SBM for reconstructed network

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1 Introduction

2 Model

Data

- p nodes = species ($1 \leqslant i, j \leqslant n$)
- K clusters ($1 \leqslant k, \ell \leqslant K$)
- Z_i = cluster of node i, $Z_{ik} = \mathbb{I}\{Z_i = k\}$, $Z = (Z_i)$
- $G_{ij} = \mathbb{I}\{i \sim k\}$ = connection between nodes i and j, $G = (G_{ij})$ = unobserved network
- S_{ij} = score of edge between nodes i and j, $S = (S_{ij})$ = observed score matrix

Parameters

- $\pi = (\pi_k)$ = cluster proportions
- $\gamma = (\gamma_{k\ell})$ = between cluster connection probabilities
- ψ_0 = parameter of the score distribution for absent edge $p(S_{ij} \mid G_{ij} = 0)$ (idem ψ_1 for present edge), $\psi = (\psi_0, \psi_1)$
- $\theta = (\pi, \gamma, \psi)$

Model

- (Z_i) iid,
$$Z_i \sim \mathcal{M}(1, \pi)$$
- (G_{ij}) independent conditionally on Z,
$$(G_{ij} \mid Z_i = k, Z_j = \ell) \sim \mathcal{B}(\gamma_{k\ell})$$
- (S_{ij}) independent conditionally on G,
$$(S_{ij} \mid G_{ij} = u) \sim F(\cdot; \psi_u), \qquad u = 0, 1$$

We further denote $F_u(\cdot) = F(\cdot; \psi_u)$ and $f_u(\cdot)$ the corresponding pdf.

Properties and definitions

- S and Z independent conditionally on G :
$$p(Z \mid G, S) = p(Z \mid G), \qquad p(S \mid G, Z) = p(S \mid G)$$
- Distribution of G_{ij}
$$P(G_{ij} = 1 \mid S_{ij}, Z_i = k, Z_j = \ell) = \frac{\gamma_{k\ell} f_1(S_{ij})}{\gamma_{k\ell} f_1(S_{ij}) + (1 - \gamma_{k\ell}) f_0(S_{ij})} =: \eta_{ij}^{k\ell}$$
$$\widetilde{P}(G_{ij} = 1 \mid S_{ij}) = \sum_{k,\ell} \tau_{ik} \tau_{j\ell} \eta_{ij}^{k\ell} =: \overline{\eta}_{ij}$$
- Kullback-Leibler divergence
$$\begin{aligned} KL(q(U); p(U)) &= \mathbb{E}_q(\log q(U) - \log p(U)) \\ KL(q(U,V); p(U,V)) &= \mathbb{E}_{q(U,V)}(\log q(U) + \log(q(V \mid U) - \log p(U) - \log p(V \mid U) \\ &= KL(q(U); p(U)) + \mathbb{E}_{q(U)} KL(q(V \mid U), p(V \mid U)) \end{aligned}$$

3 Inference

3.1 Loss function

Log-likelihood

$$\begin{aligned}
\log p(Z,G,S) &= \log p(Z;\pi) + \log p(G \mid Z;\gamma) + \log p(S \mid G;\psi) \\
&= \sum_{i,k} Z_{ik} \log \pi_k + \sum_{i<j} \sum_{k,\ell} Z_{ik} Z_{j\ell} \left(G_{ij} \log \gamma_{k\ell} + (1-G_{ij}) \log(1-\gamma_{k\ell})\right) \\
&\quad + \sum_{i<j} G_{ij} \log f_1(S_{ij}) + (1-G_{ij}) \log f_0(S_{ij})
\end{aligned}$$

