1. 单调类定理

Review:

- $\mathscr{A}$ is a field, $\mathscr{M}$ is a monotone class. Then
$$\mathscr{A} \subset \mathscr{M} \Longrightarrow \sigma(\mathscr{A}) \subset \mathscr{M}.$$
- $\mathscr{P}$ is a π-system, $\mathscr{L}$ is a λ-system. Then
$$\mathscr{P} \subset \mathscr{L} \Longrightarrow \sigma(\mathscr{P}) \subset \mathscr{L}.$$
- measurable spaces $(E, \mathscr{F}_E), (F, \mathscr{F}_F)$, $f : (E, \mathscr{F}_E) \mapsto (F, \mathscr{F}_F)$. f is $\mathscr{F}_E/\mathscr{F}_F$-measurable if
$$\sigma(f) \triangleq f^{-1}(\mathscr{F}_F) \subset \mathscr{F}_E.$$
Call it $\mathscr{F}_E$-measurable if
$$(F, \mathscr{F}_F) = (\mathbb{R}, \mathscr{B}(\mathbb{R})).$$
- $f : (E, \mathscr{F}_E) \mapsto (F, \sigma(\mathscr{E}))$, f is $\mathscr{F}_E/\sigma(\mathscr{E})$-measurable if
$$f^{-1}(\mathscr{E}) \subset \mathscr{F}_E.$$

Def 1 (**Simple function**). *$i = 1, ..., n$, $A_i \in \mathscr{F}$ (pairwise) disjoint, $c_i \in \mathbb{R}$. f is (measurable) simple if $f = \sum_{i=1}^{n} c_i 1_{A_i}$.*

Alt. $i = 1, ..., n$, $A_i \in \mathscr{F}$, $c_i \in \mathbb{R}$ non-zero distinct, f is simple if $f = \sum_{i=1}^{n} c_i 1_{A_i}$.

$\triangleright$ 1. *$a, b \in \mathbb{R}$, g simple, then $af + bg$ simple*

Thm 1 (**Simple approximation**). (1) *$f \geqslant 0$ measurable. There exist simple $\{f_n\}$, $0 \leqslant f_n \uparrow f$, uniform if f is bounded.*

(2) *f measurable. There exist simple $\{f_n\}$, $f_n \to f$, uniform if f is bounded.*

Proof. **1**. Let

$$f_n = \frac{[2^n f]}{2^n} \wedge n = \sum_{i=0}^{n2^n-1} \frac{i}{2^n} 1_{\{i/2^n \leqslant f < (i+1)/2^n\}} + n 1_{\{f \geqslant n\}}.$$

Then

$$0 \leqslant f - f_n \leqslant \frac{1}{2^n} \text{ if } f < n;\ f_n = n \leqslant f \text{ otherwise.}$$

2. $f = f^+ - f^-$. □

Thm 2 (Doob). $f : (E, \mathscr{F}_E) \mapsto (\mathbb{R}, \mathscr{B}(\mathbb{R}))$, *$g$ measurable* $(E, \mathscr{F}_E) \mapsto (F, \mathscr{F}_F)$. *If f is $\sigma(g)$-measurable, then $f = h \circ g$ for some measurable h.*

Proof. **1**. $f = 1_A$, $A = g^{-1}(B) \in \sigma(g)$, $B \in \mathscr{F}_F$. Then $x \in A$ if and only if $g(x) \in B$, i.e.,

$$f = 1_A = 1_B \circ g.$$

2. f simple, $f = \sum_{i=1}^{n} c_i 1_{A_i}$, $c_i \in \mathbb{R}$, $A_i \in \sigma(g)$ disjoint. Let $A_i = g^{-1}(B_i)$, $B_i \in \mathscr{F}_F$, then

$$C_i = B_i \Big\backslash \Big(\bigcup\nolimits_{j<i} B_j \Big) \in \mathscr{F}_F \text{ disjoint}$$

and

$$f^{-1}(C_i) = A_i \Big\backslash \Big(\bigcup\nolimits_{j<i} A_j\Big) = A_i.$$

By step **1**,

$$f = \sum_{i=1}^{n} c_i 1_{A_i} = \sum_{i=1}^{n} c_i 1_{C_i} \circ g = \left(\sum_{i=1}^{n} c_i 1_{C_i}\right) \circ g \triangleq h \circ g.$$

3. $f \geqslant 0$ is $\sigma(g)$-measurable, there exist $\sigma(g)$-measurable simple f_n with $0 \leqslant f_n \uparrow f$. It follows $f_n = h_n \circ g$ for some h_n,

$$h \triangleq \sup_n h_n$$

is $\sigma(g)$-measurable,

$$f = \lim_n f_n = \sup_n (h_n \circ g) = \left(\sup_n h_n\right) \circ g = h \circ g.$$

4. f is $\sigma(g)$-measurable. f^+, f^- are $\sigma(g)$-measurable. Use **3**. $\square$

Thm 3. *$\mathscr{A}$ is a π-system, $\Omega \in \mathscr{A}$, $\mathcal{H}$ is a collection of real-valued functions. Suppose*

