Potential Scattering :-

for incident particles of mass m_1 being scattered by target particle of mass m_2, If the interaction potential is time-independent, then

$$\left[-\frac{\hbar^2}{2m_1}\nabla_1^2 - \frac{\hbar^2}{2m_2}\nabla_2^2 + V(r_1, r_2)\right]\psi(r_1, r_2) = E_T\psi(r_1, r_2) \quad -①$$

and $\psi(r_1, r_2, t) = \psi(r_1, r_2)\, e^{-iE_T t/\hbar}$

We can write this as

$$\left[-\frac{\hbar^2}{2\mu}\nabla^2 + V(r)\right]\psi(r) = E\psi(r) \quad -② \quad ; \quad \mu = \frac{m_1 m_2}{m_1 + m_2}$$

and $\psi(r, t) = \psi(r)\, e^{-iEt/\hbar}$

from eqn. ②;

$$\left(\nabla^2 - \frac{2\mu}{\hbar^2}V(r)\right)\psi(r) = -\frac{2\mu E}{\hbar^2}\psi(r)$$

$$\Rightarrow \left[\nabla^2 + k^2 - U(r)\right]\psi(r) = 0 \quad ; \quad U(r) = \frac{2\mu}{\hbar^2}V(r)$$

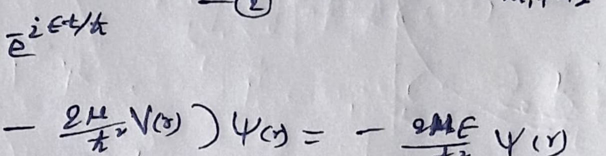
If potential $V(r)$ decreases faster than r^{-1} as $r \to \infty$ such as for $V(r) \sim \frac{1}{r^{1.5}}$, then

$V(r) \to 0$ as $r \to \infty$

$$\therefore \left[\nabla^2 + k^2\right]\psi(r) = 0$$

$$\psi(r) \xrightarrow[(r\to\infty)]{} \psi_{inc}(r) + \psi_{sc}(r)$$

In potential scattering, m_2 is replaced by $V(r)$ only. So $\mu \to m_1 = m$



$$\psi_{inc}(\vec{r}) = Ae^{i\vec{k}\cdot\vec{r}} = Ae^{ikz}$$

and $\psi_{sc}(\vec{r}) = Af(k, \theta, \phi)\dfrac{e^{ikr}}{r}$

$$\therefore \psi_k(r) \xrightarrow[(r\to\infty)]{} Ae^{i\vec{k}\cdot\vec{r}} + Af(k, \theta, \phi)\frac{e^{ikr}}{r}$$

Now, $\vec{j}(r) = \dfrac{i\hbar}{2m}\left[\psi\nabla\psi^* - \psi^*\nabla\psi\right]$

$$\therefore j_{inc} = \frac{\hbar k|A|^2}{m}$$

and $j_{sc} = |A|^2\dfrac{\hbar k}{mr^2}|f(k, \theta, \phi)|^2$

The differential cross-section is given by

$$\frac{d\sigma}{d\Omega} = \frac{\left(\frac{j_{sc} \times r^2 d\Omega}{d\Omega}\right)}{j_{inc}} = \frac{|A|^2\frac{\hbar k}{mr^2}|f(k, \theta, \phi)|^2 \times r^2}{\frac{\hbar k}{m}|A|^2}$$

$$\therefore \boxed{\frac{d\sigma}{d\Omega} = |f(k, \theta, \phi)|^2}$$

Partial wave analysis

If the central potential $V(r)$ has azimuthal symmetry for scattering, then

$$\psi_k(r,\theta) = \sum_{l=0}^{\infty} R_l(k,r) P_l(\cos\theta) \quad \text{--- (1)}$$

and $$f(k,\theta) = \sum_{l=0}^{\infty} f_l(k) P_l(\cos\theta) \quad \text{--- (2)}$$

the radial functions satisfy

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2} - U(r) + k^2\right] R_l(k,r) = 0 \quad ; \quad k = \sqrt{\frac{2mE}{\hbar^2}}$$

If $r > a$ where $a \to$ range of $V(r)$, then $U(r) \simeq 0$

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2} + k^2\right] R_l(k,r) = 0$$

$$\therefore \quad R_l(k,r) = B_l(k) j_l(kr) + C_l(k) n_l(kr)$$

for large r,

$$j_l(kr) \sim \frac{1}{kr}\sin\left(kr - \frac{l\pi}{2}\right) \quad \text{--- (3)}$$

$$n_l(kr) \sim -\frac{1}{kr}\cos\left(kr - \frac{l\pi}{2}\right) \quad \text{--- (4)}$$

So now,

$$R_l(k,r) \xrightarrow[(r\to\infty)]{} \frac{1}{kr}\left[B_l(k)\sin\left(kr - \frac{l\pi}{2}\right) - C_l(k)\cos\left(kr - \frac{l\pi}{2}\right)\right]$$

$$\Rightarrow \quad R_l(k,r) \xrightarrow[(r\to\infty)]{} A_l(k)\frac{1}{kr}\sin\left(kr - \frac{l\pi}{2} + \delta_l(k)\right) \quad \text{--- (5)}$$

where $A_l(k) = \left[B_l^2(k) + C_l^2(k)\right]^{1/2}$ --- (6) and $\delta_l(k) = -\tan^{-1}\left[\frac{C_l(k)}{B_l(k)}\right]$ --- (7)

Also note that

$$e^{i\vec{k}\cdot\vec{r}} = \sum_{l=0}^{\infty} (2l+1) i^l j_l(kr) P_l(\cos\theta)$$

$$\therefore \quad e^{i\vec{k}\cdot\vec{r}} \xrightarrow[(r\to\infty)]{} \sum_l (2l+1) i^l \frac{1}{kr}\sin\left(kr - \frac{l\pi}{2}\right) P_l(\cos\theta) \quad \text{--- (8)}$$

$$\psi_k(r,\theta) \xrightarrow[(r\to\infty)]{} e^{i\vec{k}\cdot\vec{r}} + f(k,\theta)\frac{e^{ikr}}{r} \quad \text{--- (9)}$$

using (1), (8), (5) and (9), we have

$$R_l(k,r) \longrightarrow \frac{(2l+1) i^l}{kr}\sin\left(kr - \frac{l\pi}{2}\right) + \frac{1}{r}e^{ikr} f_l(k) \quad \text{--- (10)}$$

