Planar Turán Number: Plane Graph Decomposition and Contribution Method

Zeyu Zheng

Joint work with Professor Ervin Győri and Xianzhi Wang

Fudan University

zeyuzheng19@fudan.edu.cn

September 30, 2022

Notations

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A graph G is H-free if and only if there's no subgraph of G isomorphic to H. A graph G is $\mathcal{F}$-free if and only if there's no subgraph of G isomorphic to any graph in $\mathcal{F}$.

For a planar graph G, we use v_G, e_G, f_G to denote the number of vertices, edges, faces in G, respectively.

Definition of planar Turán number

Recall the definition of Turán number $\mathrm{ex}(n, H)$:

$$\mathrm{ex}(n, H) = \max\{|E(G)| : |V(G)| = n, G \text{ is } H\text{-free}\},$$

$$\mathrm{ex}(n, \mathcal{F}) = \max\{|E(G)| : |V(G)| = n, G \text{ is } \mathcal{F}\text{-free}\}.$$

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Similarly, planar Turán number $\mathrm{ex}_{\mathcal{P}}(n, H)$ (Dowden 2016 [1]) is defined as:

$$\mathrm{ex}_{\mathcal{P}}(n, H) = \max\{|E(G)| : |V(G)| = n, G \text{ is planar and } H\text{-free}\},$$

$$\mathrm{ex}_{\mathcal{P}}(n, \mathcal{F}) = \max\{|E(G)| : |V(G)| = n, G \text{ is planar and } \mathcal{F}\text{-free}\}.$$

Roughly speaking, planar Turán number is the maximum number of edges in a H-free / $\mathcal{F}$-free planar graph of n vertices.

Classical results of Turán number

Theorem (Turán [6])

$$\mathrm{ex}(n, K_{r+1}) = (1 - \frac{1}{r})\frac{n^2}{2} - O(n).$$

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Theorem (Erdős-Simonovits [2])

If H is an $(r+1)$-chromatic graph, then

$$\mathrm{ex}(n, H) = (1 + o(1))\frac{n^2}{2}(1 - \frac{1}{r}).$$

Connections between Turán number and planar Turán number

Theorem (Kuratowski [4])

A graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$.

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By Kuratowski's theorem, it's not hard to find that planar Turán number is just a special case of Turán number.

Proposition

Denote all the subdivisions of K_5 and $K_{3,3}$ by $\mathcal{G}$. We have

$$\mathrm{ex}_{\mathcal{P}}(n, \mathcal{F}) = \mathrm{ex}(n, \mathcal{F} \cup \mathcal{G}).$$

Examples

Example 1

If H is a non-planar graph, then

$$\mathrm{ex}_{\mathcal{P}}(n, H) = 3n - 6.$$

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This is equivalent to say, the maximum number of edges in a planar graph on n vertices. For any planar graph G, if we plug in $3f_G \leq 2e_G$ to the Euler's formula $v_G - e_G + f_G = 2$, we will get $\mathrm{ex}_{\mathcal{P}}(n, H) \leq 3n - 6$. The equality comes from a trivial extremal construction.

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$$\mathrm{ex}_{\mathcal{P}}(n, K_3) = 2n - 4.$$

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For any planar graph G, this time we plug in $4f_G \leq 2e_G$ to the Euler's formula $v_G - e_G + f_G = 2$ instead, we will get $\mathrm{ex}_{\mathcal{P}}(n, H) \leq 2n - 4$. The equality comes from the extremal construction $K_{2,n-2}$.

Current progress in this topic

In 2016, Dowden initiated the study of planar Turán number. He proved:

Theorem (Dowden 2016 [1])

$\mathrm{ex}_{\mathcal{P}}(n, C_4) \leq 15(n-2)/7$ for all $n \geq 4$.
$\mathrm{ex}_{\mathcal{P}}(n, C_5) \leq (12n-33)/5$ for all $n \geq 11$.
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Theorem (Ghosh, Győri, Martin, Paulos and Xiao 2020 [3])

$\mathrm{ex}_{\mathcal{P}}(n, C_6) \leq \frac{5n}{2} - 7$ for all $n \geq 18$.
This bound is sharp.

