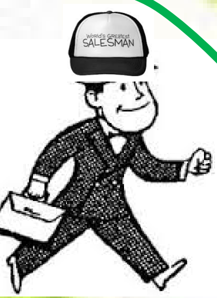
Shanghai, October 22 2024,
Fudan University seminar

感谢您的邀请！



The Salesman and the Postman : Frontiers and Crossroads of and in within **Combinatorial Optimization**

András Sebő
CNRS, G-SCOP, Univ. Grenoble

Shanghai, 2024 October 22
Frontiers of Mathematics

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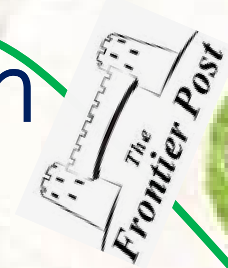
The Salesman and the Postman : Frontiers and Crossroads of and in within **Combinatorial Optimization**



András Sebő
CNRS, G-SCOP, Univ. Grenoble

Your future: Various Frontiers

1. Efficiently Solvable	Intractable
2. Real Life	Abstract
3. Discrete	Continuous
4. Exact	Approximation
5. Deterministic	Random
6. Combinatorial	Geometric
7. Linear	Nonlinear
8. Graphic	Number Theoretic
9. Elementary	Algebraic
10. Uni-Focus: Atomic	Multifocus: Synthetic

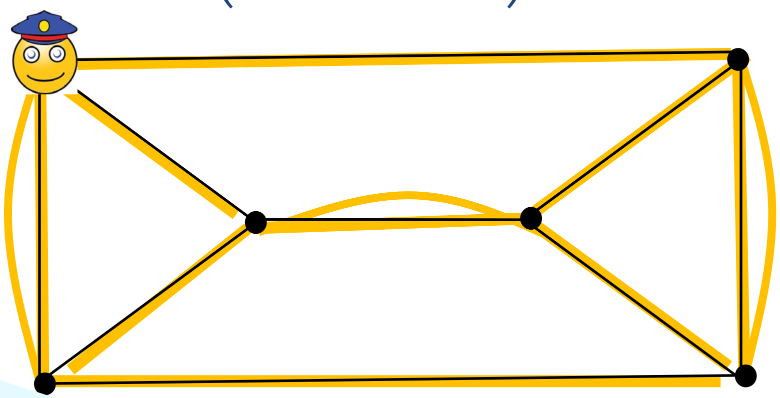
7

1. The Complexity Frontier

Minimize the roads :

中国邮递员 (管梅谷)

The (Chinese) Postman (Meigu Guan 1960)



Edges = streets

Do all the streets

and come back !

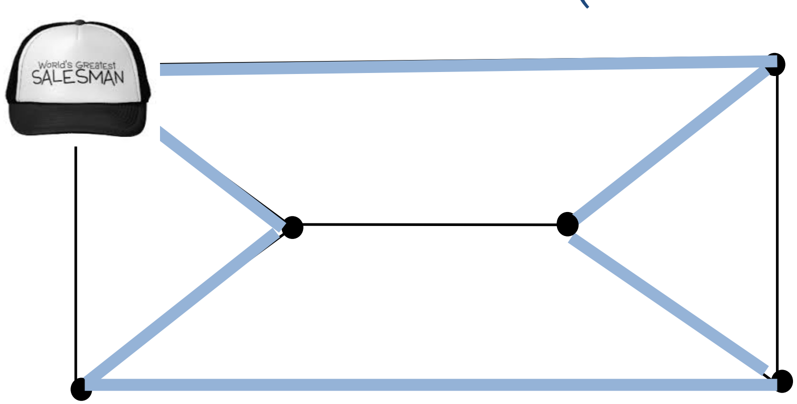
In P, that is, tractable

(Edmonds 1965)

Efficiently Solvable

Intractable

The (Travelling) Salesman



Nodes = Cities

Do all the cities

and come back !

NP-hard, that is intractable

(Karp, 1972)

2. Modellisation: Bridges over River Frontiers

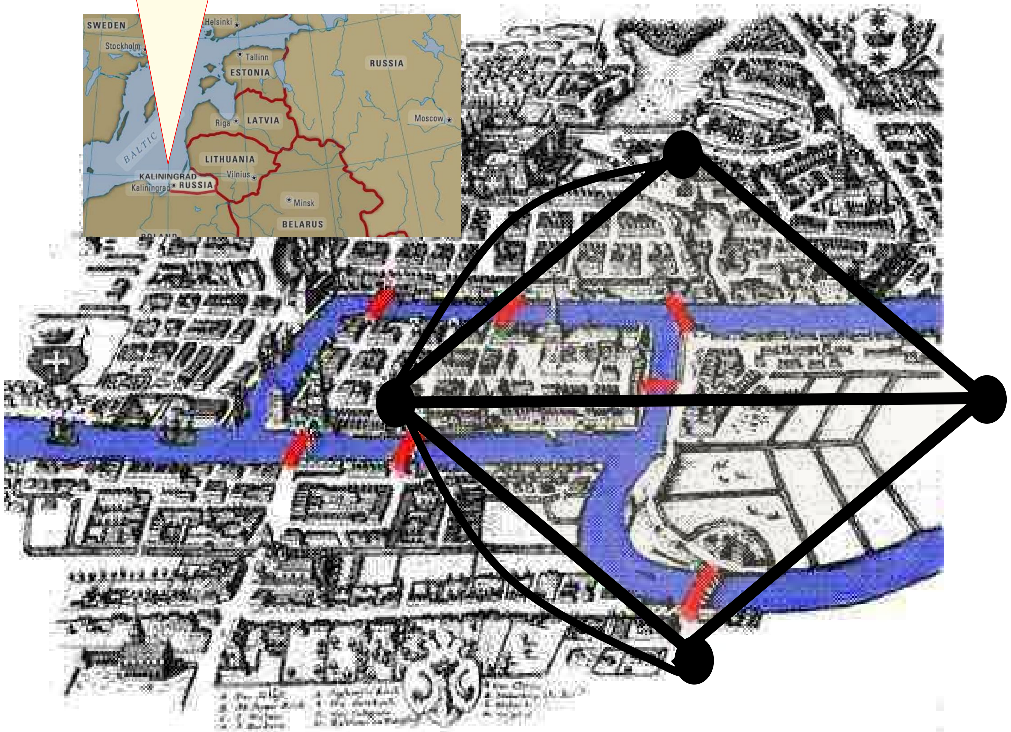
Bridges: "Real Life"

graphs: "abstract"

