Kalman Filtering: A PGM derivation

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Abstract

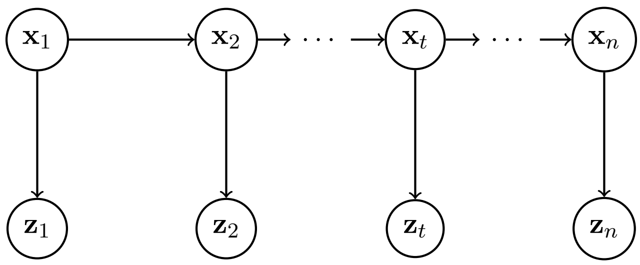

Figure 1.1: A Bayes Net for a linear Kalman filter.

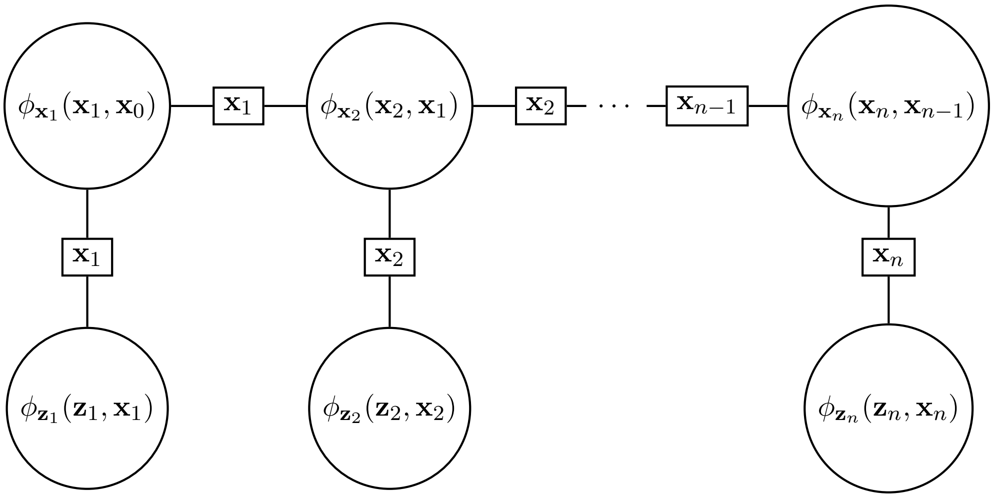

Figure 1.2: The junction tree resulting from the Bayes Net in Figure 1.1.

1. Linear Gaussian Systems

1.1 Bayes Net representation

1.2 Cluster Graph representation

1.2.1 Message Passing

Sum-Product Algorithm

Integral-Product Algorithm

2. Kalman Filtering

$$\begin{aligned}
\delta_{\phi_{\boldsymbol{x}_t} \to \phi_{\boldsymbol{x}_{t+1}}}(\boldsymbol{x}_t) &= \int \phi_{\boldsymbol{x}_t}(\boldsymbol{x}_t, \boldsymbol{x}_{t-1}) \delta_{\phi_{\boldsymbol{z}_t} \to \phi_{\boldsymbol{x}_t}}(\boldsymbol{x}_t) \delta_{\phi_{\boldsymbol{x}_{t-1}} \to \phi_{\boldsymbol{x}_t}}(\boldsymbol{x}_{t-1}) d\boldsymbol{x}_{t-1} \\
&= \underbrace{\delta_{\phi_{\boldsymbol{z}_t} \to \phi_{\boldsymbol{x}_t}}(\boldsymbol{x}_t)}_{\textit{Measurement update}} \underbrace{\int \phi_{\boldsymbol{x}_t}(\boldsymbol{x}_t, \boldsymbol{x}_{t-1}) \delta_{\phi_{\boldsymbol{x}_{t-1}} \to \phi_{\boldsymbol{x}_t}}(\boldsymbol{x}_{t-1}) d\boldsymbol{x}_{t-1}}_{\textit{Prediction}} \qquad (2.1)
\end{aligned}$$

2.1 Part 1: Prediction

$$\Psi(\boldsymbol{x}_t) = \int \phi_{\boldsymbol{x}_t}(\boldsymbol{x}_t, \boldsymbol{x}_{t-1}) \delta_{\phi_{\boldsymbol{x}_{t-1}} \to \phi_{\boldsymbol{x}_t}}(\boldsymbol{x}_{t-1}) d\boldsymbol{x}_{t-1} \qquad (2.2)$$

2.1.1 Representation in canonical form

The initial potential, $\phi_{\boldsymbol{x}_t}(\boldsymbol{x}_t, \boldsymbol{x}_{t-1})$

