COMPLEMENT TO TAUTOLOGICAL CLASSES ON MODULI SPACES OF HYPER-KÄHLER MANIFOLDS

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Abstract. In this note we prove the cohomological tautological conjecture on moduli spaces of $K3$ and $K3^{[2]}$-type hyper-Kähler manifolds. That this result holds was first asserted in [3]. However the proof given there contains a gap, see [2]. Here we give a more direct proof.

1. Introduction

A compact Kähler complex manifold X of complex dimension $2n$ is said to be *hyper-Kähler* if it is simply connected and $\mathrm{H}^0(X, \Omega_X^2) = \mathbf{C} \cdot \eta$, where η is a non-degenerate symplectic form. The free abelian group $\mathrm{H}^2(X, \mathbf{Z})$ is then endowed with a canonical quadratic form q_X — the Beauville–Bogomolov–Fujiki form) which makes it into a lattice with signature $(3, b_2(X) - 3)$.

The lattice $(\mathrm{H}^2(X, \mathbf{Z}), q_X)$ is a deformation invariant and we will consider hyper-Kähler manifolds of a fixed deformation type. Fix Λ a non-degenerate lattice of rank $b+3 \geq 3$ and signature $(3, b)$. Pick a primitive class $h \in \Lambda$ with $h^2 > 0$. We consider pairs (X, H) where H is a primitive ample line bundle on a (smooth projective) hyper-Kähler manifolds X such that there is an isometry $(H^2(X, \mathbf{Z}), q_X) \cong \Lambda$ which takes H to h. The corresponding moduli stack is then a smooth separated Deligne–Mumford stack. The moduli stack can be coarsely represented by a quasi-projective variety $\mathcal{F}_h$. The latter only depends on the $\mathrm{O}(\Lambda)$-orbit of h, called *polarization type*. The variety $\mathcal{F}_h$ is not smooth and may have several connected components. One may introduce level structures to help rigidify this moduli problem, see [3, §3.4].

In the following we fix some sufficiently large positive integer ℓ and let $\mathcal{F}_h^\ell$ be a connected component of the moduli space of h-polarised (or h-ample) hyper-Kähler manifolds of dimension $2n$ and with second Betti number $b_2 = b + 3$, with a full ℓ-level structure.

Now fix a primitive embedding $\Sigma \hookrightarrow \Lambda$ of a lattice of signature $(1, \mathrm{rank}(\Sigma) - 1)$ and a primitive class $h \in \Sigma$ such that $h^2 > 0$. As in [3, §3.7] one may consider (Σ, h)-polarized hyper-Kähler manifolds (X, H), those with an embedding $\Sigma \hookrightarrow \mathrm{Pic}(X)$ that maps h to H. We denote by $\mathcal{F}_{\Sigma,h}^\ell$ the moduli space for these. We have a natural forgetful map

$$\iota_\Sigma : \mathcal{F}_{\Sigma,h}^\ell \to \mathcal{F}_h^\ell. \tag{1}$$

Now let

$$\pi_h^\ell : \mathcal{U}_h^\ell \to \mathcal{F}_h^\ell \quad \text{and} \quad \pi_{\Sigma,h}^\ell : \mathcal{U}_{\Sigma,h}^\ell \to \mathcal{F}_{\Sigma,h}^\ell$$

be the corresponding universal families, let $r+1$ be the Picard number of the generic fiber of $\pi^{\ell}_{\Sigma,h}$, and let

$$\mathcal{B}^{\ell}_{\Sigma} = \{\mathcal{L}_0, \ldots, \mathcal{L}_r\} \subset \mathrm{Pic}_{\mathbf{Q}}(\mathcal{U}^{\ell}_{\Sigma,h})$$

be a collection of line bundles whose images in $\mathrm{Pic}_{\mathbf{Q}}(\mathcal{U}^{\ell}_{\Sigma,h}/\mathcal{F}^{\ell}_{\Sigma,h})$ form a basis.

We define the following subalgebras in $\mathrm{CH}^{\bullet}(\mathcal{F}^{\ell}_h)$:

- $\mathrm{NL}^{\bullet}(\mathcal{F}^{\ell}_h)$ is the subalgebra generated by irreducible components of the images of the maps (1) as one varies Σ;
- the *tautological ring* $\mathrm{R}^{\bullet}(\mathcal{F}^{\ell}_h)$ is the subalgebra generated by the κ-*cycles*

$$(\iota_{\Sigma} \circ \pi^{\ell}_{\Sigma,h})_* \left(\prod_{i=0}^{r} c_1(\mathcal{L}_i)^{a_i} \prod_{j=1}^{2n} c_j(\mathcal{T}_{\pi^{\ell}_{\Sigma,h}})^{b_j} \right);$$

- the *special tautological ring* $\mathrm{DR}^{\bullet}(\mathcal{F}^{\ell}_h)$ is the subalgebra generated by the *special* κ-*cycles*

$$(\iota_{\Sigma} \circ \pi^{\ell}_{\Sigma,h})_* \left(\prod_{i=0}^{r} c_1(\mathcal{L}_i)^{a_i} \right).$$

We add the subscript *hom* to denote the images of the the corresponding rings in $\mathrm{H}^{\bullet}(\mathcal{F}^{\ell}_h, \mathbf{Q})$ via the cycle class map. e.g. $\mathrm{CH}^k_{\mathrm{hom}}(\mathcal{F}^{\ell}_h)$ denotes the image of $\mathrm{CH}^k(\mathcal{F}^{\ell}_h)$ in $\mathrm{H}^{2k}(\mathcal{F}^{\ell}_h, \mathbf{Q})$.

There are natural inclusions

$$\mathrm{NL}^{\bullet}(\mathcal{F}^{\ell}_h) \subset \mathrm{DR}^{\bullet}(\mathcal{F}^{\ell}_h) \subset \mathrm{R}^{\bullet}(\mathcal{F}^{\ell}_h).$$

In [3, Conjecture 4] we proposed the following.

Conjecture 1 (Hyper-Kähler Tautological Conjecture)**.** *We have*

$$\mathrm{NL}^{\bullet}(\mathcal{F}^{\ell}_h) = \mathrm{R}^{\bullet}(\mathcal{F}^{\ell}_h).$$

Conjecture 2 (Cohomological Tautological Conjecture)**.** *We have*

$$\mathrm{NL}^{\bullet}_{\mathrm{hom}}(\mathcal{F}^{\ell}_h) = \mathrm{R}^{\bullet}_{\mathrm{hom}}(\mathcal{F}^{\ell}_h).$$

Theorem 4.3.1 [3] asserts that the cohomological tautological conjecture holds for moduli spaces of $K3$ and $K3^{[2]}$-type hyper-Kähler manifolds. However the proof given there relies on [3, Theorem 8.2.1] and Thorsten Beckman and Mirko Mauri have pointed to us a fatal gap in the proof of the latter. The goal of this note is to give a direct proof of the cohomological tautological conjecture for moduli spaces of $K3$ and $K3^{[2]}$-type hyper-Kähler manifolds. In other words we prove:

Theorem 3. *The cohomological tautological conjecture holds for both the moduli spaces of $K3$ and $K3^{[2]}$-type hyper-Kähler manifolds.*

Notation. If $\mathcal{U} = \mathcal{U}_h^\ell$ or $\mathcal{U}_{\Sigma,h}^\ell$, we denote by $\mathrm{H}^\bullet_{\leq d}(\mathcal{U})$ the subalgebra of $\mathrm{H}^\bullet(\mathcal{U}, \mathbf{Q})$ generated by the classes in

$$\mathrm{Hdg}(\mathcal{U})^{2k} := (W_{2k}\mathrm{H}^{2k}(\mathcal{U}))^{k,k} \quad \text{with } k \leq d$$

when $d \leq 2n-1$. Here $W_{2k}\mathrm{H}^{2k}(\mathcal{U})$ is the lowest weight subspace in the mixed Hodge structure of $\mathrm{H}^{2k}(\mathcal{U})$.

