

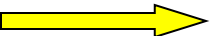
# 抽样分布的某些结论

## (I) 一个正态总体

设  $X \sim N(\mu, \sigma^2)$   $E(X) = \mu$ ,  $D(X) = \sigma^2$

总体的样本为  $(X_1, X_2, \dots, X_n)$ , 则

$$\left. \begin{aligned} \bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right) &\Rightarrow \frac{\bar{X} - \mu}{\sigma / \sqrt{n}} \sim N(0, 1) \\ \frac{(n-1)S^2}{\sigma^2} = \sum_{i=1}^n \left( \frac{X_i - \bar{X}}{\sigma} \right)^2 &\sim \chi^2(n-1) \end{aligned} \right\} \begin{aligned} &\frac{(n-1)S^2}{\sigma^2} \text{ 与 } \bar{X} \\ &\text{相互独立} \end{aligned} \quad (1)$$

  $\frac{\bar{X} - \mu}{\sigma / \sqrt{n}} \div \frac{S}{\sigma} = \frac{\bar{X} - \mu}{S / \sqrt{n}} \sim T(n-1) \quad \dots (2)$

## (II) 两个正态总体

设 $(X_1, X_2, \dots, X_n)$ 是来自正态总体  
的一个简单随机样本

$$X \sim N(\mu_1, \sigma_1^2)$$

$(Y_1, Y_2, \dots, Y_m)$ 是来自正态总体  
的一个简单随机样本

$$Y \sim N(\mu_2, \sigma_2^2)$$

它们相互独立.

$$\text{令 } \bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$$

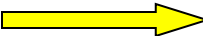
$$\bar{Y} = \frac{1}{m} \sum_{j=1}^m Y_j$$

$$S_1^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2$$

$$S_2^2 = \frac{1}{m-1} \sum_{j=1}^m (Y_j - \bar{Y})^2$$

则

$$\frac{(n-1)S_1^2}{\sigma_1^2} \sim \chi^2(n-1) \quad \frac{(m-1)S_2^2}{\sigma_2^2} \sim \chi^2(m-1)$$


$$\frac{\frac{S_1^2}{\sigma_1^2}}{\frac{S_2^2}{\sigma_2^2}} = \frac{\frac{(n-1)S_1^2}{\sigma_1^2} / n-1}{\frac{(m-1)S_2^2}{\sigma_2^2} / m-1} \sim F(n-1, m-1) \quad \text{--- (3)}$$

若  $\sigma_1 = \sigma_2$  则  $\frac{S_1^2}{S_2^2} \sim F(n-1, m-1)$

设 $(X_1, X_2, \dots, X_n)$ 是来自正态总体  $X \sim N(\mu_1, \sigma^2)$   
的一个简单随机样本

$(Y_1, Y_2, \dots, Y_m)$ 是来自正态总体  $Y \sim N(\mu_2, \sigma^2)$   
的一个简单随机样本，它们相互独立.

则  $\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i \sim N(\mu_1, \frac{\sigma^2}{n})$      $\bar{Y} = \frac{1}{m} \sum_{j=1}^m Y_j \sim N(\mu_2, \frac{\sigma^2}{m})$

$\longrightarrow \bar{X} - \bar{Y} \sim N(\mu_1 - \mu_2, \frac{\sigma^2}{n} + \frac{\sigma^2}{m})$

$\longrightarrow \frac{(\bar{X} - \bar{Y}) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma^2}{n} + \frac{\sigma^2}{m}}} \sim N(0, 1)$

$$\frac{(n-1)S_1^2}{\sigma^2} \sim \chi^2(n-1) \quad \frac{(m-1)S_2^2}{\sigma^2} \sim \chi^2(m-1)$$

$$\longrightarrow \frac{(n-1)S_1^2}{\sigma^2} + \frac{(m-1)S_2^2}{\sigma^2} \sim \chi^2(n+m-2)$$