Current progress in this topic

Θ_k graph means a cycle with a chord. In 2019, Lan, Shi and Song proved:

Theorem (Lan, Shi and Song 2019 [5])

$ex_{\mathcal{P}}(n, \Theta_4) \leq \frac{12(n-2)}{5}$ for all $n \geq 4$.
$ex_{\mathcal{P}}(n, \Theta_5) \leq \frac{5(n-2)}{2}$ for all $n \geq 5$. These bounds are sharp.
$ex_{\mathcal{P}}(n, C_6) \leq ex_{\mathcal{P}}(n, \Theta_6) \leq \frac{18(n-2)}{7}$ for all $n \geq 6$.

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Introduction of a useful tool

In 2020, Ghosh, Győri, Martin, Paulos and Xiao introduced the triangular block decomposition in the proof of the sharp upper bound of $\mathrm{ex}_{\mathcal{P}}(n, C_6)$. It's not surprising for us to find that it's also applicable for $\mathrm{ex}_{\mathcal{P}}(n, C_5)$.

Now, we give a short alternative proof of Dowden's result on $\mathrm{ex}_{\mathcal{P}}(n, C_5)$ using this method.

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Now, we give a short alternative proof of Dowden's result on $\mathrm{ex}_{\mathcal{P}}(n, C_5)$ using this method.

Before we start, it's not hard to note that if there is a vertex of degree at most 2 in a C_5-free planar graph G, then we can delete it and finish by induction. Also, if there is a cut vertex in G, we can also finish by induction by considering the blocks of the graph.

Homogenizatin of the inequality and idea of decomposition

If we plug in the Euler's formula $v_G - e_G + f_G = 2$ to the target inequality

$$e_G \leq \frac{12v_G - 33}{5},$$

we will get, equivalently,

$$9v_G - 23e_G + 33f_G \leq 0,$$

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The idea is, to find a proper way to decompose the graph into small "pieces", so that the homogeneous inequality holds for the contribution of each "piece".

Triangular block

Definition of triangular blocks

$B \leftarrow (V(e), e)$;
while *there exists an edge in* B *such that it is in a bounded* 3*-face of* G *which is not contained in* B **do**
  add all the edges of such bounded 3-faces to B;
end
Output B;

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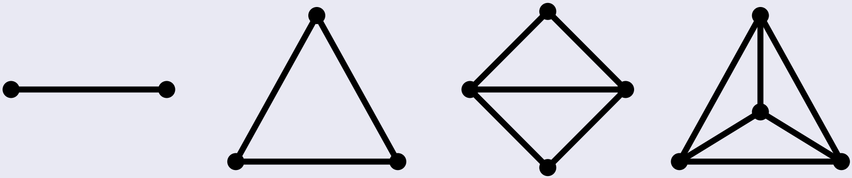
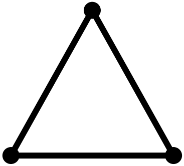
Triangular blocks in a C_5-free planar graph



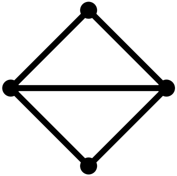
K_2 K_3 Θ_4 K_4

Notations of triangular blocks

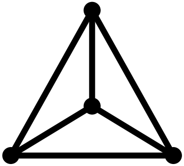
K_2



K_3



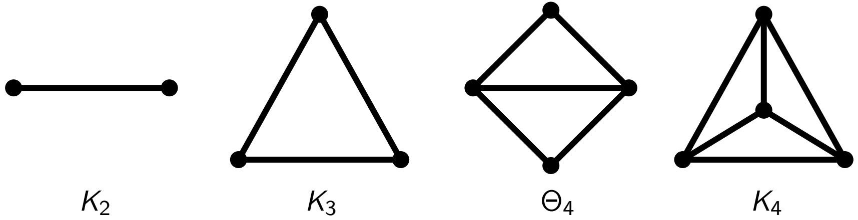
Θ_4



K_4

A vertex shared by at least two triangular blocks is called a junction vertex, otherwise it's a non-junction vertex.

Notations of triangular blocks



A vertex shared by at least two triangular blocks is called a junction vertex, otherwise it's a non-junction vertex.

A bounded face contained by a triangular block B is called an interior face of B. A face intersecting B but is not contained by B is called an exterior face of B.