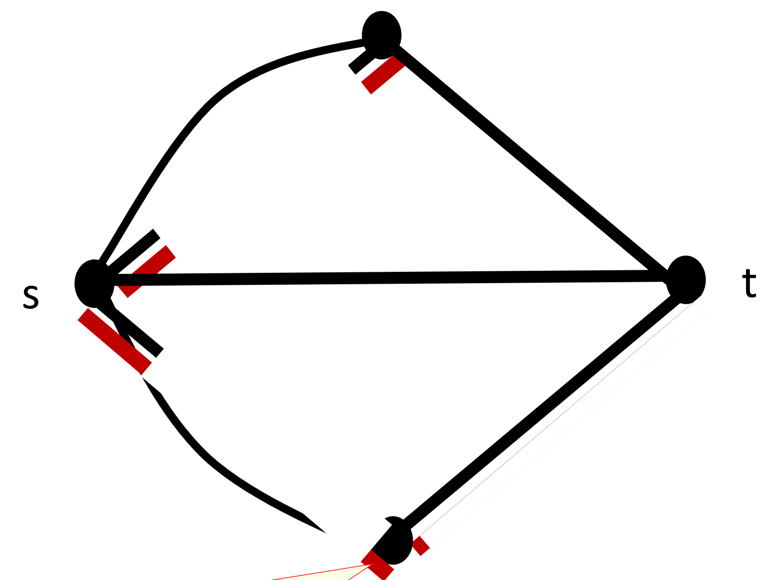
Königsberg

Real Life

Abstract



The Frontier Post



deletion was explored in a clever way by Mömke & Svensson (2011)

Euler's theorem : Given a graph

there exists a tour using every edge exactly once ⇔

The graph is **connected**, and

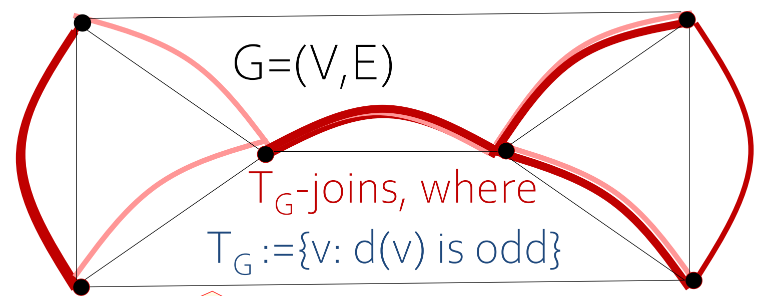
there are an **even number** of edges incident to nodes.

If s≠ t for stating and endpoints s and t then an **odd number** of edges incident to s and t .

Efficient Algorithm for the Postman

$F \subseteq E(G)$ is a *T-join,* if

T = vertices of odd degree of F.



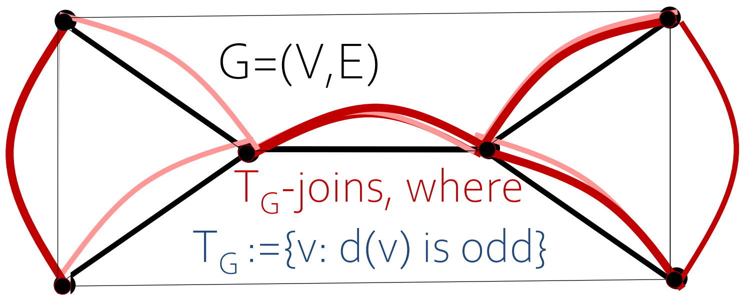
Euler: one can go through all the streets exactly once $\Leftrightarrow$ G conn., $\forall$degree even

Easy facts about T-joins : G connected, |T| even $\Rightarrow$ $\exists$ T-join ;
min weight «Eulerian replication» = duplication of a min weight T_G-join

G=(V,E), w: E $\rightarrow$ IR minimum weight T-joins $\leftrightarrow$
minimum distance-weighted perfect matchings of T
Edmonds (1965) Edmonds, Johnson (1973)

The Chinese Postman's help for the Salesman

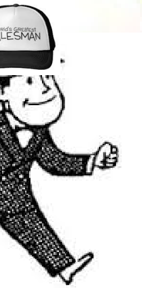
来自中国邮递员的帮助



tour : connected (on V = spanning) sub-multigraph of G, **even degrees**

Non-negative, triangle inequality, otherwise non-approximable

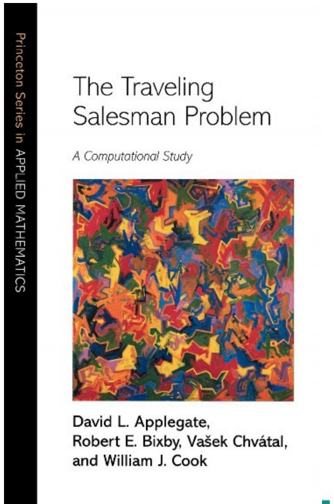
Min Hamiltonian cycle for metrics in complete graphs $\leftrightarrow$ Min tour for arbitrary positive weights in arbitrary graphs