Approximate distribution $q(Z,G) \approx p(Z,G \mid S)$

$$q(Z,G) = q(Z)q(G \mid Z) := q(Z)p(G \mid Z,S) \tag{1}$$

where

$$p(G \mid Z,S) = \prod_{i,j} p(G_{ij} \mid Z_i, Z_j, S_{ij})$$

and

$$q(Z) = \prod_i q_i(Z_i) = \prod_{i,k} \tau_{ik}^{Z_{ik}}.$$

Divergence $KL(q(Z,G); p(Z,G \mid S))$

$$\begin{aligned}
KL(q(Z,G); p(Z,G \mid S)) &= KL(q(Z)p(G \mid Z,S); p(Z \mid S)p(G \mid Z,S)) \\
&= KL(q(Z); p(Z \mid S)) + \mathbb{E}_{q(Z)} \underbrace{KL(p(G \mid Z,S); p(G \mid Z,S))}_{=0}
\end{aligned}$$

Still, the conditional entropy of $q(G \mid Z)$ contributes to the lower bound.

Entropy

$$\mathcal{H}(q(G,Z)) = \mathbb{E}_{q(Z)}[\mathcal{H}p(G|Z,Y)] + \mathcal{H}(q(Z)) \tag{2}$$

$$= -\sum_{i,k} \tau_{ik} \log \tau_{ik} - \sum_{ijk\ell} \tau_{ik}\tau_{j\ell} \left[\eta_{ij}^{k\ell} \log(\eta_{ij}^{k\ell}) + (-1\eta_{ij}^{k\ell}) \log(1-\eta_{ij}^{k\ell})\right] \tag{3}$$

Lower bound $J(\theta, q)$

$$\begin{aligned}
J(\theta,q) &= \log p_\theta(S) - KL(q(Z,G); p(Z,G \mid S)) \\
&= \mathbb{E}_q \log p_\theta(Z,G,S) + H(q(Z)) + \mathbb{E}_q H(q(G \mid Z)) \qquad (4) \\
&= \sum_{i,k} \tau_{ik} \log \pi_k + \sum_{i<j} \sum_{k,\ell} \tau_{ik}\tau_{j\ell} \left(\eta_{ij}^{k\ell} \log \gamma_{k\ell} + (1-\eta_{ij}^{k\ell}) \log(1-\gamma_{k\ell})\right) \\
&\quad + \sum_{i<j} \sum_{k,\ell} \tau_{ik}\tau_{j\ell} \left(\eta_{ij}^{k\ell} \log f_1(S_{ij}) + (1-\eta_{ij}^{k\ell}) \log f_0(S_{ij})\right) \\
&\quad - \sum_{i,k} \tau_{ik} \log \tau_{ik} - \sum_{i<j} \sum_{k,\ell} \tau_{ik}\tau_{j\ell} \left(\eta_{ij}^{k\ell} \log \eta_{ij}^{k\ell} + (1-\eta_{ij}^{k\ell}) \log(1-\eta_{ij}^{k\ell})\right) \qquad (5)
\end{aligned}$$

3.2 Estimation equation

We set

$$q_{\tau,\eta}(Z,G) = \prod_{i=1}^{n} \prod_{k=1}^{K} \tau_{ik}^{Z_{ik}} \prod_{i<j} \prod_{k,\ell} \eta_{ijk\ell}^{Z_{ik}Z_{j\ell}G_{ij}} (1-\eta_{ijk\ell})^{Z_{ik}Z_{j\ell}(1-G_{ij})}$$

where $\eta_{ijkl} = P_q(G_{ij} = 1|Z_i = k, Z_j = \ell)$ We define

$$\begin{aligned} J(\theta, q_{\tau,\eta}) &= \log p_\theta(S) - KL(q_{\tau,\eta}(Z,G); p_\theta(Z,G \mid S)) \\ &= \mathbb{E}_{q_{\tau,\eta}}[\log p_\theta(Z,G,S)] + \mathcal{H}(q_\tau(Z)) + \mathbb{E}_{q_\tau}\mathcal{H}(q_\eta(G \mid Z)) \end{aligned}$$

Iteration (t) the EM is as follows : from a current value of $\theta^{(t-1)}$

- *(V)E-step*

$$(\tau^{(t)}, \eta^{(t)}) = \arg\max_{\tau,\eta} J(\theta^{(t-1)}, q_{\tau,\eta})$$

