4. 收敛

4.1. 可测函数的收敛. $(E, \mathscr{F}, \mu)$ a measure space, $f_n \in \mathscr{F}$, $i = 1, 2, \ldots$, $f \in \mathscr{F}$

Def 16. *Almost everywhere convergence, $f_n \xrightarrow{a.e.} f$:*

$$\mu\left(\lim_n f_n \neq f\right) = 0.$$

Def 17. *Convergence in measure, $f_n \xrightarrow{\mu} f$:* $\forall \varepsilon > 0$,

$$\lim_n \mu(|f_n - f| > \varepsilon) = 0.$$

Evidently

$$\begin{aligned} f_n \xrightarrow{a.e.} f &\iff \forall \varepsilon > 0, \mu\left(\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} \{|f_m - f| > \varepsilon\}\right) = 0 \\ &\iff \forall \varepsilon > 0, \mu(\{|f_n - f| > \varepsilon\} \text{ i.o.}) = 0. \end{aligned}$$

Recall

$$x \in \limsup A_n \iff x \in A_n \text{ i.o.}$$

Thm 15. *If μ is finite, then*

$$f_n \xrightarrow{a.e.} f \implies f_n \xrightarrow{\mu} f.$$

PROOF. Indeed,

$$\mu(|f_n - f| > \varepsilon) \leqslant \mu\left(\bigcup_{m=n}^{\infty} \{|f_m - f| > \varepsilon\}\right), \ \forall n.$$

Let $n \to \infty$ and use continuity from above (requires finiteness of μ)

$$\begin{aligned} \limsup_n \mu(|f_n - f| > \varepsilon) &\leqslant \lim_n \mu\left(\bigcup_{m=n}^{\infty} \{|f_m - f| > \varepsilon\}\right) \\ &= \mu\left(\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} \{|f_m - f| > \varepsilon\}\right) = 0. \end{aligned}$$

$\square$

Def 18. *Almost uniform convergence,* $f_n \xrightarrow{a.u.} f$*:* $\forall \varepsilon > 0$*, there is* $A_\varepsilon \in \mathscr{F}$ *so that* $\mu(A_\varepsilon) < \varepsilon$*,*

$$\lim_n \sup_{x \notin A_\varepsilon} |f_n - f| = 0.$$

Compare with Egoroff's Theorem on *finite* measure!

Thm 16. $f_n \xrightarrow{a.u.} f$ *if and only if* $\forall \varepsilon > 0$,

$$\lim_n \mu\left(\bigcup_{m=n}^{\infty} \{|f_m - f| > \varepsilon\}\right) = 0.$$

PROOF. **1**. "$\Longrightarrow$". $\forall \varepsilon > 0$, there is A_ε so that $\mu(A_\varepsilon) < \varepsilon$ and

$$\lim_m \sup_{x \notin A_\varepsilon} |f_m - f| = 0.$$

So, $\forall \varepsilon' > 0$, there is $n_0 \in \mathbb{N}$ such that

$$\sup_{x \notin A_\varepsilon} |f_m - f| \leqslant \varepsilon', \ \forall m \geqslant n_0.$$

This translates to

$$\bigcup_{m=n_0}^{\infty} \{|f_m - f| > \varepsilon'\} \subset A_\varepsilon.$$

Therefore

$$\mu\left(\bigcup_{m=n_0}^{\infty} \{|f_m - f| > \varepsilon'\}\right) \leqslant \mu(A_\varepsilon) < \varepsilon.$$

2. " $\Longleftarrow$ ". $\forall \varepsilon > 0$ and $k \in \mathbb{N}$, there is $n_{\varepsilon,k} \in \mathbb{N}$ such that

$$\mu\left(\bigcup_{m=n_{\varepsilon,k}}^{\infty} \left\{|f_m - f| > \frac{1}{k}\right\}\right) < \frac{\varepsilon}{2^k},\ \forall m \geqslant n_{\varepsilon,k}.$$

Denote (the set of all possible divergence points! measurable!)

$$A_\varepsilon = \bigcup_{k=1}^{\infty} \bigcup_{m=n_{\varepsilon,k}}^{\infty} \left\{|f_m - f| > \frac{1}{k}\right\}.$$

Then

$$\mu(A_\varepsilon) \leqslant \sum_{k=1}^{\infty} \mu\left(\bigcup_{m=n_{\varepsilon,k}}^{\infty} \left\{ |f_m - f| > \frac{1}{k} \right\} \right) < \varepsilon.$$

Moreover

$$x \notin A_\varepsilon \iff \forall k,\ \exists\ n_k \geqslant n_{\varepsilon,k}\colon\ |f_m(x) - f(x)| \leqslant \frac{1}{k},\ \forall m > n_k.$$

That is $\forall k$, $\exists\ n_k$,

$$\sup_{x \notin A_\varepsilon} |f_m - f| \leqslant \frac{1}{k},\ \forall m > n_k.$$

$\square$

We have proved:

Thm 17. (1)

$$f_n \xrightarrow{a.u.} f \implies f_n \xrightarrow{a.e.} f \ \textit{and}\ f_n \xrightarrow{\mu} f$$

(2) *If μ is finite, then*

$$f_n \xrightarrow{a.u.} f \iff f_n \xrightarrow{a.e.} f \implies f_n \xrightarrow{\mu} f$$

E.g. 3.

