协方差和相关系数的性质

协方差的性质

- $\operatorname{cov}(X,Y)=\operatorname{cov}(Y,X)=E(XY)-E(X)E(Y)$

证

$$\operatorname{cov}(X,Y)=E\left[(X-EX)(Y-EY)\right]$$

$$=E\left[XY-X\cdot E(Y)-Y\cdot E(X)+E(X)\cdot E(Y)\right]$$

$$=E(XY)-E(X)\cdot E(Y)-E(Y)\cdot E(X)+E(X)\cdot E(Y)$$

$$=E(XY)-E(X)E(Y)$$

$$=\operatorname{cov}(Y,X)$$

- $\operatorname{cov}(aX, bY) = ab\operatorname{cov}(X, Y)$

证 $\operatorname{cov}(aX, bY) = E\left[(aX - E(aX))(bY - E(bY))\right]$

$$= E\left[a(X - E(X)) \cdot b(Y - E(Y))\right]$$

$$= abE\left[(X - E(X)) \cdot (Y - E(Y))\right]$$

$$= ab\operatorname{cov}(X, Y)$$

❑ $\operatorname{cov}(X+Y,Z)=\operatorname{cov}(X,Z)+\operatorname{cov}(Y,Z)$

证

$$
\begin{aligned}
\operatorname{cov}(X+Y,Z) &= E\left\{\left[(X+Y)-E(X+Y)\right](Z-EZ)\right\} \\
&= E\left\{\left[(X-EX)+(Y-EY)\right](Z-EZ)\right\} \\
&= E\left[(X-EX)(Z-EZ)+(Y-EY)(Z-EZ)\right] \\
&= E\left[(X-EX)(Z-EZ)\right]+E\left[(Y-EY)(Z-EZ)\right] \\
&= \operatorname{cov}(X,Z)+\operatorname{cov}(Y,Z)
\end{aligned}
$$

- $\operatorname{cov}(X, X) = D(X)$

证 $\operatorname{cov}(X, X) = E\left[(X - EX)(X - EX)\right]$

$$= D(X)$$

- $|\operatorname{cov}(X,Y)|^2 \le D(X)D(Y)$

当$D(X)>0,\ D(Y)>0$ 时，当且仅当

$$P(Y-E(Y)=t_0(X-E(X)))=1$$

时，等式成立

—Cauchy-Schwarz不等式

证 5 令

$$g(t)=E[(X-E(X))\cdot t-(Y-E(Y))]^2$$

$$=E[(X-E(X))^2\cdot t^2-2(X-E(X))(Y-E(Y))\cdot t+(Y-E(Y))^2]$$

$$=D(X)\cdot t^2-2\operatorname{cov}(X,Y)\cdot t+D(Y)$$

对任何实数 t,

$$g(t)\geq 0 \quad \Longrightarrow$$

$$4\operatorname{cov}^2(X,Y)-4D(X)D(Y)\leq 0$$

即 $|\operatorname{cov}(X,Y)|^2\leq D(X)D(Y)$

等号成立 $\Longleftrightarrow$ $g(t)=0$ 有两个相等的实零点

等号成立，即 $|\operatorname{cov}(X,Y)|^2 = D(X)D(Y)$

此时，零点为

$$t_0 = -\frac{-2\operatorname{cov}(X,Y)}{2D(X)} = \frac{\operatorname{cov}(X,Y)}{D(X)}$$

$$= \frac{\sqrt{D(X)\cdot D(Y)}}{D(X)} \quad \left(或 \frac{-\sqrt{D(X)\cdot D(Y)}}{D(X)} \right)$$

$$= \sqrt{\frac{D(Y)}{D(X)}} \quad \left(或 -\sqrt{\frac{D(Y)}{D(X)}} \right)$$

此时，

$g(t_0)=0$ 即

$$\text{可以证明}\quad \left.\begin{array}{l} E[(Y-E(Y))-t_0(X-E(X))]^2=0 \\ E[(Y-E(Y))-t_0(X-E(X))]=0 \end{array}\right\}$$