- *M-step*

$$\theta^{(t)} = \arg\max_{\theta} J(\theta, q_{\tau^{(t)},\eta^{(t)}})$$

VE step

$$\begin{aligned} (\hat{\tau}, \hat{\eta}) &= \arg\max_{\tau,\eta} J(\theta, q_{\tau,\eta}) \\ &= \arg\min_{\tau,\eta} KL(q_{\tau,\eta}(Z,G); p_\theta(Z,G \mid S)) \\ &= \arg\min_{\tau,\eta} \left\{ KL(q_\tau(Z); p_\theta(Z \mid S)) + \mathbb{E}_{q_\tau(Z)}\left[KL(q_\eta(G \mid Z,S); p_\theta(G \mid Z,S))\right] \right\} \end{aligned}$$

- So from the previous equation, we have

$$\hat{\eta} = \arg\min_{\eta} \mathbb{E}_{q_\tau(Z)}\left[KL(q_\eta(G \mid Z,S); p_\theta(G \mid Z,S))\right] \quad (6)$$

  Using the independencies, we have

$$\hat{\eta}_{ij\cdot\cdot} = \arg\min_{\eta_{ij\cdot\cdot}} \mathbb{E}_{q_\tau(Z)} KL(q_\eta(G_{ij} \mid Z_i, Z_j); p_\theta(G_{ij} \mid Z_i, Z_j, S_{ij})) \quad (7)$$

  $KL(q_\eta(G_{ij} \mid Z_i, Z_j); p_\theta(G_{ij} \mid Z_i, Z_j, S_{ij}))$ is minimal $(= 0)$ for

$$\eta_{ijk\ell} = P_\theta(G_{ij} = 1|Z_i = k, Z_j = l, S_{ij}) = \eta_{ij}^{k\ell}$$

  Morover, for i, j, k, l

$$P_\theta(G_{ij} = 1|Z_i = k, Z_j = l, S_{ij}) = \frac{\gamma_{k\ell} f_1(S_{ij})}{\gamma_{k\ell} f_1(S_{ij}) + (1-\gamma_{k\ell}) f_0(S_{ij})}$$

  In that case

$$KL(q_\eta(G \mid Z); p(G \mid Z,S)) = 0$$

  and so

$$\mathbb{E}_{q_\tau(Z)}\left[KL(q_\eta(G_{ij} \mid Z_i, Z_j, S_{ij}); p_\theta(G_{ij} \mid Z_i, Z_j, S))\right] = 0$$

  (minimal) It does not depend on τ. So it can be done before optimizing in τ.
- Now for fixed η we will minimize $J(\theta, q_{\tau,\eta})$ by a fixed point equation.

Denoting

$$\log A_{ijk\ell} = \eta_{ij}^{k\ell}\left(\log\gamma_{k\ell} + \log f_1(S_{ij}) - \log\eta_{ij}^{k\ell}\right) + (1-\eta_{ij}^{k\ell})\left(\log(1-\gamma_{k\ell}) + \log f_0(S_{ij}) - \log(1-\eta_{ij}^{k\ell})\right)$$

Then the lower bound is :

$$J(\theta, \eta, \tau) = \sum_{i,k} \tau_{ik}\log\pi_k + \sum_{i<j}\sum_{k,\ell} \tau_{ik}\tau_{j\ell}\log A_{ijkl} - \sum_{i,k}\tau_{ik}\log\tau_{ik}$$

setting the derivative wrt τ_{ik} to zero with the contraint $\sum_k \tau_{ik} = 0$ gives

$$\log\tau_{ik} = \log\pi_k + \sum_{j,\ell}\tau_{j\ell}\log A_{ijk\ell} + \text{cst} \qquad \Leftrightarrow \qquad \tau_{ik} \propto \pi_k \prod_{j,\ell}(A_{ijk\ell})^{\tau_{j\ell}}$$