The potential, $\phi_{\boldsymbol{x}_t}(\boldsymbol{x}_t, \boldsymbol{x}_{t-1})$, is the CPD:

$$\begin{aligned}\phi_{\boldsymbol{x}_t}(\boldsymbol{x}_t, \boldsymbol{x}_{t-1}) &= \mathcal{N}\left(\boldsymbol{x}_t | A\boldsymbol{x}_{t-1} + B\boldsymbol{x}_{t-1}, R\right) \\ &= \frac{1}{|(2\pi)^n R|^{1/2}} \exp\{-\frac{1}{2}\left(\boldsymbol{x}_t - A\boldsymbol{x}_{t-1} - B\boldsymbol{x}_{t-1}\right)^T R^{-1}\left(\boldsymbol{x}_t - A\boldsymbol{x}_{t-1} - B\boldsymbol{x}_{t-1}\right)\}\end{aligned} \tag{2.3}$$

The CPD can be represented as a joint density function through the following rearrangement:

$$\begin{aligned}&-\frac{1}{2}\left(\boldsymbol{x}_t - A\boldsymbol{x}_{t-1} - B\boldsymbol{x}_{t-1}\right)^T R^{-1}\left(\boldsymbol{x}_t - A\boldsymbol{x}_{t-1} - B\boldsymbol{x}_{t-1}\right) \\ &= -\frac{1}{2}\begin{bmatrix}(\boldsymbol{x}_t - B\boldsymbol{u}_t)^T & \boldsymbol{x}_{t-1}^T\end{bmatrix}\begin{bmatrix}R^{-1} & -R^{-1}A \\ -A^T R^{-1} & A^T R^{-1} A\end{bmatrix}\begin{bmatrix}(\boldsymbol{x}_t - B\boldsymbol{u}_t) \\ \boldsymbol{x}_{t-1}\end{bmatrix} \\ &= -\frac{1}{2}\left(\boldsymbol{X}_t - \boldsymbol{M}_t\right)^T P_t \left(\boldsymbol{X}_t - \boldsymbol{M}_t\right)\end{aligned} \tag{2.4}$$

Where,

$$\boldsymbol{X}_t = \begin{bmatrix}\boldsymbol{x}_t \\ \boldsymbol{x}_{t-1}\end{bmatrix} \tag{2.5}$$

$$\boldsymbol{M}_t = \begin{bmatrix}B\boldsymbol{u}_t \\ \mathbf{0}\end{bmatrix} \tag{2.6}$$

$$P_t = \begin{bmatrix}R^{-1} & -R^{-1}A \\ -A^T R^{-1} & A^T R^{-1} A\end{bmatrix} \tag{2.7}$$

Now $\phi_{\boldsymbol{x}_t}(\boldsymbol{x}_t, \boldsymbol{x}_{t-1})$ can be compactly represented in canonical form:

$$\begin{aligned}\phi_{\boldsymbol{x}_t}(\boldsymbol{x}_t, \boldsymbol{x}_{t-1}) &= \mathcal{N}\left(\boldsymbol{X} | \boldsymbol{M}, P\right) \\ &= \mathcal{C}_{\boldsymbol{X}_t}\left(\boldsymbol{X}_t; P_t, \boldsymbol{h}_t, g_t\right)\end{aligned} \tag{2.8}$$

Where,

$$\boldsymbol{h}_t = P_t \boldsymbol{M}_t \tag{2.9}$$

$$g_t = \boldsymbol{M}^T P \boldsymbol{M} - \ln\left\{|(2\pi)^n R|^{1/2}\right\} \tag{2.10}$$

The incoming message, $\delta_{\phi_{\boldsymbol{x}_{t-1}} \rightarrow \phi_{\boldsymbol{x}_t}}(\boldsymbol{x}_{t-1})$

$\delta_{\phi_{\boldsymbol{x}_{t-1}} \rightarrow \phi_{\boldsymbol{x}_t}}(\boldsymbol{x}_{t-1})$ is some unknown distribution which can be represented generally in canonical form:

$$\delta_{\phi_{\boldsymbol{x}_{t-1}} \rightarrow \phi_{\boldsymbol{x}_t}}(\boldsymbol{x}_{t-1}) = \mathcal{C}_{\boldsymbol{X}_{t-1}}\left(\boldsymbol{X}_{t-1}; P_{t-1}, \boldsymbol{h}_t, g_{t-1}\right) \tag{2.11}$$

Where,

$$\boldsymbol{X}_{t-1} = \boldsymbol{x}_{t-1} \tag{2.12}$$

$$P_{t-1} = \Sigma_{t-1}^{-1} \tag{2.13}$$

$$\mathbf{h}_{t-1} = \Sigma_{t-1}^{-1}\boldsymbol{\mu}_{t-1} \tag{2.14}$$

$$g_{t-1} = \boldsymbol{\mu}^T \Sigma_{t-1}^{-1}\boldsymbol{\mu} - \ln\{\eta_{t-1}\} \tag{2.15}$$

η_{t-1} is to address any constant multipliers. It not strictly necessary for the distribution to normalized, it is only required that it is expressible in canonical form.