Contributions of each block

Definition of contributions

We denote the contribution of a triangular block to the number of vertices, edges and faces by $v(B)$, $e(B)$ and $f(B)$, respectively, and definie them as follows:

$$v(B) = \sum_{v \in V(B)} \frac{1}{\#\ \textit{triangular blocks in } G \textit{ containing } v},$$

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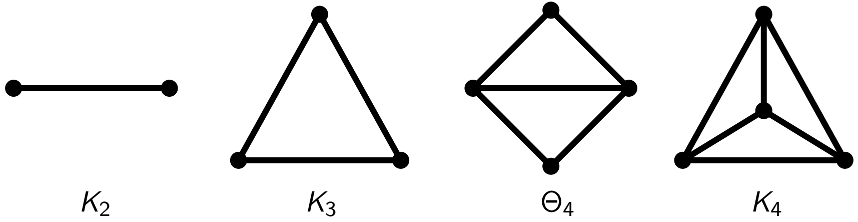
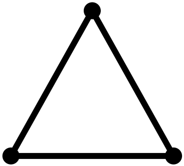
Contribution calculation

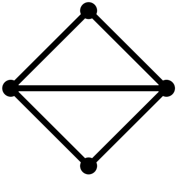


Contribution calculation

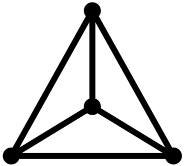
K_2



K_3



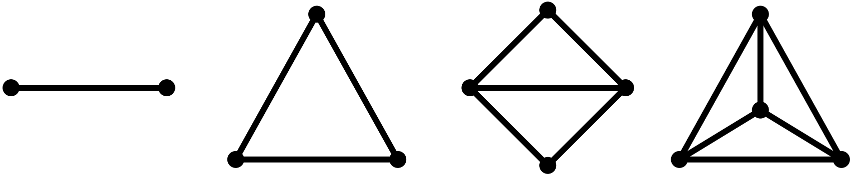
Θ_4



K_4

There's a problem!!!

Contribution calculation



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There's a problem!!!

How to fix it?

Definition of a pseudoface

Each face in G is either an interior face of a unique triangular block or an exterior face of some triangular blocks.

Definition

For an exterior face f, if its boundary contains two consecutive exterior edges of a triangular block B that is a K_4, then we replace them by the other exterior edge of B to get a smaller cycle recursively until there's no consecutive edges contained in the cycle. We call the resulting cycle an **exterior pseudoface**. We denote it by C_f, and its length by $l'(f)$.

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Note that if there is no triangular block being K_4 satisfying the previous description, an exterior pseudoface is just an exterior face.

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Problem solved, which finishes the proof as an upper bound.

What about the extremal construction?

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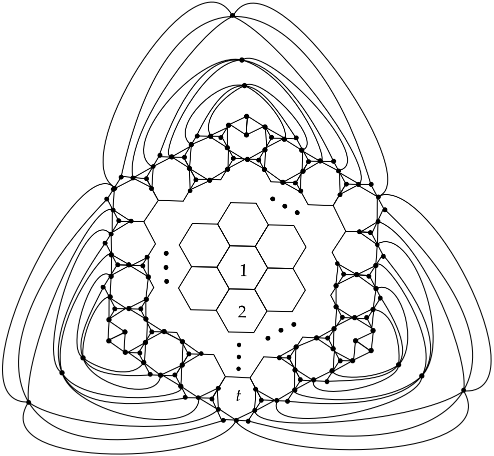
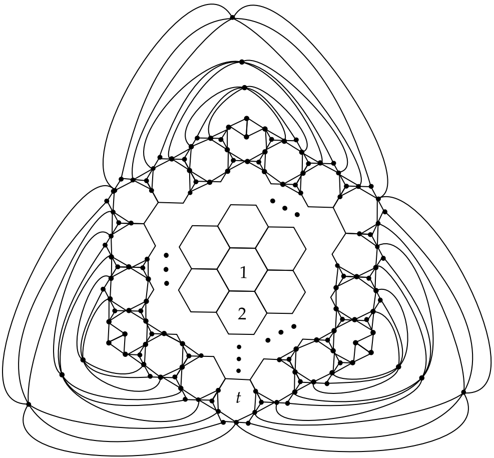
What about the extremal construction?