M step Setting the derivative wrt to each parameter gives

$$\widehat{\pi}_{ik} = \sum_i \tau_{ik} / n \; , \qquad \widehat{\gamma}_{k\ell} = \sum_{i<j} \sum_{k,\ell} \tau_{ik}\tau_{j\ell}\eta_{ij}^{k\ell} \Bigg/ \sum_{i<j} \sum_{k,\ell} \tau_{ik}\tau_{j\ell} \; .$$

Furthermore, if $f(\cdot, \psi_u) = \mathcal{N}(\cdot, \mu_u, \sigma_u^2)$ (i.e $\psi_u = (\mu_u, \sigma_u^2)$),

$$\widehat{\mu}_0 = \sum_{i<j} (1 - \overline{\eta}_{ij}) S_{ij} \Bigg/ \sum_{i<j} (1 - \overline{\eta}_{ij}) \qquad \widehat{\sigma}_0^2 = \sum_{i<j} (1 - \overline{\eta}_{ij})(S_{ij} - \widehat{\mu}_0)^2 \Bigg/ \sum_{i<j} (1 - \overline{\eta}_{ij})$$

$$\widehat{\mu}_1 = \sum_{i<j} \overline{\eta}_{ij} S_{ij} \Bigg/ \sum_{i<j} \overline{\eta}_{ij} S_{ij} \qquad \widehat{\sigma}_1^2 = \sum_{i<j} \overline{\eta}_{ij} (S_{ij} - \widehat{\mu}_0)^2 \Bigg/ \sum_{i<j} \overline{\eta}_{ij} S_{ij}$$

The case of non-parametric version of f_0 and f_1 is considered in Apprendix A.1

By-product The conditional probability for an edge to be part of G is denoted ψ_{ij}^1 :

$$\psi_{ij}^1 := \widetilde{P}\{G_{ij} = 1\} = \sum_{k,\ell} \tau_{ik}\tau_{j\ell}\eta_{ij}^{k\ell}$$

and we denote $\psi_{ij}^0 = 1 - \psi_{ij}^1$.

3.3 Model selection

Penalty for symetric networks and d scores with multivariate Gaussian distirbutions.

$$Pen(\mathcal{M}) = -\frac{1}{2} \left[(K-1) \log p + \left(\frac{K(K+1)}{2} + 2d + 2\frac{d(d+1)}{2} \right) \log \left(\frac{p(p-1)}{2} \right) \right]$$

4 Identifiability

4.1 Review of the literature

Notes on identifiability based on papers :

— [?] : "Allman, Elizabeth S. and Matias, Catherine and Rhodes, John A." : *Identifiability of parameters in latent structure models with many observed variables*

— [?] "Allman, Elizabeth S. and Matias, Catherine and Rhodes, John A." : *Parameter identifiability in a class of random graph mixture models*

— [?]"Teicher, Henry" : *Identifiability of Finite Mixtures*

— [?] "Teicher, Henry" : *Identifiability of Mixtures of product measures*

What is done in [?] : identifiability in weighted SBM

$$\begin{array}{rcl} S_{ij}|Z_i = k, Z_j = \ell & \sim & \mu_{k\ell} \\ \mu_{k\ell} & = & (1-\gamma_{k\ell})\delta_{\{0\}} + +\gamma_{kl}F_{k\ell}(\cdot) \end{array}$$

for uni dimensional S and symmetric with

— $F_{k\ell}(\cdot)$ parametric (Theorem 12 of [?]) : $F(\cdot;\theta_{k\ell})$ under the following assumptions :

[A1] The $K(K+1)/2$ parameter values $\theta_{k\ell}$ are distinct

[A2] The family of measures $\mathcal{M} = \{F(\cdot;\theta)|\theta \in \Theta\}$ is such that

[A2 (i)] all elements of $\mathcal{M}$ have no point mass at 0

[A2 (ii)] the parameters of finite mixtures of measures of $\mathcal{M}$ are identifiable (up to label switching) i.e.

$$\sum_{m=1}^{M} \alpha_m F(\cdots,\theta_m) = \sum_{m=1}^{M} \alpha'_m F(\cdots,\theta'_m) \Rightarrow \sum_{m=1}^{M} \alpha_m \delta_{\theta_m} = \sum_{m=1}^{M} \alpha'_m \delta_{\theta'_m}$$

In particular : true for Gaussian ([?]) and Laplace.

