概率论与数理统计

Function of r.v. - 正太分布

$$X \sim \mathcal{N}(0,1),\ \ p(x) = \frac{1}{(2\pi)^{1/2}} \exp\left(-\frac{1}{2}x^2\right)$$

$$Y = \frac{X-\mu}{\sigma}$$

$$Y \sim \frac{1}{(2\pi)^{1/2}\sigma} \exp\left[-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2\right]$$

Function of r.v. - Gamma

$$X \sim \mathcal{N}(0,1) \Longrightarrow X^2 \sim Gamma\left(\frac{1}{2}, \frac{1}{2}\right)$$

$$p(x) = \begin{cases} \frac{\left(\frac{1}{2}\right)^{1/2}}{\Gamma\left(\frac{1}{2}\right)} x^{-\frac{1}{2}} e^{-\frac{1}{2}x}, x > 0 \\ 0, x \leqslant 0 \end{cases}$$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty x^{-1/2} e^{-x} dx = \underbrace{\int_{-\infty}^{\infty} e^{-y^2} dy}_{y=\sqrt{x}} = \sqrt{\pi}$$

Sum of Poisson dist.

$$S_n = \underbrace{X_1 + X_2 + \cdots + X_n}_{X_i \sim Poisson(\lambda_i),\, i=1,\cdots,n, \text{ independent}}$$

$$\sim Poisson(\lambda_1 + \lambda_2 + \cdots + \lambda_n)$$

$$
\begin{aligned}
P(X_1 + X_2 = m) &= \sum_{k=0}^{m} P(X_1 = k, X_2 = m - k) \\
&= \sum_{k=0}^{m} P(X_1 = k) P(X_2 = m - k) \\
&= \sum_{k=0}^{m} e^{-\lambda_1} \frac{\lambda_1^k}{k!} \cdot e^{-\lambda_2} \frac{\lambda_2^{m-k}}{(m-k)!} \\
&= \frac{e^{-(\lambda_1+\lambda_2)}}{m!} \sum_{k=0}^{m} \binom{m}{k} \lambda_1^k \lambda_2^{m-k} \\
&= \frac{e^{-(\lambda_1+\lambda_2)}}{m!} (\lambda_1 + \lambda_2)^m
\end{aligned}
$$