By comparing (5) and (10), we get

$$f_l(k) = \frac{(2l+1)}{2ik}\left\{\exp[2i\delta_l(k)] - 1\right\} \quad \text{--- (11)} \quad ; \quad \delta_l(k) \to \text{called phase shift}$$

put (11) in (2), we get

$$f(k,\theta) = \frac{1}{2ik}\sum_l (2l+1)\exp[2i\delta_l(k) - 1] P_l(\cos\theta) \quad \text{--- (12)}$$

$$= \sum_l f_l(k) P_l(\cos\theta)$$

Now, eqn. (5) can be rewritten as

$$R_l(k,r) \rightarrow \frac{1}{2ik} A_l(k)\exp[-i\delta_l(k)]\left[\underbrace{S_l\frac{e^{i(kr-l\pi/2)}}{r}}_{\text{outgoing wave}} - \underbrace{\frac{e^{-i(kr-l\pi/2)}}{r}}_{\text{incoming wave}}\right]$$

where

$$S_l(k) = e^{2i\delta_l(k)} \rightarrow \text{called S-matrix element.}$$

* the potential produces phase difference b/w incoming and outgoing waves
* azimuthal symmetry exists for only central potential

Now,

$$\sigma_{Tot} = \int |f(k,\theta)|^2 d\Omega = 2\pi\int_{-1}^{1} f^* f \, d(\cos\theta)$$

$$= 2\pi\sum_l \int |f_l(k)|^2 P_l(\cos\theta) P_{l'}(\cos\theta)\, d(\cos\theta)$$

$$= 2\pi\sum_l\left[|f_l(k)|^2 \frac{2}{2l+1}\right]$$

$$= \frac{4\pi}{2}\sum_l \frac{(2l+1)^{\cancel{2}}}{4k^2}\frac{1}{\cancel{(2l+1)}}\left(e^{2i\delta_l}-1\right)\left(e^{-2i\delta_l}-1\right)$$

$$= \frac{\pi}{k^2}\sum_l (2l+1)\left(1-e^{2i\delta_l}-e^{-2i\delta_l}+1\right)$$

$$= \frac{\pi}{k^2}\sum_l (2l+1)\left(-2\cos 2\delta_l + 2\right) = \frac{\pi}{k^2}\sum_l (2l+1)\times 2\times 2\sin^2\delta_l$$

$$\therefore \boxed{\sigma_{Tot} = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l}$$

or

$$\sigma_{Tot} = \sum_l \sigma_l$$

where

$$\sigma_l = \frac{4\pi}{(2l+1)}|f_l(k)|^2$$

Also, $\boxed{\sigma_{Tot} = \frac{4\pi}{k}\operatorname{Im} f(k,\theta=0)}$ → optical theorem

* The no. of partial waves (values of l) increases as the energy of incident particles increases.
* Partial waves method is not applicable for Coulomb potential.

* The scattering length is defined by

$$\boxed{\alpha = -\lim_{k\to 0}\frac{\tan\delta_0(k)}{k}}$$

$$f \xrightarrow[\substack{(k\to 0 \\ s\text{-wave})}]{} , \quad \frac{d\sigma}{d\Omega} \rightarrow \alpha^2$$

$$\sigma_{Tot} \rightarrow 4\pi\alpha^2$$

The Integral equation of potential scattering

We know that for potential scattering

$$[\nabla^2 + k^2 - U(r)]\psi(r) = 0$$

$$[\nabla^2 + k^2]\psi(r) = U(r)\psi(r) \quad \text{--- (1)}$$

its general solution can be written as

$$\psi(\vec{r}) = \phi(\vec{r}) + \int G_0(k, \vec{r}, \vec{r}\,')U(\vec{r}\,')\psi(\vec{r}\,')\,d\vec{r}\,' \quad \text{--- (2)}$$

where $\phi(\vec{r})$ satisfies $(\nabla^2 + k^2)\phi(r) = 0 \Rightarrow \phi_k(r) = e^{i\vec{k}\cdot\vec{r}}$ --- (3)

and $(\nabla^2 + k^2)G_0(k, r, r') = \delta(\vec{r} - \vec{r}\,')$ --- (4)

$$\delta(\vec{r} - \vec{r}\,') = \frac{1}{(2\pi)^3}\int \exp[i\vec{K}\cdot(\vec{r} - \vec{r}\,')]\,d\vec{K} \quad \text{--- (5)} \qquad \vec{K} = (K, \Theta, \Phi)$$

and $G_0(k, \vec{r}, \vec{r}\,') = \frac{1}{(2\pi)^3}\int \exp(i\vec{K}\cdot\vec{r})\,g_0(k, K, r')\,dK$ --- (6)

putting (5) and (6) in (4), we get

$$\frac{1}{(2\pi)^3}\int(-K^2 + k^2)\exp(i\vec{K}\cdot\vec{r})\,g_0(k, K, r')\,dK = \frac{1}{(2\pi)^3}\int \exp(i\vec{K}\cdot(\vec{r} - \vec{r}\,'))\,dK$$

$$\Rightarrow g_0(k, K, r') = \frac{\exp(-i\vec{K}\cdot\vec{r}\,')}{k^2 - K^2}$$

putting this in (6);

$$\therefore G_0(k, \vec{r}, \vec{r}\,') = -\frac{1}{(2\pi)^3}\int \frac{\exp[i\vec{K}\cdot(\vec{r} - \vec{r}\,')]}{K^2 - k^2}\,d\vec{K} \quad \text{--- (7)}$$