— $F_{k\ell}(\cdot)$ non-parametric (Theorem 14 of [?]) : if the $\mu_{k\ell}$ are *linearly independent* (to be detailed)

About the demonstrations

— *Parametric case* It is done from the distribution of a triplet (S_{ij}, S_{ik}, S_{jk}) and using [?]. How to adapt it to our case ?

— *Nonparametric case* : only depends on the linear independancy of the $\mu_{k\ell}$. We have to precise it for our case ?

4.2 Proof in the parametric uni-dimensional context

I tried to mimic/extend the proof of [?] but I don't think we are in the same scope.

Distribution of the S_{ij}

$$\begin{aligned} \mathbb{P}(S_{ij}) &= \sum_{q,\ell} \pi_q\pi_\ell[(1-\gamma_{q\ell})F_0(S_{ij}) + \gamma_{q\ell}F_1(S_{ij})] \\ &= \left[1 - \sum_{q,\ell} \pi_\ell\pi_q\gamma_{q,\ell}\right] F_0(S_{ij}) + \left[\sum_{q,\ell} \pi_q\pi_\ell\gamma_{q,\ell}\right] F_1(S_{ij}) \end{aligned}$$

So **assuming that F_0 and F_1 are such that any mixture of those two distributions is identifiable**, we obtain the identifiability of θ_0, θ_1 and $\sum_{q,\ell} \pi_\ell\pi_q\gamma_{q,\ell}$.

So we have identifiability of $\pi'\gamma\pi$. It seems to me that once we have identified θ_0 and θ_1 we will be able to apply to proof of Célisse & al. [?], which is the one I know better. Which is the thing you said : meaning that once we have identified to high level, we are identifiable just like any binary SBM.

Distribution of the triplet (S_{ij}, S_{ik}, S_{jk})

$$
\begin{aligned}
\mathbb{P}(S_{ij}, S_{ik}, S_{jk}) &= \sum_{q,\ell,m} \pi_q\pi_\ell\pi_m[(1-\gamma_{q\ell})F_0(S_{ij}) + \gamma_{q\ell}F_1(S_{ij})][(1-\gamma_{qm})F_0(S_{ik}) + \gamma_{qm}F_1(S_{ik})] \\
&\quad [(1-\gamma_{\ell m})F_0(S_{jk}) + \gamma_{\ell m}F_1(S_{jk})] \\
&= \sum_{q,\ell,m} \sum_{(u,v,w)\in\{0,1\}^3} \eta_{q,\ell,m,u,v} F_u(S_{ij})F_v(S_{ik})F_w(S_{jk}) \\
&= \sum_{(u,v,w)\in\{0,1\}^3} \left(\sum_{q,\ell,m} \eta_{q,\ell,m,u,v}\right) F_u(S_{ij})F_v(S_{ik})F_w(S_{jk}) \\
&= \sum_{(u,v,w)\in\{0,1\}^3} \left(\sum_{q,\ell,m} \eta_{q,\ell,m,u,v}\right) F_{u,v,w}(S_{ij}, S_{ik}, S_{jk})
\end{aligned}
$$

with

$$
\eta_{q,\ell,m,u,v} = \pi_q\pi_\ell\pi_m(1-\gamma_{q\ell})^{1-u}\gamma_{q\ell}^u(1-\gamma_{q\ell})^{1-u}\gamma_{q\ell}^u(1-\gamma_{qm})^{1-v}\gamma_{qm}^v(1-\gamma_{\ell m})^{1-w}\gamma_{\ell m}^w.
$$