When $d = 2n$, we define $\mathrm{H}^\bullet_{\leq 2n}(\mathcal{U})$ to be the subalgebra generated by $\mathrm{H}^\bullet_{\leq 2n-1}(\mathcal{U})$ and the relative Chern class $c_{2n}(\mathcal{T}_\pi)$.

Recall from [3, §2.5] that if X is a hyper-Kähler manifold we denote by $\mathrm{BV}^\bullet(X)$ the subalgebra of $\mathrm{CH}^\bullet(X)$ generated by all divisors and Chern classes $c_i(T_X)$. Similarly we define the Beauville–Voisin ring

$$\mathrm{BV}^\bullet_{\mathrm{hom}}(\mathcal{U}) \subset \mathrm{CH}^\bullet_{\mathrm{hom}}(\mathcal{U})$$

to be the subalgebra generated by the cycle classes of $c_1(\mathcal{L}_i)$ and $c_j(\mathcal{T}_{\pi^\ell_{\Sigma,h}})$ and denote by $\mathrm{DCH}^\bullet(\mathcal{U})$ the subring generated by the divisor classes $c_1(\mathcal{L}_i)$; to ease notation we will often use the symbol $\mathcal{L}$ to designate both a line bundle and its first Chern class $c_1(\mathcal{L})$ in CH^1 or in H^2 we hope this won't create any confusion.

Remark that $\mathrm{BV}^\bullet_{\mathrm{hom}}(\mathcal{U})$ is a subring of $\mathrm{H}^\bullet_{\leq 2n}(\mathcal{U})$.

We finally define

$$\widetilde{\mathrm{R}}^*_{\mathrm{hom}}(\mathcal{F}_h^\ell) \subseteq \mathrm{H}^\bullet(\mathcal{F}_h^\ell, \mathbf{Q})$$

to be the subring generated by all the pushforwards

$$(\iota_\Sigma \circ \pi^\ell_{\Sigma,h})_*(x) \quad \text{with} \quad x \in \mathrm{H}^\bullet_{\leq 2n}(\mathcal{U}^\ell_{\Sigma,h}),$$

where we let $\pi^\ell_{\Sigma,h} : \mathcal{U}^\ell_{\Sigma,h} \to \mathcal{F}^\ell_{\Sigma,h}$ vary over all Σ. By definition, we have

$$\mathrm{DR}^\bullet_{\mathrm{hom}}(\mathcal{F}_h^\ell) \subseteq \mathrm{R}^*_{\mathrm{hom}}(\mathcal{F}_h^\ell) \subseteq \widetilde{\mathrm{R}}_{\mathrm{hom}}(\mathcal{F}_h^\ell).$$

2. The Leray spectral sequence and cup products

In this section we simply denote by $\pi : \mathcal{U} \to \mathcal{F}$ a universal family of lattice polarized hyper-Kähler manifolds $\pi^\ell_{\Sigma,h} : \mathcal{U}^\ell_{\Sigma,h} \to \mathcal{F}^\ell_{\Sigma,h}$.

Consider the local systems $\mathbf{H}^j = R^j\pi_*\mathbf{Q}$. It follows from [5, Theorem 1.1.1] that there is a splitting of the Leray filtration in the category of mixed Hodge structure, the degeneration of the Leray spectral sequence for π therefore gives an isomorphism of mixed Hodge structure

$$\mathrm{H}^k(\mathcal{U}, \mathbf{Q}) \cong \bigoplus_{i+j=k} \mathrm{H}^i(\mathcal{F}, \mathbf{H}^j). \tag{2}$$

The LHS of (2) is endowed with the standard cup product. On the other hand, the cup product maps $\mathbf{H}^i \otimes \mathbf{H}^j \to \mathbf{H}^{i+j}$ induce a cup product on the Leray spectral sequence, and therefore a cup product on the RHS of (2). However the isomorphism (2) does not preserve the ring structure in general; see e.g. [6, Prop. 0.4 or Prop. 0.6].

In [3, Theorem 8.2.1], we claimed that the two cup products on (2) agree on the subalgebra generated by low degree classes (degree $< \frac{1}{2}\dim\mathcal{F}$) up to removing some Noether–Lefschetz loci. Based on this result, we proved Theorem 8.3.1 and Theorem 4.3.1 in [3], which implies that

$$\mathrm{R}^\bullet_{\mathrm{hom}}(\mathcal{F}_h) = \mathrm{NL}^\bullet_{\mathrm{hom}}(\mathcal{F}_h)$$

when $n < \frac{b_2-3}{8}$. In particular, the cohomological tautological conjecture holds for K3 surfaces and $K3^{[2]}$-type hyper-Kähler manifolds. However, in our 'proof', we only show that

Proposition 4. *Let α_1 and α_2 be two classes in $\mathrm{H}^\bullet(\mathcal{U}, \mathbf{Q})$. Assume that the sum of the degrees of α_1 and α_2 is $< \frac{1}{2}\dim \mathcal{F}$, then*

$$\alpha_1 \wedge_{\mathrm{LHS}} \alpha_2 = \alpha_1 \wedge_{\mathrm{RHS}} \alpha_2, \tag{3}$$

up to removing some Nother-Lefschetz components.

Proof. This is a direct consequence of the cohomological generalized Franchetta Conjecture [3, Theorem 8.1.1]. □

For the entire algebra $\mathrm{H}^\bullet_{<\frac{1}{2}\dim\mathcal{F}}(\mathcal{U})$, we actually do not know if (3) holds (up to removing some NL loci).

3. On the cohomological tautological conjecture

It remains open whether the two cup products coincides (up to some classes supported on NL loci) on the entire subalgebra of $\mathrm{CH}^\bullet_{\mathrm{hom}}(\mathcal{U})$ generated by cycles of small co-dimension. This causes a problem in the proof of [3, Theorem 8.3.1] and hence of [3, Theorem 4.3.1].

As we will show in the next sections we can however prove the following theorem. The bound on n is weaker but it is good enough to apply to $K3$ surfaces and $K3^{[2]}$-type hyper-Kähler manifolds.

Recall that $\pi_h^\ell : \mathcal{U}_h^\ell \to \mathcal{F}_h^\ell$ denotes a smooth family of h-polarized hyper-Kähler manifolds over an irreducible quasi-projective variety $\mathcal{F}_h^\ell$ of dimension b. We define the following conditions to state our main result:

(∗) For every lattice-polarized universal family $\mathcal{U}^\ell_{\Sigma,h} \to \mathcal{F}^\ell_{\Sigma,h}$ with $\dim \mathcal{F}^\ell_{\Sigma,h} \geq \frac{b}{2}$, the group of Hodge classes of **degree 2n** of the very general fiber is spanned by Beauville–Voisin classes.