2.1.2 Belief update

$$\begin{aligned}
\phi_{\boldsymbol{x}_t}(\boldsymbol{x}_t, \boldsymbol{x}_{t-1})\delta_{\phi_{\boldsymbol{x}_{t-1}} \to \phi_{\boldsymbol{x}_t}}(\boldsymbol{x}_{t-1}) &= \mathcal{C}_{\boldsymbol{X}_t}\left(\boldsymbol{X}_t; P_t, \boldsymbol{h}_t, g_t\right) \cdot \mathcal{C}_{\boldsymbol{X}_{t-1}}\left(\boldsymbol{X}'_{t-1}; P'_{t-1}, \boldsymbol{h}'_t, g_{t-1}\right) \\
&= \mathcal{C}_{\boldsymbol{X}_t}\left(\boldsymbol{X}_t; P_t + P'_{t-1}, \boldsymbol{h}_t + \boldsymbol{h}'_{t-1}, g_t + g_{t-1}\right) \\
&= \mathcal{C}_{\boldsymbol{X}_t}\left(\boldsymbol{X}_t; \hat{P}_t, \hat{\boldsymbol{h}}_t, \hat{g}_t\right) \qquad (2.16)
\end{aligned}$$

$$\begin{aligned}
\hat{P}_t &= P_t + P'_{t-1} \\
&= \begin{bmatrix} R^{-1} & -R^{-1}A \\ -A^T R^{-1} & A^T R^{-1} A \end{bmatrix} + \begin{bmatrix} 0 & -0 \\ 0 & \Sigma_{t-1}^{-1} \end{bmatrix} \\
&= \begin{bmatrix} R^{-1} & -R^{-1}A \\ -A^T R^{-1} & A^T R^{-1} A + \Sigma_{t-1}^{-1} \end{bmatrix} \qquad (2.17) \\
\hat{\boldsymbol{h}}_t &= \boldsymbol{h}_t + \boldsymbol{h}'_{t-1} \\
&= \begin{bmatrix} R^{-1}B\boldsymbol{u}_t \\ -A^T R^{-1} B\boldsymbol{u}_t \end{bmatrix} + \begin{bmatrix} \boldsymbol{0} \\ \Sigma_{t-1}^{-1}\boldsymbol{\mu}_{t-1} \end{bmatrix} \\
&= \begin{bmatrix} R^{-1}B\boldsymbol{u}_t \\ -A^T R^{-1} B\boldsymbol{u}_t + \Sigma_{t-1}^{-1}\boldsymbol{\mu}_{t-1} \end{bmatrix} \qquad (2.18) \\
\hat{g}_t &= g_t + g_{t-1} \\
&= \boldsymbol{M}^T P \boldsymbol{M} - \ln\left\{|(2\pi)^n R|^{1/2}\right\} + \boldsymbol{\mu}^T \Sigma_{t-1}^{-1} \boldsymbol{\mu} - \ln\left\{\eta_{t-1}\right\} \qquad (2.19)
\end{aligned}$$

Marginalisation

$$\overline{P}_t = R^{-1} - \left(A^T R^{-1}\right)^T \left(A^T R^{-1} A + \Sigma_{t-1}^{-1}\right)^{-1} \left(A^T R^{-1}\right) \qquad (2.20)$$

$$\overline{\boldsymbol{h}}_t = R^{-1} B\boldsymbol{u}_t + R^{-1} A \left(A^T R^{-1} A + \Sigma_{t-1}^{-1}\right)^{-1} \left(-A^T R^{-1} B\boldsymbol{u}_t + \Sigma_{t-1}^{-1}\boldsymbol{\mu}_{t-1}\right) \qquad (2.21)$$

$$\overline{g}_t = \hat{g}_t - \frac{1}{2}\left(-A^T R^{-1} B\boldsymbol{u}_t + \Sigma_{t-1}^{-1}\boldsymbol{\mu}_{t-1}\right)^T \left(A^T R^{-1} A + \Sigma_{t-1}^{-1}\right)^{-1} \left(-A^T R^{-1} B\boldsymbol{u}_t + \Sigma_{t-1}^{-1}\boldsymbol{\mu}_{t-1}\right)$$

$$\Psi(\boldsymbol{x}_t) = \mathcal{C}_{\boldsymbol{X}_t}\left(\boldsymbol{X}_t; \overline{P}_t, \overline{\boldsymbol{h}}_t, \overline{g}_t\right) \qquad (2.22)$$