The distribution of (S_{ij}, S_{ik}, S_{jk}) is a mixture (weights $= \sum_{q,\ell,m}\eta_{q,\ell,m,u,v}$) of the following distributions

$$
F(s) = F_u(s_1,\theta_u)F_v(s_1,\theta_v)F_w(s_1,\theta_w)
$$

where $F \in \mathcal{F}$ with

$$
\mathcal{F} = \{F(s;\theta_0,\theta_1) : F(s;\theta_0,\theta_1) = F_u(s_1,\theta_u), F_v(s_2,\theta_v)F_w(s_3,\theta_w), (u,v,w) \in \{0,1\}^3, \theta_0, \in \Theta_0, \theta_1 \in \Theta_1\}
$$

Asumptions; [A1] we assume that any mixtures of elements of $\mathcal{F}$ is identifiable. (to develop to get assumptions on F_0 and F_1).

The, under assumption [A1], we have :

Then using Theorem 1 of [?] we have the identifiability of any mixture of the product measures.

4.3 Notes from the 20/11/19

Preliminary remarks. Let $nSBM(\pi,\gamma,F_0,F_1)$ denote the noisy SBM model and $SBM(\pi,\alpha)$ the standard binary SBM. [?] showed that $SBM(\pi,\alpha)$ is identifiable provided that all $\overline{\alpha}_k = \sum_\ell \pi_\ell\alpha_{k\ell}$ are different.

Lemma 1. *Let $S \sim nSBM(\pi,\gamma,F_0,F_1)$ and define $B_{ij} = \mathbb{I}\{S_{ij} \leqslant t\}$. We have that*

$$
B(t) := [B_{ij}(t)] \sim SBM(\pi,\alpha(t))
$$

where, denoting $\Delta(t) = F_1(t) - F_0(t)$,

$$
\alpha_{k\ell}(t) = \gamma_{k\ell}F_1(t) + (1-\gamma_{k\ell})F_0(t) = F_0(t) + \gamma_{k\ell}\Delta(t).
$$

Lemma 2. *If the model $SBM(\pi,\gamma)$ is identifiable then the model $SBM(\pi,\alpha(t))$ is identifiable as soon as $\Delta F(t) \neq 0$.*

The proof follows : the identifiability of $SBM(\pi,\gamma)$ means that all $\overline{\gamma}_k = \sum_\ell \pi_\ell\gamma_{k\ell}$ are different, so because all

$$
\overline{\alpha}_k(t) = \sum_\ell \pi_\ell\alpha_{k\ell}(t) = F_0(t) + \overline{\gamma}_k\Delta F(t)
$$

are different as soon as $\Delta F(t) \neq 0$.

Parametric case Suppose that F_0 and F_1 belong to a same parametric family, the mixture of which are identifiable. If $S \sim nSBM(\pi,\gamma,F_0,F_1)$, then the marginal distribution is the mixture

$$
S_{ij} \sim \overline{\overline{\gamma}}F_1 + (1-\overline{\overline{\gamma}})F_0,
$$

which is identifiable so $\overline{\overline{\gamma}}$, F_0 and F_1 are identifiable.

Assuming that $SBM(\pi,\gamma)$ is identifiable, we use Lemma 2, picking a threshold t such that $F_0(t) \neq F_1(t)$, to prove the identifiability of π and $\alpha(t)$. A VERIFIER : γ can then be retrieved by solving the $K(K+1)/2$ equations relating each $\gamma_{k\ell}$ with each $\alpha_{k\ell}(t)$.

Non-parametric case Our aim is to show that if $nSBM(\pi,\gamma,F_0,F_1)$ and $nSBM(\pi',\gamma',F_0',F_1')$ yields the same distribution, then necessarily, $\pi=\pi'$, $\gamma=\gamma'$, $F_0=F_0'$, $F_1=F_1'$. We assume that $F_1>F_0$, so that $\Delta(t)\neq 0$ for all t. If we further assume that $SBM(\pi,\gamma)$ is identifiable, Lemma 2 ensures the identifiability of $SBM(\pi,\alpha(t))$ for all t.