We need to find a graph which only contains those "0-contribution" blocks!

Extremal construction



Extremal construction



In the extremal construction, there are $15t^2 - 6$ vertices and $36t^2 - 21$ edges, which satisfy that $e_G = (12v_G - 33)/5$.

Some thoughs of quadrangular blocks

Now that we have finished the proof and extremal construction using triangular blocks. How to extend this method to other problems? What about quadrangular blocks?

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In this case, we need to exclude 3-faces. Naturally, we thought of bipartite / triangle-free planar graphs as the host graph and we can consider the Turán-type problem for even cycles.

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However, $K_{n-2,2}$ shows that we need something more to avoid triviality, since this graph only contains 4-cycles and the number of edges has already reached the maximum as a bipartite/ triangle-free planar graph.

Ideas to avoid triviality

1. To restrict the maximum degree
   Motivation: in the "bad" construction, $\Delta(G) = n - 2$.
   Problem: How to restrict the maximum degree?
   $n/2$? $\log n$? constant?

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2. To restrict the number of degree-2 vertices
   Motivation: in the "bad" construction, there are $n - 2$ degree-2 vertices.
   A natural way: let $\delta(G) \geq 3$.

Quadrangular blocks

Definition of quadrangular blocks

$B \leftarrow (V(e), e)$;
while *there exists an edge in B such that it is in a bounded 4-face of G which is not contained in B* **do**
  add all the edges of such bounded 4-faces to B;
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Bipartite planar graph forbidding C_6

In this case, degree-2 vertices are unavoidable.

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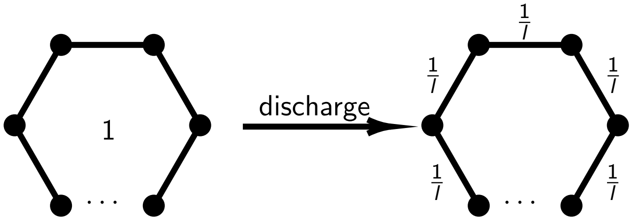
Proof sketch:

Suppose the contrary that there exists such a graph $G = (V, E)$.

"Discharging method": We assign charge 1 to each face, and discharge $1/l$ to each edge of it, where l is the length of the face.



The existence of degree-2 vertices

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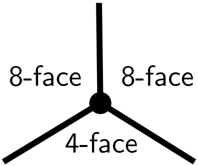
Hence, for any degree-3 vertex, the three edges incident to it will have total charge at most $3/8 + 3/8 + 1/4 = 1$.



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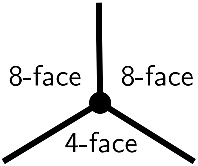

For any other vertex of degree $d \geq 4$, the edges incident to it will have total charge $\leq 3d/8$.
Let $\chi(e)$ be the charge on edge e after the discharging, we have $\sum_{v \in V} \sum_{e \ni v} \chi(e) \leq n_3 + \frac{3}{8} \sum_{i \geq 4} i n_i$, where n_i stands for the number of degree-i vertices.

The existence of degree-2 vertices

Note that in $\sum_{v\in V}\sum_{e\ni v}\chi(e)$ we counted the charge on each edge twice, we have

$$2f = \sum_{v\in V}\sum_{e\ni v}\chi(e) \leq n_3 + \frac{3}{8}\sum_{i\geq 4} i n_i$$

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$$\iff 4\leq n_3+\frac{3}{8}\sum_{i\geq 4}in_i+2\sum_{i\geq 3}n_i-\sum_{i\geq 3}in_i=\sum_{i\geq 4}(2-\frac{5}{8}i)n_i$$

The existence of degree-2 vertices

Note that in $\sum_{v\in V}\sum_{e\ni v}\chi(e)$ we counted the charge on each edge twice, we have

$$2f=\sum_{v\in V}\sum_{e\ni v}\chi(e)\le n_3+\frac{3}{8}\sum_{i\ge 4} in_i$$

$$\iff 4-2n+2e=2f\le n_3+\frac{3}{8}\sum_{i\ge 4} in_i$$

$$\iff 4-2\sum_{i\ge 3} n_i+\sum_{i\ge 3} in_i\le n_3+\frac{3}{8}\sum_{i\ge 4} in_i$$

