CIS4930 - Math for Machine Learning

Homework 1

Fernando Sckaff

January 23$^{\text{rd}}$, 2023

Question 1

Let c be a number $\neq 0$. and v an element of V. Prove that if $cv = O$. then $v = O$.

Solution: Given $c \neq 0$, multiply both sides of $cv = 0$ by $\frac{1}{c}$.

$$\begin{aligned} \frac{1}{c} \cdot cv &= 0 \cdot \frac{1}{c} \\ 1 \cdot v &= \frac{0}{c} \\ v &= 0 \quad 😄 \end{aligned}$$

Question 2

Generalize EXERCISE 6 (and this refers to exercise 1.6 or the question immediately above this one in Serge Lang, chapter 1, section 1), and prove: Let $A_1, \ldots, A_r$ be vectors in $\mathbb{R}^n$. Let W be the set of vectors B in $\mathbb{R}^n$ such that $B \cdot A_i = 0$ for every $i = 1, \ldots, r$. Show that W is a subspace of $\mathbb{R}^n$

Solution:

1. $O \cdot W = 0 \therefore$ O is in the subspace
2. Let b_1, b_2 perpendicular to A, and $\{b_1, b_2 \in B\}$:
   $(b_1 + b_2) \cdot A = b_1 \cdot A + b_2 \cdot A = 0$
3. Let $c \in \mathbb{R}$:
   $(cB) \cdot A = c(B \cdot A) = 0$

All basic properties of a subset holds. Therefore, W, the set of vectors B, is a subset of $\mathbb{R}^n$

Question 3

Let (a, b) and (c, d) be two vectors in the plane. If $ad - bc = 0$, show that they are linearly dependent. If $ad - bc \neq 0$, show that they are linearly independent.

Solution:
Let: $\begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$; Assume linear dependency, and $k \in \mathbb{R}$, then $c = ka$ and $d = kb$:

$$\begin{bmatrix} a & ka \\ b & kb \end{bmatrix} \therefore a \cdot kb - b \cdot ka = 0 \therefore kab - kab = 0 \therefore 0 = 0$$

Similarly, let's assume linear independence: $x, y \in \mathbb{R}, x \neq y, c = xa, d = yb$:

$$\begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \to a \cdot yb - b \cdot xa \neq 0 \therefore yab - xab \neq 0 \therefore ab(y - x) \neq 0$$

Assuming $a \neq 0$, and $b \neq 0$, then $x - y \neq 0 \therefore y \neq x$ 😄

Question 4

Let $A_1, \ldots, A_r$ be vectors in R^n and assume that they are mutually perpendicular (i.e. any two of them are perpendicular), and that none of them is equal to 0. Prove that they are linearly independent.

Solution:

· Assume $A_1, \ldots, A_r$ are linearly dependent.
· Thus there exists a $x_1 A_1, \ldots, x_r A_r = 0$ with $\{x_i \in \mathbb{R} | x_i \neq 0\}$
· Multiply both sides by $A_1 \therefore x_i A_i A_i = 0$
· Since $x_i \neq 0$, A_i must be 0, which is a contradiction.

Question 5

Let v, w be element of a vector space and assume that $v \neq 0$. If v, w are linearly dependent, show that there is a number a such that $w = av$.

Solution:
Linear dependence of v and $w \implies \alpha_1 v + \alpha_2 w = 0$ such that $\{\alpha_1, \alpha_2 \in \mathbb{R}\}$

$$\begin{aligned} \alpha_2 w &= -\alpha_1 v \\ w &= \frac{-\alpha_1}{\alpha_2} v \\ a &= \frac{-\alpha_1}{\alpha_2} \end{aligned}$$

With $w = av$. This is an equivalence. 😄

Question 6

Write a program to determine the plane in which 500 random vectors in $\mathbf{R}^3$ are projected into by the action of a projection operator given by

$$P = A(A^T A)^{-1} A^T$$

where $A = [v_1, v_2]$ and v_1, v_2 are two independent, random vectors in $\mathbf{R}^3$. (The matrix P is a projection matrix which confines vectors Pv into a subspace spanned by v_1 and v_2.) Confirm that Pv returns a vector in $\mathbf{R}^3$ for $v \in \mathbf{R}^3$ and therefore all 500 random vectors retain their "three dimensional" nature. Despite this, they actually live in a 2D subspace. (The uploaded python script on the Canvas page accomplishes projection into a 2D subspace.)

Solution: on Google Colab, Link:

$$https://colab.research.google.com/drive/1Ez_gXqOtgsWfKD-uR5piDP1cVzRTzQ3t?usp=sharing$$

F.S.

Note:-

- Question 1: Correct
- Question 2: Correct
- Question 3: Correct
- Question 4: Correct
- Question 5: Correct
- Question 6: 5/20 Programming Exercise: OK, we have a problem that no one seems to know what the equation of a plane is in general. Or equivalently the equation of a line in 3D going through the origin which is (I-P)x=O where x lies on the line. This is different from the equation of a plane in 3D going through the origin which is ax+by+cz=0.