

§4.2 方差

引例 甲、乙两射手各打了10发子弹，每发子弹击中的环数分别为：

甲	10, 6, 7, 10, 8, 9, 9, 10, 5, 10
乙	8, 7, 9, 10, 9, 8, 7, 9, 8, 9

有六个不同数据

仅有四个不同数据

问哪一个射手的技术较好？

解 首先比较平均环数

$$\bar{甲} = 8.4, \quad \bar{乙} = 8.4$$

再比较稳定程度

$$\sum_{i=1}^{10} (x_i - \bar{x})$$

$$\sum_{i=1}^{10} |x_i - \bar{x}|$$

$$\sum_{i=1}^{10} (x_i - \bar{x})^2$$

$$\sum_{i=1}^{10} (x_i - \bar{x})^2$$

$$\begin{aligned}\text{甲} : & 4 \times (10 - 8.4)^2 + 2 \times (9 - 8.4)^2 + (8 - 8.4)^2 \\ & + (7 - 8.4)^2 + (6 - 8.4)^2 + (5 - 8.4)^2 \\ & = 30.4\end{aligned}$$

$$\begin{aligned}\text{乙} : & (10 - 8.4)^2 + 4 \times (9 - 8.4)^2 \\ & + 3 \times (8 - 8.4)^2 + 2 \times (7 - 8.4)^2 \\ & = 6.44\end{aligned}$$

乙比甲技术稳定

进一步比较平均偏离平均值的程度

$$\begin{aligned}\text{甲} \quad & \frac{1}{10} \{4 \times (10 - 8.4)^2 + 2 \times (9 - 8.4)^2 + (8 - 8.4)^2 \\ & + (7 - 8.4)^2 + (6 - 8.4)^2 + (5 - 8.4)^2\} \\ & = 3.04 \triangleq \sum_{k=1}^6 (x_k - E(X))^2 p_k\end{aligned}$$

$$\begin{aligned}\text{乙} \quad & \frac{1}{10} \{(10 - 8.4)^2 + 4 \times (9 - 8.4)^2 \\ & + 3 \times (8 - 8.4)^2 + 2 \times (7 - 8.4)^2\} \\ & = 0.644 \triangleq \sum_{k=1}^4 (x_k - E(X))^2 p_k\end{aligned}$$

● 方差的概念

定义 若 $E((X - E(X))^2)$ 存在，则称其为随机变量 X 的**方差**，记为 $D(X)$

$$D(X) = E((X - E(X))^2)$$

称 $\sqrt{D(X)}$ 为 X 的**均方差**.

$(X - E(X))^2$ —— 随机变量 X 的取值偏离平均值的情况，是 X 的函数，也是随机变量

$E(X - E(X))^2$ —— 随机变量 X 的取值偏离平均值的平均偏离程度—— 数

若 X 为离散型 r.v. , 概率分布为

$$P(X = x_k) = p_k, \quad k = 1, 2, \dots$$

$$D(X) = \sum_{k=1}^{+\infty} (x_k - E(X))^2 p_k$$

若 X 为连续型 , 概率密度为 $f(x)$

$$D(X) = \int_{-\infty}^{+\infty} (x - E(X))^2 f(x) dx$$

常用的计算方差的公式：

$$D(X) = E(X^2) - E^2(X)$$

$$D(X) = E((X - E(X))^2)$$

$$= E(X^2 - 2X \cdot E(X) + E^2(X))$$

$$= E(X^2) - E(X) \cdot 2 \cdot E(X) + E^2(X)$$

$$= E(X^2) - E^2(X)$$

方差计算

例1 设 $X \sim P(\lambda)$, 求 $D(X)$.