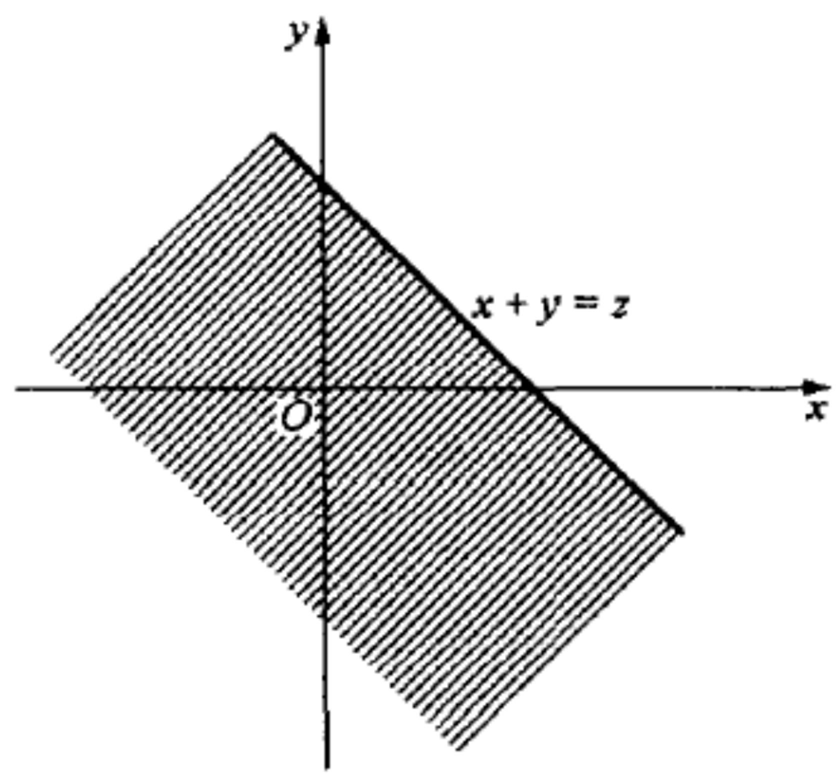
Convolution

定理 4.1 设(X,Y)有联合密度 $p(x,y)$，$Z=X+Y$，则 Z 的分布密度为

$$p_Z(z)=\int_{-\infty}^{+\infty}p(x,z-x)\mathrm{d}x. \tag{4.2}$$

证明 先求出 Z 的分布函数. 令

$$A=\{(x,y):x+y\leqslant z\},$$



$$P(Z\leqslant z)=P((X,Y)\in A)=\iint\limits_{\{x+y\leqslant z\}}p(x,y)\mathrm{d}x\mathrm{d}y.$$

利用二重积分和累次积分的关系，有

$$\iint\limits_{\{x+y\leqslant z\}}p(x,y)\mathrm{d}x\mathrm{d}y=\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{z-x}p(x,y)\mathrm{d}y\right]\mathrm{d}x$$

$$=\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{z}p(x,u-x)\mathrm{d}u\right]\mathrm{d}x\ (\text{利用变量替换 } u=y+x)$$

$$=\int_{-\infty}^{z}\left[\int_{-\infty}^{+\infty}p(x,u-x)\mathrm{d}x\right]\mathrm{d}u.$$

因此

$$P(Z\leqslant z)=\int_{-\infty}^{z}\left[\int_{-\infty}^{+\infty}p(x,u-x)\mathrm{d}x\right]\mathrm{d}u.$$

Sum of normal dist.

$$(X,Y) \sim \mathcal{N}(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$$

$$\implies$$

$$X+Y \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2)$$

$$X \sim \mathcal{N}(\mu_1, \sigma_1^2), Y \sim \mathcal{N}(\mu_2, \sigma_2^2) \text{ independent}$$

$$\implies$$

$$X+Y \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$

$$p_Z(z) = \int_{-\infty}^{+\infty} p(x, z-x)\mathrm{d}x,$$

其中

$$p(x, z-x) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\underline{\left(\frac{x-\mu_1}{\sigma_1}\right)^2} \right.\right.$$
$$\left.\left.\underline{-2\rho\frac{(x-\mu_1)(z-x-\mu_2)}{\sigma_1\sigma_2}+\left(\frac{z-x-\mu_2}{\sigma_2}\right)^2}\right]\right\}.$$

令$\frac{x-\mu_1}{\sigma_1}=u$,则 $x=\mu_1+\sigma_1 u$

$$\left(\frac{x-\mu_1}{\sigma_1}\right)^2-2\rho\frac{(x-\mu_1)(z-x-\mu_2)}{\sigma_1\sigma_2}+\left(\frac{z-x-\mu_2}{\sigma_2}\right)^2$$

$$=u^2-2\rho u\frac{z-\sigma_1 u-\mu_1-\mu_2}{\sigma_2}+\left(\frac{z-\sigma_1 u-\mu_1-\mu_2}{\sigma_2}\right)^2$$

$$=\left(1+2\rho\frac{\sigma_1}{\sigma_2}+\left(\frac{\sigma_1}{\sigma_2}\right)^2\right)u^2-2u\left(\frac{z-\mu_1-\mu_2}{\sigma_2}\right)\left(\rho+\frac{\sigma_1}{\sigma_2}\right)$$

$$+\left(\frac{z-\mu_1-\mu_2}{\sigma_2}\right)^2$$

$$\xlongequal{\text{记为}} Au^2-2Bu+C^2,$$

其中

$$A=1+2\rho\frac{\sigma_1}{\sigma_2}+\left(\frac{\sigma_1}{\sigma_2}\right)^2,\quad B=\left(\rho+\frac{\sigma_1}{\sigma_2}\right)C,\quad C=\frac{z-\mu_1-\mu_2}{\sigma_2}.$$

$$A=1+2\rho\frac{\sigma_1}{\sigma_2}+\left(\frac{\sigma_1}{\sigma_2}\right)^2,\quad B=\left(\rho+\frac{\sigma_1}{\sigma_2}\right)C,\quad C=\frac{z-\mu_1-\mu_2}{\sigma_2}.$$

