Triangles in the Plane

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Graph Theory Seminar at Shanghai Center for Mathematical Science

This is partially joint work with József Balogh and Adrian Dumitrescu.

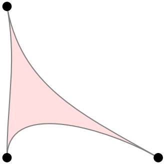
October 22th, 2024

Hypergraph Turán Theory

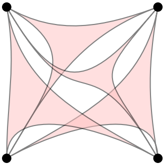
$\mathrm{ex}(n, H) =$ The maximum number of edges in an n-vertex k-graph G which does not contain H as a copy.

The Turán density of H: $$\pi(H) = \lim_{n\to\infty} \frac{\mathrm{ex}(n, H)}{\binom{n}{k}}$$

H_1



K_4^3



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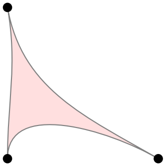

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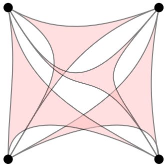

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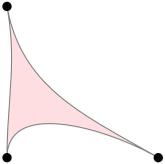
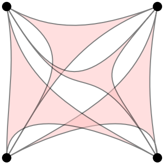


K_4^3



$$\mathrm{ex}(n, H_1) = 0$$
$$\pi(H_1) = 0$$

$$\mathrm{ex}(n, K_4^3) = ?$$
$$\frac{5}{9} \le \pi(K_4^3) \le 0.5615$$

Turán's Tetrahedron Conjecture (1961): $\pi(K_4^3) = \frac{5}{9}$ (500\$)

Hypergraph Turán Theory

Some questions one might ask:

- Given a k-graph H, determine $\pi(H)$.
- What can be said about $\{\pi(H) : H \text{ is finite } k\text{-graph}\}$?

Hypergraph Turán Theory

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- What can be said about $\{\pi(H) : H \text{ is finite } k\text{-graph}\}$?

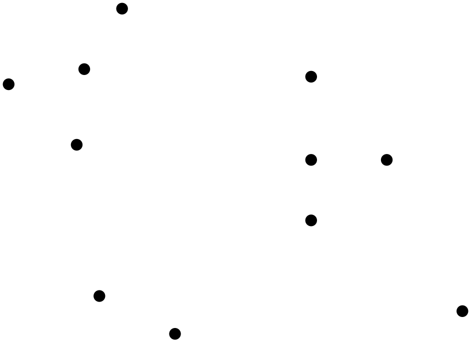
Let $\mathcal{H}$ be a family of k-graphs.

$\mathrm{ex}(n, \mathcal{H}) =$ The maximum number of edges in an n-vertex k-graph G which does not contain any $H \in \mathcal{H}$ as a copy.

The Turán density of $\mathcal{H}$: $$\pi(\mathcal{H}) = \lim_{n \to \infty} \frac{\mathrm{ex}(n, \mathcal{H})}{\binom{n}{k}}$$

Application 1: The unit distance problem

n points in $\mathbb{R}^2$



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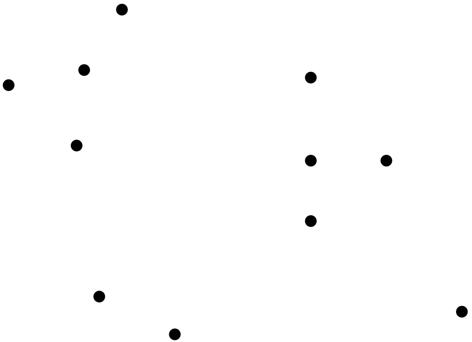

Question (Erdős, 1946)

What is the maximum number of times that the unit distance can occur among n points in the plane?

$$u(n) := \max_{P \subseteq \mathbb{R}^2,\ |P| = n} \Big| \big\{ \{u, v\} \subset P : |u - v| = 1 \big\} \Big|.$$

Application 1: The unit distance problem

- Erdős (1946): $n^{1+c_1/\log\log n} \leq u(n) \leq O(n^{3/2})$
- Józsa and Szemerédi (1975): $u(n) = o(n^{3/2})$
- Beck and Spencer (1984): $u(n) = O(n^{13/9})$
- Spencer, Szemerédi, Trotter (1984): $u(n) = O(n^{4/3})$

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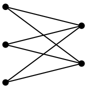
Erdős' proof of $u(n) = O(n^{3/2})$:

Let $G = (V, E)$ be the graph with $V = P$ and edges $e = xy \in E(G)$ iff $|x - y| = 1$. The graph G is $K_{3,2}$-free.