TSP: Min weight tours for nonneg weights

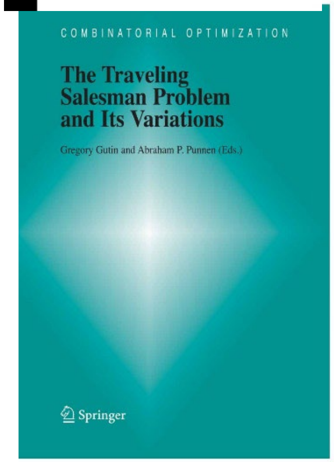
- *Graph* TSP: all 1 weights, i.e. cardinality
- Use of polyhedra
- Sparsity

3. Exact Approximation



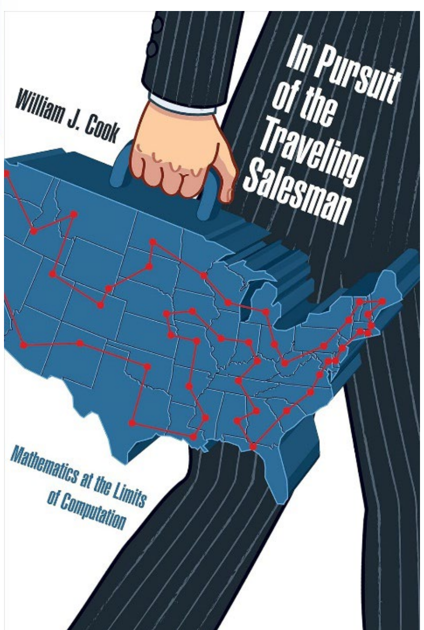
D. Applegate, R. Bixby, V. Chvátal, W.J. Cook

(2006)



G. Gutin and A. P. Punnen

(2002)



W. Cook

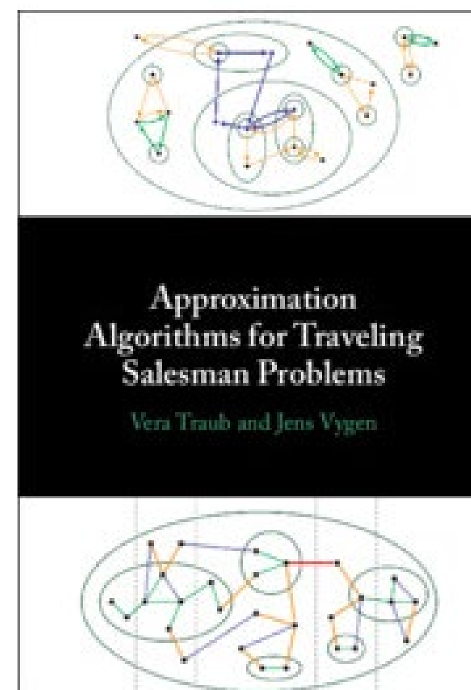
(2012)

Williamson, Shmoys

(2010)

Traub, Vygen

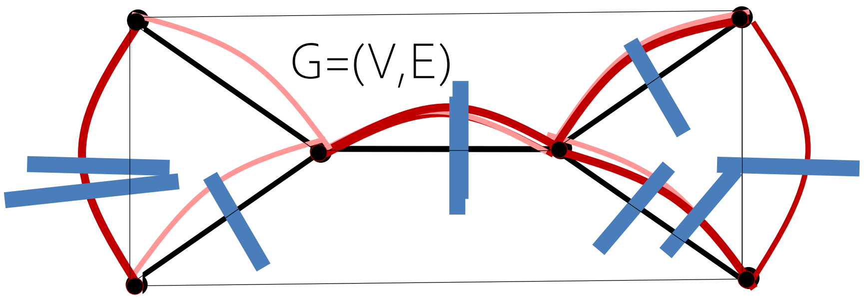
(2024)



Approximate Solution for the Postman

Kruskal's connectivity + the Postman';s parity correction

$= 1 + ½ = 3/2$ OPT



Christofides, Serdyukov (1976)

3/2 approximation

Connectivity

Minimum weight spanning tree : Greedy algorithm (Kruskal 1956)

Minimum T-join : Minimum weight matching on T

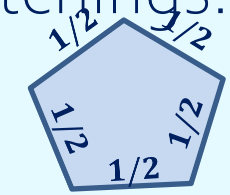
Weight of edge ab= distance

Parity correction

4. Discrete Continuous

Edmonds(1965)
Perfect Matching Polytope PM(G):=

$x \in \mathbb{R}^E$, conv hull of perf. matchings:
$x \geq 0$, $x(\delta(v)) = 1$,
$x(\delta(U)) \geq 1$ $U \subseteq V$, $|U|$ odd



matching : a set of vertex-disjoint edges.

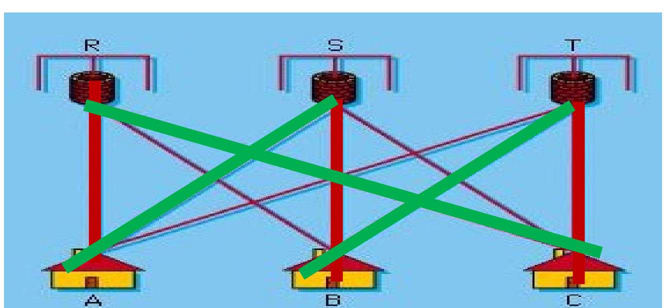
$\delta(U)$ is the set of edges with 1 end in $U \subseteq V$

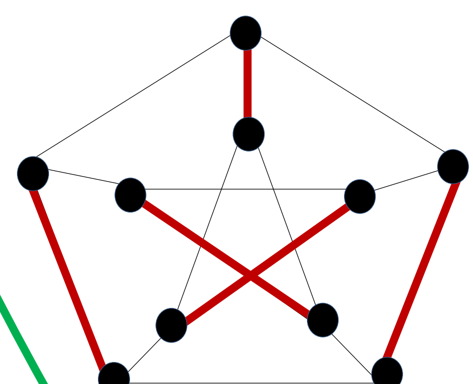
Every vertex has 3 incident edges. König (1931) $\Rightarrow$

Theorem: G cubic, bipartite
$\Rightarrow$ G is 3-edge-colorable i.e
E(G) partitioned to 3 matchings.