(1) *If $A \in \mathscr{A}$, then $1_A \in \mathcal{H}$*
(2) *If $f, g \in \mathcal{H}$, $c \in \mathbb{R}$, then $f + g$, $cg \in \mathcal{H}$*
(3) *If $f_n \in \mathcal{H}$, $0 \leqslant f_n \uparrow f$ with f bounded, then $f \in \mathcal{H}$*

Then

$$\{f : f \text{ bounded } \sigma(\mathscr{A})\text{-measurable}\} \subset \mathcal{H}$$

Proof. The system of sets

$$\mathscr{G} = \{A : 1_A \in \mathcal{H}\}$$

is a λ-system and $\mathscr{A} \subset \mathscr{G}$. Hence

$$\sigma(\mathscr{A}) \subset \mathscr{G}.$$

(2) implies that $\mathcal{H}$ contains all $\sigma(\mathscr{A})$-measurable simple functions, (3) implies that $\mathcal{H}$ contains all bounded $\sigma(\mathscr{A})$-measurable functions. $\square$

$\triangleright$ 2. *$\mathscr{A}$ is a π-system, $\Omega \in \mathscr{A}$, $\mathcal{H}$ is a collection of real-valued functions. Suppose*

(1) *If $A \in \mathscr{A}$, then $1_A \in \mathcal{H}$*

(2) *If* $f, g \in \mathcal{H}$, $a, b \geqslant 0$, *then* $af + bg \in \mathcal{H}$

(3) *If* $f, g \in \mathcal{H}$ *are bounded,* $f \geqslant g$, *then* $f - g \in \mathcal{H}$

(4) *If* $f_n \in \mathcal{H}$, $0 \leqslant f_n \uparrow f$, *then* $f \in \mathcal{H}$

Then

$$\{f : f \text{ nonnegative } \sigma(\mathscr{A})\text{-measurable}\} \subset \mathcal{H}$$

2. 集函数与测度

2.1. 集函数. $\mathscr{E}$ is a collection of subsets of E.

Def 2. *Set function,* $\mu : \mathscr{E} \mapsto \mathbb{R} \cup \{\pm\infty\}$.

Def 3. *Nonnegative set function,* $\mu : \mathscr{E} \mapsto \mathbb{R} \cup \{\infty\}$.

Def 4. μ *is finite if,* $\forall A \in \mathscr{E}$, $|\mu(A)| < \infty$.

Def 5. μ *is* σ*-finite on* $\mathscr{E}$ *if,* $\forall A \in \mathscr{E}$, *there exist* $\{A_n\} \subset \mathscr{E}$, $A = \bigcup_n A_n$ *with* $|\mu(A_n)| < \infty$.

Def 6. μ *is additive if,* $\forall A, B \in \mathscr{E}$, $AB = \varnothing$,

$$\mu(A + B) = \mu(A) + \mu(B).$$

Def 7. μ *is countably additive if,* $\forall A_i \in \mathscr{E}$, $i = 1, 2, \ldots$, *disjoint,*

$$\mu\left(\sum_i A_i\right) = \sum_i \mu(A_i).$$

Def 8. $\varnothing \in \mathscr{E}$. *$\mu$ is a measure on $\mathscr{E}$ if it is nonnegative, countably additive, $\mu(\varnothing) = 0$.*

E.g. 1. *$(X, \mathscr{F})$ measurable space, $x \in X$,*

$$\delta_x(A) = 1_A(x),\ \forall A \in \mathscr{F}.$$

$x_1, ..., x_n \in X$,

$$\mu(A) = \sum_i \delta_{x_i}(A),\ \forall A \in \mathscr{F}.$$

E.g. 2. *F real-valued nonnegative, non-decreasing, right continuous. Semi-ring on $\mathbb{R}$,*

$$\mathscr{A} = \{(a, b] : a, b, \in \mathbb{R}\}.$$

Then

$$\mu((a, b]) = F(b) - F(a)$$

defines a measure $\mathscr{A}$. It is unique on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$.

PROOF. **1**. Additivity. $(a_i, b_i]$, $i = 1, ..., n$, disjoint, $(a, b] = \bigcup_i^n (a_i, b_i]$, then

$$\mu((a, b]) = \sum_{i=1}^{n} \mu((a_i, b_i]).$$

2. $(a_i, b_i]$, $i = 1, ...,$ disjoint, $\bigcup_i (a_i, b_i] \subset (a, b]$, then

$$\sum_{i=1}^{\infty} \mu((a_i, b_i]) \leqslant \mu((a, b]).$$

3. $(a_i, b_i]$, $i = 1, ..., n$, $(a, b] \subset \bigcup_i^n (a_i, b_i]$, then

$$\mu((a, b]) \leqslant \sum_{i=1}^{n} \mu((a_i, b_i]).$$

4. $(a_i, b_i]$, $i = 1, ...,$ disjoint, $\bigcup_i (a_i, b_i] = (a, b]$, then

$$\mu((a, b]) = \sum_{i=1}^{\infty} \mu((a_i, b_i]).$$

$\forall \varepsilon > 0$, there is $\delta_i > 0$,

$$F(b_i + \delta_i) - F(b_i) < \frac{\varepsilon}{2^i}.$$

$\forall \theta > 0$, $\{(a_i, b_i + \delta_i) : i\}$ is an open cover of $[a + \theta, b]$, there exists n_0

$$(a + \theta, b] \subset \bigcup_i^{n_0} (a_i, b_i + \delta_i].$$

By **3**.,

$$
\begin{aligned}
\mu((a+\theta,b]) &\leqslant \sum_{i=1}^{n_0}\mu((a_i,b_i+\delta_i]) \\
&= \sum_{i=1}^{n_0}(F(b_i+\delta_i)-F(b_i)) \\
&\leqslant \sum_{i=1}^{n_0}(F(b_i)-F(b_i))+\sum_{i=1}^{n_0}\frac{\varepsilon}{2^i} \\
&\leqslant \sum_{i=1}^{\infty}(F(b_i)-F(b_i))+\varepsilon.
\end{aligned}
$$