于是

$$\begin{aligned}
p_Z(z) &= \frac{\sigma_1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \\
&\quad \cdot\int_{-\infty}^{+\infty}\exp\left\{-\frac{1}{2(1-\rho^2)}(Au^2-2Bu+C^2)\right\}\mathrm{d}u \\
&= \frac{1}{2\pi\sigma_2\sqrt{1-\rho^2}}\int_{-\infty}^{+\infty}\exp\left\{-\frac{A}{2(1-\rho^2)}\left(u-\frac{B}{A}\right)^2\right. \\
&\quad \left.+\frac{1}{2(1-\rho^2)}\left(\frac{B^2}{A}-C^2\right)\right\}\mathrm{d}u
\end{aligned}$$

$$=\frac{1}{2\pi\sigma_2\sqrt{1-\rho^2}}\mathrm{e}^{\frac{1}{2(1-\rho^2)}\left(\frac{B^2}{A}-C^2\right)}\int_{-\infty}^{+\infty}\mathrm{e}^{-\frac{A}{2(1-\rho^2)}u^2}\mathrm{d}u$$

$$=\frac{1}{2\pi\sigma_2\sqrt{1-\rho^2}}\mathrm{e}^{\frac{1}{2(1-\rho^2)}\left(\frac{B^2}{A}-C^2\right)}\cdot\sqrt{\frac{2\pi}{A/(1-\rho^2)}}.$$

由于 $\rho^2+2\rho\frac{\sigma_1}{\sigma_2}+\left(\frac{\sigma_1}{\sigma_2}\right)^2$ $A=1+2\rho\frac{\sigma_1}{\sigma_2}+\left(\frac{\sigma_1}{\sigma_2}\right)^2$

$$B^2-AC^2=\left(\left(\rho+\frac{\sigma_1}{\sigma_2}\right)^2-A\right)C^2=(\rho^2-1)C^2$$

$$=(\rho^2-1)\left(\frac{z-\mu_1-\mu_2}{\sigma_2}\right)^2,$$

所以

$$p_Z(z)=\frac{1}{\sqrt{2\pi(\sigma_1^2+2\rho\sigma_1\sigma_2+\sigma_2^2)}}\exp\left\{-\frac{(z-\mu_1-\mu_2)^2}{2(\sigma_1^2+2\rho\sigma_1\sigma_2+\sigma_2^2)}\right\}.$$

Sum of Exp. dist.

$$S_n = \underbrace{T_1 + T_2 + \cdots + T_n}_{\text{independent Exp}(\lambda)} \sim Gamma(n, \lambda)$$

$$p(x|n, \lambda > 0) = \begin{cases} \dfrac{\lambda^n}{\Gamma(n)} x^{n-1} e^{-\lambda x}, x > 0 \\ 0, x \leqslant 0 \end{cases}$$

$$Gamma(n, \lambda) + Exp(\lambda) \sim \int_0^z \left[\frac{\lambda^n}{\Gamma(n)} x^{n-1} e^{-\lambda x} \cdot \lambda e^{-\lambda(z-x)} \right] dx$$

$$= \frac{\lambda^{n+1}}{\Gamma(n)} \int_0^z x^{n-1} e^{-\lambda z} dx$$

$$= \frac{\lambda^{n+1}}{\Gamma(n+1)} z^n e^{-\lambda z} \sim Gamma(n+1, \lambda)$$

Min. of Exp. dist.

$$T_{min} = \underbrace{\min\{T_1, T_2, \cdots, T_n\}}_{T_i \sim \mathrm{Exp}(\lambda_i),\ i=1,\cdots,n,\ \text{independent}}$$

$$\sim \mathrm{Exp}(\lambda_1 + \lambda_2 + \cdots + \lambda_n)$$

$$P(T_{min} > t) = \prod_{i=1}^{n} P(T_i > t) = e^{-(\lambda_1+\lambda_2+\cdots+\lambda_n)t}$$

Sum of Gamma dist.