$\bar{X} - \bar{Y}$  与  $\frac{(n-1)S_1^2}{\sigma^2} + \frac{(m-1)S_2^2}{\sigma^2}$  相互独立



$$\frac{(\bar{X} - \bar{Y}) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma^2}{n} + \frac{\sigma^2}{m}}}$$
$$\sqrt{\frac{\frac{(n-1)S_1^2}{\sigma^2} + \frac{(m-1)S_2^2}{\sigma^2}}{n+m-2}}$$

$$= \frac{(\bar{X} - \bar{Y}) - (\mu_1 - \mu_2)}{\sqrt{\frac{1}{n} + \frac{1}{m}} \sqrt{\frac{(n-1)S_1^2 + (m-1)S_2^2}{n+m-2}}} \sim t(n+m-2)$$

..... (4)

**例3** 设总体 $X \sim N(72, 100)$ ,为使样本均值大于70的概率不小于90%,则样本容量至少应为多少?

**解** 设样本容量为  $n$ , 则  $\bar{X} \sim N(72, \frac{100}{n})$

故  $P(\bar{X} > 70) = 1 - P(\bar{X} \leq 70)$

$$= 1 - \Phi\left(\frac{70 - 72}{\frac{10}{\sqrt{n}}}\right) = 1 - \Phi(-0.2\sqrt{n}) = \Phi(0.2\sqrt{n})$$

$$\text{令 } \Phi(0.2\sqrt{n}) \geq 0.9 \quad \text{得 } 0.2\sqrt{n} \geq 1.29$$

$$\text{即 } n \geq 41.6025 \quad \text{所以取 } n = 42$$

**例4** 从正态总体  $X \sim N(\mu, \sigma^2)$  中, 抽取了  $n = 20$  的样本  $(X_1, X_2, \dots, X_{20})$

$$(1) \text{ 求 } P\left(0.37\sigma^2 \leq \frac{1}{20} \sum_{i=1}^{20} (X_i - \bar{X})^2 \leq 1.76\sigma^2\right)_{X_1, X_2, \dots, X_{20}}$$

$$(2) \text{ 求 } P\left(0.37\sigma^2 \leq \frac{1}{20} \sum_{i=1}^{20} (X_i - \mu)^2 \leq 1.76\sigma^2\right)$$



解 (1)  $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$

即  $\frac{1}{\sigma^2} \sum_{i=1}^{20} (X_i - \bar{X})^2 = \frac{19S^2}{\sigma^2} \sim \chi^2(19)$

故  $P\left(0.37\sigma^2 \leq \frac{1}{20} \sum_{i=1}^{20} (X_i - \bar{X})^2 \leq 1.76\sigma^2\right)$   
 $= P\left(7.4 \leq \frac{1}{\sigma^2} \sum_{i=1}^{20} (X_i - \bar{X})^2 \leq 35.2\right)$   
 $= P\left(\frac{1}{\sigma^2} \sum_{i=1}^{20} (X_i - \bar{X})^2 \geq 7.4\right) - P\left(\frac{1}{\sigma^2} \sum_{i=1}^{20} (X_i - \bar{X})^2 \geq 35.2\right)$

查表

$$= 0.99 - 0.01 = 0.98$$

$$(2) \quad \sum_{i=1}^{20} \left( \frac{X_i - \mu}{\sigma} \right)^2 \sim \chi^2(20)$$

$X_i$ 与 $X$ 同分布，即

$$X_i \sim N(\mu, \sigma^2)$$

$$\frac{X_i - \mu}{\sigma} \sim N(0, 1)$$

$$\left( \frac{X_i - \mu}{\sigma} \right)^2 \sim \chi^2(1)$$

$$\sum_{i=1}^{20} \left( \frac{X_i - \mu}{\sigma} \right)^2 \sim \chi^2(20)$$

$$\begin{aligned}
\text{故 } & P\left(0.37\sigma^2 \leq \frac{1}{20} \sum_{i=1}^{20} (X_i - \mu)^2 \leq 1.76\sigma^2\right) \\
&= P\left(7.4 \leq \sum_{i=1}^{20} \left(\frac{X_i - \mu}{\sigma}\right)^2 \leq 35.2\right) \\
&= P\left(\sum_{i=1}^{20} \left(\frac{X_i - \mu}{\sigma}\right)^2 \geq 7.4\right) - P\left(\sum_{i=1}^{20} \left(\frac{X_i - \mu}{\sigma}\right)^2 \geq 35.2\right) \\
&= 0.995 - 0.025 = 0.97
\end{aligned}$$