Simplifications

$$\left(A^T R^{-1} A + \Sigma_{t-1}^{-1}\right)^{-1} = \left(\Sigma_{t-1} - \Sigma_{t-1} A^T \left(R + A\Sigma_{t-1}A^T\right)^{-1} A\Sigma_{t-1}\right) \qquad (2.23)$$

Let,

$$\overline{\Sigma}_t = R + A\Sigma_{t-1}A^T \qquad (2.24)$$

$$
\begin{aligned}
\overline{P}_t &= R^{-1} - \left(A^T R^{-1}\right)^T \left(\Sigma_{t-1} - \Sigma_{t-1} A^T \left(R + A\Sigma_{t-1}A^T\right)^{-1} A\Sigma_{t-1}\right)\left(A^T R^{-1}\right) \\
&= R^{-1} - \left(A^T R^{-1}\right)^T \left(\Sigma_{t-1} - \Sigma_{t-1} A^T \overline{\Sigma}_t^{-1} A\Sigma_{t-1}\right)\left(A^T R^{-1}\right) \\
&= R^{-1} - R^{-1}\left(A\Sigma_{t-1}A^T\right) R^{-1} + R^{-1}\left(A\Sigma_{t-1}A^T\right)\overline{\Sigma_t^{-1}}\left(A\Sigma_{t-1}A^T\right) R^{-1} \\
&= R^{-1} - R^{-1}\left(\overline{\Sigma}_t - R\right) R^{-1} + R^{-1}\left(\overline{\Sigma}_t - R\right)\overline{\Sigma_t^{-1}}\left(\overline{\Sigma}_t - R\right) R^{-1} \\
&= R^{-1} - R^{-1}\overline{\Sigma}_t R^{-1} - R^{-1} - R^{-1}\left(I - R\overline{\Sigma}_t^{-1}\right)\left(I - \left(R\overline{\Sigma}_t\right)^{-1}\right) \\
&= 2R^{-1} - R^{-1}\overline{\Sigma}_t R^{-1} + R^{-1}\left(I - R\overline{\Sigma}_t^{-1} - \left(R\overline{\Sigma}_t\right)^{-1} + I\right) \\
&= 2R^{-1} - R^{-1}\overline{\Sigma}_t R^{-1} - 2R^{-1} + \overline{\Sigma}_t^{-1} + R^{-1}\overline{\Sigma}_t R^{-1} \\
&= \overline{\Sigma}_t^{-1}
\end{aligned}
\tag{2.25}
$$