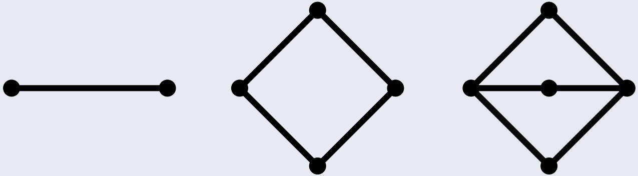
$$\iff 4\le n_3+\frac{3}{8}\sum_{i\ge 4} in_i+2\sum_{i\ge 3} n_i-\sum_{i\ge 3} in_i=\sum_{i\ge 4}(2-\frac{5}{8}i)n_i$$

While $2-i\cdot 5/8<0$ for $i\ge 4$, a contradiction. Thus, we know that $\delta(G)\le 2$.

A step further

Since there must be degree-2 vertices in a C_6-free bipartite planar graph G. It's natural to consider including the number of degree 2 vertices (denoted by k) in our bound. Now all the possible quadrangular blocks are:

Quadrangular blocks in a C_6-free bipartite planar graph



K_2 C_4 $K_{2,3}$

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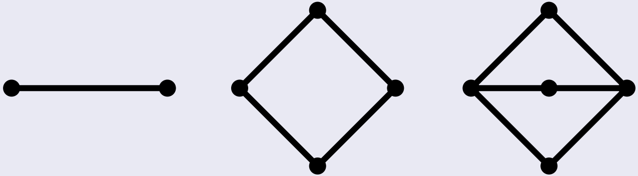

K_2 $\quad$ C_4 $\quad$ $K_{2,3}$

Whereas, when we only include k in the bound, we can't really find a sharp construction. The problematic block is the $K_{2,3}$ one. We finally resolved this problem by introducing $e_{2,3}$, the number of edges connecting a degree 2 vertex and a degree 3 vertex to our bound.

A result on C_6-free bipartite planar graphs

Theorem

Let G be a C_6-free planar bipartite graph on n vertices. Then $\delta(G) \leq 2$, and if any degree 2 vertex v in G has a neighbor of degree at most 3, then

$$e_G \leq \frac{3}{2}n + \frac{1}{2}k + \frac{1}{4}e_{2,3} - 4,$$

for all $n \geq 6$, where k is the number of degree 2 vertices in G and $e_{2,3}$ is the number of edges xy in G such that $d(x) = 2$ and $d(y) = 3$.

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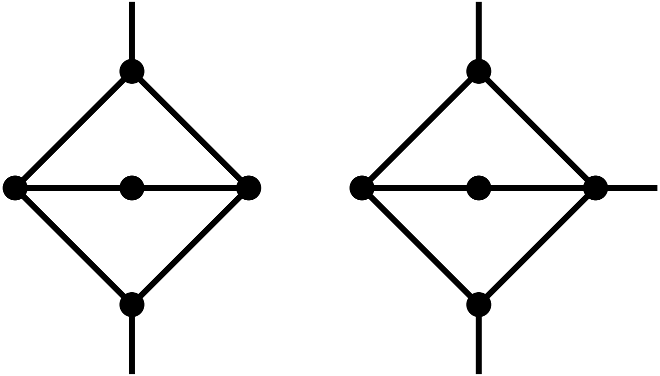
Of course we need more rigorous definitions to finish the proof.(contribution of k, $e_{2,3}$, etc.) But once we finished the set-up and use the method introduced here, the proof part is pretty much the same as before. The more tricky part is to find an extremal construction.

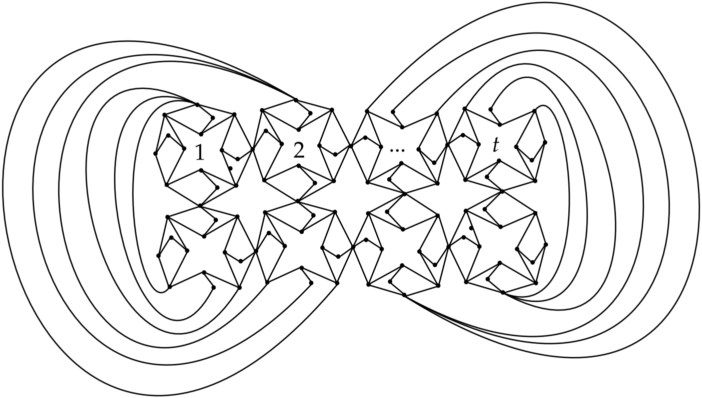
0-contribution cases

The two 0-contribution cases:



Here, we need to specify that, the 0-contribution cases not only include the block itself, but also all the surrounding information we used in the inequality.

