概率论与数理统计

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统计量，样本分布

$$\mathbf{X} = (X_1, ..., X_n)^T, \overline{\mathbf{X}} = \left(\overline{X}_n, ..., \overline{X}_n\right)^T, \mathbf{1} = (1, ..., 1)^T, \mathbf{a} = (a, ..., a)^T$$

$$\left(\mathbf{X} - \overline{\mathbf{X}}\right)^T \cdot \mathbf{1} = 0,$$

$$\|\mathbf{X} - \mathbf{a}\|^2 = \left\|\left(\mathbf{X} - \overline{\mathbf{X}}\right) - \left(\mathbf{a} - \overline{\mathbf{X}}\right)\right\|^2 = \left\|\mathbf{X} - \overline{\mathbf{X}}\right\|^2 + \left\|\mathbf{a} - \overline{\mathbf{X}}\right\|^2.$$

$$\sum_{i=1}^{n} (X_i - a)^2 = \sum_{i=1}^{n} \left(X_i - \overline{X}_n\right)^2 + n\left(\overline{X}_n - a\right)^2, \ \forall a.$$

$$\sum_{i=1}^{n} \left(X_i - \overline{X}_n\right)^2 = \sum_{i=1}^{n} X_i^2 - n\overline{X}_n^2$$

大数定律

$$\frac{1}{n}\sum_{i=1}^{n} X_i^k \approx E\left(X^k\right),\ k = 1, 2, \ldots$$

样本均值,

$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

令

$$S_n^2 = \sum_{i=1}^{n}\left(X_i - \overline{X}_n\right)^2$$

样本方差

$$\frac{S_n^2}{n} = \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \overline{X}_n\right)^2,\ \frac{S_n^2}{n-1} = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \overline{X}_n\right)^2$$

容易验证若 $X_1, ..., X_n$ 是来自期望为 μ, 方差为 σ^2 的独立同分布样本 , 那么

$$E\left(\overline{X}_n\right) = \mu,\ Var\left(\overline{X}_n\right) = \frac{\sigma^2}{n},\ E\left(\frac{S_n^2}{n-1}\right) = \sigma^2.$$

若总体为正态分布, 利用统计学基本定理

$$Var\left(\frac{S_n^2}{n-1}\right) = Var\left(\frac{\sigma^2}{n-1}\frac{S_n^2}{\sigma^2}\right) = \frac{2\sigma^4}{n-1}.$$

回顾 Gamma 分布

$$X \sim Gamma\left(\alpha, \beta\right) = \begin{cases} \dfrac{\beta^{\alpha}}{\Gamma\left(\alpha\right)} x^{\alpha-1} e^{-\beta x}, & x \geqslant 0, \\ 0 & otherwise. \end{cases}$$

$$EX = \frac{\beta^{\alpha}}{\Gamma\left(\alpha\right)} \int_0^{\infty} x^{\alpha} e^{-\beta x} dx = \frac{\beta^{\alpha}}{\Gamma\left(\alpha\right)} \frac{\Gamma\left(\alpha+1\right)}{\beta^{\alpha+1}} = \frac{\alpha}{\beta}.$$

$$Var\left(X\right) = E\left(X^2\right) - \left(EX\right)^2 = \frac{\beta^{\alpha}}{\Gamma\left(\alpha\right)} \frac{\Gamma\left(\alpha+2\right)}{\beta^{\alpha+2}} - \frac{\alpha^2}{\beta^2} = \frac{\alpha}{\beta^2}.$$

$$Gamma\left(1, \beta\right) = Exp\left(\beta\right),\ Gamma\left(n, \beta\right) = \underbrace{Exp\left(\beta\right) + \cdots + Exp\left(\beta\right)}_{n \text{ independent copies}},$$

$$\underbrace{Gamma\left(\alpha_1, \beta\right) + \cdots + Gamma\left(\alpha_n, \beta\right)}_{n \text{ independent copies}} = Gamma\left(\alpha_1 + \cdots + \alpha_n, \beta\right)$$

$$X_1, \ldots, X_n \sim \text{iid}\, \mathcal{N}\left(0, 1\right), X_1^2 + \cdots + X_n^2 \sim Gamma\left(\frac{n}{2}, \frac{1}{2}\right) \triangleq \chi_n^2.$$

$$E\left(\chi_n^2\right) = n,\ Var\left(\chi_n^2\right) = 2n.$$

Theorem 3. 若 $X_1, \ldots, X_n \sim \mathcal{N}(\mu, \sigma^2)$ 为独立同分布随机样本，那么

(1) 相互独立

$$\overline{X}_n,\ S_n^2 = \sum_{i=1}^{n} \left(X_i - \overline{X}_n\right)^2$$

(2)

$$\overline{X}_n \sim \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right)$$

(3)

$$\frac{S_n^2}{\sigma^2} = \frac{1}{\sigma^2} \sum_{i=1}^{n} \left(X_i - \overline{X}_n\right)^2 \sim \chi^2_{n-1}$$

$Y \sim \chi_n^2, Z \sim \mathcal{N}(0,1)$ 相互独立,

$$\frac{Z}{(Y/n)^{1/2}}$$

的分布称为自由度为 n 的 t-分布, 记为 t_n.

若 $X_1, \dots, X_n \sim \mathcal{N}(\mu, \sigma^2)$ 为独立同分布随机样本, 那么

$$\frac{(n(n-1))^{1/2}(\overline{X}_n - \mu)}{\left(\sum_{i=1}^n (X_i - \overline{X}_n)^2\right)^{1/2}} = \frac{(\overline{X}_n - \mu)/(\sigma/\sqrt{n})}{\left(\frac{1}{\sigma^2}\sum_{i=1}^n (X_i - \overline{X}_n)^2/(n-1)\right)^{1/2}} \sim t_{n-1}$$

$Y \sim \chi_m^2, W \sim \chi_n^2$ 相互独立,

$$\frac{Y/m}{W/n}$$

的分布称为自由度为 (m,n) 的 F-分布, 记为 $F_{m,n}$.

置信区间

待估参数	其他参数	枢轴量 V	V 的分布
μ	σ^2 已知	$\dfrac{\overline{X}_n - \mu}{\sigma/\sqrt{n}}$	$\mathcal{N}(0,1)$
μ	σ^2 未知	$\dfrac{\overline{X}_n - \mu}{\hat{\sigma}/\sqrt{n}}$	t_{n-1}
σ^2	μ 已知	$\displaystyle\sum_{i=1}^{n}\left(\frac{X_i - \mu}{\sigma}\right)^2$	χ_n^2
σ^2	μ 未知	$\displaystyle\sum_{i=1}^{n}\left(\frac{X_i - \overline{X}_n}{\sigma}\right)^2$	χ_{n-1}^2

这里

$$\hat{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \overline{X}_n\right)^2$$