抽样分布的某些结论

（Ⅰ）一个正态总体

设 $X \sim N(\mu,\sigma^2)$ $E(X)=\mu,$ $D(X)=\sigma^2$

总体的样本为$(X_1, X_2,\ldots ,X_n)$，则

$$\overline{X} \sim N(\mu,\frac{\sigma^2}{n}) \Rightarrow \frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

$$\boxed{\frac{(n-1)S^2}{\sigma^2}=\sum_{i=1}^{n}\left(\frac{X_i-\overline{X}}{\sigma}\right)^2 \sim \chi^2(n-1)}$$

$\frac{(n-1)S^2}{\sigma^2}$ 与 $\overline{X}$ 相互独立

(1)

$$\frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \div \frac{S}{\sigma} = \boxed{\frac{\overline{X}-\mu}{S/\sqrt{n}} \sim T(n-1)}$$

(2)

（II）两个正态总体

设$(X_1, X_2, \ldots, X_n)$是来自正态总体 $X \sim N(\mu_1, \sigma_1^2)$ 的一个简单随机样本

$(Y_1, Y_2, \ldots, Y_m)$是来自正态总体 $Y \sim N(\mu_2, \sigma_2^2)$ 的一个简单随机样本

它们相互独立.

令 $\overline{X} = \dfrac{1}{n}\sum_{i=1}^{n} X_i$ $\quad \overline{Y} = \dfrac{1}{m}\sum_{j=1}^{m} Y_j$

$$S_1^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2 \qquad S_2^2 = \frac{1}{m-1}\sum_{j=1}^{m}(Y_j - \overline{Y})^2$$

则

$$\frac{(n-1)S_1^2}{\sigma_1^2} \sim \chi^2(n-1) \qquad \frac{(m-1)S_2^2}{\sigma_2^2} \sim \chi^2(m-1)$$

$$\boxed{\frac{S_1^2 \Big/ \sigma_1^2}{S_2^2 \Big/ \sigma_2^2} = \frac{\dfrac{(n-1)S_1^2}{\sigma_1^2} \Big/ n-1}{\dfrac{(m-1)S_2^2}{\sigma_2^2} \Big/ m-1} \sim F(n-1, m-1)} \quad \text{(3)}$$

若 $\sigma_1 = \sigma_2$ 则 $\dfrac{S_1^2}{S_2^2} \sim F(n-1, m-1)$

设$(X_1, X_2, \ldots, X_n)$是来自正态总体 $X \sim N(\mu_1, \boxed{\sigma^2})$ 的一个简单随机样本

$(Y_1, Y_2, \ldots, Y_m)$是来自正态总体 $Y \sim N(\mu_2, \boxed{\sigma^2})$ 的一个简单随机样本，它们相互独立.

则 $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \sim N(\mu_1, \frac{\sigma^2}{n})$ $\quad \overline{Y} = \frac{1}{m}\sum_{j=1}^{m} Y_j \sim N(\mu_2, \frac{\sigma^2}{m})$

$$\Longrightarrow \quad \overline{X} - \overline{Y} \sim N(\mu_1 - \mu_2, \frac{\sigma^2}{n} + \frac{\sigma^2}{m})$$

$$\Longrightarrow \quad \frac{(\overline{X} - \overline{Y}) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma^2}{n} + \frac{\sigma^2}{m}}} \sim N(0,1)$$

$$\frac{(n-1)S_1^2}{\sigma^2} \sim \chi^2(n-1) \qquad \frac{(m-1)S_2^2}{\sigma^2} \sim \chi^2(m-1)$$

$$\Longrightarrow \frac{(n-1)S_1^2}{\sigma^2} + \frac{(m-1)S_2^2}{\sigma^2} \sim \chi^2(n+m-2)$$