Corollary : G cubic, bipartite $\Rightarrow$
$\exists$ *p.m. matching of weight at most 1/3 of the sum of the weights.*

Is this true for non-bipartite graphs ?





Thm: G cubic
No bridge $\Rightarrow$
$\underline{1}/3 \in PM(G)$

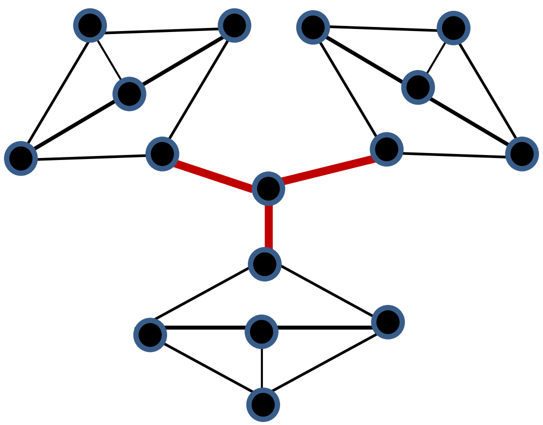
Theorem: Petersen (1891) G is a cubic graph
G has no bridge $\Rightarrow$ G has a perfect matching.

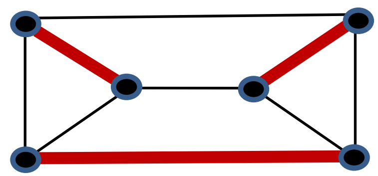
桥

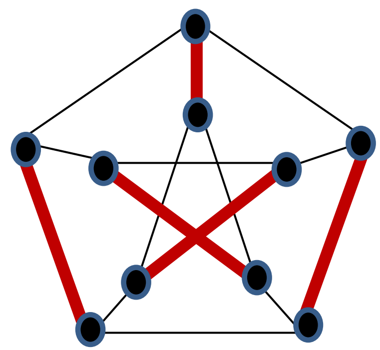
Bridges

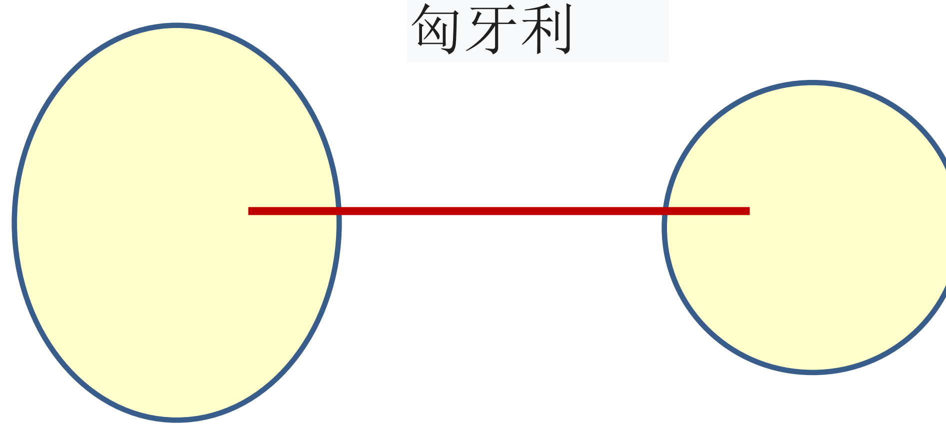
布达佩斯

匈牙利









Bridgeless, cubic, but not partitionable into three matchings

G cubic, no bridge $\Rightarrow \underline{1}/3 \in PM(G)$

Theorem: $G=(V,E)$ cubic, bridgeless (or bipartite),
$w: E \rightarrow \mathbb{R}$. $\exists$ matching M: $w(M) \leq w(E)/3$

LP for the TSP

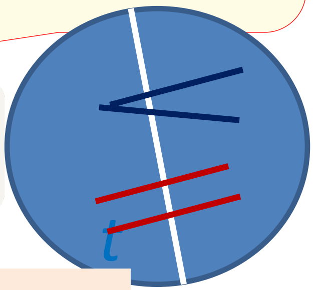
subtour elimination constraints

But the convex hull of tours is hopeless

OPT := min tour

tour OPT_{LP}:=min $\{ c^T x : x \in LP(G)\}$

$LP(G) := \{x \in IR_+^E : x(\delta(W)) \geq 2, \quad \text{for all } \emptyset \neq W \subset V\}$



connectivity

Parity correction

Edmonds, Johnson (1973)

conv(spanning trees) + $\mathbb{R}^n_+ \supseteq$ LP(G) **+** conv (T-joins) + $\mathbb{R}^n_+ \supseteq \frac{1}{2}$LP(G)

Wolsey (1980) , Cunningham (1986) : **=** conv(tours) $\supseteq \frac{3}{2}$ LP

$\Rightarrow$ OPT $\leq \frac{3}{2} OPT_{LP}(w)$; **4/3 conjecture:** $\leq \frac{4}{3} OPT_{LP}(w)$?

4/3 conjecture : conv(tours) $\supseteq \frac{4}{3}$ LP ?

Isn't this too much to ask ?