**例5** 设随机变量 $X$ 与 $Y$ 相互独立,  $X \sim N(0,16)$ ,  $Y \sim N(0,9)$ ,  $X_1, X_2, \dots, X_9$ 与 $Y_1, Y_2, \dots, Y_{16}$ 分别是取自 $X$ 与 $Y$ 的简单随机样本, 求统计量

$$\frac{X_1 + X_2 + \dots + X_9}{\sqrt{Y_1^2 + Y_2^2 + \dots + Y_{16}^2}}$$

所服从的分布

**解**  $X \sim N(0,16)$ ,  $X_i \sim N(0,16)$

$$X_1 + X_2 + \dots + X_9 \sim N(0, 9 \times 16)$$

$$\frac{1}{3 \times 4} (X_1 + X_2 + \dots + X_9) \sim N(0, 1)$$

$$Y \sim N(0,9), \quad Y_i \sim N(0,9)$$

$$\frac{1}{3}Y_i \sim N(0,1) \quad , i = 1, 2, \dots, 16$$

$$\left(\frac{1}{3}Y_i\right)^2 \sim \chi^2(1) \quad \sum_{i=1}^{16} \left(\frac{1}{3}Y_i\right)^2 \sim \chi^2(16)$$

从而

$$\frac{\frac{1}{3 \times 4} (X_1 + X_2 + \dots + X_9)}{\sqrt{\frac{\sum_{i=1}^{16} \left(\frac{1}{3}Y_i\right)^2}{16}}} \sim t(16)$$

即

$$\frac{X_1 + X_2 + \dots + X_9}{\sqrt{Y_1^2 + Y_2^2 + \dots + Y_{16}^2}} \sim t(16)$$

**例6** 设总体  $X \sim N(0,1)$ ,  $X_1, X_2, \dots, X_6$  为总体  $X$  的样本,  $Y = (X_1 + X_2 + X_3)^2 + (X_4 + X_5 + X_6)^2$  试确定常数  $c$  使  $cY$  服从  $\chi^2$  分布.

**解**  $X_1 + X_2 + X_3 \sim N(0,3), \quad X_4 + X_5 + X_6 \sim N(0,3)$

$$\frac{1}{\sqrt{3}}(X_1 + X_2 + X_3) \sim N(0,1), \quad X \sim N(0,1)$$

$$\frac{1}{\sqrt{3}}(X_4 + X_5 + X_6) \sim N(0,1)$$

$$\text{故} \quad \left[ \frac{1}{\sqrt{3}}(X_1 + X_2 + X_3) \right]^2 + \left[ \frac{1}{\sqrt{3}}(X_4 + X_5 + X_6) \right]^2$$

$$= \frac{1}{3}Y \sim \chi^2(2)$$

$$\text{因此} \quad c = \frac{1}{3}$$

**例7**  $X_1, X_2, \dots, X_n$ 是来自正态总体 $N(\mu, \sigma^2)$ 的简单随机样本,  $\bar{X}$ 是样本均值,

$$S_1^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2, \quad S_2^2 = \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2,$$

$$S_3^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \mu)^2, \quad S_4^2 = \frac{1}{n} \sum_{i=1}^n (X_i - \mu)^2,$$

则服从自由度为  $n - 1$  的  $t$  分布的随机变量为:

$$(A) \frac{\bar{X} - \mu}{S_1} \sqrt{n-1} \quad (B) \frac{\bar{X} - \mu}{S_2} \sqrt{n-1}$$

$$(C) \frac{\bar{X} - \mu}{S_3} \sqrt{n} \quad (D) \frac{\bar{X} - \mu}{S_4} \sqrt{n}$$

**解**  $\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1) \quad \frac{1}{\sigma^2} \sum_{i=1}^n (X_i - \bar{X})^2 \sim \chi^2(n-1)$

$$\frac{\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}}}{\sqrt{\frac{\frac{1}{\sigma^2} \sum_{i=1}^n (X_i - \bar{X})^2}{n-1}}} = \frac{\sqrt{n(n-1)}(\bar{X} - \mu)}{\sqrt{\sum_{i=1}^n (X_i - \bar{X})^2}} \sim t(n-1)$$

故应选(B)