$\square$

2.2. 半环上非负集函数. $\mathscr{E}$ is a collection of subsets of E, μ is a nonnegative set function on $\mathscr{E}$.

Def 9. *Monotonicity:* $\forall A \subset B \in \mathscr{E}$,

$$\mu(A) \leqslant \mu(B).$$

Def 10. *Countably subadditive:* $\forall A_i \in \mathscr{E}$, $i = 1, 2, ...,$ $\bigcup_{i=1}^{\infty} A_i \in \mathscr{E}$,

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leqslant \sum_{i=1}^{\infty} \mu(A_i).$$

Def 11. *Continuity from below:* $A_i \in \mathscr{E}$, $A_i \uparrow A \in \mathscr{E}$,

$$\lim_n \mu(A_i) = \mu(A).$$

Def 12. *Continuity from above:* $A_i \in \mathscr{E}$, $A_i \downarrow A \in \mathscr{E}$, $\mu(A_1) < \infty$,

$$\lim_n \mu(A_i) = \mu(A).$$

REMARK 1. ***Note*** *finiteness is part of the defintion of continuity from above.*

$\mathscr{S}$ is a semi-ring on E, μ is a nonnegative set function on $\mathscr{S}$.

Suppose μ is **additive**.

1. $\mu(\varnothing) = 0, +\infty$.

PROOF. $\varnothing \in \mathscr{S}$. By additivity

$$\mu(\varnothing) = \sum_{i=1}^{n} \mu(\varnothing).$$

$\mu(\varnothing)$ equals 0, or ∞. $\square$

2. Monotonicity.

PROOF. $A, B \in \mathscr{S}$, $A \subset B$. There exist disjoint $C_1, ..., C_k \in \mathscr{S}$,

$$B \backslash A = \bigcup_{i=1}^{k} C_i.$$

$$B = A \cup (B \backslash A) = A \cup \left(\bigcup_{i=1}^{k} C_i \right).$$

By additivity

$$\mu(B) = \mu(A) + \sum_{i=1}^{k} \mu(C_i) \geqslant \mu(A).$$

$\square$

Suppose μ is **countably additive**.

3. Continuity from below.

PROOF. $A_i \in \mathscr{S}$, $A_i \uparrow A \in \mathscr{S}$. There exist disjoint $C_{n,1}, ..., C_{n,k_n} \in \mathscr{S}$,

$$B_n \triangleq A_n \backslash A_{n-1} = \bigcup_{i=1}^{k_n} C_{n,i}.$$

$(A_0 = \varnothing)$

$$
\begin{aligned}
\mu(A) &= \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \mu\left(\bigcup_{n=1}^{\infty}\bigcup_{i=1}^{k_n} C_{n,i}\right) \\
&= \sum_{n=1}^{\infty}\sum_{i=1}^{k_n} \mu(C_{n,i}) = \lim_N \sum_{n=1}^{N}\sum_{i=1}^{k_n} \mu(C_{n,i}) \\
&= \lim_N \mu\left(\bigcup_{n=1}^{N}\bigcup_{i=1}^{k_n} C_{n,i}\right) = \lim_n \mu(A_n).
\end{aligned}
$$

$\square$

4. Continuity from above.

Proof. (**WRONG PROOF**)$A_i \in \mathscr{S}$, $A_i \downarrow A \in \mathscr{S}$, $\mu(A_1) < \infty$. Clearly

$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right) \leqslant \mu(A_i) \leqslant \mu(A_1) < \infty.$$

$$\lim_n \mu(A_n) = \mu\left(\bigcap_{n=1}^{\infty} A_n\right)$$

$$\Longleftrightarrow$$

$$\mu(A_1) - \lim_n \mu(A_n) = \mu(A_1) - \mu\left(\bigcap_{n=1}^{\infty} A_n\right)$$

$$\Longleftrightarrow$$

$$\lim_n \mu(A_1 \backslash A_n) = \mu\left(A_1 \Big\backslash \bigcap_{n=1}^{\infty} A_n\right) = \mu\left(\bigcup_{n=1}^{\infty} (A_1 \backslash A_n)\right).$$

□

5. Subadditivity.

PROOF. Analogous to continuity from below. $\square$

2.3. 环上非负集函数.

Thm 4. *$\mathscr{R}$ is a ring. μ is nonnegative additive.*
(1) μ *countably additive*

$$\Longleftrightarrow$$

(2) μ *countably subadditive*

$$\Longleftrightarrow$$

(3) μ *continuity from below*

$$\Longrightarrow$$

(4) μ *continuity from above*

$$\Longrightarrow$$

(5) μ *continuity from above at $\varnothing$.*

If μ is finite, (5) implies (1).