Now, let $\vec{R} = \vec{r} - \vec{r}\,'$ along z-axis, then

$$G_0(k, R) = -\frac{1}{(2\pi)^3}\int_0^\infty K^2\,dK\int_0^\pi \sin\Theta\,d\Theta\int_0^{2\pi}\frac{\exp(iKR\cos\Theta)}{K^2 - k^2}\,d\Phi$$

$$= -\frac{2\pi}{(2\pi)^3}\int_0^\infty \frac{K^2}{K^2 - k^2}\,dK\int_0^\pi e^{i(KR\cos\Theta)}\sin\Theta\,d\Theta$$

let $x = \cos\Theta \Rightarrow dx = -\sin\Theta\,d\Theta$

$\Theta = 0,\ x = 1$

$\Theta = \pi,\ x = -1$

$$G_0(k, R) = \frac{-1}{4\pi^2}\int_{-1}^{1} e^{iKRx}\,dx\int_0^\infty \frac{K^2}{K^2 - k^2}\,dK$$

$$= -\frac{1}{4\pi^2 R}\int_{-\infty}^{\infty}\frac{K\sin(KR)}{K^2 - k^2}\,dK$$

[due to even function limit changes from 0 to ∞ to −∞ to +∞]

since $\sin(KR) = \dfrac{e^{iKR} - e^{-iKR}}{2i}$

$$G_0(k,R) = \frac{-1}{4\pi^2 R}\int_0^\infty \frac{K}{K^2-k^2} \times \left(\frac{e^{iKR} - e^{-iKR}}{2i}\right) dK$$

$$= \frac{i}{8\pi^2 R}\int_0^\infty \frac{\left(e^{iKR} - e^{-iKR}\right)}{2}\left[\frac{1}{K-k} + \frac{1}{K+k}\right] dK$$

$$= \frac{i}{16\pi^2 R}\left(I_1 - I_2\right)$$

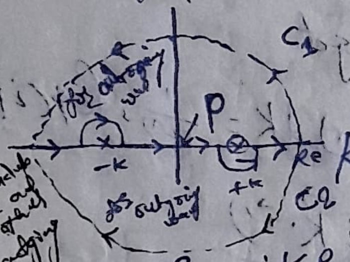
where

$$I_1 = \int_{-\infty}^{\infty} e^{iKR}\left(\frac{1}{K-k} + \frac{1}{K+k}\right) dK$$

and

$$I_2 = \int_{-\infty}^{\infty} e^{-iKR}\left(\frac{1}{K-k} + \frac{1}{K+k}\right) dK$$

there are two poles at $K = \pm k$.



$$I_1 = \oint_C e^{iKR}\left(\frac{1}{K-k} + \frac{1}{K+k}\right) dK$$

Contour C consists of path P and C_1.

*We include one pole and exclude other to get outgoing wave only. If both poles are included we get both incoming and outgoing which are not desired.

$$I_1 = \underbrace{\int_{C_1} e^{iKR}\frac{2K}{K^2-k^2}dK}_{0} + \int_P \frac{2K e^{iKR}}{K^2-k^2} dK$$

$$= 2\pi i \times R(K=k)$$

$$= 2\pi i \times \left[\lim_{K\to k} \cancel{(K-k)}\frac{e^{iKR}\,2K}{(K+k)\cancel{(K-k)}}\right]$$

$$I_1 = 2\pi i \exp(ikR)$$

Similarly $I_2 = -2\pi i \exp(ikR)$ [along path $P + C_2$]

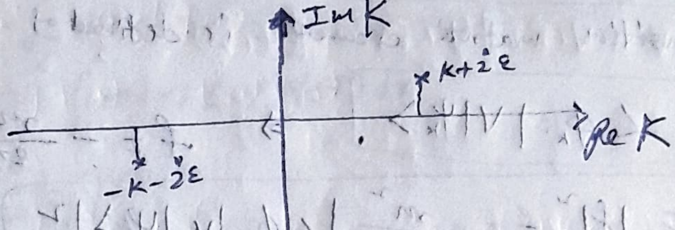
$$G_0(k,R) = \frac{i}{16\pi^2 R}\exp(ikR)\left(2\pi i + 2\pi i\right)$$

$$= -\frac{1}{4\pi}\frac{\exp(ikR)}{R}$$

or

$$\boxed{G_0(k,\vec{r},\vec{r}\,') = -\frac{1}{4\pi}\frac{\exp[ik|\vec{r}-\vec{r}\,'|]}{|\vec{r}-\vec{r}\,'|}} \quad \text{—⑧}$$

* Choosing the path P is equivalent to keeping the integration path along the real axis, and shifting the two poles at $K = \pm k'$ off the real axis.



so two poles are located at $K = \pm(k + i\varepsilon')$; $\varepsilon' \to 0^+$ $(\varepsilon' > 0)$

$K^2 = k^2 - \varepsilon'^2 + 2i\varepsilon' k \simeq k^2 + i\varepsilon$; $\varepsilon' \to 0$

therefore, using (7); $\epsilon = 2k\varepsilon'$

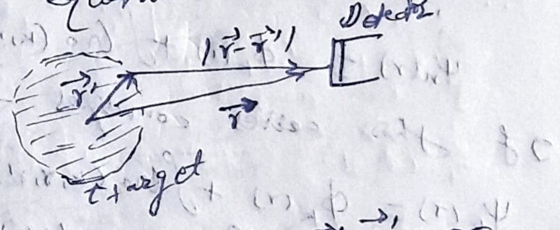
$$G_0(k, r, r') = \frac{-1}{(2\pi)^3} \lim_{\varepsilon \to 0^+} \int \frac{\exp[i\vec{K}\cdot(\vec{r}-\vec{r}')]}{K^2 - k^2 - i\varepsilon} d\vec{K}$$

↑ this, when integrating along P, give equ. (8).

Now, we can have

$$\boxed{\psi_k(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} - \frac{1}{4\pi}\int \frac{\exp[ik|\vec{r}-\vec{r}'|]}{|\vec{r}-\vec{r}'|} U(\vec{r}')\psi_k(\vec{r}')\, d\vec{r}'}$$

↳ Lippmann - Schwinger equation.