So assume that, any $t\in\mathbb{R}$, $\alpha(t)=\alpha'(t)$, then,

$$\begin{array}{lrcl} & (1-\gamma_{k\ell})F_0(t)+\gamma_{k\ell}F_1(t) &=& (1-\gamma'_{k\ell})F_0'(t)+\gamma'_{k\ell}F_1'(t)\\ \Leftrightarrow & F_0(t)+\gamma_{k\ell}\Delta(t) &=& F_0'(t)+\gamma'_{k\ell}\Delta'(t)\\ \text{where} & \Delta(t) &=& F_1(t)-F_0(t)\end{array}$$

This equality is true for any (k,ℓ,t).

As a consequence

$$\begin{aligned}\gamma_{k\ell} &= \frac{F_0'(t)-F_0(t)}{\Delta(t)}+\gamma'_{k\ell}\frac{\Delta'(t)}{\Delta(t)}\\ &= A(t)+\gamma'_{k\ell}B(t),\quad \forall t\in\mathbb{R}\end{aligned}$$

(we used the fact that $\forall t\in\mathbb{R}$, $\Delta(t)=F_1(t)-F_0(t)\neq 0$.)

Let us consider two pairs (k,ℓ) and (k',ℓ'), we have

$$\begin{aligned}\gamma_{k\ell} &= A(t)+\gamma'_{k\ell}B(t)\\ \gamma_{k'\ell'} &= A(t)+\gamma'_{k'\ell'}B(t)\end{aligned}$$

So, if $\gamma'_{k\ell}\neq\gamma'_{k'\ell'}$

$$B(t)=\frac{\gamma_{k\ell}-\gamma_{k'\ell'}}{\gamma'_{k\ell}-\gamma'_{k'\ell'}}$$

So $B(t)$ is a constant function : $B(t)=\frac{\Delta'(t)}{\Delta(t)}=\frac{F_1'(t)-F_0'(t)}{F_1(t)-F_0(t)}=B$ and $B>0$.

Moreover we get

$$\gamma_{k\ell}-\gamma_{k'\ell'}=B(\gamma'_{k\ell}-\gamma'_{k'\ell'}),\quad \forall(k,\ell,k',\ell').$$

Hence

$$F_1(t)-F_0(t)=B(F_1'(t)-F_0'(t))$$

So

$$F_0(t)=F_1(t)-BF_1'(t)+BF_0'(t)$$

Since $t\mapsto B(t)$ is constant then $t\mapsto A(t)$ is also a constant. So

$$A(t)=\frac{F_0'(t)-F_0(t)}{\Delta(t)}=A$$

so

$$\begin{aligned}F_0'(t) &= F_0(t)+A(F_1(t)-F_0(t))\\ F_0'(t) &= (1-A)F_0(t)+AF_1(t)\\ F_0(t) &= \frac{1}{1-A}F_0'(t)-\frac{A}{1-A}F_1(t)\\ F_0(t) &= BF_0'(t)+F_1(t)-BF_1'(t)\end{aligned}$$

As a consequence,

$$\begin{aligned}\frac{1}{1-A}F_0'(t)-\frac{A}{1-A}F_1(t)-BF_0'(t)-F_1(t)+BF_1'(t) &= 0\\ \frac{1}{1-A}F_0'(t)-\left(1+\frac{A}{1-A}\right)F_1(t)-BF_0'(t)+BF_1'(t) &= 0\\ \frac{1}{1-A}F_0'(t)-\frac{1}{1-A}F_1(t)-BF_0'(t)+BF_1'(t) &= 0\end{aligned}$$

So

$$
\begin{aligned}
F_1(t) &= (1-B(1-A))F_0'(t) + B(1-A)F_1'(t) \\
F_0(t) &= BF_0'(t) + (1-B(1-A))F_0'(t) + B(1-A)F_1'(t) - BF_1'(t) \\
&= (1-B(1-A)+B)F_0'(t) - ABF_1'(t) \\
&= (1+AB)F_0'(t) - ABF_1'(t) \\
&= F_0'(t) - AB(F_1'(t) - F_0'(t)) \\
F_0(t) - F_0'(t) &= AB(F_0'(t) - F_1'(t))
\end{aligned}
$$

A FINIR : il va falloir jouer sur le support des F_1 et F_0.