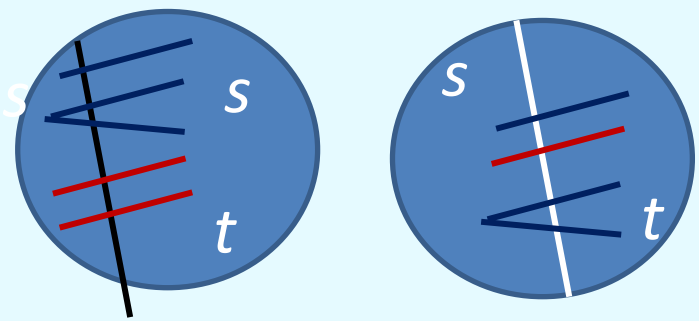
For the s,t -path TSP

But the convex hull of tours is hopeless

$LP(V,s,t)=\{x\in \mathbb{R}_+^E: x(\delta(W)) \geq 2, \emptyset \neq W \subset V, s, t \in W$ or $\notin$

1, if s,t separated by W

on vertices (1 for s, t, else 2).



$\frac{1}{2}$ LP does not correct the parity !

3/2 **conjecture:** $OPT \leq \frac{3}{2} OPT_{LP}(w)$?

3/2 conjecture : $\frac{3}{2}LP \subseteq$ conv(tours) ?

Isn't this too much to ask ?

The gateway

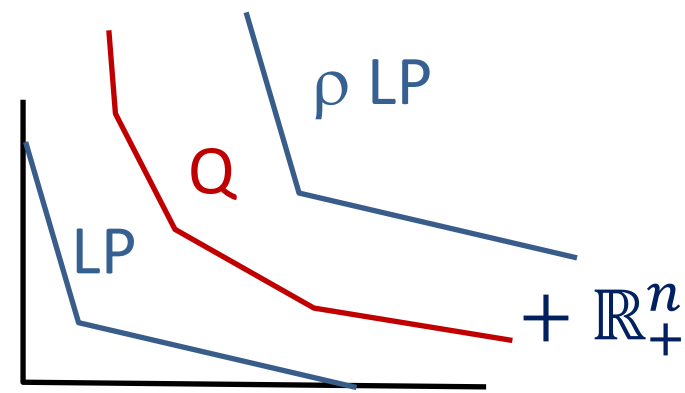
Discrete | Continuous

No, equivalent ! Goemans (1995), Carr, Vempala (2004)

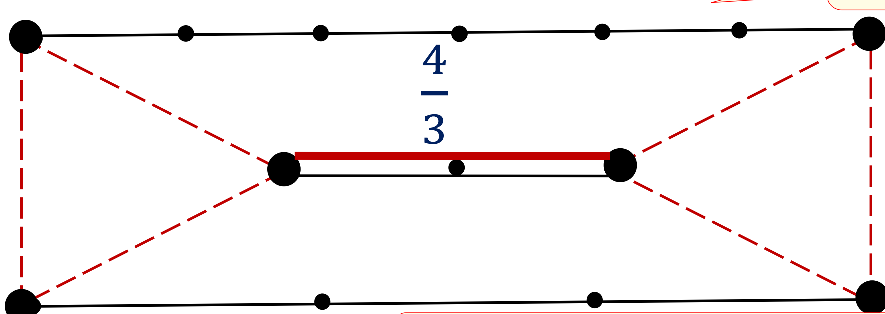
Integrality ratio

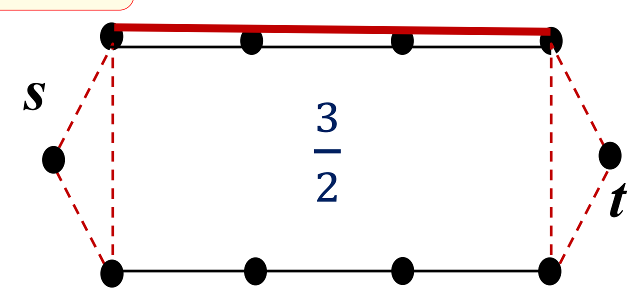
Thm: For ρ $(\geq 1) \in \mathrm{IR}$: $OPT(w) \leq \rho OPT_{LP}(w)$ $\forall w \geq 0 \Leftrightarrow \mathrm{conv}(\mathrm{tours}) + \mathbb{R}^n_+ \supseteq \rho LP$

Proof: Farkas' lemma



Lower bound coefficients





Lower bound for approximability: 1+ ε (Karpinsky, Lampis, Schmied, … ,2015)