If $r \gg r'$, then

$k|\vec{r}-\vec{r}'| = k\sqrt{r^2 - 2rr' + r'^2}$

$\simeq k\sqrt{r^2\left(1 - \frac{2r'}{r}\right)}$

$\simeq kr\left(1 - \frac{r'}{r}\right) = k(r - \hat{r}\cdot\vec{r}') = kr - \vec{k}'\cdot\vec{r}'$; $\boxed{\vec{k}' = k\hat{r}}$

and $\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{r\left|1 - \frac{\vec{r}\cdot\vec{r}'}{r^2}\right|} \simeq \frac{1}{r}\left(1 + \frac{\vec{r}\cdot\vec{r}'}{r^2}\right) \simeq \frac{1}{r}$

$\therefore \psi_k(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} - \frac{1}{4\pi}\int \frac{e^{ikr}\, e^{-i\vec{k}'\cdot\vec{r}'}}{r} U(r')\psi_k(r')\, dr'$

$$\boxed{\psi_k(r) = e^{i\vec{k}\cdot\vec{r}} - \frac{e^{ikr}}{4\pi r}\int e^{-i\vec{k}'\cdot\vec{r}'} U(\vec{r}')\psi_k(\vec{r}')\, d\vec{r}'} \quad (r \to \infty)$$

Since $\psi_k(r) = e^{i\vec{k}\cdot\vec{r}} + f(k, \theta, \phi)\frac{e^{ikr}}{r}$, therefore,

$f(k, \theta, \phi) = -\frac{1}{4\pi}\int e^{-i\vec{k}'\cdot\vec{r}} U(r)\psi_k(r)\, dr$

$= -\frac{1}{4\pi}\langle \phi_{k'}|\hat{U}|\psi_k\rangle$

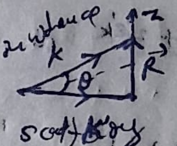
where $\phi_{k'}(r) = e^{i\vec{k}'\cdot\vec{r}}$; $\vec{k}' \to$ final wave vector since $V(r) = \frac{\hbar^2}{2m} U(r)$, eq

$$\boxed{f = -\frac{m}{2\pi\hbar^2} \langle \phi_{k'}^{(0)} | V | \psi_k^{(+)} \rangle}$$

→ The transition matrix element is defined as

$$\boxed{T_{k'k} = \langle \phi_{k'} | V | \psi_k \rangle}, \Rightarrow f = -\frac{m}{2\pi\hbar^2} T_{k'k}$$

$$\boxed{\frac{d\sigma}{d\Omega} = |f|^2 = \frac{m^2}{4\pi^2\hbar^4} |\langle \phi_{k'} | V | \psi_k \rangle|^2}$$



The Born Series

→ If the potential is weak or the energy is higher, perturbation theory can be used.

→ for zeroth order approximation and subsequent orders,

$$\psi_0(r) = \phi_k(r) = e^{i\vec{k}\cdot\vec{r}}$$

$$\psi_1 = \phi_k(r) + \int G_0(k, r, r')\, U(r')\, \psi_0(r')\, dr'$$

$$\psi_n(r) = \phi_k(r) + \int G_0(k, r, r')\, U(r')\, \psi_{n-1}(r')\, dr'$$

If this series converges to exact wave function ψ_K, then

$$\psi_K(r) = \phi_K(r) + \int G_0(k, r, r')\, U(r')\, \phi_K(r')\, dr' + \int G_0(k, r, r')\, U(r')\, G_0(k, r', r'')\, U(r'')\, \phi_K(r'')\, dr'\, dr'' + \cdots$$

$$\Rightarrow f = -\frac{1}{4\pi} \langle \phi_{k'} | U + U G_0 U + U G_0 U G_0 U + \cdots | \phi_k \rangle$$

* for 1st Born approximation,

$$f^B = -\frac{1}{4\pi} \langle \phi_{k'} | U | \phi_k \rangle = -\frac{m}{2\pi\hbar^2} T^B_{k'k}$$

$$T^B_{k'k} = \langle \phi_{k'} | V | \phi_k \rangle$$

* for convergence of Born series, $\frac{|U_0| a}{2k} \ll 1$.

Now,

$$f^B = -\frac{1}{4\pi} \int \exp(-i\vec{k}'\cdot\vec{r})\, U(\vec{r})\, \exp(i\vec{k}\cdot\vec{r})\, d^3\vec{r}$$

$$= -\frac{1}{4\pi} \int \exp(i\vec{\Delta}\cdot\vec{r})\, U(\vec{r})\, d^3\vec{r} \quad ; \quad \vec{\Delta} = \vec{k} - \vec{k}'$$

$$\equiv -\frac{\mu}{2\pi\hbar^2} \int e^{i\vec{q}\cdot\vec{r}}\, V(\vec{r})\, d^3\vec{r}$$

$$\vec{q} = \vec{k}' - \vec{k}$$

for the elastic scattering. $k = k'$ and $\vec{k}\cdot\vec{k}' = k^2\cos\theta$

$$\Delta = |\vec{k}-\vec{k}'| = \sqrt{k^2+k^2-2k^2\cos\theta} = 2k\sin\left(\frac{\theta}{2}\right)$$

also, $$T^B_{k'k} = \int \exp(-i\vec{k}'\cdot\vec{r})V(\vec{r})\exp(i\vec{k}\cdot\vec{r})\,d^3\vec{r}$$
$$= \int \exp(i\vec{\Delta}\cdot\vec{r})V(\vec{r})\,d^3\vec{r}$$

and $$\frac{d\sigma^B}{d\Omega} = |f^B|^2$$

* for central potential, $U(\vec{r}) = U(r)$ let $\vec{r} = (r, \theta', \phi')$

$$f^B = -\frac{1}{4\pi}\int_0^{2\pi} d\phi' \int_0^\infty r^2 U(r)\,dr \int_0^\pi d\theta' \sin\theta' \exp(i\Delta r\cos\theta')$$

$$\boxed{f^B = -\frac{2m}{\hbar^2\Delta}\int_0^\infty r\sin(\Delta r)V(r)\,dr} = -\frac{2\mu}{\hbar^2 q}\int_0^\infty r'V(r')\sin(qr')\,dr'$$

and $$\boxed{\frac{d\sigma^B}{d\Omega} = |f^B|^2 = \frac{4m^2}{\hbar^4 q^2}\left|\int_0^\infty r'V(r')\sin(qr')\,dr'\right|^2}$$