PROOF. **1**. Already have: (1) $\Longrightarrow$ (2), (1) $\Longrightarrow$ (3), (1) $\Longrightarrow$ (4), (4) $\Longrightarrow$ (5).

2. (2) $\Longrightarrow$ (1). Suppose $A_i \in \mathscr{R}$, $i = 1, 2, ...$, disjoint, $\bigcup\limits_{i=1}^{\infty} A_i \in \mathscr{R}$.

By countable subadditivity,

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leqslant \sum_{i=1}^{\infty} \mu(A_i).$$

By monotonicity and additivity,

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \geqslant \mu\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu(A_i),\ \forall n.$$

Sending $n \to \infty$,

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \geqslant \sum_{i=1}^{\infty} \mu(A_i).$$

3. (3) $\Longrightarrow$ (1). Suppose $A_i \in \mathscr{R}$, $i = 1, 2, ...$, disjoint, $\bigcup_{i=1}^{\infty} A_i \in \mathscr{R}$. Since

$$\bigcup_{i=1}^{n} A_i \uparrow \bigcup_{i=1}^{\infty} A_i,$$

by continuity from below,

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_n \mu\left(\bigcup_{i=1}^{n} A_i\right) = \lim_n \sum_{i=1}^{n} \mu(A_i) = \sum_{i=1}^{\infty} \mu(A_i).$$

4. (5) $\Longrightarrow$ (1). Suppose $A_i \in \mathscr{R}$, $i = 1, 2, ...$, disjoint, $\bigcup_{i=1}^{\infty} A_i \in \mathscr{R}$. Then, $\forall n$,

$$\bigcup_{i=1}^{n} A_i \in \mathscr{R} \text{ and } \bigcup_{i=n+1}^{\infty} A_i = \bigcup_{i=1}^{\infty} A_i \Big\backslash \bigcup_{i=1}^{n} A_i \in \mathscr{R}.$$

By additivity

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \mu\left(\bigcup_{i=1}^{n} A_i\right) + \mu\left(\bigcup_{i=n+1}^{\infty} A_i\right).$$

Since μ is finite

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) < \infty.$$

The continuity from above at $\varnothing$ yields,

$$\lim_{n} \mu\left(\bigcup_{i=n+1}^{\infty} A_i\right) = 0.$$

Hence

$$
\begin{aligned}
\mu\left(\bigcup_{i=1}^{\infty} A_i\right) &= \lim_n \mu\left(\bigcup_{i=1}^{n} A_i\right) + \lim_n \mu\left(\bigcup_{i=n+1}^{\infty} A_i\right) \\
&= \lim_n \sum_{i=1}^{n} \mu(A_i) = \sum_{i=1}^{\infty} \mu(A_i).
\end{aligned}
$$

$\square$

3. Carathéodory's 延拓

3.1. 外测度.

Def 13. *μ^* is an outer measure on E if*
(1) $\mu^*(\varnothing) = 0$
(2) $\forall A, B \in 2^E$, *if* $A \subset B$, *then*

$$\mu^*(A) \leqslant \mu^*(B)$$

(3) *If* $A_i \in 2^E$,$i = 1, 2, ...,$

$$\mu^*\left(\bigcup_{i=1}^{\infty} A\right) \leqslant \sum_{i=1}^{\infty} \mu^*(A_i)$$

Thm 5. *Let $\mathscr{E}$ be a collection of sets on E, $\varnothing \in \mathscr{E}$. μ is a nonnegative set function on $\mathscr{E}$ with $\mu(\varnothing) = 0$. Define, $\forall A \in 2^E$,*

$$\mu^*(A) = \inf\left\{\sum_{i=1}^{\infty} \mu(A_i) : A_i \in \mathscr{E},\ A \subset \bigcup_{i=1}^{\infty} A_i\right\}.$$

Then $\mu^*(A)$ is an outer measure.

PROOF. **1**. $\mu^*(\varnothing) = 0$ since $\varnothing \in \mathscr{E}$, $\varnothing \subset \bigcup\limits_{i=1}^{\infty} \varnothing$.

2. If $A \subset B$, $B \subset \bigcup\limits_{i=1}^{\infty} B_i$, then $A \subset \bigcup\limits_{i=1}^{\infty} B_i$, from the definition $\mu^*(A) \leqslant \mu^*(B)$.

3. Let $A_i \in 2^E, i = 1, 2, \ldots$, $\varepsilon > 0$. There are $A_{i,k} \in \mathscr{E}$, $A_i \subset \bigcup\limits_{k=1}^{\infty} A_{i,k}$,

$$\sum_{k=1}^{\infty} \mu(A_{i,k}) \leqslant \mu^*(A_i) + \frac{\varepsilon}{2^i}, \ \forall i.$$

Since

$$\bigcup_{i=1}^{\infty} A_i \subset \bigcup_{i=1}^{\infty} \bigcup_{k=1}^{\infty} A_{i,k},$$

$$\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leqslant \sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\mu(A_{i,k})$$
$$\leqslant \sum_{i=1}^{\infty}\left[\mu^*(A_i)+\frac{\varepsilon}{2^i}\right] \leqslant \sum_{i=1}^{\infty}\mu^*(A_i)+\varepsilon.$$

$\square$

Def 14. *μ^* is an outer measure on E. $A \in 2^E$ is μ^*-measurable if*

$$\mu^*(D) = \mu^*(D\cap A) + \mu^*(D\cap A^c),\ \forall D \in 2^E.$$

The class of μ^*-measurable sets is denoted by $\mathscr{F}_\mu^*$.