(∗∗) For every lattice-polarized universal family $\mathcal{U}^\ell_{\Sigma,h} \to \mathcal{F}^\ell_{\Sigma,h}$ with $\dim \mathcal{F}^\ell_{\Sigma,h} \geq \frac{b}{2}$, the group of Hodge classes of **degree $\leq$ 2n** of the very general fiber is spanned by Beauville–Voisin classes.

Then we have

Theorem 5. *Suppose $b \geq 16n - 12$. Then*

$$\mathrm{DR}^\bullet_{\mathrm{hom}}(\mathcal{F}_h^\ell) = \mathrm{NL}^\bullet_{\mathrm{hom}}(\mathcal{F}_h^\ell).$$

Moreover, we have $\mathrm{R}^\bullet_{\mathrm{hom}}(\mathcal{F}^\ell_h) = \mathrm{NL}^\bullet_{\mathrm{hom}}(\mathcal{F}^\ell_h)$ *if the condition* $(*)$ *holds and*

$$\mathrm{NL}^\bullet_{\mathrm{hom}}(\mathcal{F}^\ell_h) = \mathrm{R}^\bullet_{\mathrm{hom}}(\mathcal{F}^\ell_h) = \widetilde{\mathrm{R}}^\bullet_{\mathrm{hom}}(\mathcal{F}^\ell_h), \tag{4}$$

if either $b \geq 16n-8$ *or condition* $(**)$ *holds. In particular, the cohomological tautological conjecture holds for both* $K3$ *surfaces and* $K3^{[2]}$*-type hyper-Kähler manifolds.*

4. Proof of Theorem 5

Before embarquing into the proof we first collect some results that quickly follow from theorems proved in [3].

4.1. **Noether–Lefschetz locus.**

Definition 6. Let α be a class in $\mathrm{Hdg}^{2k}(\mathcal{U}^\ell_{\Sigma,h})$. We say that α is *supported on the Noether–Lefschetz locus* if there exist finitely many lattices $\Sigma_j \supset \Sigma$ with $\mathrm{rank}(\Sigma_j) = \mathrm{rank}(\Sigma)+1$ and classes $\gamma_j \in \mathrm{Hdg}^{2k-2}(\mathcal{U}^\ell_{\Sigma_j,h})$ such that

$$\alpha = \sum_j r_j (\rho_j)_*(\gamma_j),$$

where $r_j \in \mathbf{Q}$ and ρ_j denotes the map $\mathcal{U}^\ell_{\Sigma_j,h} \to \mathcal{U}^\ell_{\Sigma,h}$ corresponding to the forgetful map $\mathcal{F}^\ell_{\Sigma_j,h} \to \mathcal{F}^\ell_{\Sigma,h}$.

Let $d = \dim \mathcal{F}^\ell_{\Sigma,h}$. The main results in [4] and [3] imply the following

Theorem 7. *(i) For every degree* $k < \frac{d+1}{3}$ *or* $k > \frac{2d-1}{3}$*, we have*

$$\mathrm{Hdg}^{2k}(\mathcal{F}^\ell_{\Sigma,h}) = \mathrm{NL}^k_{\mathrm{hom}}(\mathcal{F}^\ell_{\Sigma,h}).$$

(ii) Let $\alpha \in \mathrm{Hdg}^{2k}(\mathcal{U}^\ell_{\Sigma,h})$ *with* $k < \min\left\{\frac{d}{4}+1, \frac{d+1}{3}\right\}$*. If the restriction of* α *to the very general fiber of* $\pi^\ell_{\Sigma,h}$ *is zero, then* α *is supported on the Noether–Lefschetz locus.*

Proof. The first part is precisely the main result of [4].

Te prove the second part recall that if Σ' is a lattice containing Σ the forgetful map

$$\iota_{\Sigma',\Sigma} : \mathcal{F}^\ell_{\Sigma',h} \to \mathcal{F}^\ell_{\Sigma,h} \quad (\text{with } \Sigma \subset \Sigma' \text{ and } \mathrm{rank}(\Sigma') = \mathrm{rank}(\Sigma)+i)$$

defines a codimension i cycle. Now consider a finite dimensional $\mathrm{SO}(2,d;\mathbb{R})$-representation E and let $\mathbf{E}$ be the associated local system on $\mathcal{F}^\ell_{\Sigma,h}$. We can assign coefficients to the cycle class of $\mathcal{F}^\ell_{\Sigma',h}$ (see [3, § 5.4] for more details): given a parallel vector $\mathbf{v}$ in the local system $\mathbf{E}$, form the corresponding cycle class

$$(\iota_{\Sigma',\Sigma})_*\left([\mathcal{F}^\ell_{\Sigma',h}] \otimes \mathbf{v}\right) \in \mathrm{H}^{2i}(\mathcal{F}^\ell_{\Sigma,h}, \mathbf{E}).$$

We call these *decorated special cycle classes* and denote by $\mathrm{SC}^i_{\mathrm{hom}}(\mathcal{F}^\ell_{\Sigma,h}, \mathbf{E})$ the subspace they span as Σ' and v vary. Theorem 6.4.1 of [3] implies that for all $i < \frac{d+1}{3}$, we in fact have

$$\mathrm{H}^{i,i}(\mathcal{F}^\ell_{\Sigma,h}, \mathbf{E}) = \mathrm{SC}^i_{\mathrm{hom}}(\mathcal{F}^\ell_{\Sigma,h}, \mathbf{E}).$$

(Note that it follows from Zucker's conjecture, proved by Looijenga, Saper and Stern, that the mixed Hodge structure of $\mathrm{H}^{<d-1}(\mathcal{F}^\ell_{\Sigma,h}, \mathbf{E})$ is pure.)

On the other hand it follows from [3, §6.2] that for all $i < \frac{d}{4}$, we have

$$\mathrm{H}^{2i}(\mathcal{F}^\ell_{\Sigma,h}, \mathbf{E}) = \mathrm{H}^{i,i}(\mathcal{F}^\ell_{\Sigma,h}, \mathbf{E}).$$

We conclude that

(†) for all $i < \frac{d}{4}$, the space $\mathrm{H}^{2i}(\mathcal{F}^\ell_{\Sigma,h}, \mathbf{E})$ is spanned by decorated special cycle classes in $\mathrm{SC}^i_{\mathrm{hom}}(\mathcal{F}^\ell_{\Sigma,h}, \mathbf{E})$.

The proof of (ii) now proceeds as that of [3, Theorem 8.1.1]: under the hypotheses of the theorem, the class α belongs to

$$\bigoplus_{i=1}^{k} \mathrm{H}^{2i}(\mathcal{F}^\ell_{\Sigma,h}, \mathbf{H}^{2(k-i)}).$$

If $i < k$ then by hypothesis we have $i < \frac{d}{4}$ and (†), applied to $\mathbf{E} = \mathbf{H}^{2(k-i)}$, implies that the component of α in $\mathrm{H}^{2i}(\mathcal{F}^\ell_{\Sigma,h}, \mathbf{H}^{2(k-i)})$ is a linear combination of decorated Noether–Lefschetz cycle classes. Finally if $i = k$ the local system $\mathbf{H}^{2(k-i)}$ is trivial and we can similarly apply (i) to conclude that all the components of α can be decomposed as linear combinations of decorated special cycle classes.