On a $F_1(t) < F_0(t)$ si on veut que en moyenne les valeurs sous F_1 soient plus grandes que celles sous F_0.

Si il existe τ tel que $F_0(\tau) = 1 = F_0'(\tau)$ alors en ce point, $1 = 1 - AB(F_1'(\tau) - F_0'(\tau))$ donc $AB(F_1'(\tau) - 1)$; Or $F_1'(\tau) < F_0'(\tau)$ donc $AB = 0$. Or $B > 0$ donc $A = 0$ Donc $F_0'(t) = F_0(t)$.

On peut peut-être seulement mettre des vitesses sur les queues de distributions et travailler en limite ?

5 Simulation study

5.1 Simulation design

Data simulation.

— $p = 20, 30, 50, 80$ nodes
— $n = 20, 50, 100, 200$ replicates
— $K = 3$ clusters
— $\pi = (1/6; 1/3, 1/2)$
— γ higher for smaller clusters, density $\overline{\gamma} = \pi^\intercal \gamma \pi = 1.5 \log(p)/p$
— $G \sim SBM(p, \pi, \gamma)$ conditional on G connected
— $\Omega = \text{Laplacian}(G)$ (+ increases the diagonal until positive-definite)
— $(Y_i)_{i=1\dots n}$ iid $\sim \mathcal{N}(0, \Omega^{-1})$

Inference methods.

oracle : SBM fit on (unobserved) G

vemGlasso : proposed VEM on glasso scores

vemMB : proposed VEM on M-B scores

vemTree : proposed VEM on tree-based edge probalities

sbmGlasso : pipe-line = SBM on $\widehat{G}_{glasso}$ (with $eBIC$ selection)

vemMB : pipe-line = SBM on $\widehat{G}_{MB}$ (with ric selection)

vemTree : pipe-line = SBM on $\widehat{G}_{Tree}$ (with edge proba $> 2/p$ selection)

6 Illustrations

A Appendix

A.1 Non-parametric emission distributions

Non-parametric estimates. Given a kernel function κ (s.t. $\int \kappa(x)\ \mathrm{d}x = 1$), we propose to estimate the conditional score pdf f_u ($u = 0, 1$) as

$$\widehat{f}_u(s) = \sum_{a<b} w^u_{ab} \kappa(s - S_{ab}), \qquad \text{with } \sum_{a<b} w^u_{ab} = 1.$$

For each $u = 0, 1$, the maximisation of the lower bound (4) wrt $w^u = (w^u_{ab})_{a<b}$ is equivalent to the maximization of

$$\sum_{i<j} h^u_{ij} \log \widehat{f}_u(S_{ij}) - \lambda^u \sum_{a<b} w^u_{ab}$$

with $h^1_{ij} = \sum_{k,\ell} \tau_{ik}\tau_{j\ell}\eta^{k\ell}_{ij}$ and $h^0_{ij} = \sum_{k,\ell} \tau_{ik}\tau_{j\ell}(1 - \eta^{k\ell}_{ij})$. The derivative wrt w^u_{ab} is zero when

$$\sum_{i<j} h^u_{ij} \frac{\kappa(S_{ij} - S_{ab})}{\widehat{f}_u(S_{ij})} - \lambda^u = 0,$$

which has no close form solution. However, close-form updates that increase the log-likelihood are provided in Propr. 3 of [**?**]. We may check if they still hold for the VEM lower bound.

Alternatively, a pragmatic, unjustified choice is to simply set $w^u_{ab} = \psi^u_{ab}$, that is to let each pair (a, b) contribute to the estimation f_1 (resp. f_0) proportionally to the probability for the edge G_{ab} to be equal to 1 (resp. 0).