# A step towards a general density Corrádi–Hajnal Theorem

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2023.03.30

# Hypergraph

***r*-graph**: A collection of $r$-subsets of some finite set $V$.

We identify an $r$-graph $\mathscr{H}$ with its **edge set** and use $V(\mathscr{H})$ to denote its **vertex set**. The **size** of $V(\mathscr{H})$ is denoted by $v(\mathscr{H})$.

Given two $r$-graphs $F$ and $\mathscr{H}$ we use $\nu(F,\mathscr{H})$ to denote the maximum of $k \in \mathbb{N}$ such that there exist $k$ vertex-disjoint copies of $F$ in $\mathscr{H}$. We call $\nu(F,\mathscr{H})$ the ***F*-matching number** of $\mathscr{H}$.

# Turán problem

Fix an $r$-graph $F$, we say another $r$-graph $\mathscr{H}$ is $F$**-free** if $\nu(F, \mathscr{H}) = 0$.

- **Turán number** $\mathrm{ex}(n, F)$ : the maximum number of edges in an $F$-free $r$-graph on $n$ vertices
- $\mathrm{EX}(n, F)$ : the collection of all $n$-vertex $F$-free $r$-graphs with exactly $\mathrm{ex}(n, F)$ edges
- **Turán density** $\pi(F)$ :

$$\pi(F) := \lim_{n\to\infty} \frac{\mathrm{ex}(n, F)}{\binom{n}{r}}$$

An $r$-graph $F$ is called **nondegenerate** if $\pi(F) > 0$.

## Problem 1

Given an $r$-graph $F$ and integers $n, t \in \mathbb{N}$:

*What kinds of constraints on an $n$-vertex $r$-graph $\mathscr{H}$ force it to satisfy $\nu(F, \mathscr{H}) \geq t+1$?*

**Theorem 2 (Erdős–Gallai Theorem, 1959)**

*For all integers $n, \ell \in \mathbb{N}$ with $t+1 \le n/2$ and for every $n$-vertex graph $G$,*

$$|G| > \max\left\{\binom{2t+1}{2}, \binom{n}{2} - \binom{n-t}{2}\right\} \quad \Rightarrow \quad \nu(G) \ge t+1.$$

**Conjecture 3 (Erdős, 1965)**

*Suppose that $n, t, r \in \mathbb{N}$ satisfy $r \ge 3$ and $t+1 \le n/r$. Then for every $n$-vertex $r$-graph $\mathscr{H}$,*

$$|\mathscr{H}| > \max\left\{\binom{r(t+1)-1}{r}, \binom{n}{r} - \binom{n-t}{r}\right\} \quad \Rightarrow \quad \nu(\mathscr{H}) \ge t+1.$$

**Theorem 4 (Corrádi–Hajnal 1963)**

*Suppose that $n, t \in \mathbb{N}$ are integers with $t \le n/3$. Then for every n-vertex graph G,*

$$\delta(G) \ge t + \left\lfloor \frac{n-t}{2} \right\rfloor \quad \Rightarrow \quad \nu(K_3, G) \ge t.$$

*In particular, if $3 \mid n$, then every n-vertex graph G with $\delta(G) \ge 2n/3$ contains a $K_3$-factor.*

**Theorem 5 (Hajnal–Szemerédi Theorem, 1970)**

*For all integers $n \ge \ell \ge 2$, $t \le \lfloor n/(\ell+1) \rfloor$, and for every n-vertex graph G,*

$$\delta(G) \ge t + \left\lfloor \frac{\ell-1}{\ell}(n-t) \right\rfloor \quad \Rightarrow \quad \nu(K_{\ell+1}, G) \ge t.$$

Given two $r$-graphs $\mathscr{G}$ and $\mathscr{H}$ whose vertex sets are disjoint.

we define the **join** $\mathscr{G} \boxtimes \mathscr{H}$ of $\mathscr{G}$ and $\mathscr{H}$ to be the $r$-graph obtained from $\mathscr{G} \sqcup \mathscr{H}$ (the vertex-disjoint union of $\mathscr{G}$ and $\mathscr{H}$) by adding all $r$-sets that have nonempty intersection with both $V(\mathscr{G})$ and $V(\mathscr{H})$.

For a family $\mathscr{F}$ of $r$-graphs, $\mathscr{H} \boxtimes \mathscr{F} := \{\mathscr{H} \boxtimes \mathscr{G} : \mathscr{G} \in \mathscr{F}\}$.