Def 15. *Let μ be a measure on a σ-field $\mathscr{F}$ of E, the measure space $(E,\mathscr{F},\mu)$ is complete if*

$$A \in \mathscr{F},\ \mu(A) = 0 \Longrightarrow B \in \mathscr{F},\ \forall B \subset A.$$

Thm 6 (Carathéodory). *Let $\mathscr{E}$ be a collection of sets on E, $\varnothing \in \mathscr{E}$. μ is a nonnegative set function on $\mathscr{E}$ with $\mu(\varnothing) = 0$.*

(1) $\mathscr{F}_\mu^*$ is a σ-field.

(2) $\left(E, \mathscr{F}_\mu^*, \mu^*\right)$ is a complete measure space.

PROOF. **1**. Obviously, $E \in \mathscr{F}_\mu^*$ and $A^c \in \mathscr{F}_\mu^*$ if $A \in \mathscr{F}_\mu^*$.

2. If $A_1, A_2 \in \mathscr{F}_\mu^*$, then $A_1 \cup A_2$, $A_1 \cap A_2 \in \mathscr{F}_\mu^*$.

$\forall D \in 2^E$, we note

$$D \cap (A_1 \cup A_2) = (D \cap A_1) \cup (D \cap A_1^c \cap A_2).$$

Then

$$\begin{aligned}
&\mu^*(D \cap (A_1 \cup A_2)) + \mu^*(D \cap (A_1 \cup A_2)^c) \\
&\leqslant \mu^*(D \cap A_1) + \mu^*(D \cap A_1^c \cap A_2) + \mu^*(D \cap A_1^c \cap A_2^c) \text{ (subadditivity)} \\
&\leqslant \mu^*(D \cap A_1) + \mu^*(D \cap A_1^c) \; (A_2 \in \mathscr{F}_\mu^*) \\
&= \mu^*(D) \; (A_1 \in \mathscr{F}_\mu^*).
\end{aligned}$$

Hence

$$A_1 \cup A_2 \in \mathscr{F}_\mu^*.$$

It follows that

$$(A_1 \cap A_2)^c = A_1^c \cup A_2^c \in \mathscr{F}_\mu^*.$$

3. Finite additivity. If $A_1, ..., A_n \in \mathscr{F}_\mu^*$ disjoint, then $\forall D \in 2^E$,

$$\mu^* \left(D \cap \left(\bigcup_{i=1}^{n} A_i \right) \right) = \sum_{i=1}^{n} \mu^*(D \cap A_i).$$

Indeed, since $A_1 \in \mathscr{F}_\mu^*$,

$$\begin{aligned}
&\mu^*\left(D \cap \left(\bigcup_{i=1}^n A_i\right)\right) \\
&= \mu^*\left(D \cap \left(\bigcup_{i=1}^n A_i\right) \cap A_1\right) + \mu^*\left(D \cap \left(\bigcup_{i=1}^n A_i\right) \cap A_1^c\right) \\
&= \mu^*(D \cap A_1) + \mu^*\left(D \cap \left(\bigcup_{i=2}^n A_i\right)\right) = \cdots = \sum_{i=1}^n \mu^*(D \cap A_i)
\end{aligned}$$

4. If $A_1, A_2, ..., \in \mathscr{F}_\mu^*$, then $A \triangleq \bigcup\limits_{i=1}^{\infty} A_i \in \mathscr{F}_\mu^*$.

We can assume that $A_1, A_2, ..., \in \mathscr{F}_\mu^*$ are disjoint. Indeed, by **1** and **2**, $B_i = A_i \Bigg\backslash \left(\bigcup\limits_{j<i} A_j\right) \in \mathscr{F}_\mu^*$, are disjoint and $\bigcup_{i=1}^n B_i = \bigcup_{i=1}^n A_i$,

$\forall n$. Let

$$C_n = \bigcup_{i=1}^{n} A_i \in \mathscr{F}_{\mu}^*, \ \forall n.$$

Since $A_1, A_2, ...$ are disjoint, we can use **3** (the finite additivity). $\forall D \in 2^E$,

$$\begin{aligned}
\mu^*(D) &= \mu^*(D \cap C_n) + \mu^*(D \cap C_n^c) \\
&= \sum_{i=1}^{n} \mu^*(D \cap C_i) + \mu^*(D \cap C_n^c) \\
&\geqslant \sum_{i=1}^{n} \mu^*(D \cap C_i) + \mu^*(D \cap A^c), \ \forall n.
\end{aligned}$$

Let $n \to \infty$, note $A \subset \bigcup_{i=1}^{\infty} C_i$ and use subadditivity of outer measure

$$\mu^*(D) \geqslant \sum_{i=1}^{\infty} \mu^*(D \cap C_i) + \mu^*(D \cap A^c) \geqslant \mu^*(D \cap A) + \mu^*(D \cap A^c).$$

5. Countable additivity.

If $A_1, A_2, ..., \in \mathscr{F}_\mu^*$ are disjoint, use **3** and send $n \to \infty$,

$$\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \geqslant \mu^*\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu^*(A_i),\ \forall n.$$

The opposite inequality is subadditivity of outer measure.