* for Yukawa potential,

$$U(r) = U_0\frac{e^{-r/a}}{r} = U_0\frac{e^{-\alpha r}}{r}$$

$$f^B = -\frac{U_0}{\alpha^2+\Delta^2} \quad \text{and} \quad \frac{d\sigma^B}{d\Omega} = \frac{U_0^2}{(\alpha^2+\Delta^2)^2}$$

for Coulomb potential, $V_c(r) = \dfrac{q_1 q_2}{4\pi\epsilon_0 r}$

$$U_c(r) = \frac{U_0}{r}; \quad U_0 = \frac{2m}{\hbar^2}\frac{q_1 q_2}{4\pi\epsilon_0}$$

when $\alpha \to 0$, then from Yukawa,

* $$f^B_c \approx -\frac{U_0}{\Delta^2} \quad \text{and} \quad \frac{d\sigma^B_c}{d\Omega} = \frac{U_0^2}{\Delta^4} = \left(\frac{\gamma}{2k}\right)^2\frac{1}{\sin^4(\theta/2)} \quad \left[\because \Delta = 2k\sin\frac{\theta}{2}\right]$$

$$= \left(\frac{q_1 q_2}{4\pi\epsilon_0}\right)^2\frac{1}{16E^2\sin^4(\theta/2)}$$ Rutherford

where $\gamma = \dfrac{q_1 q_2}{4\pi\epsilon_0\hbar v} = \dfrac{U_0}{2k}$

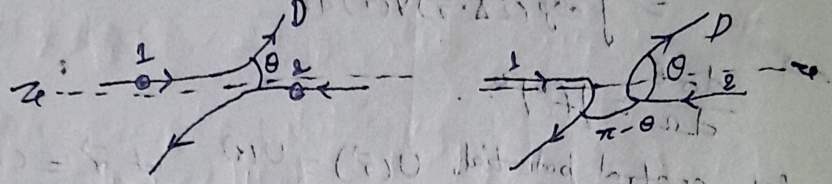
$$\frac{d\sigma^B_c}{d\Omega} \propto \frac{1}{E^2}$$

at $\theta \to 0$, $\dfrac{d\sigma^B_c}{d\Omega} \to \infty$.

Scattering of two spinless identical bosons

for two bosons of mass m, in their COM frame.

$$\left[-\frac{\hbar^2}{2\mu}\nabla^2 + V(\vec{r})\right]\psi(\vec{r}) = E\psi(\vec{r}) \quad ; \quad \vec{r} = \vec{r}_1 - \vec{r}_2$$



for classical result:

$$\frac{d\sigma_{cl}}{d\Omega} = |f(\theta,\phi)|^2 + |f(\pi-\theta, \phi+\pi)|^2$$

$$\psi_k(\vec{r}) \xrightarrow[(r\to\infty)]{} e^{i\vec{k}\cdot\vec{r}} + f(\theta,\phi)\frac{e^{ikr}}{r}$$

If we interchange the coordinates, $\vec{r}_1 \leftrightarrow \vec{r}_2$, then.

$\vec{r} \to -\vec{r}$ so $(r,\theta,\phi) \to (r, \pi-\theta, \pi+\phi)$,

$\psi_+(\vec{r}) = \psi_k(\vec{r}) + \psi_k(-\vec{r})$ is symmetric for bosons

$$\psi_+(-\vec{r}) = \psi_+(\vec{r})$$

$$\therefore \quad \psi_+(\vec{r}) \xrightarrow[(r\to\infty)]{} \left[e^{i\vec{k}\cdot\vec{r}} + e^{-i\vec{k}\cdot\vec{r}}\right] + \left[f(\theta,\phi) + f(\pi-\theta, \pi+\phi)\right]\frac{e^{ikr}}{r}$$

So the amplitude of outgoing wave is

$$f_+(\theta,\phi) = f(\theta,\phi) + f(\pi-\theta, \phi+\pi)$$

$$\frac{d\sigma}{d\Omega} = |f(\theta,\phi) + f(\pi-\theta, \phi+\pi)|^2$$

This differs from the classical result.

* If the potential is symmetric, then f_+ will be independent of ϕ, and so

$$\frac{d\sigma}{d\Omega} = |f(\theta) + f(\pi-\theta)|^2$$

If it is symmetric about $\theta = \pi/2$, then

$$\frac{d\sigma(\theta=\pi/2)}{d\Omega} = 4|f(\theta=\pi/2)|^2$$

Non-relativistic Reduction of Dirac equation:

The dirac eqn. for +ve energy case is

$$i\frac{\partial\psi^{(1,2)}_{(x)}}{\partial t} = \left[\hat{\alpha}\cdot(\vec{p}-q\vec{A}) + \hat{\beta}m + q\,\phi(x)\right]\psi^{(1,2)}_{(x)} \quad -①$$

where, $\hat{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}$ and $\hat{\beta} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$ $-②$

Now,

$$\psi^{(1,2)}(x) = \begin{pmatrix} \tilde{\phi}(x) \\ \tilde{\chi}(x) \end{pmatrix} \quad -③$$

put ② and ③ in ①;

$$i\frac{\partial}{\partial t}\begin{pmatrix} \tilde{\phi} \\ \tilde{\chi} \end{pmatrix} = \left[\begin{pmatrix} 0 & \vec{\sigma}\cdot(\vec{p}-q\vec{A}) \\ \vec{\sigma}\cdot(\vec{p}-q\vec{A}) & 0 \end{pmatrix}_{4\times4} + \begin{pmatrix} m\mathbb{1} & 0 \\ 0 & -m\mathbb{1} \end{pmatrix}_{4\times4} + q\,\phi(x)\begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}_{4\times4}\right]\begin{pmatrix} \tilde{\phi} \\ \tilde{\chi} \end{pmatrix}$$

$$= \begin{pmatrix} \vec{\sigma}\cdot(\vec{p}-e\vec{A})\tilde{\chi} \\ \vec{\sigma}\cdot(\vec{p}-e\vec{A})\tilde{\phi} \end{pmatrix} + m\begin{pmatrix} \tilde{\phi} \\ -\tilde{\chi} \end{pmatrix} + e\phi\begin{pmatrix} \tilde{\phi} \\ \tilde{\chi} \end{pmatrix}$$

$$\Rightarrow i\frac{\partial\tilde{\phi}}{\partial t} = \vec{\sigma}\cdot(\vec{p}-e\vec{A})\tilde{\chi} + m\tilde{\phi} + e\phi\tilde{\phi} \quad -④$$

$$i\frac{\partial\tilde{\chi}}{\partial t} = \vec{\sigma}\cdot(\vec{p}-e\vec{A})\tilde{\phi} - m\tilde{\chi} + e\phi\tilde{\chi} \quad -⑤$$

from eqn ③;

$$\psi^{(1,2)}(x) = \begin{pmatrix} \tilde{\phi}(x) \\ \tilde{\chi}(x) \end{pmatrix} = \begin{pmatrix} \phi_0(x) \\ \chi_0(x) \end{pmatrix} e^{-iE_p t}$$