## Theorem 6 (Erdős, 1962)

*Suppose that* $n, t \in \mathbb{N}$ *and* $t \leq \sqrt{n/400}$. *Then*

$$\mathrm{EX}\left(n, (t+1)K_3\right) = \{K_t \boxtimes T(n-t, 2)\}.$$

## Theorem 7 (Moon, 1968)

*Suppose that integers* $n, t, \ell \in \mathbb{N}$ *satisfy* $\ell \geq 2$, $t \leq \frac{2n-3\ell^2+2\ell}{\ell^3+2\ell^2+\ell+1}$, *and* $\ell \mid (n-t)$. *Then*

$$\mathrm{EX}\left(n, (t+1)K_{\ell+1}\right) = \{K_t \boxtimes T(n-t, \ell)\}.$$

**Definition 8 (Boundedness)**

Let $f_1, f_2 \colon \mathbb{N} \to \mathbb{R}$ be two nonnegative functions. An $r$-graph $F$ is $(f_1,f_2)$-bounded if every $F$-free $r$-graph $\mathcal{H}$ on $n$ vertices with

$$d(\mathcal{H}) \geq \frac{r \cdot \mathrm{ex}(n,F)}{n} - f_1(n) \quad \Rightarrow \quad \Delta(\mathcal{H}) \leq \frac{r \cdot \mathrm{ex}(n,F)}{n} + f_2(n).$$

**Definition 9 (Smoothness)**

Let $g \colon \mathbb{N} \to \mathbb{R}$ be a nonnegative function. The Turán function $\mathrm{ex}(n,F)$ of an $r$-graph $F$ is $g$-smooth if

$$\left| \mathrm{ex}(n,F) - \mathrm{ex}(n-1,F) - \frac{r \cdot \mathrm{ex}(n-1,F)}{n-1} \right| \leq g(n) \quad \text{holds for all } n \in \mathbb{N}.$$

## Theorem 10

*Fix integers $m \geq r \geq 2$ and a nondegenerate r-graph $F$ on $m$ vertices. Suppose that there exists a constant $c > 0$ such that for all sufficiently large $n \in \mathbb{N}$:*

(1) *$F$ is $\left(c\binom{n}{r-1}, \frac{1-\pi(F)}{4m}\binom{n}{r-1}\right)$-bounded, and*

(2) *$\mathrm{ex}(n,F)$ is $\frac{1-\pi(F)}{8m}\binom{n}{r-1}$-smooth.*

*Then there exists $N_0$ such that for all integers $n \geq N_0$ and $t \leq \min\left\{\frac{c}{4erm}n, \frac{1-\pi(F)}{64rm^2}n\right\}$, we have*

$$\mathrm{EX}\left(n,(t+1)F\right) = K_t^r \boxtimes \mathrm{EX}(n-t,F),$$

*and, in particular,*

$$\mathrm{ex}\left(n,(t+1)F\right) = \binom{n}{r} - \binom{n-t}{r} + \mathrm{ex}(n-t,F).$$

# Applications

**Theorem 11**

*Suppose that $F$ is an edge-critical graph with $\chi(F) \geq 3$. Then there exist constants $N_0$ and $c_F > 0$ such that for all integers $n \geq N_0$ and $t \in [0, c_F n]$ we have*

$$\mathrm{EX}(n, (t+1)F) = \{K_t \boxtimes T(n-t, \chi(F)-1)\}.$$

# Applications

**Theorem 12**

*There exist constants $N_0$ and $c_{\mathbb{F}} > 0$ such that for all integers $n \geq N_0$ and $t \in [0, c_{\mathbb{F}} n]$ we have*

$$\mathrm{EX}(n, (t+1)\mathbb{F}) = \left\{ K_t^3 \boxtimes B_3(n-t) \right\}.$$

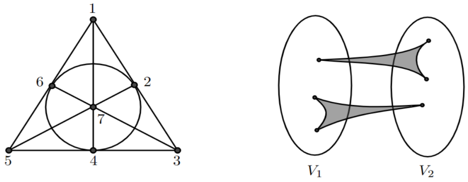


Figure: The Fano plane and the complete bipartite 3-graph $B_3(n)$.

# Applications

**Theorem 13**

*There exist constants $N_0$ and $c_{\mathrm{T}_3}$ such that for all integers $n \geq N_0$ and $t \in [0, c_{\mathrm{T}_3} n]$ we have*

$$\mathrm{EX}(n, (t+1)\mathrm{T}_3) = \left\{ K_t^3 \boxtimes T_3(n-t, 3) \right\}.$$

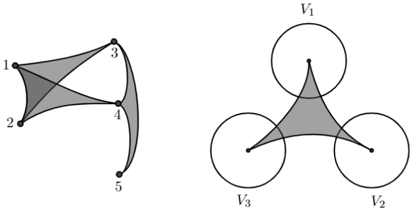


Figure: The generealized triangle $\mathrm{T}_3$ and the Turán 3-graph $T_3(n,3)$.

