THOUGHTS ON THE BLACK-SCHOLES MODEL

I got even more confused reading Hull so I decided to take matters into my own hands

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on my couch
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1 Derivation

Step 1: Risk-neutral pricing

Under risk-neutral probability, the expected return on the stock equals the risk-free rate, r. This implies that the discounted expected value of the option's payoff at expiration equals the discounted expected value of the option price. For a call option, this relationship can be written as:

$$C(S,t) = e^{-r(T-t)} E\left[\max(S_T - X, 0)\right]$$

where:

- $C(S,t)$ is the call option price.
- S is the current stock price.
- t is the current time.
- T is the time to expiration.
- r is the risk-free interest rate.
- S_T is the stock price at expiration.
- X is the option strike price.
- $E[\cdot]$ denotes the expected value.

Step 2: Apply Ito's Lemma

Ito's Lemma states that if a function G follows a stochastic process, we can find its differential, dG, in terms of the underlying process. The stock price, S, follows a geometric Brownian motion:

$$dS = \mu S dt + \sigma S dW$$

where:

- dS is the change in the stock price.
- μ is the expected return on the stock.
- σ is the stock price volatility.
- dt is the change in time.
- dW is a Wiener process (a random process representing the "noise" in the stock price movement).

Applying Ito's Lemma to the option price $C(S,t)$ gives us a partial differential equation (PDE) to solve for $C(S,t)$:

$$dC = \left(\frac{\partial C}{\partial t} + \mu S \frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}\right) dt + \sigma S \frac{\partial C}{\partial S} dW$$

Step 3: Solve the PDE

To solve the PDE, we use a change of variables and some mathematical tricks. Let $M = ln(S)$ and $N = T - t$. Then, we have:

$$\frac{\partial C}{\partial S} = \frac{\partial C}{\partial M}\frac{\partial M}{\partial S} = \frac{1}{S}\frac{\partial C}{\partial M}$$

$$\frac{\partial^2 C}{\partial S^2} = \frac{\partial}{\partial S}\left(\frac{1}{S}\frac{\partial C}{\partial M}\right) = -\frac{1}{S^2}\frac{\partial C}{\partial M} + \frac{1}{S^2}\frac{\partial^2 C}{\partial M^2}$$

Substituting these expressions back into the PDE and rearranging, we get:

$$\frac{\partial C}{\partial t} + \left(\mu - \frac{1}{2}\sigma^2\right) S\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2} = rC$$

Now, we use the risk-neutral pricing formula to eliminate μ:

$$\frac{\partial C}{\partial t} + \left(r - \frac{1}{2}\sigma^2\right) S\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2} = rC$$

This is the Black-Scholes PDE. Solving it gives us the famous Black-Scholes equation for the price of a European call option:

$$C(S,t) = SN(d_1) - Xe^{-r(T-t)}N(d_2)$$

where:

$$d_1 = \frac{ln(\frac{S}{X}) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

A similar process can be followed to derive the equation for the price of a European put option:

$$P(S,t) = Xe^{-r(T-t)}N(-d_2) - SN(-d_1)$$