$$
\begin{aligned}
\overline{\boldsymbol{h}}_t &= R^{-1}B\boldsymbol{u}_t + R^{-1}A\left(\Sigma_{t-1} - \Sigma_{t-1}A^T\left(R + A\Sigma_{t-1}A^T\right)^{-1} A\Sigma_{t-1}\right)\left(-A^T R^{-1}B\boldsymbol{u}_t + \Sigma_{t-1}^{-1}\boldsymbol{\mu}_{t-1}\right) \\
&= R^{-1}B\boldsymbol{u}_t + R^{-1}A\left(\Sigma_{t-1} - \Sigma_{t-1}A^T\overline{\Sigma}_t^{-1}A\Sigma_{t-1}\right)\left(-A^T R^{-1}B\boldsymbol{u}_t + \Sigma_{t-1}^{-1}\boldsymbol{\mu}_{t-1}\right) \\
&= R^{-1}B\boldsymbol{u}_t - R^{-1}\left(A\Sigma_{t-1}A^T\right)R^{-1}B\boldsymbol{u}_t + R^{-1}\left(A\Sigma_{t-1}A^T\right)\overline{\Sigma}_t^{-1}\left(A\Sigma_{t-1}A^T\right)R^{-1}B\boldsymbol{u}_t \\
&\quad + R^{-1}A\left(\Sigma_{t-1}\Sigma_{t-1}^{-1}\right)\boldsymbol{\mu}_{t-1} - R^{-1}\left(A\Sigma_{t-1}A^T\right)\bar{(\Sigma)}_t^{-1}A\left(\Sigma_t\Sigma_t^{-1}\right)\boldsymbol{\mu}_{t-1} \\
&= R^{-1}B\boldsymbol{u}_t - R^{-1}\left(\overline{\Sigma}_t - R\right)R^{-1}B\boldsymbol{u}_t + R^{-1}\left(\overline{\Sigma}_t - R\right)\overline{\Sigma}_t^{-1}\left(\overline{\Sigma}_t - R\right)R^{-1}B\boldsymbol{u}_t \\
&\quad + R^{-1}A\boldsymbol{\mu}_{t-1} - R^{-1}\left(\overline{\Sigma}_t - R\right)\overline{\Sigma}_t^{-1}A\boldsymbol{\mu}_{t-1} \\
&= R^{-1}\left(A\boldsymbol{\mu}_{t-1} + B\boldsymbol{u}_t\right) - R^{-1}\left(\overline{\Sigma}_t - R\right)\left(I - \overline{\Sigma}_t^{-1}\left(\overline{\Sigma}_t - R\right)\right)R^{-1}B\boldsymbol{u}_t - R^{-1}\left(\overline{\Sigma}_t - R\right)\overline{\Sigma}_t^{-1}A\boldsymbol{\mu}_t \\
&= R^{-1}\left(A\boldsymbol{\mu}_{t-1} + B\boldsymbol{u}_t\right) - R^{-1}\left(\overline{\Sigma}_t - R\right)\overline{\Sigma}_t^{-1}\left(RR^{-1}\right)B\boldsymbol{u}_t - R^{-1}\left(\overline{\Sigma}_t - R\right)\overline{\Sigma}_t^{-1}A\boldsymbol{\mu}_t \\
&= R^{-1}\left(A\boldsymbol{\mu}_{t-1} + B\boldsymbol{u}_t\right) - R^{-1}\left(\overline{\Sigma}_t - R\right)\overline{\Sigma}_t^{-1}\left(A\boldsymbol{\mu}_{t-1} - B\boldsymbol{u}_t\right) \\
&= R^{-1}\left(I - \left(\overline{\Sigma}_t - R\right)\overline{\Sigma}_t^{-1}\right)\left(A\boldsymbol{\mu}_{t-1} + B\boldsymbol{u}_t\right) \\
&= \left(R^{-1}R\right)\overline{\Sigma}_t^{-1}\left(A\boldsymbol{\mu}_{t-1} + B\boldsymbol{u}_t\right) \\
&= \overline{\Sigma}_t^{-1}\left(A\boldsymbol{\mu}_{t-1} + B\boldsymbol{u}_t\right)
\end{aligned}
\tag{2.26}
$$

From the definition of the information vector, it can be seen that the mean of $\Psi(\boldsymbol{x}_t)$ is:

$$
\boldsymbol{\mu}_t = A\boldsymbol{\mu}_{t-1} + B\boldsymbol{u}_t \tag{2.27}
$$

Lemma 1 (Specialised Woodbury Inversion Identity[a]) *For any invertible quadratic matrices R and Q and any matrix P with appropriate dimensions, the following holds true*

$$
(R + PQP^T)^{-1} = R^{-1} - R^{-1}P(Q^{-1} + P^T R^{-1} P)^{-1} P^T R^{-1}
$$

Proof: *Define $\Psi = (Q^{-1} + P^T R^{-1} P)^{-1}$. It suffices to show that*

$$
(R^{-1} - R^{-1}P\Psi P^T R^{-1})(R + PQP) = I
$$

This is shown through a series of transformations

$$
\begin{aligned}
&= R^{-1}R - R^{-1}PQP^T - R^{-1}P\Psi P^T R^{-1}R + R^{-1}P\Psi P^T R^{-1}PQP^T \\
&= I + R^{-1}PQP^T - R^{-1}P\Psi P^T - R^{-1}P\Psi P^T R^{-1}PQP^T \\
&= I + R^{-1}P\left[QP^T - \Psi P^T - \Psi P^T R^{-1}PQP^T\right] \\
&= I + R^{-1}P\left[QP^T - \Psi Q^{-1}QP^T - \Psi P^T R^{-1}PQP^T\right] \\
&= I + R^{-1}P\left[QP^T - \Psi\left[Q^{-1} + P^T R^{-1}P\right]QP^T\right] \\
&= I + R^{-1}P\left[QP^T - \Psi\Psi^{-1}QP^T\right] \\
&= I + R^{-1}P\left[I - I\right]QP^T \\
&= I
\end{aligned}
$$

[a] This is directly stolen, with a few added steps, from [].

2.2 Part 2: Measurement Update

References

Appendix

A. The Canonnical Form Representation