# Applications

**Theorem 14**

*Fix integers $\ell \geq r \geq 2$. There exist constants $N_0$ and $c = c(\ell, r) > 0$ such that for all integers $n \geq N_0$ and $t \in [0, cn]$ we have*

$$\mathrm{EX}(n, (t+1)H^r_{\ell+1}) = \{K^r_t \boxtimes T_r(n-t, \ell)\}.$$

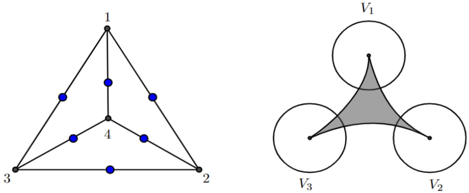


Figure: The expansion $H^3_4$ of $K_4$ and the Turán 3-graph $T_3(n,3)$.

# Applications

**Theorem 15**

*For every integer $r \geq 2$ there exist constants $N_0$ and $c > 0$ such that for all integers $n \geq N_0$ and $t \in [0, cn]$, we have*

$$\mathrm{EX}\left(n, (t+1)\mathscr{C}_3^{2r}\right) \subset K_t^{2r} \boxtimes \left\{B_{2r}^{\mathrm{odd}}(n-t, m) : m \in \left[0, \sqrt{2r(n-t)}\right]\right\}.$$

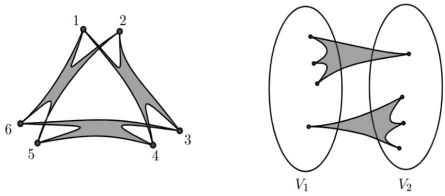


Figure: The 4-graph $\mathscr{C}_3^4$ (expanded triangle) and the 4-graph $B_4^{\mathrm{odd}}(n)$.

# Applications

**Theorem 16**

*There exist constants $N_0$ and $c > 0$ such that for all integers $n \geq N_0$ and $t \in [0, cn]$, we have*

$$\mathrm{EX}(n, (t+1)F_7) = \left\{ K_t^4 \boxtimes B_4^{\text{even}}(n-t) \right\}.$$

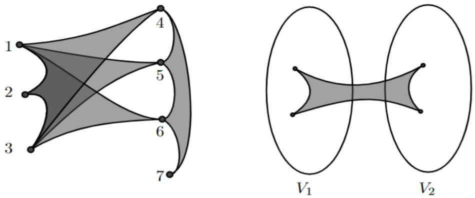


Figure: The 4-graph $F_7$ (4-book with 3 pages) and the 4-graph $B_4^{\text{even}}(n)$.

# Applications

**Theorem 17**

*There exist constants $N_0$ and $c > 0$ such that for all integers $n \geq N_0$ and $t \in [0, cn]$, we have*

$$\mathrm{EX}(n, (t+1)\mathbb{F}_{4,3}) \subset K_t^4 \boxtimes \left\{ B_4^{\mathrm{odd}}(n-t, m) : m \in [0, \sqrt{4(n-t)}] \right\}.$$

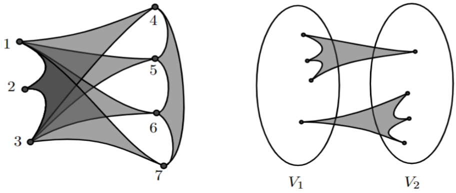


Figure: The 4-graph $\mathbb{F}_{4,3}$ and the 4-graph $B_4^{\mathrm{odd}}(n)$.

# Applications

## Theorem 18

*There exist constants $N_0$ and $c > 0$ such that for all integers $n \geq N_0$ and $t \in [0, cn]$, we have*

$$\mathrm{EX}\left(n, (t+1)\mathbb{F}_{3,2}\right) = \left\{K_t^r \boxtimes S_3(n-t)\right\}.$$

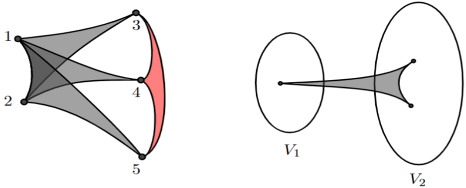


Figure: The 3-graph $\mathbb{F}_{3,2}$ and the semibipartite 3-graph $S_3(n)$.

# Problems

## Problem 19

*Let $r \geq 2$ be an integer and $F$ be a nondegenerate $r$-graph with $m$ vertices. For large $n$ determine $\mathrm{ex}(n,(t+1)F)$ for all $t \leq n/m$.*

## Problem 20

*Let $r \geq 2$ be an integer and $F$ be an $r$-graph with $m$ vertices. For large $n$ determine the maximum value of $s(n,F)$ such that*

$$\mathrm{ex}(n,(t+1)F) = \binom{n}{r} - \binom{n-t}{r} + \mathrm{ex}(n-t,F)$$

*holds for all $t \in [0, s(n,F)]$.*