6. Completeness. If $A \in \mathscr{F}_\mu^*$, $\mu^*(A) = 0$ and $B \subset A$, then $\mu^*(B) = 0$. $\forall D \in 2^E$,

$$\mu^*(D) \geqslant \mu^*(D \cap B^c) = \mu^*(D \cap B) + \mu^*(D \cap B^c).$$

So $B \in \mathscr{F}_\mu^*$. □

3.2. 域上测度的延拓.

Thm 7. *If μ is a measure on a field $\mathscr{A}$ with the generated outer measure μ^*. Then*

(1) $\mathscr{A} \subset \mathscr{F}_\mu^*$ *thus* $\sigma(\mathscr{A}) \subset \mathscr{F}_\mu^*$.
(2) μ^* *is an extension of* μ *to* $\sigma(\mathscr{A})$ *in the sense that*

$$\mu(A) = \mu^*(A), \ \forall A \in \mathscr{A}.$$

PROOF. **1**. Let $A \subset \mathscr{A}$. If $A_i \in \mathscr{A}$, $A \subset \bigcup_{i=1}^{\infty} A_i$, then

$$\mu(A) \leqslant \sum_{i=1}^{\infty} \mu(A_i). \tag{3.1}$$

Indeed,

$$\mu\left(A \cap \bigcup_{i=1}^{n} A_i\right) \leqslant \mu\left(\bigcup_{i=1}^{n} A_i\right) \leqslant \sum_{i=1}^{n} \mu(A_i).$$

Let $n \to \infty$ and use that μ is a measure to get (3.1). So

$$\mu(A) \leqslant \mu^*(A).$$

Since $A \subset \mathscr{A}$, $A_1 = A$, $A_2 = A_3 ... = \varnothing$ form a countable cover of A, so

$$\mu^*(A) \leqslant \mu(A).$$

2. Fix $A \subset \mathscr{A}$, will prove $A \in \mathscr{F}^*_\mu$. $\forall D \in 2^E$, it is enough to show that

$$\mu^*(D) \geqslant \mu^*(D \cap A) + \mu^*(D \cap A^c).$$

There is nothing to prove if $\mu^*(D) = \infty$, so we assume that $\mu^*(D) < \infty$. Then, $\forall \varepsilon > 0$, there exist $A_i \in \mathscr{A}$, $D \subset \bigcup\limits_{i=1}^{\infty} A_i$ so that

$$\sum_{i=1}^{\infty} \mu(A_i) \leqslant \mu^*(D) + \varepsilon.$$

Since $\mathscr{A}$ is a field,

$$A_i \cap A, A_i \cap A^c \in \mathscr{A}.$$

By **1** and the additivity of μ,

$$\begin{aligned}\mu(A_i) &= \mu(A_i \cap A) + \mu(A_i \cap A^c) \\ &= \mu^*(A_i \cap A) + \mu^*(A_i \cap A^c).\end{aligned}$$

Summing over i gives

$$\begin{aligned}\sum_{i=1}^{\infty} \mu(A_i) &= \sum_{i=1}^{\infty} \mu^*(A_i \cap A) + \sum_{i=1}^{\infty} \mu^*(A_i \cap A^c) \\ &\geqslant \mu^*(D \cap A) + \mu^*(D \cap A^c).\end{aligned}$$

So

$$\mu^*(D) + \varepsilon \geqslant \sum_{i=1}^{\infty} \mu(A_i) \geqslant \mu^*(D \cap A) + \mu^*(D \cap A^c).$$

□

Thm 8 (Uniqueness)**.** *Let $\mathscr{P}$ be a π-system on E, μ and ν measures on $\sigma(\mathscr{P})$. Assume that*

(1) *μ and ν agree on $\mathscr{P}$.*

(2) *There are* $B_i \in \mathscr{P}$, $i = 1, 2, ...,$ *disjoint so that* $\bigcup_{i=1}^{\infty} B_i = E$ *and* $\mu(B_i) < \infty$.

Then μ *and* ν *are equal on* $\sigma(\mathscr{P})$.

PROOF. **1**. Let $B \in \mathscr{P}$ have $\mu(B) < \infty$. Define

$$\mathscr{L} = \{A \in \sigma(\mathscr{P}) : \mu(A \cap B) = \nu(A \cap B)\}.$$

$\mathscr{L}$ is a λ-system (finiteness is needed to justify sets subtraction!), $\mathscr{P} \subset \mathscr{L}$. So

$$\sigma(\mathscr{P}) \subset \mathscr{L},$$

i.e.

$$\mu(A \cap B) = \nu(A \cap B), \ \forall A \in \sigma(\mathscr{P}).$$

2. $\forall A \in \sigma(\mathscr{P})$, use (2) to write it as disjoint union,

$$A = \bigcup_{i=1}^{\infty} (A \cap B_i), \ \mu(A \cap B_i) \leqslant \mu(B_i) < \infty.$$

Then, by **1**,

$$\begin{aligned}\mu(A) &= \mu\left(\bigcup_{i=1}^{\infty}(A\cap B_i)\right) = \sum_{i=1}^{\infty}\mu(A\cap B_i)\\ &= \sum_{i=1}^{\infty}\nu(A\cap B_i) = \nu\left(\bigcup_{i=1}^{\infty}(A\cap B_i)\right) = \nu(A).\end{aligned}$$

$\square$

$\triangleright$ 3. *The condition Therem 8 (2) can be replaced with either one of the following:*

(2′) *$\mathscr{P}$ is a semi-ring, $E\in\mathscr{P}$ and μ is σ-finite on $\mathscr{P}$.*

(2″) *there are $B_1, B_2, \ldots \in \mathscr{P}$, so that $B_i \uparrow E$ and $\mu(B_i) < \infty$.*

3.3. 半环上测度的延拓.