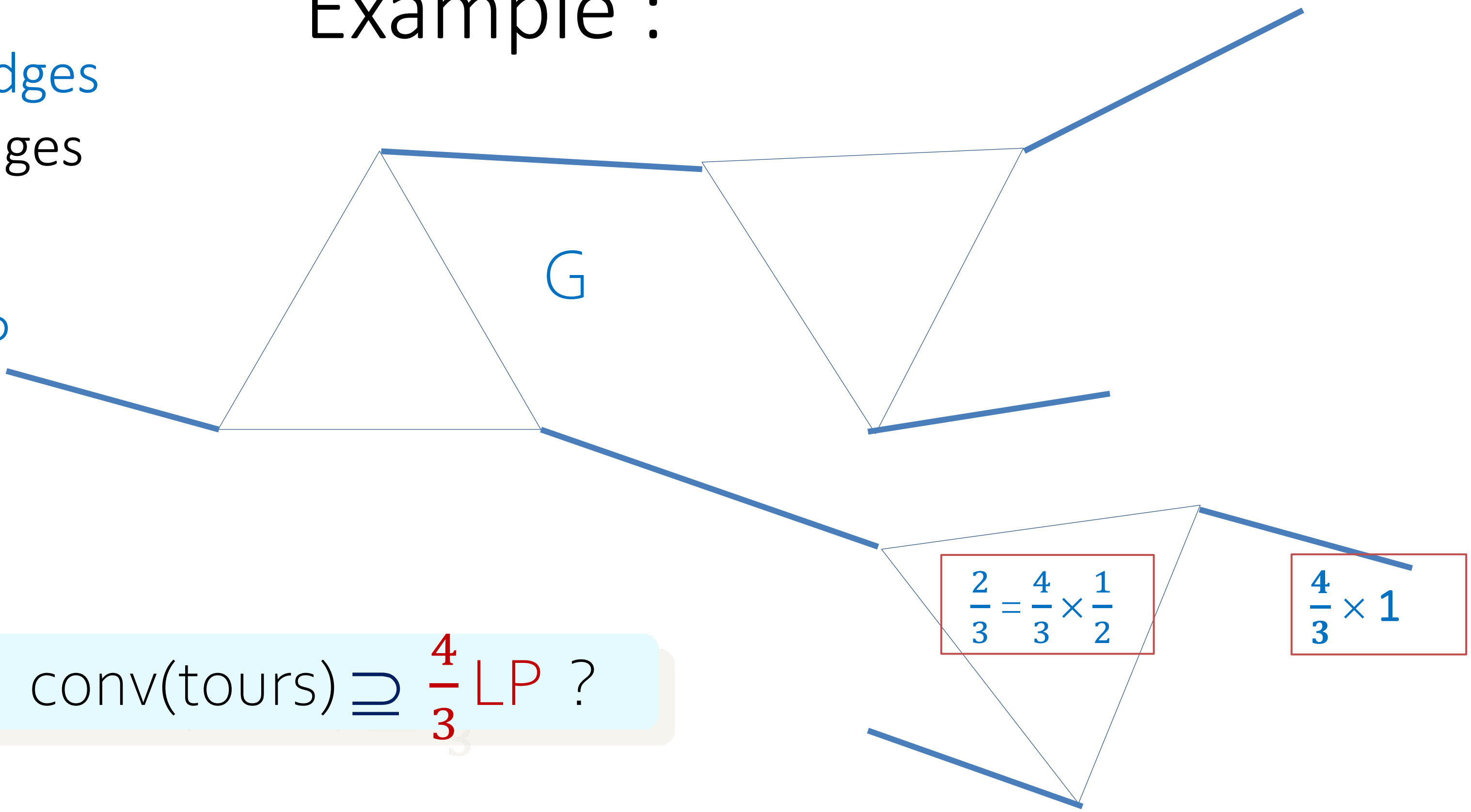
Example :

x:= ——— ½ -edges

━━━ 1-edges

Assume $x \in$ LP

i.e. no bridge



4/3 conjecture : conv(tours) $\supseteq \frac{4}{3}$ LP ?

If the ½-edges form partitioning triangles: conv(tours) $\supseteq \frac{4}{3}$ LP ?

Proof: No bridge, so $\underline{1}/3 \in$ PM(G) ! For each matching in the convex combination, delete its 1/2 –edges and double its 1-edges. The graph remains connected, so tours: the conv comb is $\frac{2}{3}$ on ___ , $\frac{4}{3}$ on ___ $= \frac{4}{3}x$

5. Deterministic — Probabilistic

$x = \sum_{M \in \mathfrak{M}} \lambda_M M$

$(\lambda_M \geq 0, \sum_{M \in M} \lambda_M = 1)$

$\mathfrak{M}$ a set of incidences of perfect matchings
In the example $E[\mathfrak{M}]=1/3$ …

or $\mathfrak{F}$ a set of tours,…

$\Pr(\mathfrak{M} = M) = \lambda_M$
$\Pr(e \in \mathfrak{M}) = x(e)$
$E[\mathfrak{M}] = x$

The Frontier Post

Integrality ratios:

Max of $-\sum_{F \in \mathfrak{F}} \lambda_F \lambda_F \log_2 \lambda_F$ on trees

Christofides, Serdyukov (1976) $\frac{3}{2}$

Oveis Gharan, Saberi, Singh (2011) $\frac{3}{2}-\varepsilon$

with maximum entropy distribution

graph TSP

Sebő, Vygen (2014) $\frac{7}{5}$

Mömke Svensson (2012): **Removable pairs**

The Frontier Post

Hoogeveen (1991) $\frac{5}{3}$

Not LP

path TSP

An, Kleinberg, Shmoys (2011) $\frac{1+\sqrt{5}}{2}$

Zenklusen(2019) $\frac{3}{2}$

Sebő (2013) $\frac{8}{5}$

Traub Vygen, Zenklusen (2019)
s,t tour $= OPT_{\text{not LP}}$ tour $+ \varepsilon$

Sebő, van Zuylen (2019) $OPT_{LP} \leq \frac{3}{2}+\varepsilon$

Traub (2020) $+ \varepsilon'$

Karlin, Klein, Oveis Gharan 2020-2023, $\frac{3}{2} - \varepsilon$ **with max entropy, in general**

Deterministic (2023) … with lower bound (2024) using Boyd, Sebő (2021)