Extremal Construction



It only contains the 0-contribution quadrangular blocks.

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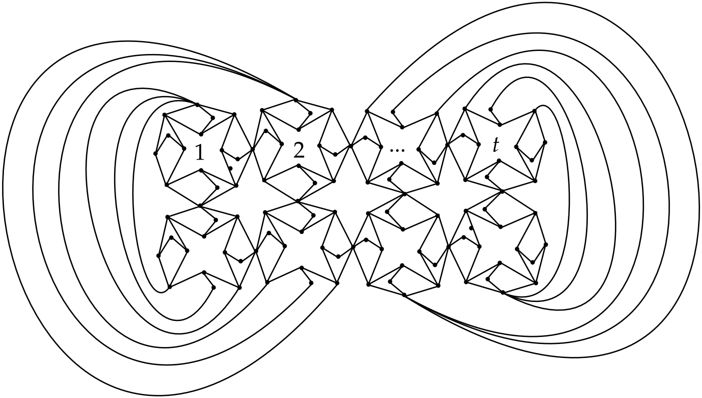

It only contains the 0-contribution quadrangular blocks. There are $28t + 2$ vertices, $48t$ edges, $8t$ degree 2 vertices, and $8t + 4$ edges joining a degree 2 vertex and a degree 3 vertex, which satisfy that
$e_G = 3v_G/2 + k/2 + e_{2,3}/4 - 4$.

C_8-free bipartite planar graphs

Now, we can let $\delta(G) \geq 3$.

Theorem

Let G be a C_8-free planar bipartite graph with $\delta(G) \geq 3$ on n vertices. Then

$$e_G \leq \frac{5}{3}n - \frac{10}{3}.$$

The equality holds for infinitely many integers n.

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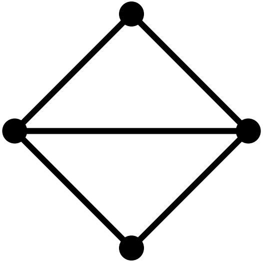
Again, the proof is pretty much the same. Note that here, the homogeneous inequality equivalent to the bound is $5f(B) - 2e(B) \leq 0$, while $v(B)$ does not appear. In this case, the contribution method is a rewording of the classic discharging method, one can easily give a proof using discharging method by assigning an unbalanced discharging according to different quadrangular blocks. Thus, in some sense, the contribution method is an extension of the classic discharging method.

Extremal Construction

There's only one 0-contribution block:



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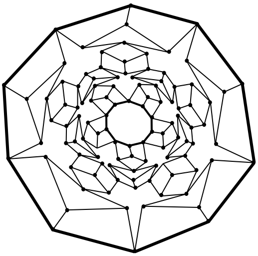

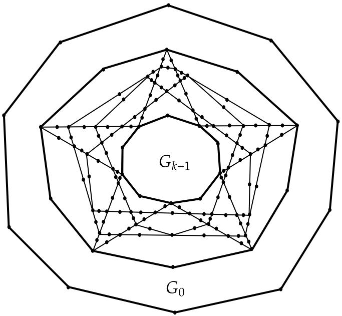

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How to get a bound?

Actually, in the last result, we first obtained the bound by discharging method. One problem is, to use contribution method, you have to have a conjectured bound. How do we get such a promising bound?

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Actually, in the last result, we first obtained the bound by discharging method. One problem is, to use contribution method, you have to have a conjectured bound. How do we get such a promising bound?

We use undetermined coefficients! Since the planar Turán number is linear, we can start from $e_G \leq a \cdot v_G + b$, and plug in the Euler's formula to get a homogenous inequality with a and b. We apply this to each block, and get some linear restrictions from each block. Now it becomes a linear optimization problem and the objective function we seek to minimize is just a.

Okay, now there's no secrete of this method. Let's let's see what else we have got.

