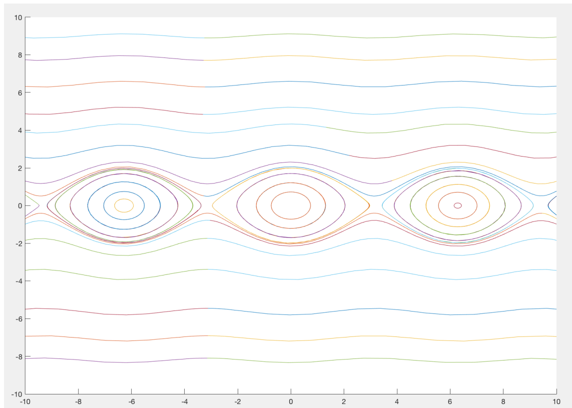
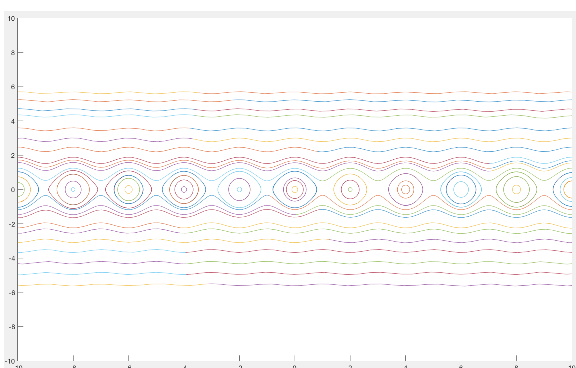
Application to Cherry's Problem (355) states that $\dot{x_1} = x_2, \quad \dot{x_2} = -\Phi(x)$, which is equivalent to $\ddot{x} + \Phi(x) = 0$.
First, we let $\Phi(x) = sin(x)$ and we obtain the phase portrait below:



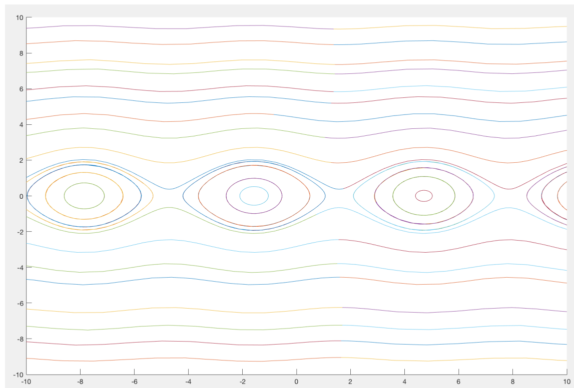
From the phase portrait, we can observe that every $(k\pi, 0), k \in \mathbb{Z}$, is a critical point. More over, every $(2k\pi, 0)$ is a center and $((2k+1)\pi, 0)$ is a saddle point. This is consistent with Theorem 4.1 (355) which states that the equation has an orbit $(\phi(x), \phi'(x))$ connecting saddle points $({u_1}^-, 0)$ and $({u_1}^+, 0)$.

Next, we let $\phi(x) = sin(2\pi x/T)$ where T is the period of the function to analyze the change in the critical points with the period of $\phi(x)$ changing. Letting $T = 2$, we get $\phi(x) = sin(\pi x)$ and it produces:



From this diagram, we can see that the critical points change depending on the period of the function. It seems that the center points lie at $(kT, 0), k \in \mathbb{Z}$, and the saddle points at $((2k+1)T/2, 0)$.

We now let $\phi(x) = cos(x)$. We then obtain:

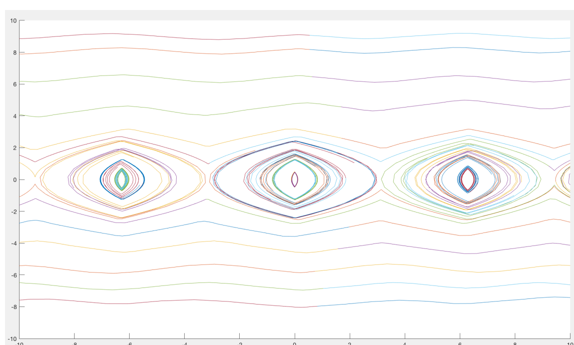


Comparing it to $\phi(x) = sin(x)$, we can see that the the phase portrait shifted $\pi/2$ in the negative direction. Since $cos(x) = sin(x - \pi/2)$, this makes sense.

We now choose a different T-periodic function, and see if it acts as we would expect it. We let $\phi(x)$ be defined piecewise as

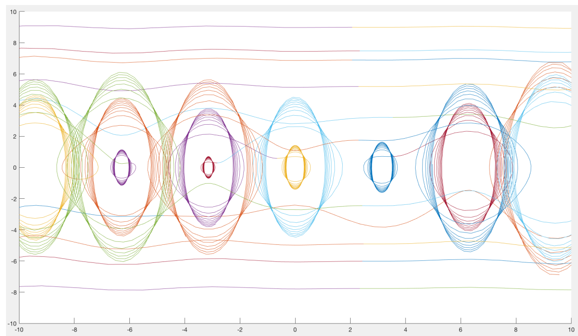
$$\phi(x) = \begin{cases} 1 & x \in (0, \pi) \\ -1 & x \in (\pi, 2\pi) \\ 0 & x = 0 \end{cases}$$

and $\phi(x + T) = \phi(x)$, with $T = 2\pi$ in this case.



Although this function is not continuous, it can still be transformed as a Fourier series. We see that the key characteristics of its critical points remain the same as when $\phi(x) = sin(x)$

We now look at system of equations $\dot{x_1} = x_2, \quad \dot{x_2} = -\lambda p(t)\Phi(x)$, which is equivalent to $\ddot{x} + \lambda p(t)\Phi(x) = 0$.
We let $\lambda p(t) = t$ and $\phi(x) = sin(x)$ and it produces the phase portrait below:



From this portrait, I'm not completely sure if the portrait I produced is correct, or if I made a mistake when adding a new function of t. It almost seems as though the graph should be visualized with 3 dimensions.