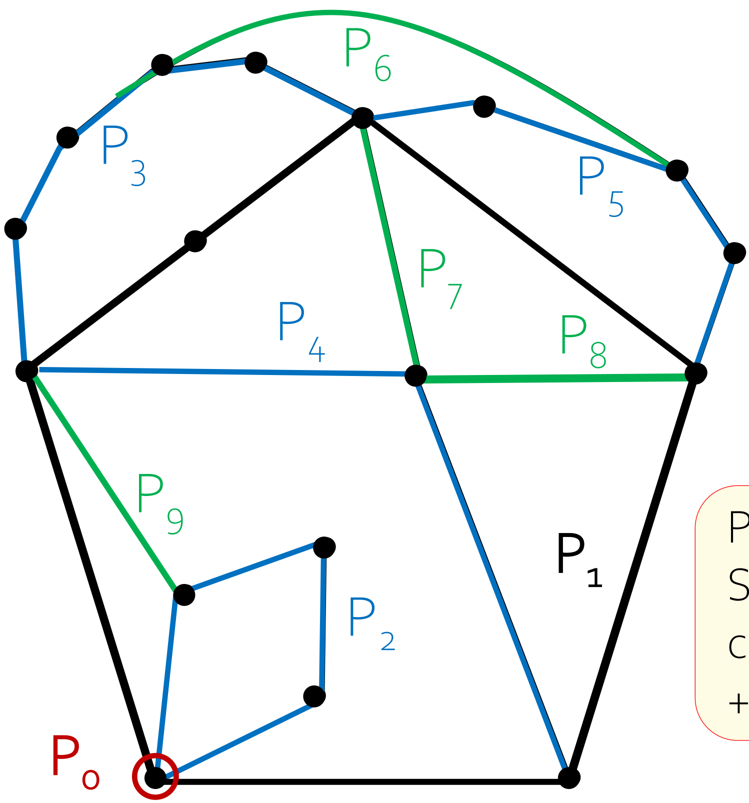
Opening the frontier between connectivity and parity connection

Simultaneous connectivity and parity correction made possible by **matroid intersection** in the best bounds:

- for the graph TSP
- for "fundamental vertices"
- for general path TSP/LP
-

Parity correction

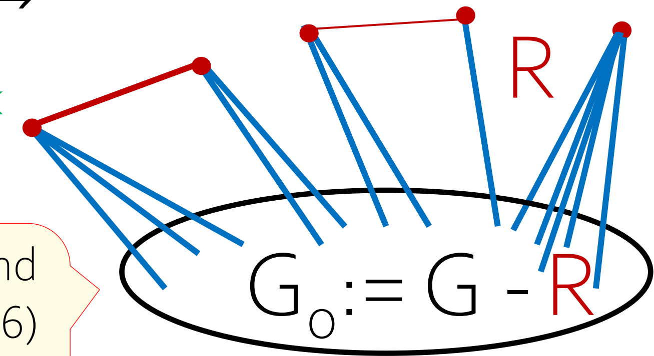
Matroid intersection:
Max weight of common elements of two particular hypergraphs, solved in polynomial time by Edmonds (1979)



The longer the ears, the smaller the quotient

G 2-edge-connected $\Rightarrow$

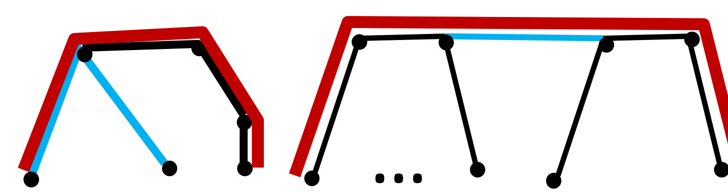
$G = P_0 + P_1 + P_2 + \ldots + P_k$



Probabilities through Mömke and Svensson's ingenious Lemma (2016) combined with the ear-decomposition, + matroid intersection, Frank's joins,

Lemma : (Whitney, Cheriyan, Sebő, Szigeti, Vygen, 1932-2012) If G is 2-connected, then there exists a nice open ear-decomposition, i.e.

- 1-ears last, 2-ears, 3-ears « before the last »
- no edges between their inner vertices,
- min number of even ears



Best **integrality and approx ratio** for the graph TSP :

Thm: Sebő, Vygen (2014): Tour in polytime of cardinality $\leq$ 7/5 OPT_{LP}

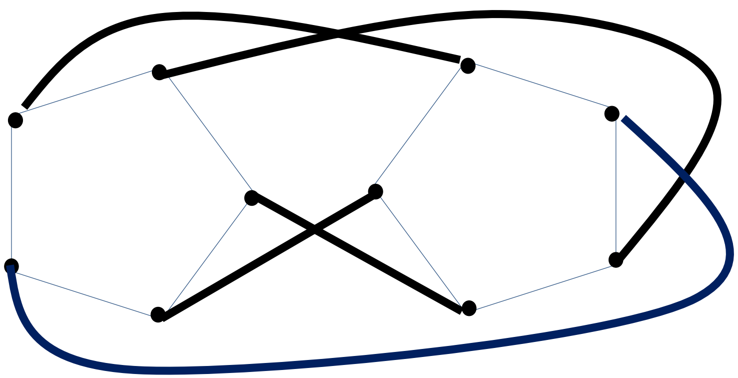
Recent trials multiple methods : Fundamental vertices, ½-vertices, uniform covers