Thm 9. *Let μ be a measure on the semi-ring $\mathscr{S}$ with the generated outer measure μ^*. Then*

(1) *$\mathscr{S} \subset \mathscr{F}_\mu^*$ thus $\sigma(\mathscr{S}) \subset \mathscr{F}_\mu^*$.*

(2) *μ^* is an extension of μ to $\sigma(\mathscr{S})$ in the sense that*

$$\mu(A) = \mu^*(A), \ \forall A \in \mathscr{S}. \tag{3.2}$$

(3) *Assume that there are $B_i \in \mathscr{S}$, $i = 1, 2, ...,$ disjoint so that $\bigcup_{i=1}^{n} B_i = E$ and $\mu(B_i) < \infty$, then the extension of μ to $\sigma(\mathscr{S})$ is unique.*

Proof. Let $\bar{\mu}$ be the outer measure generated by μ.

1. $\bar{\mu}$ agrees with μ on $\mathscr{S}$.

The proof is identical to Theorem 7 (1).

2. Fix $A \subset \mathscr{S}$, will prove $A \in \mathscr{F}_\mu^*$.

The proof is identical to Theorem 7 (2). The difference is $A_i \cap A^c$ is replaced with disjoint union of sets in $\mathscr{S}$.

3. Uniqueness. Apply Theorem 8 to conclude. $\square$

3.4. Approximating $\mu^*|_{\mathscr{F}_\mu^*}$ by $\mu^*|_{\sigma(\mathscr{S})}$.

Thm 10. *Let μ be a measure on the semi-ring $\mathscr{S}$ with the generated outer measure μ^*. Suppose $E \in \mathscr{S}$.*

(1) $\forall A \in \mathscr{F}_\mu^*$, there is $B \in \sigma(\mathscr{S})$ such that $A \subset B$ and

$$\mu^*(A) = \mu^*(B).$$

(2) If μ is σ-finite on $\mathscr{S}$, then $\forall A \in \mathscr{F}_\mu^*$, there is $B \in \sigma(\mathscr{S})$ such that $A \subset B$ and

$$\mu^*(B \backslash A) = 0.$$

Proof.

1. There is nothing to prove if $\mu^*(A) = \infty$, we assume that $\mu^*(A) < \infty$. There are $B_{n,i} \in \mathscr{S}$, $A \subset \bigcup_{i=1}^{\infty} B_{n,i}$,

$$\sum_{i=1}^{\infty} \mu(B_{n,i}) < \mu^*(A) + \frac{1}{n}.$$

Set

$$B = \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{\infty} B_{n,i}.$$

Then $A \subset B \in \sigma(\mathscr{S})$,

$$\mu^*(A) \leqslant \mu^*(B).$$

Moreover

$$\mu^*(B) \leqslant \mu^*\left(\bigcup_{i=1}^{\infty} B_{n,i}\right) \leqslant \sum_{i=1}^{\infty} \mu(B_{n,i}) \leqslant \mu^*(A) + \frac{1}{n}.$$

It follows that

$$\mu^*(B) \leqslant \mu^*(A).$$

2. If μ is *finite* on $\mathscr{S}$, then by **1**, $\forall A \in \mathscr{F}_\mu^*$, there is $B \in \sigma(\mathscr{S})$ such that $A \subset B$ and

$$\mu^*(A) = \mu^*(B).$$

Since μ^* is a measure on $\mathscr{F}_\mu^*$, this gives

$$\mu^*(B \backslash A) = 0.$$

The σ-finite case follows from similar argument as in step **3** of Theorem 9. $\square$

3.5. Approximating $\mu|_{\sigma(\mathscr{A})}$ by $\mu|_{\mathscr{A}}$.

Thm 11. *Let μ be a measure on the field $\mathscr{A}$ with the generated outer measure μ^*. For any $A \in \sigma(\mathscr{A})$ with $\mu^*(A) < \infty$, $\forall \varepsilon > 0$, there is $B \in \mathscr{A}$ such that $\mu^*(A \Delta B) < \varepsilon$.*

If, in the last Theorem, the measure μ is defined on $\sigma(\mathscr{A})$ and σ-finite on $\mathscr{A}$, then μ must equal μ^* on $\sigma(\mathscr{A})$ by uniqueness, we can use μ in place of μ^* in the conclusion.