We are therefore reduced to proving that a decorated special cycle class represents a class in $\mathrm{H}^\bullet(\mathcal{U}^\ell_{\Sigma,h})$ that is supported on the Noether–Lefschetz locus. To do so recall the decorated special cycle classes were defined as

$$(\iota_{\Sigma',\Sigma})_* \left([\mathcal{F}^\ell_{\Sigma',h}] \otimes \mathbf{v} \right) \in \mathrm{H}^{2i}(\mathcal{F}^\ell_{\Sigma,h}, \mathbf{H}^{2a}) \subset \mathrm{H}^{2(i+a)}(\mathcal{U}^\ell_{\Sigma,h}).$$

where $\mathbf{v}$, a parallel vector in a local system $\mathbf{H}^{2a}$ corresponds to a class γ in $\mathrm{Hdg}^{2a}(\mathcal{U}^\ell_{\Sigma',h})$ whose restriction to the very general fiber is precisely $\mathbf{v}$.

By induction on d and up to classes supported on the Noether–Lefschetz locus of $\mathcal{U}^\ell_{\Sigma',h}$ the class γ is equal to

$$[\mathcal{F}^\ell_{\Sigma',h}] \otimes \mathbf{v} \in \mathrm{H}^0(\mathcal{F}^\ell_{\Sigma',h}, \mathbf{H}^a)$$

in any given decomposition (2) of $\mathrm{H}^\bullet(\mathcal{U}^\ell_{\Sigma',h})$. Since $(\iota_{\Sigma',\Sigma})_*\gamma$ is obviously supported on the Noether–Lefschetz locus, it follows that

$$(\iota_{\Sigma',\Sigma})_* \left([\mathcal{F}^\ell_{\Sigma',h}] \otimes \mathbf{v} \right),$$

and therefore α, are also supported on the Noether–Lefschetz locus. □

Another key result is the following

Corollary 8. *Assume that $d < \frac{b}{2}+1$. Then for any $\alpha \in \mathrm{CH}^k(\mathcal{F}^\ell_{\Sigma,h})$, the cohomology class $(\iota_\Sigma)_*[\alpha]$ is lying in $\mathrm{NL}^{k+b-d}_{\mathrm{hom}}(\mathcal{F}^\ell_h)$.*

Proof. By assumption, the class $[\alpha]$ belongs to $\mathrm{Hdg}^{2k}(\mathcal{F}^\ell_{\Sigma,h})$ and its pushforward image

$$(\iota_\Sigma)_*[\alpha] \in \mathrm{Hdg}^{2(k+b-d)}(\mathcal{F}^\ell_h).$$

We then have distinguish two cases.

(1) If $k+b-d > \frac{2b-1}{3}$, it follows from Theorem 7 (i), applied to $\mathcal{F}^\ell_h$, that $(\iota_\Sigma)_*[\alpha]$ lies in $\mathrm{NL}^{k+b-d}_{\mathrm{hom}}(\mathcal{F}^\ell_h)$.
(2) If $k+b-d \leq \frac{2b-1}{3}$, then $k \leq d - \frac{b+1}{3} < \frac{d+1}{3}$ and it follows from Theorem 7 (i), applied to $\mathcal{F}^\ell_{\Sigma,h}$, that $[\alpha] \in \mathrm{NL}^k_{\mathrm{hom}}(\mathcal{F}^\ell_{\Sigma,h})$. The assertion then follows from the fact that $(\iota_\Sigma)_*(\mathrm{NL}^\bullet(\mathcal{F}^\ell_{\Sigma,h})) \subseteq \mathrm{NL}^\bullet(\mathcal{F}^\ell_h)$.

□

4.2. **Inductive step.** Throughout this subsection, we simply denote by $\pi : \mathcal{U} \to \mathcal{F}$ the universal family $\mathcal{U}^\ell_{\Sigma,h} \to \mathcal{F}^\ell_{\Sigma,h}$. We make use of Theorem 7(ii) to investigate the difference between the rings

$$\mathrm{DCH}^\bullet_{\mathrm{hom}}(\mathcal{U}) \subseteq \mathrm{BV}^\bullet_{\mathrm{hom}}(\mathcal{U}) \subseteq \mathrm{H}^\bullet_{\leq 2n}(\mathcal{U}).$$

The following result shows that when $d = \dim \mathcal{F}$ is large enough, they only differ by some classes supported on the Noether–Lefschetz loci.

Theorem 9. *Let $\alpha = \prod_i \alpha_i \in \mathrm{H}^{2k}_{\leq 2n}(\mathcal{U})$ with $k \geq 2n$, where each α_i belongs to $\mathrm{Hdg}^{2k_i}(\mathcal{U})$ with $k_i \leq 2n$ and $\alpha_i = c_{2n}(\mathcal{T}_\pi)$ if $k_i = 2n$. Suppose one of the following conditions holds*

(1) $\dim \mathcal{F} \geq \sup\{8n-3, 6n\}$, *or*
(2) $\dim \mathcal{F} \geq \sup\{8n-7, 6n-3\}$, *$k_i < 2n$ for all i, and $k_{i_0} \neq n$ for some i_0.*

Then, there exists $\beta \in \mathrm{DCH}^{2k}_{\mathrm{hom}}(\mathcal{U})$ such that

$$\alpha - \beta = \sum_j r_j (\rho_j)_*(\gamma_j), \tag{5}$$

with $r_j \in \mathbf{Q}$ and $\gamma_j \in \mathrm{H}^{2k-2}_{\leq 2n-1}(\mathcal{U}_{\Sigma_j})$ for some lattices $\Sigma_j \supset \Sigma$ with $\mathrm{rank}(\Sigma_j) = \mathrm{rank}(\Sigma) + 1$.

Moreover, in Case (1), β can be chosen of the form $a\mathcal{L}^k$ for some relative ample line bundle $\mathcal{L}$, while in Case (2), β can be chosen of the form $\mathcal{L}^{k-1}\mathcal{L}'$ for some $\mathcal{L}, \mathcal{L}' \in \mathrm{Pic}(\mathcal{U})$ with $\mathcal{L}$ relative ample.

Proof. Let

$$\delta = \min\left\{\frac{d}{4}+1, \frac{d+1}{3}\right\}$$

be the constant appearing in Theorem 7(ii).

Lemma 10. *Let $\alpha \in \mathrm{Hdg}^{2k}(\mathcal{U})$ with $k \in]n, 2n]$. Suppose furthermore $k < \delta$. Then, there exist $\mathcal{L} \in \mathrm{Pic}(\mathcal{U})$ relative ample and $\beta \in \mathrm{Hdg}^{4n-2k}(\mathcal{U})$ such that the difference $\alpha - \mathcal{L}^{2k-2n}\beta$ is supported on the Noether–Lefschetz locus.*

Proof. According to the relative Hard Lefschetz isomorphism

$$\mathrm{H}^0(\mathcal{F}, \mathbf{H}^{2k}) \cong \mathrm{H}^0(\mathcal{F}, \mathbf{H}^{4n-2k}),$$

we can find a class $\mathcal{L}^{2k-2n}\beta$ with $\mathcal{L} \in \mathrm{Pic}(\mathcal{U})$ relative ample and

$$\beta \in \mathrm{Hdg}^{4n-2k}(\mathcal{U})$$

such that the restriction of $\alpha - \mathcal{L}^{2k-2n}\beta$ to each fiber is zero. Now by hypothesis we have $k < \delta$ and we can apply Theorem 7(ii) to $\alpha - \mathcal{L}^{2k-2n}\beta$. We conclude that the latter is supported on the Noether–Lefschetz locus of $\mathcal{F}$. □

We will now make repeated use of this lemma to prove the theorem in each of the two cases.