Boyd, Sebő 2018-24

Alternating between $\frac{1}{k}, \frac{k-1}{k}$

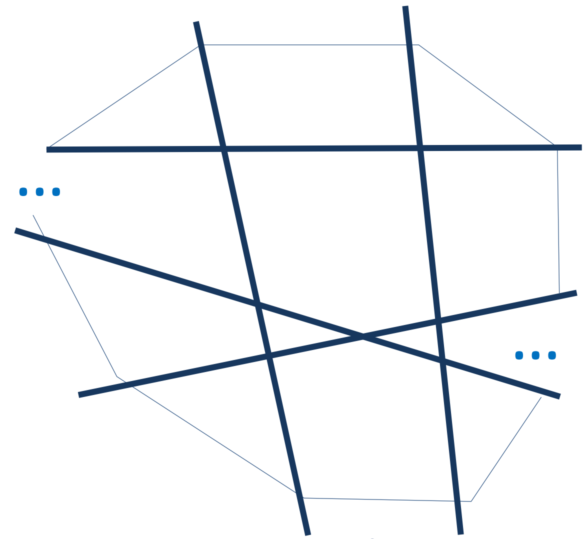
for k=2 : ½ -edges

1-edges



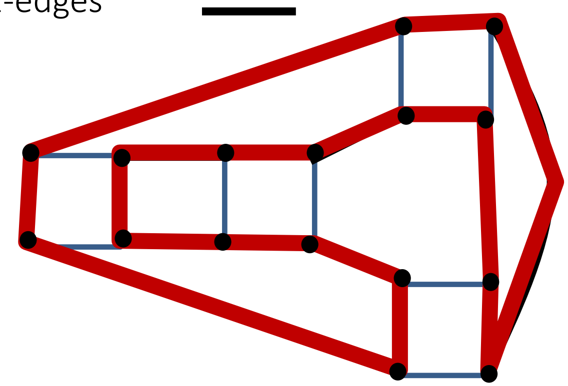
Generalized Prism

Conjectures open for n ≥ 9



Carr-Vempala fundamental vertex

The reduction keeps ½ integrality



Boyd-Carr square fundamental vertex

10/7 for ½ -integer vertices of subtour elimination (BS '21)

Uses delta-matroids (Bouchet), polyhedra, matroid intersection, elementary probabilities; *½ not kept*

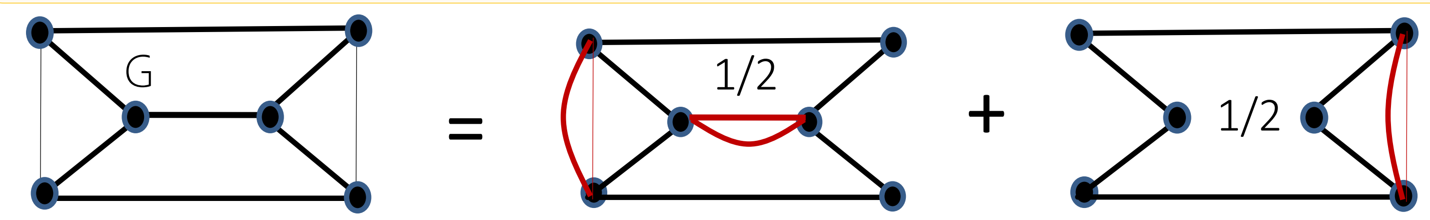
Not true, but $\frac{1}{2}$ seems to be still sufficient

Conjecture: (Schalekamp, Williamson, van Zuylen 2014) Largest ratio : $\frac{1}{2}$ - integer

Uniform Covers by Tours

Remark (A. S. 2018) : G= (V,E) 3-edge conn. Then

$1 \in \mathrm{conv}\,(\, x \;:\; x \in \{0,1,2\}E$ is the incidence vector of a tour}



Proof : $\underline{2}/3$ dominates a point of the *spanning tree polytope*

$\underline{1}/3$ dominates a point the *T –join polyhedron* $\forall$ T.

E (tree F + parity correction for F, i.e. a T_F-join) $\leq \underline{2}/3 + \underline{1}/3$

solved

unsolved

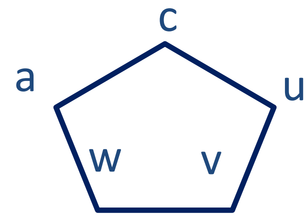
4/3 x $\underline{2}/3$ = $\underline{8}/9$

Conjecture (A. S. 2018) : G= (V,E) cubic,3-edge connected. Then 8/9 $\in \mathrm{conv}\,(\, x \;:\; x \in \{0,1,2\}^E$ is the incidence vector of a tour}

Haddadad, Newman, Ravi (2019): $\frac{18}{19}$

Boyd, Sebő (2021): $\frac{6}{7}$ for square 1/2

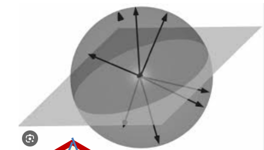
6. Combinatorial

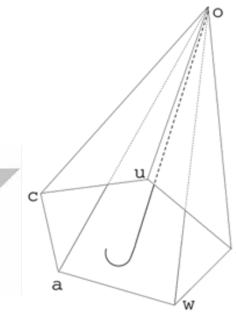


Coloring

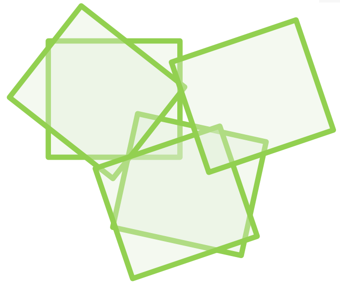
Geometric (Lovász 1979)

Topology ...

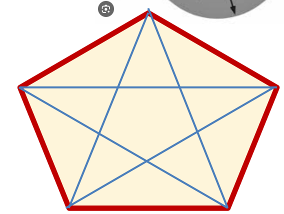




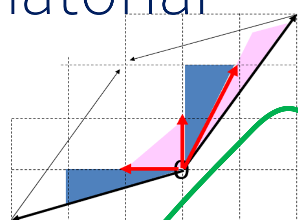
Geometric



Combinatorial



7 Linear, Combinatorial



Nonlinear: Semidef. programming (Goemans-Williamson, max cut, 1995)

Entropy maximization

8. Combinatorial

Number Theoretic: Hilbert bases, TDI, width 1990-2025

9. Elementary

Algebraic or Analytic: Polynomials, Gröbner bases, Nullstellensatz (colorings, graph factors), 1990-

10 . Unique focus

Multifocus, Various tools simultaneously (eg above) eg Olver, Shepherd some forTSP, 2010-

We are HAPPY but it is not yet the END

Matroid intersection

connectivity

Parity correction

感谢您的关注

Thank you for your attention