$\overline{X}-\overline{Y}$ 与 $\dfrac{(n-1)S_1^2}{\sigma^2} + \dfrac{(m-1)S_2^2}{\sigma^2}$ 相互独立

$$\Longrightarrow \quad \frac{\dfrac{(\overline{X}-\overline{Y})-(\mu_1-\mu_2)}{\sqrt{\dfrac{\sigma^2}{n}+\dfrac{\sigma^2}{m}}}}{\sqrt{\dfrac{\dfrac{(n-1)S_1^2}{\sigma^2}+\dfrac{(m-1)S_2^2}{\sigma^2}}{n+m-2}}}$$

$$\boxed{=\frac{(\overline{X}-\overline{Y})-(\mu_1-\mu_2)}{\sqrt{\dfrac{1}{n}+\dfrac{1}{m}}\sqrt{\dfrac{(n-1)S_1^2+(m-1)S_2^2}{n+m-2}}} \sim t(n+m-2)}$$

$$\text{-·-·-·-·-·} \quad \textbf{(4)}$$

例3 设总体X~N(72, 100),为使样本均值大于70的概率不小于90%，则样本容量至少应为多少?

解 设样本容量为 n，则 $\overline{X} \sim N(72, \frac{100}{n})$

故 $P(\overline{X} > 70) = 1 - P(\overline{X} \le 70)$

$$= 1 - \Phi\left(\frac{70-72}{\frac{10}{\sqrt{n}}}\right) = 1 - \Phi\left(-0.2\sqrt{n}\right) = \Phi\left(0.2\sqrt{n}\right)$$

令 $\Phi\left(0.2\sqrt{n}\right) \ge 0.9$ 得 $0.2\sqrt{n} \ge 1.29$

即$n \ge 41.6025$ 所以取$n=42$

例4 从正态总体 $X \sim N(\mu, \sigma^2)$ 中，抽取了 $n = 20$的样本$(X_1, X_2, \dots, X_{20})$

(1) 求$P\left(0.37\sigma^2 \le \frac{1}{20}\sum_{i=1}^{20}\left(X_i - \overline{X}\right)^2 \le 1.76\sigma^2\right)$

$X_1, X_2, \cdots, X_{20}$

(2) 求$P\left(0.37\sigma^2 \le \frac{1}{20}\sum_{i=1}^{20}\left(X_i - \mu\right)^2 \le 1.76\sigma^2\right)$

解 (1) $\dfrac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$

即 $\dfrac{1}{\sigma^2}\sum_{i=1}^{20}\left(X_i-\overline{X}\right)^2=\dfrac{19S^2}{\sigma^2}\sim\chi^2(19)$

故 $P\left(0.37\sigma^2 \le \dfrac{1}{20}\sum_{i=1}^{20}\left(X_i-\overline{X}\right)^2 \le 1.76\sigma^2\right)$

$$=P\left(7.4 \le \frac{1}{\sigma^2}\sum_{i=1}^{20}\left(X_i-\overline{X}\right)^2 \le 35.2\right)$$

$$=P\left(\frac{1}{\sigma^2}\sum_{i=1}^{20}\left(X_i-\overline{X}\right)^2 \ge 7.4\right)-P\left(\frac{1}{\sigma^2}\sum_{i=1}^{20}\left(X_i-\overline{X}\right)^2 \ge 35.2\right)$$

$$\overset{\text{查表}}{=} 0.99-0.01=0.98$$

(2) $$\sum_{i=1}^{20}\left(\frac{X_i-\mu}{\sigma}\right)^2 \sim \chi^2(20)$$

X_i与X同分布，即

$$X_i \sim N(\mu,\sigma^2)$$

$$\frac{X_i-\mu}{\sigma} \sim N(0,1)$$

$$\left(\frac{X_i-\mu}{\sigma}\right)^2 \sim \chi^2(1)$$

$$\sum_{i=1}^{20}\left(\frac{X_i-\mu}{\sigma}\right)^2 \sim \chi^2(20)$$

故 $$P\left(0.37\sigma^2 \le \frac{1}{20}\sum_{i=1}^{20}\left(X_i-\mu\right)^2 \le 1.76\sigma^2\right)$$

$$=P\left(7.4 \le \sum_{i=1}^{20}\left(\frac{X_i-\mu}{\sigma}\right)^2 \le 35.2\right)$$

$$=P\left(\sum_{i=1}^{20}\left(\frac{X_i-\mu}{\sigma}\right)^2 \ge 7.4\right)-P\left(\sum_{i=1}^{20}\left(\frac{X_i-\mu}{\sigma}\right)^2 \ge 35.2\right)$$