解
$$E(X) = \sum_{k=0}^{+\infty} k \cdot \frac{\lambda^k e^{-\lambda}}{k!} = \lambda e^{-\lambda} \sum_{k=1}^{+\infty} \frac{\lambda^{k-1}}{(k-1)!}$$

$$= \lambda e^{-\lambda} \sum_{k-1=0}^{+\infty} \frac{\lambda^{k-1}}{(k-1)!}$$

$$= \lambda e^{-\lambda} \sum_{m=0}^{+\infty} \frac{\lambda^m}{m!} = \lambda e^{-\lambda} \cdot e^{\lambda}$$

$$= \lambda$$

$$= \lambda$$

$$E(X^2) = E(X(X-1)) + E(X)$$

$$E(X(X-1)) = \sum_{k=0}^{+\infty} k(k-1) \cdot \frac{\lambda^k e^{-\lambda}}{k!}$$

$$= \lambda^2 e^{-\lambda} \sum_{k=2}^{+\infty} \frac{\lambda^{k-2}}{(k-2)!} = \lambda^2$$

$$\longrightarrow E(X^2) = \lambda^2 + \lambda$$

$$D(X) = E(X^2) - E^2(X) = (\lambda^2 + \lambda) - \lambda^2 = \lambda$$

例3 设 $X \sim N(\mu, \sigma^2)$, 求 $D(X)$

$$\text{解 } D(X) = \int_{-\infty}^{+\infty} (x - \mu)^2 \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$\begin{aligned} & \stackrel{\text{令 } \frac{x-\mu}{\sigma}=y}{=} \int_{-\infty}^{+\infty} \sigma^2 y^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \end{aligned}$$

$$= \sigma^2 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} y^2 e^{-\frac{y^2}{2}} dy = \sigma^2 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} y^2 \frac{1}{-y} de^{-\frac{y^2}{2}}$$

$$= -\sigma^2 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} y de^{-\frac{y^2}{2}} = -\sigma^2 \frac{1}{\sqrt{2\pi}} \left[ye^{-\frac{y^2}{2}} \Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} e^{-\frac{y^2}{2}} dy \right]$$

$$= \sigma^2 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{y^2}{2}} dy = \sigma^2$$

常见随机变量的方差

分布	概率分布	方差
参数为 p 的 0-1分布	$P(X = 1) = p$ $P(X = 0) = 1 - p$	$P(1-p)$
$B(n,p)$	$P(X = k) = C_n^k p^k (1 - p)^{n-k}$ $k = 0, 1, 2, \dots, n$	$np(1-p)$
$P(\lambda)$	$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$ $k = 0, 1, 2, \dots$	λ

分布	概率密度	方差
区间 (a,b) 上的 均匀分布	$f(x) = \begin{cases} \frac{1}{b-a}, & a < x < b, \\ 0, & \text{其它} \end{cases}$	$\frac{(b-a)^2}{12}$
$E(\lambda)$	$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0, \\ 0, & \text{其它} \end{cases}$	$\frac{1}{\lambda^2}$
$N(\mu, \sigma^2)$	$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	σ^2

方差

方差的性质

$$\left. \begin{array}{l} \square D(C) = 0 \\ \square D(aX) = a^2 D(X) \end{array} \right\} D(aX + b) = a^2 D(X)$$

性质 1 : $D(C) = 0$

证明 : $D(C) = E(C - E(C))^2 = 0$

性质 2 : $D(aX) = a^2 D(X)$

证明 :
$$\begin{aligned} D(aX + b) &= E((aX + b) - E(aX + b))^2 \\ &= E(a(X - E(X)) + (b - E(b)))^2 \\ &= E(a^2 (X - E(X))^2) = a^2 D(X) \end{aligned}$$

性质 3 : $D(X \pm Y) = D(X) + D(Y)$
 $\pm 2E((X - E(X))(Y - E(Y)))$

证明：

$$\begin{aligned}
 D(X \pm Y) &= E((X \pm Y) - E(X \pm Y))^2 \\
 &= E((X - E(X)) \pm (Y - E(Y)))^2 \\
 &= E(X - E(X))^2 + E(Y - E(Y))^2 \\
 &\quad \pm 2E((X - E(X))(Y - E(Y))) \\
 &= D(X) + D(Y) \\
 &\quad \pm 2E((X - E(X))(Y - E(Y)))
 \end{aligned}$$

注意到 , $E((X - E(X))(Y - E(Y)))$
 $= E(XY) - E(X)E(Y)$

$$\begin{aligned}
D(X \pm Y) &= D(X) + D(Y) \\
&\quad \pm 2E((X - E(X))(Y - E(Y))) \\
&= D(X) + D(Y) \\
&\quad \pm 2(E(XY) - EX \cdot EY)
\end{aligned}$$