$$f_n(x) = \begin{cases} 1, & x \in (0, 1/n), \\ 0, & x \in [1/n, 1]. \end{cases}$$

E.g. 4.

$$f_n(x) = x^n, x \in [0, 1]$$

$\triangleright$ 4. *Let $f = 0$ and $f_n = 1_{A_n}$. Then $f_n \xrightarrow{\mu} f$ is equivalent to $\mu(A_n) \to 0$ and $\left(\lim_n f_n \neq f\right) = (A_n \text{ i.o.})$.*

Any sequence $\{A_n\}$ so that $\mu(A_n) \to 0$ but $\mu(A_n \text{ i.o.}) > 0$ gives an exmple that $f_n \xrightarrow{\mu} f \not\Rightarrow f_n \xrightarrow{a.e.} f$. It is enough to have $\mu(A_n) \to 0$ and

$$\sum_{i=1}^{\infty} 1_{A_n}(x) = \infty, \ \sum_{i=1}^{\infty} 1_{A_n^c}(x) = \infty.$$

E.g. 5. *For each* $n = 1, 2, ...$ *there is a unique decomposition* $n = k(k-1)/2 + i$ *with* $k = 1, 2, ...,$ $i = 1, 2, ..., k.$

$$f_n(x) = \begin{cases} 1, & x \in (((i-1)/k, i/k]), \\ 0, & \textit{otherwise.} \end{cases}$$

E.g. 6. *Consider*

$$A_k^i = \left[\frac{i-1}{k}, \frac{i}{k}\right],\ h_k^i(x) = 1_{A_k^i}(x),\ i = 1, ..., k.$$

Let f_n *be the sequence*

$$\left\{h_1^1; h_2^1, h_2^2; h_3^1, h_3^2; h_3^3; ...\right\}$$

4.2. 随机变量的分布函数.

Def 19. $(\Omega, \mathscr{F}, P)$ *is a probability space if* P *is a nonnegative measure on the* σ*-field* $\mathscr{F}$ *with* $P(\Omega) = 1$.

Def 20. *A random variable (r.v.)* X *on* $(\Omega, \mathscr{F}, P)$ *is a real-valued mapping,* $X : \omega \in \Omega \mapsto X(\omega) \in \mathbb{R}$.

Def 21. *The distribution function of a r.v. X is*

$$F(x) = P(X \leqslant x).$$

Denoted by $X \sim F$.

Thm 18. *Any distribution function F has the following properties.*
(1) *non-decreasing, $F(-\infty) = 0$ and $F(\infty) = 1$*
(2) *right continuity: $\lim_{y \downarrow x} F(y) = F(x)$.*
(3) *left limit exists: $F(x-) = \lim_{y \uparrow x} F(y) = P(X < x)$.*
(4) $P(X = x) = F(x) - F(x-)$.

The **inverse of the distrbution function** F is defined as below. $\forall z \in (0, 1)$,

$$F^{-1}(z) = \inf\{x \in \mathbb{R} : F(x) \geqslant z\}. \tag{4.1}$$

$\triangleright$ 5. *Also equivalently defined as,*

$$F^{-1}(z) = \sup\{x \in \mathbb{R} : F(x) < z\}. \tag{4.2}$$

Lemma 19. *F^{-1} has the properties,*

(1) F^{-1} *is real-valued, left-continuous and has right limit.*
(2) $F^{-1}(F(x)) \leqslant x$, $F\big(F^{-1}(z)\big) \geqslant z$.
(3) $F^{-1}(z) \leqslant x$ *iff* $F(x) \geqslant z$.

PROOF. Exercise. $\square$

Thm 20. *If F satisfies* (1)(2)(3) *of Theorem 18, there is a r.v. X with distribution F.*

PROOF. Let $\Omega = (0,1)$, $\mathscr{F} = \mathscr{B}_{(0,1)}$, $P =$ Lebesgue measure. Define

$$X(\omega) = F^{-1}(\omega).$$

Then

$$\begin{aligned} P(\omega : X(\omega) \leqslant x) &= P(\omega : F(x) \geqslant \omega) \\ &= \text{Lebesgue measure of } (0, F(x)) = F(x). \end{aligned}$$

So X is a r.v. with distribution function F. $\square$

$\triangleright$ 6. *Another construction of a r.v. X with distribution F is to take $\Omega = (\mathbb{R}, \mathscr{B})$, $P =$ the Lebesgue measure induced by F and consider the coordinate map $X(\omega) = \omega$.*

4.3. 随机变量的收敛. Probability space $(\Omega, \mathscr{F}, P)$, r.v. X_n, X,

$$X_n \xrightarrow{a.s.} X \iff P(X_n = X) = 1.$$

$$X_n \xrightarrow{P} X \iff \forall \varepsilon > 0, \ \lim_n P(|X_n - X| > \varepsilon) = 0.$$

Def 22. $X_n \sim F_n$, $X \sim F$. *Convergence in distribution (weak convergence): $F_n(x) \to F(x)$ for all x where F is continuous, written $X_n \xrightarrow{d} X$.*

Thm 21. *$X_n \sim F_n$, $X \sim F$.*

$$X_n \xrightarrow{a.s.} X \implies X_n \xrightarrow{P} X \implies X_n \xrightarrow{d} X.$$

Proof. **1**. The first implication is a special case of Theorem 15.

2. Check the second implication. $\square$