[where in the non-relative case, $E_p \approx \frac{p^2}{2m} + mc^2 = K + m$
$= \frac{p^2}{2m} + m$ $(c=1)$

$$\begin{pmatrix} \tilde{\phi}(x) \\ \tilde{\chi}(x) \end{pmatrix} = \begin{pmatrix} \phi_0(x) \\ \chi_0(x) \end{pmatrix} e^{-i(K+m)t}$$

$$\approx \begin{pmatrix} u_A(x) \\ u_B(x) \end{pmatrix} e^{-imt} \quad (p\to 0)$$

$$\Rightarrow \tilde{\phi}(x) = u_A(x)\,e^{-imt} \quad -⑥$$

$$\tilde{\chi}(x) = u_B(x)\,e^{-imt} \quad -⑦$$

Put ⑥ and ⑦ in ④ and ⑤, we get

$$i(-im)\cancel{u_A e^{-imt}} + i\frac{\partial u_A}{\partial t}\cancel{e^{-imt}}$$

$$= \vec{\sigma}\cdot(\vec{p}-e\vec{A})u_B\cancel{e^{-imt}} + \cancel{m u_A e^{-imt}} + e\phi u_A\cancel{e^{-imt}}$$

$$\Rightarrow \quad i\frac{\partial u_A}{\partial t} = \vec{\sigma}\cdot(\vec{p}-e\vec{A})u_B + e\phi u_A \quad — (8)$$

and

$$i\frac{\partial u_B}{\partial t}e^{-imt} + i(-im)u_B e^{-imt} = \vec{\sigma}\cdot(\vec{p}-e\vec{A})u_A e^{-imt} - mu_B e^{-imt} + e\phi u_B e^{-imt}$$

$$\Rightarrow \quad i\frac{\partial u_B}{\partial t} = \vec{\sigma}\cdot(\vec{p}-e\vec{A})u_A - 2mu_B + e\phi u_B \quad — (9)$$

for -ve negative solution in non-relativistic case,

$$|u_B| << |u_A| \text{ and } \left|i\frac{\partial u_B}{\partial t}\right| << |mu_B|, \text{ so}$$

$$|e\phi u_B| << |mu_B|$$

$$\therefore \quad i\frac{\partial u_B}{\partial t} \to 0 \text{ and } e\phi u_B \to 0 \text{ (in (9))}$$

$$u_B = \frac{\vec{\sigma}\cdot(\vec{p}-e\vec{A})u_A}{2m}$$

put this in equ. (8), we get

$$i\frac{\partial u_A}{\partial t} = \left[\frac{[\vec{\sigma}\cdot(\vec{p}-e\vec{A})][\vec{\sigma}\cdot(\vec{p}-e\vec{A})]}{2m} + e\phi\right]u_A \quad — (10)$$

Here,

$$\{\vec{\sigma}\cdot(\vec{p}-e\vec{A})\}\{\vec{\sigma}\cdot(\vec{p}-e\vec{A})\}u_A$$

$$= \{(\vec{p}-e\vec{A})\cdot(\vec{p}-e\vec{A}) + i\vec{\sigma}\cdot[(\vec{p}-e\vec{A})\times(\vec{p}-e\vec{A})]\}u_A$$

$$= \{(\vec{p}-e\vec{A})^2 + i\vec{\sigma}\cdot[\vec{p}\times\vec{p} - e\vec{p}\times\vec{A} - e\vec{A}\times\vec{p} + e^2\vec{A}\times\vec{A}]\}u_A$$

$$= (\vec{p}-e\vec{A})^2 u_A + i\vec{\sigma}\cdot[\underbrace{(-i)^2\vec{\nabla}\times\vec{\nabla}}_{0} + ie\vec{\nabla}\times\vec{A} + ie\vec{A}\times\vec{\nabla}]u_A$$

$$= (\vec{p}-e\vec{A})^2 u_A - e\vec{\sigma}\cdot[\vec{\nabla}\times\vec{A}u_A + \vec{A}\times\vec{\nabla}u_A]$$

$$= (\vec{p}-e\vec{A})^2 u_A - e\vec{\sigma}\cdot(\cancel{\nabla u_A\times\vec{A}} + u_A\nabla\times\vec{A} + \cancel{\vec{A}\times\nabla u_A})$$

$$= (\vec{p}-e\vec{A})^2 u_A - e\vec{\sigma}\cdot(\vec{\nabla}\times\vec{A})u_A$$

$$= [(\vec{p}-e\vec{A})^2 - e\vec{\sigma}\cdot\vec{B}]u_A$$

put this in equ. (10)

$$\boxed{i\frac{\partial u_A}{\partial t} = \left[\frac{1}{2m}(\vec{p}-e\vec{A})^2 - \frac{e\vec{\sigma}\cdot\vec{B}}{2m} + e\phi\right]u_A}$$

this is called Pauli-Schrodinger equ.