First consider Case (1), then by hypothesis $2n < \delta$. Lemma 10 applies to every product $\alpha_{i_1} \cdots \alpha_{i_r}$ with $k_{i_1} + \ldots + k_{i_r} \in]n, 2n]$. Repeating this we get a class $\beta' \in \mathrm{Hdg}^{2m}(\mathcal{U})$ with $m \leq n$ such that the difference $\alpha - \mathcal{L}^{k-m}\beta'$ satisfies (5). Note that the restrictions of $\mathcal{L}^{2n-m}\beta'$ and $\mathcal{L}^{2n}$ to the very general fiber are proportional, Lemma 10 therefore implies that $\mathcal{L}^{2n-m}\beta' - a\mathcal{L}^{2n}$ is supported on the NL locus for some $a \in \mathbf{Q}$. As a result, we have

$$\mathcal{L}^{k-m}\beta' - a\mathcal{L}^{k} = \mathcal{L}^{k-2n}(\mathcal{L}^{2n-m}\beta' - a\mathcal{L}^{2n})$$

satisfies (5).

Now consider Case (2), then by hypothesis $2n - 1 < \delta$ and there is at least one class α_{i_0} of degree $\neq n$. Applying Lemma 10 as in Case (1) we therefore conclude that there exists a class $\beta' \in \mathrm{Hdg}^{2m}(\mathcal{U})$ with $m \leq n-1$ such that the difference

$$\alpha - \mathcal{L}^{k-m}\beta' \in \mathrm{Hdg}^{2k}(\mathcal{U})$$

satisfies (5). By the relative Hard Lefschetz theorem, we can find a line bundle $\mathcal{L}' \in \mathrm{Pic}(\mathcal{U})$ such that $\mathcal{L}^{2n-m-1}\beta'$ and $\mathcal{L}^{2n-2}\mathcal{L}'$ agree on the very general fiber. It follows from Theorem 7(ii) that $\beta = \mathcal{L}^{k-1}\mathcal{L}'$ is as desired. □

We derive the following result directly from Corollary 8 and Theorem 9.

Proposition 11. *Set $b = \dim \mathcal{F}_h^{\ell}$. Suppose that $b \geq 16n-8$ and $\mathrm{DR}_{\mathrm{hom}}(\mathcal{F}_h^{\ell}) = \mathrm{NL}^{\bullet}_{\mathrm{hom}}(\mathcal{F}_h^{\ell})$, then we have*

$$\mathrm{NL}^{\bullet}_{\mathrm{hom}}(\mathcal{F}_h^{\ell}) = \mathrm{R}^{\bullet}_{\mathrm{hom}}(\mathcal{F}_h^{\ell}) = \widetilde{\mathrm{R}}^{\bullet}_{\mathrm{hom}}(\mathcal{F}_h^{\ell}).$$

Proof. To ease notation we keep using $\pi : \mathcal{U} \to \mathcal{F}$ to denote the universal family $\pi_{\Sigma}^{\ell} : \mathcal{U}_{\Sigma,h}^{\ell} \to \mathcal{F}_{\Sigma,h}^{\ell}$ and let $\iota : \mathcal{F} \to \mathcal{F}_h^{\ell}$ be the forgetful map. Under the assumption of the proposition we want to prove that

$$\iota_*(\pi_* \mathrm{H}^{\bullet}_{\leq 2n}(\mathcal{U})) \subseteq \mathrm{NL}^{\bullet}_{\mathrm{hom}}(\mathcal{F}_h^{\ell}) \tag{6}$$

for all $\mathcal{U} \to \mathcal{F}$.

According to Theorem 9, once $\dim \mathcal{F} \geq \sup\{8n-3, 6n\}$, any class in $\pi_*(\mathrm{H}^\bullet_{\leq 2n}(\mathcal{U}))$ is a linear combination of classes in $\mathrm{DR}^\bullet_{\mathrm{hom}}(\mathcal{F})$ and in the images $(\iota')_*(\pi'_*(\mathrm{H}^\bullet_{\leq 2n-1}(\mathcal{U}')))$ as

$$\begin{tikzcd} \mathcal{U}' \arrow[r, "\rho'"] \arrow[d, "\pi'"] & \mathcal{U} \arrow[d, "\pi"] \\ \mathcal{F}' \arrow[r, "\iota'"] & \mathcal{F} \end{tikzcd}$$

runs over all the universal family of sublattice polarized hyper-Kähler varieties in $\mathcal{F}$. By our assumption, the pushforward of $\mathrm{DR}^\bullet_{\mathrm{hom}}(\mathcal{F})$ is lying in $\mathrm{NL}^\bullet_{\mathrm{hom}}(\mathcal{F}^\ell_h)$. So it suffices to prove

$$(\iota \circ \iota')_*(\pi'_*(\mathrm{H}^\bullet_{\leq 2n-1}(\mathcal{U}'))) \subseteq \mathrm{NL}^\bullet_{\mathrm{hom}}(\mathcal{F}^\ell_h), \tag{7}$$

with $\dim \mathcal{F}' < \dim \mathcal{F}$. This allows us to cut the dimension of $\mathcal{F}$ whenever $\dim \mathcal{F} \geq \sup\{8n-3, 6n\}$ and we are reduced to prove that (6) holds as long as $\dim \mathcal{F} \leq \sup\{8n-4, 6n-1\}$. This follows from Corollary 8 and our hypothesis. Indeed: when $n > 1$ we have $b \geq 16n - 8$ which implies $\dim \mathcal{F} \leq \frac{1}{2} \dim \mathcal{F}^\ell_h$. When $n = 1$, $\mathcal{F}$ is the moduli space of K3 surfaces hence of dimension $b = 19$. One still has $\dim \mathcal{F} \leq 5 \leq \frac{19}{2}$. □

Note that the bound of b in Proposition 11 will be enough to prove Theorem 5 in the case of $K3$ surfaces, but it fails short to deal with the case of $K3^{[2]}$-type hyper-Kähler manifolds, where $b = 20 < 24$. From now on we will therefore suppose $n > 1$. To strengthen Proposition 11, we need the following result.

