概率论与数理统计

随机事件独立性(复习)

A, B独立：$P(A \cap B) = P(A)P(B)$

注意:

$$E, F \text{ 与独立 } A,\ E \subset F \Longrightarrow F \setminus E \text{与独立 } A$$

$$\forall E, F \Longrightarrow (F \setminus E) \cap A = (F \cap A) \setminus (E \cap A)$$

随机变量独立性

$$\{a < X < b\}, \{c < Y < d\}\text{相互独立}, \forall a, b, c, d$$

$$\Longleftrightarrow$$

$$\{a < X \leqslant b\}, \{c < Y \leqslant d\}\text{相互独立}, \forall a, b, c, d$$

$$(x, y) = \bigcup_{n \geqslant 1}^{\infty} \left(x, y - \frac{1}{n}\right], (x, y] = \bigcap_{n \geqslant 1}^{\infty} \left(x, y + \frac{1}{n}\right)$$

同理 $\Longleftrightarrow$

★

$$\{-\infty < X \leqslant b\}, \{-\infty < Y \leqslant d\}\text{相互独立}, \forall b, d$$

随机变量独立性

★ “$\Longleftarrow$”亦可利用分布函数的性质

$$
\begin{aligned}
P(a < X \leqslant b, c < Y \leqslant d) &= F(b,d) - F(a,d) - F(b,c) + F(a,c) \\
&= F_X(b)F_Y(d) - F_X(a)F_Y(d) - F_X(b)F_Y(c) + F_X(a)F_Y(c) \\
&= (F_X(b) - F_X(a))(F_Y(d) - F_Y(c))
\end{aligned}
$$

独立性等价条件

$$X \sim F_X(x),\ Y \sim F_Y(y)$$
$$F(x,y) = F_X(x)F_Y(y),\ \forall x,y$$

离散型：

$$X \in \{x_1, x_2, \cdots, \},\ Y \in \{y_1, y_2, \cdots, \}$$
$$P(X = x_i, Y = y_j) = P(X = x_i)P(Y = y_j),\ \forall i,j$$

连续型：

$$X \sim p_X(x),\ Y \sim p_Y(y)$$
$$p(x,y) = p_X(x)p_Y(y),\ a.e.\ (x,y)$$

正态分布，边际与条件分布

$$p(x,y) = \frac{1}{2\pi\sigma_1\sigma_2(1-\rho^2)^{1/2}} \cdot$$

$$\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right) + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]\right\}$$

$$
\begin{aligned}
p_X(x) &= \int_{-\infty}^{+\infty} p(x,y)\mathrm{d}y \\
&= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left(\frac{x-\mu_1}{\sigma_1}\right)^2\right\} \\
&\quad \cdot \int_{-\infty}^{+\infty} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{y-\mu_2}{\sigma_2}\right)^2\right.\right. \\
&\qquad \left.\left. - 2\rho\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2}\right]\right\}\mathrm{d}y.
\end{aligned}
$$

作变量替换 $t=\dfrac{y-\mu_2}{\sigma_2}$，得

$$
p_X(x) = \frac{1}{2\pi\sigma_1\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left(\frac{x-\mu_1}{\sigma_1}\right)^2\right\} \cdot A,
$$

其中

$$
A = \int_{-\infty}^{+\infty} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[t^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)t\right]\right\}\mathrm{d}t
$$

$$A=\int_{-\infty}^{+\infty}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[t^2-2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)t\right]\right\}\mathrm{d}t$$

$$=\int_{-\infty}^{+\infty}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(t-\rho\frac{x-\mu_1}{\sigma_1}\right)^2-\rho^2\left(\frac{x-\mu_1}{\sigma_1}\right)^2\right]\right\}\mathrm{d}t$$

$$=\exp\left\{\frac{\rho^2}{2(1-\rho^2)}\left(\frac{x-\mu_1}{\sigma_1}\right)^2\right\}$$

$$\cdot\int_{-\infty}^{+\infty}\exp\left\{-\frac{1}{2(1-\rho^2)}\left(t-\rho\frac{x-\mu_1}{\sigma_1}\right)^2\right\}\mathrm{d}t$$

$$=\exp\left\{\frac{\rho^2}{2(1-\rho^2)}\left(\frac{x-\mu_1}{\sigma_1}\right)^2\right\}\sqrt{2\pi(1-\rho^2)}.$$

于是 $$p_X(x)=\frac{1}{\sqrt{2\pi}\sigma_1}\exp\left\{-\frac{(x-\mu_1)^2}{2\sigma_1^2}\right\}.$$

$$p(x,y) = \frac{1}{2\pi\sigma_1\sigma_2(1-\rho^2)^{1/2}} \cdot$$

$$\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2\right]\right\} \cdot \exp\left\{\frac{\rho^2}{2(1-\rho^2)}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2\right]\right\} \cdot$$

$$\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{y-\mu_2}{\sigma_2} - \rho\frac{x-\mu_1}{\sigma_1}\right)^2\right]\right\}$$

$$\exp\left\{-\frac{1}{2}\left[\left(\frac{y-m}{\sigma_2(1-\rho^2)^{1/2}}\right)^2\right]\right\}$$

$$m = \mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x-\mu_1)$$