Study of Preferences and Manipulation in the Group Partitioning Problem

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Introduction

- Suppose we have n agents, each of which have a friendship or stranger relation with the other agents. The problem looks to divide these n agents into k groups of nearly equal size, such that the number of friends of each agent in its group is its utility.
- The forthcoming AAAI paper by Li *et al.*, 2023 discusses various results in this regard, which we will delve into as we go ahead.

Definitions

- **Balanced k-partition**
  A partition of n agents $\in V$ given by $X = (X_0, X_1, ...X_{k-1})$ is called a balanced k-partition if for every $i \in [k]$, $\lfloor n/k \rfloor \leq |X_i| \leq \lceil n/k \rceil$.
- **Min k-Cut**
  The cut size of a k-partition X is the number of edges between its different parts, i.e., $cut(X) = |(i, i') \in E : X(i) \neq X(i')|$. A balanced k-partition with the smallest cut size is called a balanced min k-cut.

$$cut(X) = \sum_{i \in V}(|N(i)| - u_i(X(i))) = 2|E| - \sum_{i \in V} u_i(X(i))$$

This shows that a balanced min cut also maximizes the social welfare among all agents.

Notions of Fairness

- **(α, β)-Core**
  For a given $\alpha \geq 1$ and $\beta \geq 0$, a coalition $S \subseteq V$ is called (α, β)-blocking for a balanced k-partition if
  $$u_i(S) > \alpha u_i(X(i)) + \beta$$
  for every $i \in S$. A balanced k-partition is said to be in (α, β)-core if there is no blocking coalition S such that $\lfloor n/k \rfloor \leq |S| \leq \lceil n/k \rceil$
- **Envy Free upto r**
  A balanced k-partition is called envy-free upto r ($EF - r$) if for every pair of agents $i, i' \in V$, $u_i(X(i)) \geq u_i(X(i') \cup \{i\} \setminus \{i'\}) - r$

Results - Core

- **(Open)** $k = 2 \implies$ Does every graph admit a balanced 2-partition in the core?
- $k \geq 3 \implies$ there exists an instance for each of the following where no k-partition is in
  - the $(\alpha, 0)$-core for any $\alpha \geq 1$
  - the $(1, \beta)$-core for any $\beta < k - 2$
- The proofs for these involve counterexamples that rely on some n not divisible by k.
  - **(Open)** Does every graph with n nodes admit a balanced k-partition in the core, if n is divisible by k?

Results - Core (contd.)

- $k = 2 \implies$ A balanced min 2-cut is in the $(2, 0)$-core.
- Finding a balanced min 2-cut in a general graph is NP-hard. This leads to efficiency concerns.
  - **(Open)** Can a balanced 2-partition in the $(2, 0)$-core be computed in polynomial time?
  - Aim to find balanced 2-partitions other than those in the balanced min 2-cut class - both existence and search are challenging problems.
- $n \geq k^2 + k \implies$ A balanced min k-cut is in the $(2k-1, 0)$-core and there exists a poly-time algorithm **P** for finding a balanced k-partition in the $(2k-1, 0)$-core.
- **P** also always finds a solution in the $(k, k-1)$-core. This reduces to the $(2, 1)$-core for $k = 2$.

Results - Core (contd.)

- $k \geq 3 \implies$ there exists an instance with $n \geq k^2 + k$ where some balanced min k-cut is not in the $(\alpha, 0)$ core for $\alpha < 2k - 2$.
- For $n \leq k^2 + k$,
  - Every balanced k-partition is in the $(1, k)$-core.
  - For $\beta < k/2 - 2$, there is an instance with no balanced k-partitions in the $(1, \beta)$-core.
  - $k \geq 3 \implies$ there is an instance with no balanced k-partitions in the $(\alpha, 0)$-core for $\alpha \geq 1$.
- Next, we consider a simplified version of the network - a **tree**.

Results - Core (Trees)

- Every balanced k-partition of a tree is in the $(1, 1)$-core.
- $k = 2 \implies$ Every balanced min 2-cut is in the core, one such solution can be obtained in poly-time.
- $k = 4 \implies$ There exists a tree for which no balanced k-partition is in the $(\alpha, 0)$-core for any $\alpha \geq 1$.
- **(Open)** $k = 3$

Results - Envy-Freeness

- $k = 2 \implies$ A balanced 2-partition that is $EF - 1$ does not always exists.
- $k \geq 2 \implies$ A balanced k-partition that is $EF - O(\sqrt{(\frac{n}{k} ln(k)})$ exists and can be computed in polynomial time.
- **(Open)** Does every graph admit, for all $k \geq 2$, a balanced k-partition that is $EF - 2$?

Results - Envy-Freeness (Trees)

- $k \geq 2 \implies$ A balanced $EF-1$ k-partition for a tree can be obtained by using a given poly-time algorithm $\mathbf{Q}$.
  - Note that the algorithm $\mathbf{Q}$ (from the paper) is not Pareto-Optimal.
- For any $k \geq 2$, every balanced min-k cut is $EF-1$.
- Checking if a given tree admits a balanced EF k-partition is NP-complete when k is part of the input.

Weak Core

- Current notion of a blocking coalition S for a balanced k-partition demands that $\forall i \in S$

$$u_i(S) > \alpha u_i(X(i)) + \beta$$

- The strict inequality plays a crucial role in some results mentioned above, for example -
  - Every balanced k-partition of a tree is in the $(1,1)$-core.
- One direction involves studying the problem from the weak-core perspective -

$$\forall i, u_i(S) \geq \alpha u_i(X(i)) + \beta$$
$$\exists j, u_j(S) > \alpha u_j(X(j)) + \beta$$

Tri-valued utilities

- The paper has considered that the agents have bi-valued utility. However, in most real-world applications, players also have negative opinions about some of the other participants, and would strongly believe in not forming groups with them.
- Our first direction of extension would be to look at weighted relationship graphs, where the agents now have tri-valued utilities.
- Negative utilities have been studied in the past for group partitioning problems [Barrot and Yokoo, 2019], but only when the groups were not balanced in size.

Preferences over resources

- In Kyropoulou *et al.*, 2020, the authors study the problem of fairly allocating indivisible goods between groups of agents.
- A combined model of the two has been proposed as possible future work, where first agents are partitioned into groups, and then resources are allocated to these groups, with agents having preferences towards both other agents and the goods.

Manipulation

- In a problem involving a preference of agents over other agents, falsified preferences can be used to manipulate the results in one's favour.
- Manipulation is well-studied in the stable-marriage and stable-roommates problems. One such work [Hosseini *et. al.*, 2022] discusses the probability of success through manipulations when a random stable matching is returned.
- We plan to explore such aspects in the generalized problem and look for targeted attacks by a single agent as well as a group.

Experiments

- While looking at manipulation, randomness is often used to get estimates of the probability of occurrence of different situations. This can help us in making special cases which are still highly viable so that we can form results in these scenarios.
- On the experimental side of our project, we will be looking at various randomly derived friendship graphs (or the more particular case of trees), to explore how manipulation can be involved in these settings.
- We also plan to consider other special classes of graphs which are closer to real-life group associations -
  - Disjoint collection of cliques
  - Tree over collections of cliques with one node per clique acting as a representative of the clique in the tree

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