Lemma 12. *Suppose $b \geq 16n - 12$ and $n > 1$. Let $\pi : \mathcal{U} \to \mathcal{F}$ be the universal family of a lattice polarized hyper-Kähler varieties in $\mathcal{F}^\ell_h$ with* $\dim \mathcal{F} \leq 8n - 4$. *Then for any class*

$$\alpha = (c_{2n}(\mathcal{T}_\pi))^m \prod \alpha_i \in \mathrm{H}^{2k}_{\leq 2n}(\mathcal{U})$$

where $\alpha_i \in \mathrm{Hdg}^{2k_i}(\mathcal{U})$ with $k_i < 2n$, we have

$$\iota_*(\pi_*(\alpha)) \in \mathrm{NL}^\bullet_{\mathrm{hom}}(\mathcal{F}^\ell_h) \tag{8}$$

if one of the following conditions holds

- *(i)* $\dim \mathcal{F} \leq 8n - 6$;
- *(ii) the α_i's are either Chern classes of $\mathcal{T}_\pi$ or relative ample line bundles;*
- *(iii) there exists some $k_i \neq n$ and $\sum k_i \geq 2n$.*

In particular, this implies $\iota_*(\pi_* \mathrm{BV}^\bullet_{\mathrm{hom}}(\mathcal{U})) \subseteq \mathrm{NL}^\bullet_{\mathrm{hom}}(\mathcal{F}^\ell_h)$.

Proof. As $b \geq 16n - 12$, when the condition (i) holds, we can conclude the assertion directly by Corollary 8. It remains to consider the case when $\dim \mathcal{F} = 8n - 5$ or $8n - 4$. We first consider the case where $\dim \mathcal{F} = 8n - 5$. In that case, the proof of Corollary 8 implies that (8) holds for every $k \neq$

$\frac{8n-4}{3}+2n$. It remains to deal with the case where $k=\frac{8n-4}{3}+2n$ (the latter being forced to be an integer). Write $\alpha'=\prod\alpha_i$; it is a class of degree

$$\sum 2k_i = 2k-4nm = \frac{1}{3}(4n(7-3m)-8).$$

With the assumption (ii) or (iii) holds, a simple exercise shows that we are in one of the following situations:

(a) $m=2$ and α_i are relative Chern classes or relative ample line bundles,
(b) $m<2$ and there exists some $k_i\neq n$;
(c) $n=2$ and $\alpha=[c_4(\mathcal{T}_\pi)][c_2(\mathcal{T}_\pi)]^2$ or $[c_2(\mathcal{T}_\pi)]^4$.

We now deal with each of these cases separately.

Case (a). The class α' has degree $\frac{4n-8}{3}$ and it involves at most $\frac{2n-4}{3}$ distinct line bundles $\mathcal{L}_i$. Let Σ be the lattice generated by these line bundles and let $\mathcal{F}''$ be the corresponding moduli space of h-ample Σ-polarized hyper-Kähler varieties. Then α' can be obtained as the pullback of a cohomology class $\tilde\alpha$ on the universal family above $\mathcal{F}''$, in other words considering the diagram

$$\begin{tikzcd} \mathcal{U}'' \arrow[d, "\pi''"] & \mathcal{U} \arrow[l, "\rho''"'] \arrow[d, "\pi"] \\ \mathcal{F}'' & \mathcal{F} \arrow[l, "\iota''"'] \end{tikzcd}$$

we have $\alpha'=(\rho'')^*\tilde\alpha$.

Now we have

$$\pi''_*(c_{2n}(\mathcal{T}_\pi)^2\tilde\alpha)\in \mathrm{Hdg}^{\frac{16n-8}{3}}(\mathcal{F}'')$$

and

$$\dim\mathcal{F}''\geq \dim\mathcal{F}_h^\ell - \frac{2n-4}{3}+1 > 8n-5.$$

Theorem 7 therefore implies that

$$\pi''_*(c_{2n}(\mathcal{T}_\pi)^2\tilde\alpha)\in \mathrm{NL}^\bullet_{\mathrm{hom}}(\mathcal{F}'')$$

and hence

$$\pi_*(\alpha)=(\iota'')^*(\pi''_*(c_{2n}(\mathcal{T}_\pi)^2\tilde\alpha))\in \mathrm{NL}^\bullet_{\mathrm{hom}}(\mathcal{F})$$

as pullback preserves the NL ring.

In case (b). It follows from Theorem 9 that $\alpha'-\mathcal{L}^{k-2n-1}\mathcal{L}'$ is supported on the NL loci of $\mathcal{F}$ for some $\mathcal{L},\mathcal{L}'\in \mathrm{Pic}(\mathcal{U})$. Then it suffices to show that

$$\pi_*([c_{2n}(\mathcal{T}_\pi)]^m\mathcal{L}^{k-2n-1}\mathcal{L}')\in \mathrm{NL}^{k-2n}(\mathcal{F}). \tag{9}$$

We proceed as in case (a). Let Σ be the lattice generated by $\mathcal{L}$ and $\mathcal{L}'$. The class $[c_{2n}(\mathcal{T}_\pi)]^m\mathcal{L}^{k-2n-1}\mathcal{L}'$ can be obtained as the pullback of a cohomology class on the universal family associated to the moduli space of h-ample Σ-polarized hyper-Kähler manifolds. The rest of the proof is similar.

In case (c). The class α can be obtained as the pullback of a class in $\mathrm{H}^{2k}_{\leq 2n}(\mathcal{U}^{\ell}_{h})$ and we proceed as in the first two cases.

Finally, in case $\dim \mathcal{F} = 8n-4$, (8) holds for all $k \neq [\frac{8n-1}{3}] + 2n$ from Corollary 8 and we can proceed the discussion for the case $k = [\frac{8n-1}{3}] + 2n$ similarly. □

Proposition 13. *Suppose that $b \geq 16n - 12 > 4$ and $\mathrm{DR}^{\bullet}_{\mathrm{hom}}(\mathcal{F}^{\ell}_{h}) = \mathrm{NL}^{\bullet}_{\mathrm{hom}}(\mathcal{F}^{\ell}_{h})$. Then we have*

$$\mathrm{NL}^{\bullet}_{\mathrm{hom}}(\mathcal{F}^{\ell}_{h}) = \mathrm{R}^{\bullet}_{\mathrm{hom}}(\mathcal{F}^{\ell}_{h})$$

if $(*)$ holds, and even

$$\mathrm{NL}^{\bullet}_{\mathrm{hom}}(\mathcal{F}^{\ell}_{h}) = \widetilde{\mathrm{R}}^{\bullet}_{\mathrm{hom}}(\mathcal{F}^{\ell}_{h}).$$

if $(**)$ holds.

Proof. According to the proof in Proposition 11, we have

$$\mathrm{NL}^{\bullet}_{\mathrm{hom}}(\mathcal{F}^{\ell}_{h}) = \widetilde{\mathrm{R}}^{\bullet}_{\mathrm{hom}}(\mathcal{F}^{\ell}_{h})$$

once the inclusion (6) holds for any $\mathcal{U} \to \mathcal{F}$ with $\dim \mathcal{F} \leq 8n-4$. So it suffices to check (6) when $\dim \mathcal{F} \leq 8n-4$. By Lemma 10, any class in $\mathrm{H}^{\bullet}_{\leq 2n}(\mathcal{U})$ can be expressed as a linear combination of classes which, up to a class supported on the Noether–Lefschetz locus, is a product of Hodge classes on $\mathcal{U}$ of degree $\leq 2n$ and top degree relative Chern class $c_{2n}(\mathcal{T}_{\pi})$. When $(**)$ holds, such classes satisfy either the condition (i) or the condition (ii) in Lemma 12 from which the assertion follows.

