Rightarrow -\partial_0(\frac{E_r}{c}) = 0 \Rightarrow E = E(r)$

therefore, the electric field is time independent

For $\alpha = 0 and \beta = 1, we have : \partial_1 F^{01} + \Gamma^{0}_{\mu\beta}F^{\mu\beta} + \Gamma^{\beta}_{\mu\beta}F^{0\mu} = 0 -$

$\Gamma^{\beta}_{\mu\beta}F^{0\mu} = \Gamma^{\beta}_{1\beta}F^{01 = F^{01}(\Gamma^0_{10} + \Gamma^1_{11} + \Gamma^2_{12} + \Gamma^3_{13}) = F^{01}(\frac{A'}{2A} + \frac{B'}{2B} + \frac{2}{r}) = \frac{2}{r}F^{01}}$

and since
$\frac{A'}{2A} + \frac{B'}{2B} = 0$
using this in equation (1) we have

$\frac{\partial}{\partial_r}F^{01} = -\frac{2}{r}F^{01} => F^{01} = \frac{E_r}{c} = \frac{C}{r^2}, where C is const of integra$
By Gauss' law, $E_r = \frac{Q}{4\pi\epsilon_o r^2}, therefore, C = \frac{Q}{4\pi\epsilon_o}$

$hence, E_r = \frac{Q}{4\pi\epsilon_o r^2}$ Now since T=0, we have : $R_{\alpha\beta} = \frac{8\pi G}{c^4}T_{\alpha\beta}$ using values of $\mathrm{R}_{22} and T_{22}, and solving for A, we get :$
$A = f(t) + \frac{GQ^2}{4\pi\epsilon_o c^4 r^2} + \frac{C(t}{r}$
$when Q = 0, the metric should reduce to the Schwarzschild me$
$1 - \frac{2GM}{rc^2} = f(t) + 0 + \frac{C(t)}{r} => f(t) = 1 and C(t) = -\frac{2GM}{c^2} = -r_s$
$So now we have;$

$A = 1 - \frac{r_s}{r} + \frac{(r_Q)^2}{r^2}$ and since AB=f(t)=1,we have the obtained the **Reissner-Nordstrom metric**:

$$\mathrm{g}_{\alpha\beta} = \begin{pmatrix} 1 - \frac{r_s}{r} + \frac{(r_Q)^2}{r^2} & 0 & 0 & 0 \\ 0 & -(1 - \frac{r_s}{r} + \frac{(r_Q)^2}{r^2})^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 sin^2\theta \end{pmatrix}$$

2 Applications of Reissner-Nordstrom Metric :

1. Gravitaional Time Dialation

Consider some fixed point in space. With constant r, θ and ϕ, we have that dr, $d\theta$ and $d\phi$ is zero and the metric becomes

$$ds^2 = \left(1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2}\right) c^2 dt^2.$$

Using that $ds^2 = c^2 d\tau^2$, we obtain

$$d\tau = dt\sqrt{1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2}}$$

dt can be interpreted as an infinitesimal time interval measured by an observer that is infinitely far away from a gravitational body, while $d\tau$ is the interval measured by an observer at a distance r from the center of the body.

blueWe then have that $d\tau < dt$. This means that the far-away observer will measure the clock that is closer to the body run slower by a factor of

$$\sqrt{1 - r_s/r + r_Q^2/r^2}.$$

2. Naked Singularity

Consider now the situation when $r_s < 2r_Q$. In this case there are no singularities when $r > 0$, and therefore no event horizons. The singularity at $r = 0$ does still exist, which means that there is no event horizon preventing someone far away from directly observing this singularity. A singularity with this property (i.e., no event horizon "hiding" it) is called a naked singularity. It is widely believed, but not proven, that no naked singularity (except maybe the one occurring in the Big Bang model) exists in the universe.