$$\Longleftrightarrow \quad D[(Y-E(Y))-t_0(X-E(X))]=0$$

$$\Longleftrightarrow \quad P[(Y-E(Y))-t_0(X-E(X))=0]=1$$

$$P[(Y-E(Y))-t_0(X-E(X))=0]=1$$

即

$$P[(Y-E(Y))=t_0(X-E(X))]=1$$

即Y与X有线性关系的概率等于1，这种线性关系为

$$P[(Y-E(Y))=\pm\sqrt{\frac{D(Y)}{D(X)}}(X-E(X))]=1$$

$$P\left(\frac{Y-E(Y)}{\sqrt{D(Y)}}=\pm\frac{X-E(X)}{\sqrt{D(X)}}\right)=1$$

标准化随机变量

设随机变量 X 的期望$E(X)$、方差$D(X)$都存在，且$D(X)\neq 0$，则称

$$X^* = \frac{X - E(X)}{\sqrt{D(X)}}$$

为 X 的标准化随机变量。显然，

$$E(X^*) = 0, \quad D(X^*) = 1$$

$$P\left(Y^* = \pm X^*\right) = 1$$

完全类似地可以证明

$$E^2(XY) \le E(X^2)E(Y^2)$$

当$E(X^2) > 0, E(Y^2) > 0$时，当且仅当

$$P(Y = t_0 X) = 1$$

时，等式成立

$$g(t) = E[Y - tX]^2$$

已知 $E(X^2)=0$

求证 $E(X)=0$

证明：

由 $D(X)=E(X^2)-E^2(X)\geq 0$

知 $E^2(X)\leq E(X^2)$

故当 $E(X^2)=0$ 知 $E(X)=0$

相关系数的性质

- $|\rho_{XY}| \leq 1$

证明：由 $|\operatorname{cov}(X,Y)|^2 \leq D(X)D(Y)$

及 $\rho_{XY} = \dfrac{\operatorname{cov}(X,Y)}{\sqrt{D(X)} \cdot \sqrt{D(Y)}}$

知 $\rho_{XY}^2 = \dfrac{|\operatorname{cov}(X,Y)|^2}{D(X)D(Y)} \leq 1$

故 $|\rho_{XY}| \leq 1$

- $|\rho_{XY}|=1 \Longleftrightarrow$ Cauchy-Schwarz不等式的等号成立

$\Longleftrightarrow$ 即Y与X有线性关系的概率等于1，这种线性关系为

$$P\left(\frac{Y-E(Y)}{\sqrt{D(Y)}}=\pm\frac{X-E(X)}{\sqrt{D(X)}}\right)=1$$

$$\rho_{XY}=1 \quad \Longrightarrow$$

$$t_0=-\frac{-2\operatorname{cov}(X,Y)}{2D(X)}=\frac{\operatorname{cov}(X,Y)}{D(X)}$$

$$=\frac{\rho\cdot\sqrt{D(X)\cdot D(Y)}}{D(X)}=\sqrt{\frac{D(Y)}{D(X)}}$$

$$P[(Y-E(Y))=\sqrt{\frac{D(Y)}{D(X)}}(X-E(X))]=1$$

$$P\left(\frac{Y-E(Y)}{\sqrt{D(Y)}}=\frac{X-E(X)}{\sqrt{D(X)}}\right)=1$$

$$P\left(Y^*=X^*\right)=1$$

$$\rho_{XY} = -1 \Longrightarrow$$

$$t_0 = -\frac{-2\operatorname{cov}(X,Y)}{2D(X)} = \frac{\operatorname{cov}(X,Y)}{D(X)}$$

$$= \frac{\rho \cdot \sqrt{D(X) \cdot D(Y)}}{D(X)} = -\sqrt{\frac{D(Y)}{D(X)}}$$

$$P[(Y - E(Y)) = -\sqrt{\frac{D(Y)}{D(X)}}(X - E(X))] = 1$$

$$P\left(\frac{Y - E(Y)}{\sqrt{D(Y)}} = -\frac{X - E(X)}{\sqrt{D(X)}}\right) = 1$$

$$P\left(Y^* = -X^*\right) = 1$$

- $\rho_{XY}=0 \iff X$，Y不相关

$$\iff \operatorname{cov}(X,Y)=0$$

$$\iff E(XY)=E(X)E(Y)$$

$$\iff D(X\pm Y)=D(X)+D(Y)$$

X，Y相关时，

$$D(X+Y)=D(X)+D(Y)+2\operatorname{cov}(X,Y)$$

$$D(X-Y)=D(X)+D(Y)-2\operatorname{cov}(X,Y)$$

X, Y 相互独立 $\overset{\Longrightarrow}{\not\Longleftarrow}$ X, Y 不相关

若 X, Y 服从二维正态分布，

X, Y 相互独立 $\Longleftrightarrow$ X, Y 不相关

例5 设 $(X, Y) \sim N(1,4;\ 1,4;\ 0.5)$, $Z = X + Y$, 求ρ_{XZ}

解 $E(X) = E(Y) = 1$, $\rho_{XY} = \dfrac{1}{2}$,

$D(X) = D(Y) = 4$,

$$\operatorname{cov}(X,Y) = \rho_{XY} \cdot \sqrt{D(X)} \cdot \sqrt{D(Y)} \quad = 2$$