$$=0.995-0.025=0.97$$

例5 设随机变量X与Y相互独立，$X \sim N(0,16)$, $Y \sim N(0,9)$, $X_1, X_2,\dots, X_9$ 与$Y_1, Y_2,\dots, Y_{16}$ 分别是取自X与Y的简单随机样本, 求统计量

$$\frac{X_1 + X_2 + \cdots + X_9}{\sqrt{Y_1^2 + Y_2^2 + \cdots + Y_{16}^2}}$$

所服从的分布

解

$$X \sim N(0,16), \quad X_i \sim N(0,16)$$

$$X_1 + X_2 + \cdots + X_9 \sim N(0, 9\times 16)$$

$$\frac{1}{3\times 4}(X_1 + X_2 + \cdots + X_9) \sim N(0,1)$$

$$Y \sim N(0,9), \qquad Y_i \sim N(0,9)$$

$$\frac{1}{3} Y_i \sim N(0,1) \quad, i = 1, 2, \cdots, 16$$

$$\left(\frac{1}{3} Y_i \right)^2 \sim \chi^2(1) \qquad \sum_{i=1}^{16} \left(\frac{1}{3} Y_i \right)^2 \sim \chi^2(16)$$

从而 $$\frac{\frac{1}{3\times 4}\left(X_1 + X_2 + \cdots + X_9 \right)}{\sqrt{\frac{\sum_{i=1}^{16}\left(\frac{1}{3} Y_i \right)^2}{16}}} \sim t(16)$$

即 $$\frac{X_1 + X_2 + \cdots + X_9}{\sqrt{Y_1^2 + Y_2^2 + \cdots + Y_{16}^2}} \sim t(16)$$

例6 设总体$X \sim N(0,1)$，$X_1, X_2, \ldots, X_6$为总体 X 的样本，$Y=(X_1+X_2+X_3)^2+(X_4+X_5+X_6)^2$ 试确定常数c 使cY服从 χ^2 分布.

解 $X_1+X_2+X_3 \sim N(0,3)$，$X_4+X_5+X_6 \sim N(0,3)$

$$\frac{1}{\sqrt{3}}\left(X_1+X_2+X_3\right) \sim N(0,1),$$

$$X \sim N(0,1)$$

$$\frac{1}{\sqrt{3}}\left(X_4+X_5+X_6\right) \sim N(0,1)$$

故 $$\left[\frac{1}{\sqrt{3}}\left(X_1+X_2+X_3\right)\right]^2+\left[\frac{1}{\sqrt{3}}\left(X_4+X_5+X_6\right)\right]^2$$

$$=\frac{1}{3}Y \sim \chi^2(2)$$

因此 $c=\dfrac{1}{3}$

例7 设 $X_1, X_2, \ldots, X_n$是来自正态总体$N(\mu, \sigma^2)$的简单随机样本， $\bar{X}$是样本均值，

$$S_1^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2, \quad S_2^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2,$$

$$S_3^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \mu)^2, \quad S_4^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2,$$

则服从自由度为 $n-1$ 的 t 分布的随机变量为:

(A) $\dfrac{\bar{X}-\mu}{S_1}\sqrt{n-1}$ (B) $\dfrac{\bar{X}-\mu}{S_2}\sqrt{n-1}$

(C) $\dfrac{\bar{X}-\mu}{S_3}\sqrt{n}$ (D) $\dfrac{\bar{X}-\mu}{S_4}\sqrt{n}$

解 $\dfrac{\bar{X}-\mu}{\dfrac{\sigma}{\sqrt{n}}} \sim N(0,1)$ $\quad \dfrac{1}{\sigma^2}\sum_{i=1}^{n}(X_i-\bar{X})^2 \sim \chi^2(n-1)$

$$\frac{\dfrac{\bar{X}-\mu}{\dfrac{\sigma}{\sqrt{n}}}}{\sqrt{\dfrac{\dfrac{1}{\sigma^2}\sum_{i=1}^{n}(X_i-\bar{X})^2}{n-1}}} = \frac{\sqrt{n(n-1)}(\bar{X}-\mu)}{\sqrt{\sum_{i=1}^{n}(X_i-\bar{X})^2}} \sim t(n-1)$$

故应选(B)