$$S_n = \underbrace{T_1 + T_2 + \cdots + T_n}_{T_i \sim Gamma(\alpha_i, \beta),\ i=1,\cdots,n,\ \text{independent}}$$

$$\sim Gamma(\alpha_1 + \alpha_2 + \cdots + \alpha_n, \lambda)$$

$$S_n = \underbrace{X_1^2 + X_2^2 + \cdots + X_n^2}_{X_i \sim \mathcal{N}(0,1),\, i=1,\cdots,n,\ \text{independent}}$$

$$\sim Gamma\left(\left(\frac{1}{2}\right)^n, \frac{1}{2}\right)$$

$$
\begin{aligned}
& Gamma(\alpha_1, \beta) + Gamma(\alpha_2, \beta) \sim \\
& \int_0^z \left[\frac{\beta^{\alpha_1}}{\Gamma(\alpha_1)} x^{\alpha_1 - 1} e^{-\beta x} \cdot \frac{\beta^{\alpha_2}}{\Gamma(\alpha_2)} (z-x)^{\alpha_2 - 1} e^{-\beta(z-x)} \right] dx \\
& = \frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} e^{-\beta z} \int_0^z \left[x^{\alpha_1-1}(z-x)^{\alpha_2-1} \right] dx \\
& = \frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} z^{\alpha_1+\alpha_2-1} e^{-\beta z} \int_0^1 \left[y^{\alpha_1-1}(1-y)^{\alpha_2-1} \right] dy \\
& = \frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2)} z^{\alpha_1+\alpha_2-1} e^{-\beta z} \\
& \sim Gamma(\alpha_1 + \alpha_2, \beta)
\end{aligned}
$$

$$
x = zy, \quad y \in [0, 1]
$$

$$
B(\alpha_1, \alpha_2) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1 + \alpha_2)}
$$

序统计量的分布

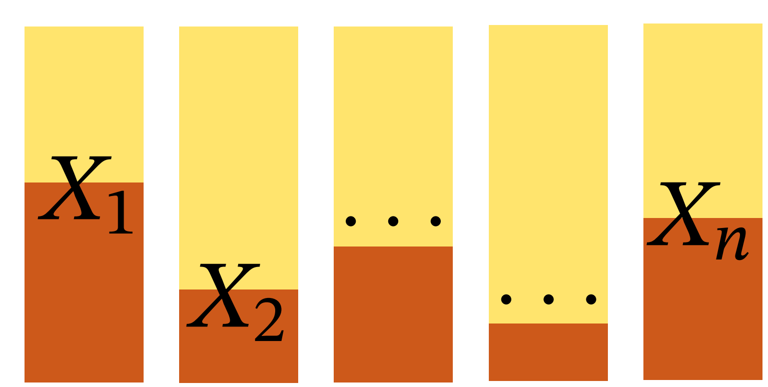
$$\underbrace{X_1, X_2, \cdots, X_n}_{X_i \sim \text{cdf } F(x),\ i=1,\cdots,n,\ \text{independent}}$$

$$F'(x) = f(x)$$

$$X_{(1)} \leqslant X_{(2)}, \cdots, \leqslant X_{(n)}$$

$$P\{X_{(i)} \leq x\} = \sum_{k=i}^{n} \binom{n}{k} (F(x))^k (1 - F(x))^{n-k}$$

至少i个值不超过x



$$
\begin{aligned}
f_{X_{(i)}}(x) = f(x) \sum_{k=i}^{n} \binom{n}{k} k(F(x))^{k-1}(1-F(x))^{n-k} \\
- f(x) \sum_{k=i}^{n} \binom{n}{k} (n-k)(F(x))^{k}(1-F(x))^{n-k-1} \\
= f(x) \sum_{k=i}^{n} \frac{n!}{(n-k)!(k-1)!}(F(x))^{k-1}(1-F(x))^{n-k} \\
\longrightarrow \qquad - f(x) \sum_{k=i}^{n-1} \frac{n!}{(n-k-1)!k!}(F(x))^{k}(1-F(x))^{n-k-1} \\
= f(x) \sum_{k=i}^{n} \frac{n!}{(n-k)!(k-1)!}(F(x))^{k-1}(1-F(x))^{n-k} \\
j = k+1 \longrightarrow \qquad - f(x) \sum_{j=i+1}^{n} \frac{n!}{(n-j)!(j-1)!}(F(x))^{j-1}(1-F(x))^{n-j} \\
= \frac{n!}{(n-i)!(i-1)!} f(x)(F(x))^{i-1}(1-F(x))^{n-i}
\end{aligned}
$$