Similarly, to prove

$$\mathrm{NL}^{\bullet}_{\mathrm{hom}}(\mathcal{F}^{\ell}_{h}) = \mathrm{R}^{\bullet}_{\mathrm{hom}}(\mathcal{F}^{\ell}_{h}),$$

it suffices to show that

$$\iota_*(\pi_*\mathrm{H}^{\bullet}_{\leq 2n-1}(\mathcal{U})) \subseteq \mathrm{NL}^{\bullet}_{\mathrm{hom}}(\mathcal{F}^{\ell}_{h}) \text{ and } \iota_*(\pi_*\mathrm{BV}^{\bullet}_{\mathrm{hom}}(\mathcal{U})) \subseteq \mathrm{NL}^{\bullet}_{\mathrm{hom}}(\mathcal{F}^{\ell}_{h}) \tag{10}$$

for all $\mathcal{U} \to \mathcal{F}$ with $\dim \mathcal{F} \leq 8n-4$. As before, we use Lemma 10 to express the classes in $\mathrm{H}^{\bullet}_{\leq 2n-1}(\mathcal{U})$, up to classes supported on the NL locus, as linear combinations of products of Hodge classes on $\mathcal{U}$ of degree $\leq 2n$. The assumption $(*)$ ensures that such product satisfies one of the conditions in Lemma 12. For classes in the second part of (10), they automatically satisfy the condition (ii) of Lemma 12. Thus we can conclude the assertion by applying Lemma 12 again.

□

4.3. **Proof of Theorem 5.** Assume that $b \geq 16n - 12$. We first prove the following

Lemma 14. *Let $\mathcal{L}$ be a universal polarization of $\mathcal{U}^{\ell}_{h} \to \mathcal{F}^{\ell}_{h}$. For all integer $k \geq 2n+1$ we have*

$$(\pi^{\ell}_{h})_*(\mathcal{L}^{k}) \in \mathrm{NL}^{k-2n}_{\mathrm{hom}}(\mathcal{F}^{\ell}_{h}). \tag{11}$$

Proof. We proceed by induction on $k \geq 2n+1$. Theorem 7(ii) implies that $\mathcal{L}^{2n+1}$ is supported on the Noether–Lefschetz locus of $\mathcal{F}_h^\ell$. Assume (11) holds for $k \leq k_0$. Now let $k = k_0 + 1 > 2n$. By frequently replacing a factor $\mathcal{L}^{2n+1}$ in $\mathcal{L}^k$ with a Hodge class supported on the Noether-Lefscehtz loci of $\mathcal{F}_h^\ell$, we are reduced to prove that for every Σ and every $\gamma \in \mathrm{Hdg}^{2j}(\mathcal{F}_{\Sigma,h}^\ell)$ with $j \leq 2n-1$ and $j + j' + b = k + \dim \mathcal{F}_{\Sigma,h}^\ell$, we have

$$(\iota_\Sigma)_*(\pi_\Sigma)_*(\mathcal{L}_\Sigma^{j'}\gamma) \in \mathrm{NL}_{\mathrm{hom}}^{k-2n}(\mathcal{F}_h^\ell), \tag{12}$$

where

$$\begin{tikzcd} \mathcal{U}_{\Sigma,h}^\ell \arrow[r, "\rho_\Sigma"] \arrow[d, "\pi_\Sigma"] & \mathcal{U}_h^\ell \arrow[d, "\pi_h^\ell"] \\ \mathcal{F}_{\Sigma,h}^\ell \arrow[r, "\iota_\Sigma"] & \mathcal{F}_h^\ell \end{tikzcd}$$

and we write $\mathcal{L}_\Sigma = \rho_\Sigma^*\mathcal{L}$ for simplicity.

The proof of (12) is similar to that of Proposition 11. We first explain how to reduce to $\mathcal{F}_{\Sigma,h}^\ell$ with $\dim \mathcal{F}_{\Sigma,h}^\ell \leq \sup\{8n-4, 6n-1\}$. Indeed: if $\dim \mathcal{F}_{\Sigma,h}^\ell \geq \sup\{8n-3, 6n\}$, Theorem 9 implies that there exists a constant a such that the difference

$$(\mathcal{L}_\Sigma)^{2n-j}\gamma - a\mathcal{L}_\Sigma^{2n} \tag{13}$$

is supported on the Noether–Lefschetz locus of $\mathcal{F}_{\Sigma,h}^\ell$. It follows that

$$\mathcal{L}_\Sigma^{j'}\gamma - a\mathcal{L}_\Sigma^{k+\dim \mathcal{F}_{\Sigma,h}^\ell - b} = \mathcal{L}_\Sigma^{k+\dim \mathcal{F}_{\Sigma,h}^\ell - b - 2n}(\mathcal{L}_\Sigma^{2n-j}\gamma - a\mathcal{L}_\Sigma^{2n}) \tag{14}$$

can be expressed as linear combinations of the pushforward of the class

$$\mathcal{L}_{\Sigma_i}^{j_i}\gamma_i \in \mathrm{H}_{\leq 2n-1}^\bullet(\mathcal{U}_{\Sigma_i,h}^\ell)$$

via ρ_{Σ_i}, where $\rho_{\Sigma_i} : \mathcal{U}_{\Sigma_i,h}^\ell \to \mathcal{U}_{\Sigma,h}^\ell$ is the universal family of some sublattice polarized hyper-Kähler varieties and $\gamma_i \in \mathrm{Hdg}^{2k_i}(\mathcal{F}_{\Sigma_i,h}^\ell)$ with $k_i < 2n$.

Since $b - \dim \mathcal{F}_{\Sigma,h}^\ell \geq 1$, the inductive hypothesis implies that

$$(\pi_h^\ell)_*\mathcal{L}^{k+\dim \mathcal{F}_{\Sigma,h}^\ell - b} \in \mathrm{NL}_{\mathrm{hom}}^\bullet(\mathcal{F}_h^\ell),$$

therefore that

$$a(\pi_\Sigma)_*(\mathcal{L}_\Sigma)^{k+\dim \mathcal{F}_{\Sigma,h}^\ell - b} \in \mathrm{NL}_{\mathrm{hom}}^\bullet(\mathcal{F}_{\Sigma,h}^\ell) \tag{15}$$

and hence that the pushforward $(\iota_\Sigma)_*$ of the class (15) is in $\mathrm{NL}_{\mathrm{hom}}^{k-2n}(\mathcal{F}_h^\ell)$.

We are therefore reduced to show that $(\iota_{\Sigma_i})_*(\pi_{\Sigma_i})_*(\mathcal{L}_{\Sigma_i}^{j_i}\gamma_i) \in \mathrm{NL}_{\mathrm{hom}}^\bullet(\mathcal{F}_h^\ell)$ and this allows us to reduce the dimension of $\mathcal{F}_{\Sigma,h}^\ell$.

It remains to prove (12) when $\dim \mathcal{F}_{\Sigma,h}^\ell \leq \sup\{8n-4, 6n-1\}$. Since $b \geq \sup\{16n-12, 12n-4\}$, Corollary 8 applies as long as $\dim \mathcal{F}_{\Sigma,h}^\ell \leq \sup\{8n-6, 6n-2\}$ and we are left with the cases where $\dim \mathcal{F}_{\Sigma,h}^\ell = 8n-4$ or $8n-5$ (if $n > 1$). In both cases, any class of the form (12) satisfies the condition (iii) of Lemma 12 and hence belongs to $\mathrm{NL}_{\mathrm{hom}}^\bullet(\mathcal{F}_h^\ell)$.