Q.11 To find $[H, \vec{S}]$, where $\vec{S} = \frac{\hbar}{2}\vec{\Sigma}$

Sol^n:-

$$[\hat{H}, \vec{S}] = [\hat{\alpha}.\vec{p} + \hat{\beta}m, \frac{\hbar}{2}\vec{\Sigma}]$$

$$= [\hat{\alpha}\cdot\vec{p}, \frac{\hbar}{2}\vec{\Sigma}] + m\cancelto{0}{[\hat{\beta}, \frac{\hbar}{2}\vec{\Sigma}]}$$ [$\because$ $\hat{\beta}$ is a const. matrix diagonal]

$[H, S^j] = \frac{\hbar}{2}[\alpha^k p^k, \Sigma^j]$ [for j-th component]

$$= \frac{\hbar}{2}[\alpha^k, \Sigma^j]p^k + \frac{\hbar}{2}\alpha^k[p^k, \Sigma^j]$$

here, $[p^k, \Sigma^j]\psi(x) = p^k(\Sigma^j\psi(x)) - \Sigma^j(p^k\psi(x))$

$$= \Sigma^j(p^k\psi(x)) - \Sigma^j(p^k\psi(x))$$

$$= 0 \quad \left[\because \Sigma^j = \begin{pmatrix} \sigma^j & 0 \\ 0 & \sigma^j \end{pmatrix}_{4\times4}\right]$$

$\therefore$ $[H, S^j] = \frac{\hbar}{2}[\alpha^k, \Sigma^j]p^k$

$$\alpha^k\Sigma^j = \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix}\begin{pmatrix} \sigma^j & 0 \\ 0 & \sigma^j \end{pmatrix} = \begin{pmatrix} 0 & \sigma^k\sigma^j \\ \sigma^k\sigma^j & 0 \end{pmatrix}$$

$$\Sigma^j\alpha^k = \begin{pmatrix} 0 & \sigma^j\sigma^k \\ \sigma^j\sigma^k & 0 \end{pmatrix}$$

$$\Rightarrow [\alpha^k, \Sigma^j] = \begin{pmatrix} 0 & [\sigma^k, \sigma^j] \\ [\sigma^k, \sigma^j] & 0 \end{pmatrix}_{4\times4}$$

We know that

$$[\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma^k$$

$$[\alpha^i, \Sigma^j] = \begin{pmatrix} 0 & 2i\epsilon^{ijk}\sigma^k \\ 2i\epsilon^{ijk}\sigma^k & 0 \end{pmatrix}$$

Now,

$$= 2i\epsilon^{ijk}\begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix} = 2i\epsilon^{ijk}\alpha^k$$

$$[H, S^j] = \frac{\hbar}{2}(2i\epsilon^{jkl}\alpha^l)p^k = i\hbar(\hat{\alpha}\times\vec{p})_j$$

$$\therefore \boxed{[\hat{H}, \vec{S}] = i\hbar(\hat{\alpha}\times\vec{p})} = i\hbar c(\vec{\alpha}\times\vec{p}) \quad (c=1)$$

Since $\vec{S} = \frac{\hbar}{2}\vec{\Sigma}$, so

$$\boxed{[\hat{H}, \vec{\Sigma}] = 2i(\hat{\alpha}\times\vec{p})}$$

Q. To find $[\hat{H}, \vec{L}]$, if
$L_j = \varepsilon_{jkl} x_k p_l$

Solution:

$[\hat{H}, \vec{L}] = [\hat{\alpha}\cdot\vec{p} + \hat{\beta} m, \vec{L}]$

$= [\hat{\alpha}\cdot\vec{p}, \vec{L}] + m\underbrace{[\hat{\beta}, \vec{L}]}^{0}$ [$\because \hat{\beta}$ is a diagonal and constant matrix]

$= [\hat{\alpha}\cdot\vec{p}, \vec{L}]$

Now, $[\alpha^i p^i, L^j] = \alpha^i [p^i, L^j] + [\alpha^i, L^j] p^i$ — ①

$[p^i, L^j] = [p^i, \varepsilon^{jkl} x^k p^l]$

$= -i\hbar \varepsilon^{jik} p^k = -i \varepsilon^{jik} p^k$ [$\hbar = 1$] — ②

$[\alpha^i, L^j] = [\alpha^i, \varepsilon^{jkl} x^k p^l]$

$= 0$ [$\because \hat{\alpha}\cdot\vec{L} = \vec{L}\cdot\hat{\alpha}$]

from ① and ②;

$[\alpha^i p^i, L^j] = \alpha^i (-i \varepsilon^{jik} p^k)$

$= -i \varepsilon^{jik} \alpha^i p^k$

$\Rightarrow [H, L^j] = -i \varepsilon^{jik} \alpha^i p^k$

$\boxed{[\hat{H}, \vec{L}] = -i(\alpha \times \vec{p})}$ [$\hbar = 1$, $c = 1$]

$= -i\hbar c(\hat{\alpha} \times \vec{p})$

Q. Show that $[\hat{H}, \vec{J}] = 0$.

We know that

$[\hat{H}, \vec{S}] = i\hbar c(\hat{\alpha} \times \vec{p})$

and $[\hat{H}, \vec{L}] = -i\hbar c(\hat{\alpha} \times \vec{p})$

$[\hat{H}, \vec{J}] = [\hat{H}, \vec{S}] + [\hat{H}, \vec{L}]$

$= i(\hat{\alpha} \times \vec{p}) - i(\hat{\alpha} \times \vec{p})$

$= 0$

Q: Proof of closure relations?

(i) $\sum_{r=1}^{2} u^{(r)}(p)\bar{u}^{(r)}(p) = \not{p} + m\mathbb{1}_{4\times 4}$

Answer:-

We know that $u^{(r)}(p) = \sqrt{E_p+m}\begin{pmatrix} \xi^{(r)} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\xi^{(r)} \end{pmatrix}$

$\xi^{(1)} = \begin{pmatrix}1\\0\end{pmatrix}$, $\xi^{(2)} = \begin{pmatrix}0\\1\end{pmatrix}$

and $\bar{u}^{(r)}(p) = u^{(r)\dagger}(p)\gamma^0$

$= \sqrt{E_p+m}\begin{pmatrix} \xi^{(r)\dagger} & \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\xi^{(r)\dagger} \end{pmatrix}\begin{pmatrix}\mathbb{1} & 0\\ 0 & -\mathbb{1}\end{pmatrix}$