特别地，若 X, Y 相互独立，则

$$D(X \pm Y) = D(X) + D(Y)$$

若 X, Y 相互独立

$$\begin{array}{c} \longrightarrow \\ \longleftarrow \end{array} D(X \pm Y) = D(X) + D(Y)$$

$$\longleftrightarrow E(XY) = E(X)E(Y)$$

若 X_1, X_2, \dots, X_n 相互独立, a_1, a_2, \dots, a_n, b 为常数

则
$$D\left(\sum_{i=1}^n a_i X_i + b\right) = \sum_{i=1}^n a_i^2 D(X_i)$$

$$\begin{aligned} D\left(\sum_{i=1}^n a_i X_i + b\right) &= D\left(\sum_{i=1}^n a_i X_i\right) \\ &= \sum_{i=1}^n D(a_i X_i) \\ &= \sum_{i=1}^n a_i^2 D(X_i) \end{aligned}$$

□ 对任意常数 C , $D(X) \leq E(X - C)^2$,
当且仅当 $C = E(X)$ 时等号成立

证明：
$$\begin{aligned} E(X - C)^2 &= E((X - E(X)) - (C - E(X)))^2 \\ &= E(X - E(X))^2 + (C - E(X))^2 \\ &= D(X) + (C - E(X))^2 \end{aligned}$$

当 $C = E(X)$ 时，显然等号成立；

当 $C \neq E(X)$ 时， $(C - E(X))^2 > 0$

$$E(X - C)^2 > D(X)$$

□ $D(X) = 0 \iff P(X = E(X)) = 1$

称为 X 依概率 1 等于常数 $E(X)$

例4 已知 X, Y 相互独立, 且都服从 $N(0, 0.5)$,
求 $E(|X - Y|)$.

解 $X \sim N(0, 0.5), \quad Y \sim N(0, 0.5)$

$$E(X - Y) = 0, \quad D(X - Y) = 1$$

故 $X - Y \sim N(0, 1)$ 记为 $Z \sim N(0, 1)$

$$E(|X - Y|) = \int_{-\infty}^{+\infty} |z| \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = 2 \int_0^{+\infty} z \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^{+\infty} z e^{-\frac{z^2}{2}} dz = -\frac{2}{\sqrt{2\pi}} \int_0^{+\infty} e^{-\frac{z^2}{2}} d\left(-\frac{z^2}{2}\right)$$

$$= -\frac{2}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \Big|_0^{+\infty} = \frac{2}{\sqrt{2\pi}} = \sqrt{\frac{2}{\pi}}$$

例5 设 X 表示独立射击直到击中目标 n 次为止所需射击的次数, 已知每次射击中靶的概率为 p , 求 $E(X)$, $D(X)$.

解 令 X_i 表示击中目标 $i-1$ 次后到第 i 次击中目标所需射击的次数, $i=1,2,\dots,n$

X_1, X_2, \dots, X_n 相互独立, 且 $X = \sum_{i=1}^n X_i$

$$P(X_i = k) = pq^{k-1}, \quad k = 1, 2, \dots$$

$$p + q = 1$$

$$E(X_i) = \sum_{k=1}^{+\infty} k p q^{k-1} = p \sum_{k=1}^{+\infty} k q^{k-1} = p \frac{1}{(1-q)^2} = \frac{1}{p}$$

$$E(X_i^2) = \sum_{k=1}^{+\infty} k^2 pq^{k-1} = \sum_{k=1}^{+\infty} k(k-1)pq^{k-1} + \sum_{k=1}^{+\infty} kpq^{k-1}$$

$$= pq \sum_{k=2}^{+\infty} k(k-1)q^{k-2} + \frac{1}{p}$$

$$= pq \frac{d^2}{dx^2} \left(\sum_{k=0}^{+\infty} x^k \right) \Big|_{x=q} + \frac{1}{p}$$

$$= pq \frac{2}{(1-x)^3} \Big|_{x=q} + \frac{1}{p} = \frac{2-p}{p^2}$$

$$D(X_i) = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}$$

故

$$E(X) = \sum_{i=1}^n E(X_i) = \frac{n}{p}$$

$$D(X) = \sum_{i=1}^n D(X_i) = \frac{n(1-p)}{p^2}$$