□

Strengthening the proof of Lemma 14 we get:

Lemma 15. *Consider* $\mathcal{F}^{\ell}_{\Sigma,h}$ *of dimension* $\geq \sup\{8n-5, 6n-2\}$. *Let* $\mathcal{L}_1, \dots, \mathcal{L}_m \in \mathrm{Pic}(\mathcal{U}^{\ell}_{\Sigma,h})$ *and* $\gamma \in \mathrm{Hdg}^{2k}(\mathcal{U}^{\ell}_{\Sigma,h})$ *with* $k < 2n$. *Suppose* $m+k > 2n$, *then we have*

$$(\iota_\Sigma)_*(\pi_\Sigma)_*(\mathcal{L}_1\mathcal{L}_2 \dots \mathcal{L}_m\gamma) \in \mathrm{NL}^{\bullet}_{\mathrm{hom}}(\mathcal{F}^{\ell}_h). \tag{16}$$

Proof. If $\dim \mathcal{F}^{\ell}_{\Sigma,h} \geq \sup\{8n-3, 6n\}$, Theorem 9 implies that there exists a constant a such that the class

$$(\prod_{i=1}^{m} \mathcal{L}_i)\gamma - a\mathcal{L}_{\Sigma}^{k+m} \tag{17}$$

is supported on the Noether–Lefschetz locus of $\mathcal{F}^{\ell}_{\Sigma,h}$. Here $\mathcal{L}_\Sigma = \rho_\Sigma^*\mathcal{L}$ still denotes the pull-back of the universal polarisation fixed in Lemma 14.

Since $m+k > 2n$, Lemma 14 implies that $(\pi^{\ell}_h)_*(\mathcal{L}^{k+m})$ belongs to $\mathrm{NL}^{\bullet}_{\mathrm{hom}}(\mathcal{F}^{\ell}_h)$. We conclude that

$$(\pi_\Sigma)_*(\mathcal{L}_\Sigma^{k+m}) \in \mathrm{NL}^{\bullet}_{\mathrm{hom}}(\mathcal{F}^{\ell}_{\Sigma,h}).$$

It remains to prove that the pushforward of (17) is lying in $\mathrm{NL}^{\bullet}_{\mathrm{hom}}(\mathcal{F}^{\ell}_h)$. By a repeated use of Theorem 9, as in the proof of Lemma 14, we are inductively reduced to prove that (16) holds when $\mathcal{F}^{\ell}_{\Sigma,h} \leq \sup\{8n-4, 6n-1\}$. The two cases we are left with follow from Lemma 12. □

Lemma 15 implies in particular that for all $\mathcal{F}^{\ell}_{\Sigma,h}$ of dimension $\geq \sup\{8n-5, 6n-2\}$ we have

$$(\iota_\Sigma)_*((\pi^{\ell}_{\Sigma,h})_*\mathrm{DCH}^{\bullet}_{\mathrm{hom}}(\mathcal{U}^{\ell}_{\Sigma,h})) \subseteq \mathrm{NL}^{\bullet}_{\mathrm{hom}}(\mathcal{F}^{\ell}_h).$$

Then, as long as $b \geq 16n-12$ we have $\mathrm{DR}^{\bullet}_{\mathrm{hom}}(\mathcal{F}^{\ell}_h) = \mathrm{NL}^{\bullet}_{\mathrm{hom}}(\mathcal{F}^{\ell}_h)$, by Corollary 8. Then we get (4) from Proposition 11 and Proposition 13.

To conclude the proof of Theorem 5, it remains to check that both $K3$ surfaces and $K3^{[2]}$-type hyper-Kähler manifolds satisfy the conditions.

For $K3$ surfaces, the second Betti number is 22 and hence $b = \dim \mathcal{F}^{\ell}_h = 19$, which is greater than both $16-8=8$ and $12-2=10$.

For $K3^{[2]}$-type hyper-Kähler manifolds, the second Betti number is 23 and $b = \dim \mathcal{F}^{\ell}_h = 20$, which is exactly $16 \times 2 - 12 = 12 \times 2 - 4$. In this case, as a representation of $\mathrm{SO}(2, b; \mathbf{R})$ we have $\mathbf{H}^4 = \mathrm{Sym}^2(\mathbf{H}^2)$. The primitive part of $\mathbf{H}^2$ is the standard representation of the real Lie group $\mathrm{SO}(h^\perp)$. As a representation of $\mathrm{SO}(\Sigma^\perp)$, the space

$$\mathbf{H}^2 = \Sigma \oplus \mathbf{H}^2_{\mathrm{prim}}$$

decomposes as the direct sum of the subspace spanned by Σ on which $\mathrm{SO}(\Sigma^\perp)$ acts trivially and the standard representation $\mathbf{H}^2_{\mathrm{prim}}$. Here, $\mathbf{H}^2_{\mathrm{prim}}$ can be viewed as the Σ-primitive part of a very general hyper-Kähler X in $\mathcal{F}^{\ell}_{\Sigma,h}$, i.e. the subspace spanned by the transcendental lattice of X.

Suppose that $\mathrm{rank}(\Sigma^\perp) > 2$ and hence $\dim \mathcal{F}^\ell_{\Sigma,h} > 1$. Then Zarhin's computation in [7] show that the Hodge group of a very general hyper-Kähler variety X in $\mathcal{F}^\ell_{\Sigma,h}$ is isomorphic to the special orthogonal group $\mathrm{SO}(\Sigma^\perp)$. The degree 4 Hodge class on X are $\mathrm{SO}(\Sigma^\perp)$-invariant classes. As a $\mathrm{SO}(\Sigma^\perp)$-representation, the trivial isotypic subspace of $\mathbf{H}^4 = \mathrm{Sym}^2(\mathbf{H}^2)$ decomposes as the direct sum of $\mathrm{Sym}^2(\langle\Sigma\rangle)$ and the unique trivial summand in $\mathrm{Sym}^2(\mathbf{H}^2_{\mathrm{prim}})$. The trivial isotypic subspace in $\mathrm{Sym}^2(\langle\Sigma\rangle)$ is spanned by the products of line bundle classes. Finally, the trivial summand in $\mathrm{Sym}^2(\mathbf{H}^2_{\mathrm{prim}})$ is a monodromy invariant class. As the monodromy invariant class of $\mathbf{H}^4$ is the second Chern class of the tangent bundle (cf. [1, Lemma 3.2]), we conclude that condition $(*)$ indeed holds.

Remark 16. From our proof, one can see that we do not need the full strength of condition $(*)$ or $(**)$. What we really need is that: the group of Hodge classes of degree $= 2n$ (or $\leq 2n$ respectively) on general fibers of $\mathcal{U}^\ell_{\Sigma,h} \to \mathcal{F}^\ell_{\Sigma,h}$ is spanned by the product of line bundles and classes which can descends to the general fiber of $\mathcal{U}^\ell_h \to \mathcal{F}^\ell_h$.

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