Thm 12. *Let $\mathscr{A}$ be a field, μ a measure on $\sigma(\mathscr{A})$ and σ-finite on $\mathscr{A}$. For any $A \in \sigma(\mathscr{A})$ with $\mu(A) < \infty$, $\forall \varepsilon > 0$, there is $B \in \mathscr{A}$ such that $\mu(A \Delta B) < \varepsilon$.*

3.6. Completion of a measure space.

Thm 13. *Let $(X, \mathscr{F}, \mu)$ be a measure space,*

$$\bar{\mathscr{F}} \triangleq \{A \cup N : A \in \mathscr{F}, N \subset B \text{ for some } B \in \mathscr{F} \text{ with } \mu(B) = 0\}.$$

Define

$$\bar{\mu}(A \cup N) = \mu(A), \ \forall A \in \bar{\mathscr{F}}.$$

Then $\left(X, \bar{\mathscr{F}}, \bar{\mu}\right)$ *is a complete measure space.*

Clearly the Theorem says

$$\bar{\mu}(A) = \mu(A), \ \forall A \in \mathscr{F}.$$

PROOF. **1**. $\bar{\mathscr{F}}$ is a σ-field.

Suppose $A \cup N \in \bar{\mathscr{F}}$ where $A \in \mathscr{F}$, $N \subset B$, $B \in \mathscr{F}$ with $\mu(B) = 0$. Then

$$(A \cup N)^c = (A^c \cap B^c) \cup (B \cap A^c \cap N^c) \in \bar{\mathscr{F}}.$$

Suppose $A_i \cup N_i \in \bar{\mathscr{F}}$ where $A_i \in \mathscr{F}$, $N_i \subset B_i$, $B_i \in \mathscr{F}$ with $\mu(B_i) = 0$. Then

$$\bigcup_{i=1}^{\infty}(A_i \cup N_i) = \left(\bigcup_{i=1}^{\infty} A_i\right) \cup \left(\bigcup_{i=1}^{\infty} N_i\right) \in \bar{\mathscr{F}},$$

since

$$\bigcup_{i=1}^{\infty} N_i \subset \bigcup_{i=1}^{\infty} B_i \in \mathscr{F}$$

and

$$\mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} \mu(B_i) = 0.$$

2. The definition of $\bar{\mu}$ nonambiguous, i.e.

$$A_1 \cup N_1 = A_2 \cup N_2 \in \bar{\mathscr{F}} \Longrightarrow \bar{\mu}(A_1 \cup N_1) = \bar{\mu}(A_2 \cup N_2).$$

Here $N_i \subset B_i$ for some $B_i \in \mathscr{F}$ with $\mu(B_i) = 0$, $i = 1, 2$.

$$\bar{\mu}(A_1 \cup N_1) = \mu(A_1) = \mu(A_1 \cup B_1 \cup B_2) \geqslant \mu(A_2) = \bar{\mu}(A_2 \cup N_2).$$

By symmetry,

$$\bar{\mu}(A_1 \cup N_1) \leqslant \bar{\mu}(A_2 \cup N_2).$$

3. Countable additivity. Suppose $A_i \cup N_i \in \bar{\mathscr{F}}$ disjoint, where $A_i \in \mathscr{F}$, $N_i \subset B_i$, $B_i \in \mathscr{F}$ with $\mu(B_i) = 0$. Then

$$\bar{\mu}\left(\bigcup_{i=1}^{\infty}(A_i \cup N_i)\right) = \mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty}\mu(A_i) = \sum_{i=1}^{\infty}\bar{\mu}(A_i \cup N_i).$$

4. Completeness. Let $A \cup N \in \bar{\mathscr{F}}$, $N \subset B$, $B \in \mathscr{F}$ with $\mu(B) = 0$ and $\bar{\mu}(A \cup N)$, then

$$\mu(A \cup B) = \mu(A) = \bar{\mu}(A \cup N) = 0.$$

So for any $C \subset A \cup N$, $C \subset A \cup B$,

$$C = \varnothing \cup C \in \bar{\mathscr{F}}.$$

□

Thm 14. *Suppose that μ is σ-finite on the semi-ring $\mathscr{S}$ with the generated outer measure μ^*. Then $\left(X, \mathscr{F}_{\mu}^{*}, \mu^*\right)$ is the completion of $(X, \sigma(\mathscr{S}), \mu^*)$.*

Proof. Let

$\bar{\mathscr{F}} \triangleq \{A \cup N : A \in \sigma(\mathscr{S}), N \subset B$ for some $B \in \sigma(\mathscr{S})$ with $\mu(B) = 0\}$.

It is enough to show that

$$\mathscr{F}_\mu^* = \bar{\mathscr{F}}.$$

Since $\left(X, \mathscr{F}_\mu^*, \mu^*\right)$ is a complete measure space,

$$\bar{\mathscr{F}} \subset \mathscr{F}_\mu^*.$$

Let $A \in \mathscr{F}_\mu^*$, by Theorem 10 there exist $B, C \in \sigma(\mathscr{S})$ so that

$$A \subset B,\ \mu^*(B \backslash A) = 0;\ B \backslash A \subset C,\ \mu^*(C) = \mu^*(B \backslash A) = 0.$$

Writing

$$A = (B \cap C^c) \cup (A \cap C),$$

we get that $B \cap C^c \in \sigma(\mathscr{S})$, $(A \cap C) \subset C$, $\mu^*(C) = 0$, so $A \in \bar{\mathscr{F}}$. $\square$