$$\operatorname{cov}(X,Z) = \operatorname{cov}(X,X) + \operatorname{cov}(X,Y)$$

$$= D(X) + \rho_{XY} \cdot \sqrt{D(X)} \cdot \sqrt{D(Y)}$$

$$= 4 + \frac{1}{2} \cdot \sqrt{4} \cdot \sqrt{4} \quad = 6$$

$$
\begin{aligned}
D(Z) &= D(X+Y) \\
&= D(X) + D(Y) + 2\operatorname{cov}(X,Y) \\
&= D(X) + D(Y) + 2\cdot\rho_{XY}\cdot\sqrt{D(X)}\cdot\sqrt{D(Y)} \\
&= 4+4+2\cdot\frac{1}{2}\sqrt{4}*\sqrt{4} = 12
\end{aligned}
$$

$$
\begin{aligned}
\rho_{XZ} &= \frac{\operatorname{cov}(X,Z)}{\sqrt{D(X)}\cdot\sqrt{D(Z)}} \\
&= \frac{6}{2\cdot\sqrt{12}} \\
&= \frac{\sqrt{3}}{2}
\end{aligned}
$$

例6 设随机变量 X 的概率密度函数为

$$f(x)=\frac{1}{2}e^{-|x|}, -\infty<x<+\infty$$

(1) $E(|X|), D(|X|)$

(2) 求$\operatorname{cov}(X,|X|)$, 问$X$与$|X|$是否不相关.

(3) 问X与$|X|$是否独立？为什么？

解 (1) $E(|X|)=\int_{-\infty}^{+\infty}|x|\frac{1}{2}e^{-|x|}dx$

$$=2\cdot\int_{0}^{+\infty}x\frac{1}{2}e^{-x}dx$$

$$=1$$

$$E(|X|^2)=\int_{-\infty}^{+\infty}|x|^2\,\frac{1}{2}e^{-|x|}dx$$

$$=2\cdot\int_0^{+\infty}x^2\,\frac{1}{2}e^{-x}dx \qquad =\int_0^{+\infty}x^2e^{-x}dx$$

$$=2$$

$$D(|X|)=E(|X|^2)-E^2(|X|)$$

$$=2-1^2=1$$

(2) $$E(X\cdot|X|)=\int_{-\infty}^{+\infty}x\cdot|x|\frac{1}{2}e^{-|x|}dx$$

$$=\frac{1}{2}\int_{-\infty}^{0}-x^2e^{x}dx+\frac{1}{2}\int_{0}^{+\infty}x^2e^{-x}dx$$

$$=\frac{1}{2}\int_{+\infty}^{0}-y^2e^{-y}d(-y)+\frac{1}{2}\int_{0}^{+\infty}x^2e^{-x}dx$$

$$=-\frac{1}{2}\int_{0}^{+\infty}y^2e^{-y}dy+\frac{1}{2}\int_{0}^{+\infty}x^2e^{-x}dx$$

$$=0$$

$$E(X)=\int_{-\infty}^{+\infty}x\frac{1}{2}e^{-|x|}dx$$

$$=\frac{1}{2}\int_{-\infty}^{0}xe^{x}dx+\frac{1}{2}\int_{0}^{+\infty}xe^{-x}dx$$

$$=0$$

已证：$E(|X|)=1$

$$\Longrightarrow \operatorname{cov}(X,|X|)=E(X|X|)-E(X)E(|X|)=0$$

X与$|X|$不相关.

(3) $P(X<-2,|X|<1)=0$

$$P(X<-2)=\int_{-\infty}^{-2} f(x)dx=\int_{-\infty}^{-2} \frac{1}{2} e^{-|x|} dx$$

$$=\frac{1}{2}\int_{-\infty}^{-2} e^{x} dx=\frac{1}{2} e^{-2}$$

$$P(|X|<1)=\int_{-1}^{1} f(x)dx=\int_{-1}^{1} \frac{1}{2} e^{-|x|} dx$$

$$=2\cdot\frac{1}{2}\int_{0}^{1} e^{-x} dx=1-e^{-1}$$

显然 $P(X<-2,|X|<1)\neq P(X<-2)P(|X|<1)$

因而 X 与 $|X|$ 不独立