$\{C_8, C_{10}\}$-free bipartite planar graphs

Theorem

Let G be a planar bipartite graph on n vertices which does not contain C_8 or C_{10} and let $\delta(G) \geq 3$. Then

$$e_G \leq \frac{18}{11}n - \frac{84}{11}.$$

The equality holds for infinitely many integers n.

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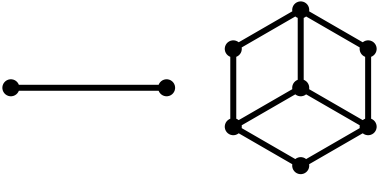
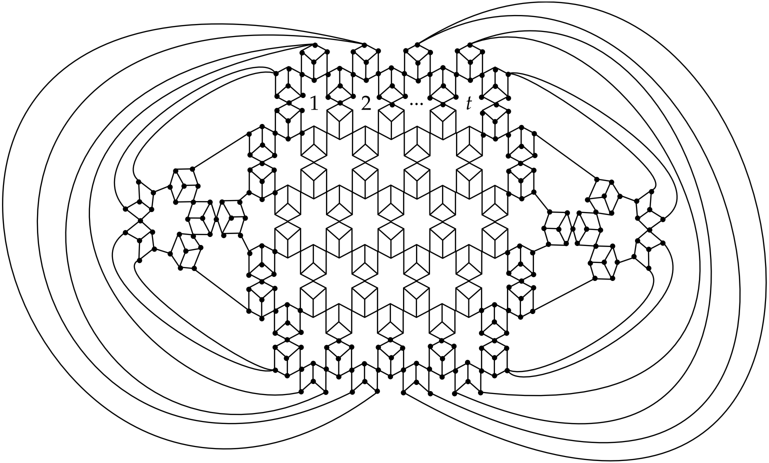


Extremal construction



C_6-free triangle-free planar graphs

Theorem

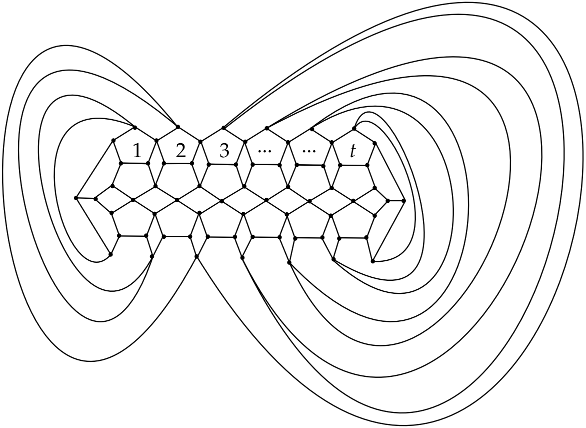
Let G be a C_6-free triangle-free planar graph with $\delta(G) \geq 3$ on n vertices. Then

$$e_G \leq \left\lfloor \frac{9}{5}n - 4 \right\rfloor.$$

The equality holds for infinitely many integers n.

Both K_2 and C_4 are contribution-0 quadrangular blocks. (There are only two possible quadrangular blocks)

Extremal construction



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Theorem

Let G be a C_8-free triangle-free planar graph with $\delta(G) \geq 3$ on n vertices. Then

$$e_G \leq \frac{81}{44}n - \frac{105}{22}.$$

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Let G be a C_8-free triangle-free planar graph with $\delta(G) \geq 3$ on n vertices. Then

$$e_G \leq \frac{81}{44}n - \frac{105}{22}.$$

Unfortunately, in this case we haven't found an extremal construction. We believe that this bound is still not sharp. We will need some further observations here.

Future steps

Conjecture (Cranston, Lidický, Liu and Shantanam 2021)

There exists a constant D such that for all k and for all sufficiently large n, we have $\mathrm{ex}_{\mathcal{P}}(n, C_k) \leq (3 - \frac{3}{Dk^{\log_2^3}})n$.

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For every graph H, there exists a non-negative integer k, such that $\mathrm{ex}_{\mathcal{P}}(n, H, \emptyset) = \Theta(n^k)$.
For all finite sets of graph $\mathcal{F}$ and for all graphs H, there exists a non-negative integer k, such that $\mathrm{ex}_{\mathcal{P}}(n, H, \mathcal{F}) = \Theta(n^k)$.

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The End