$= \sqrt{E_p+m}\begin{pmatrix} \xi^{(r)\dagger} & -\frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\xi^{(r)\dagger} \end{pmatrix}$

$u^{(r)}(p)\bar{u}^{(r)}(p) = \left(\sqrt{E_p+m}\right)^2 \begin{pmatrix} \xi^{(r)} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\xi^{(r)} \end{pmatrix}\begin{pmatrix} \xi^{(r)\dagger} & -\frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\xi^{(r)\dagger} \end{pmatrix}$

$= E_p+m \begin{pmatrix} \xi^{(r)}\xi^{(r)\dagger} & -\frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\xi^{(r)}\xi^{(r)\dagger} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\xi^{(r)}\xi^{(r)\dagger} & -\left(\frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\right)\left(\frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\right)\xi^{(r)}\xi^{(r)\dagger} \end{pmatrix}$

$\Rightarrow \sum_r u^{(r)}(p)\bar{u}^{(r)}(p) = E_p+m \begin{pmatrix} \sum \xi^{(r)}\xi^{(r)\dagger} & -\left(\frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\right)\sum \xi^{(r)}\xi^{(r)\dagger} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\sum \xi^{(r)}\xi^{(r)\dagger} & -\left(\frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\right)^2 \sum \xi^{(r)}\xi^{(r)\dagger} \end{pmatrix}$

$\boxed{\sum_r \xi^{(r)}\xi^{(r)\dagger} = \mathbb{1}}$

$= E_p+m \begin{pmatrix} \mathbb{1} & -\frac{\vec{\sigma}\cdot\vec{p}}{E_p+m} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m} & -\left(\frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\right)^2\mathbb{1} \end{pmatrix}$

$\hookrightarrow \frac{m-E_p}{E_p+m}$

$(\sigma_i p_i)(\sigma_j p_j)$

$(\sigma_i \sigma_j)(p_i p_j)$

$= p^2\mathbb{1}$

$\frac{p^2}{(E_p+m)^2}$

$\frac{E^2-m^2}{(E+m)^2}$

$\frac{\mathbb{1}(E-m)}{E+m}$

$\not{p} + \mathbb{1}m = \gamma^\mu p_\mu + \mathbb{1}m = \gamma^0 E - \vec{\gamma}\cdot\vec{p} + m\mathbb{1}$

$= \begin{pmatrix}\mathbb{1} & 0\\ 0 & -\mathbb{1}\end{pmatrix}E_p - \begin{pmatrix} 0 & \vec{\sigma}\cdot\vec{p} \\ -\vec{\sigma}\cdot\vec{p} & 0\end{pmatrix} + \begin{pmatrix} m & 0 \\ 0 & m\end{pmatrix}$

$= \begin{pmatrix} E_p+m & 0 \\ 0 & -E_p+m\end{pmatrix} - \begin{pmatrix} 0 & \vec{\sigma}\cdot\vec{p} \\ -\vec{\sigma}\cdot\vec{p} & 0\end{pmatrix}$

$= \begin{pmatrix} E_p+m & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -E_p+m\end{pmatrix} = E_p+m\begin{pmatrix} \mathbb{1} & -\frac{\vec{\sigma}\cdot\vec{p}}{E_p+m} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m} & \frac{m-E_p}{E_p+m}\end{pmatrix}$

$\therefore \boxed{\sum_{r=1}^{2} u^{(r)}(p)\bar{u}^{(r)}(p) = \not{p} + \mathbb{1}m}$

Proved

(2.)

$$\sum_{s=3}^{4} u^{(s)}(-p)\,\bar{u}^{(s)}(-p) = \not{p} - \mathbb{1} m$$

$$\Rightarrow \sum_{s} v^{(s)}(p)\,\bar{v}^{(s)}(p) = \not{p} - \mathbb{1} m$$

$$v^{(s)}(p) = \sqrt{E_p+m}\begin{pmatrix} -\frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\,\xi^{(s)} \\ \xi^{(s)} \end{pmatrix}$$

$$\bar{v}^{(s)}(p) = v^{(s)\dagger}(p)\,\gamma^0$$

$$= N\left(\xi^{(s)\dagger}\left(\frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\right)^\dagger \quad \xi^{(s)\dagger}\right)\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$$

$$= \left(\xi^{(s)\dagger}\left(\frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\right)^\dagger \quad -\xi^{(s)\dagger}\right)\sqrt{E_p+m}$$

$$\sum_{s=3}^{4} v^{(s)}(p)\,\bar{v}^{(s)}(p) = E_p+m\begin{pmatrix} -\frac{E_p - m}{E_p+m} & -\frac{\vec{\sigma}\cdot\vec{p}}{E_p+m} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m} & -\mathbb{1} \end{pmatrix}$$

$$\not{p} - m\mathbb{1} = \begin{pmatrix} E_p & 0 \\ 0 & -E_p \end{pmatrix} - \begin{pmatrix} m\mathbb{1} & 0 \\ 0 & m\mathbb{1} \end{pmatrix}$$

$$\gamma^0 p^0 - \vec{\gamma}\cdot\vec{p} - m\mathbb{1}$$

$$\begin{pmatrix} E_p - m & 0 \\ 0 & -E_p - m \end{pmatrix} - \begin{pmatrix} 0 & \vec{\sigma}\cdot\vec{p} \\ -\vec{\sigma}\cdot\vec{p} & 0 \end{pmatrix}$$

$$\begin{pmatrix} E_p - m & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -E_p - m \end{pmatrix}$$

$$= E_p+m\begin{pmatrix} \frac{E_p - m}{E_p+m} & -\frac{\vec{\sigma}\cdot\vec{p}}{E_p+m} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m} & -\mathbb{1} \end{pmatrix}$$

$$\boxed{\sum_{s=3}^{4} u^{(s)}(-p)\,\bar{u}^{(s)}(-p) = \not{p} - m\mathbb{1}}$$

(3.)

$$\sum_{s=\pm 1/2} \chi^{(s)}\chi^{(s)\dagger} = \chi^{(1/2)}\chi^{(1/2)\dagger} + \chi^{(-1/2)}\chi^{(-1/2)\dagger}$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \mathbb{1}_{2\times 2}$$