$K_{3,2}$

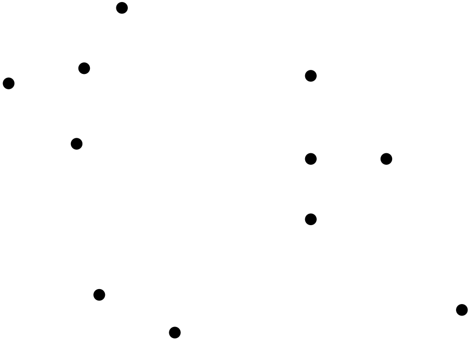
$$u(n) \le \mathrm{ex}(n, K_{3,2}) \le O(n^{3/2}). \qquad \square$$

Application 1: The unit distance problem

Combinatorial Geometry	Extremal Graph Theory
Two circles of unit radius cannot intersect in $\geq$ three points $\longrightarrow$	The unit distance graph is $K_{3,2}$-free
	$\downarrow$
UB on $u(n)$ $\longleftarrow$	UB on $\mathrm{ex}(n, K_{3,2})$

Application 2: Almost congruent unit triangles

n points in $\mathbb{R}^2$



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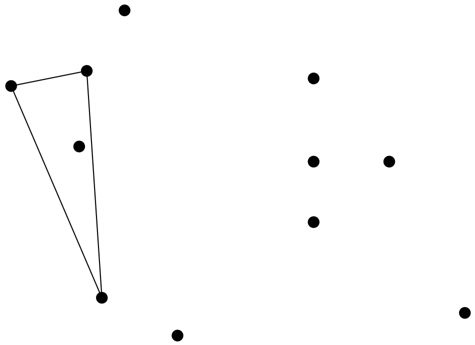

Up to $\binom{n}{3}$ triangles.

Question (Erdős, Purdy, 1975)

What is the maximum number of triangles almost congruent to the unit triangle?

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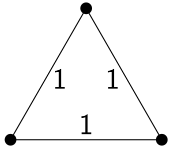
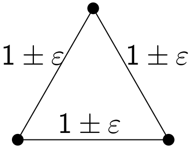


ε-unit triangle



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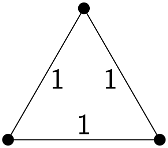

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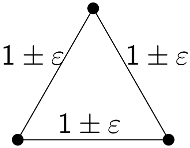
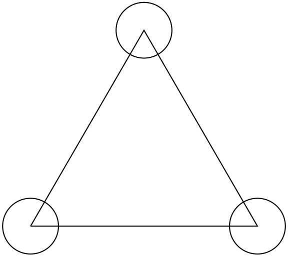


Definition

$g(n, \varepsilon) =$ The maximum number of ε-unit triangles in a point set $P \subseteq \mathbb{R}^2$ of size n.

$g(n) = \min_{\varepsilon>0} g(n, \varepsilon)$.

Almost congruent unit triangles



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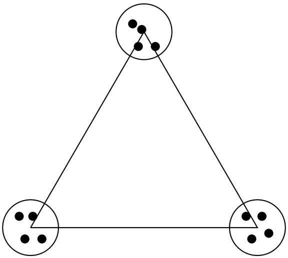

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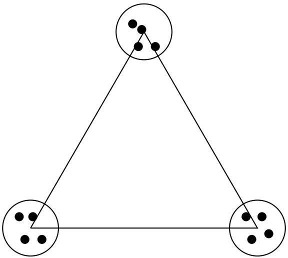
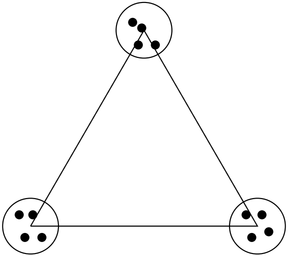


$$g(n) \geq \left\lfloor \frac{n}{3} \right\rfloor \cdot \left\lfloor \frac{n+1}{3} \right\rfloor \cdot \left\lfloor \frac{n+2}{3} \right\rfloor \approx \frac{n^3}{27}$$

Almost congruent unit triangles



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Theorem (Balogh, C., Dumitrescu, 2023+)

For every positive integer n, we have

$$g(n) = \left\lfloor \frac{n}{3} \right\rfloor \cdot \left\lfloor \frac{n+1}{3} \right\rfloor \cdot \left\lfloor \frac{n+2}{3} \right\rfloor .$$

Application 2: Almost congruent unit triangles

We say a 3-graph G is *cancellative*, if there do not exist 3 edges A, B, C with $A \triangle B \subset C$.

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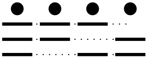
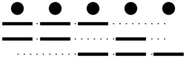
We say a 3-graph G is *cancellative*, if there do not exist 3 edges A, B, C with $A \triangle B \subset C$. Equivalently, G is cancellative iff G if $\{K_4^{3-}, F_5\}$-free.

K_4^{3-}	$123, 124, 134$	
F_5	$123, 124, 345$	

Theorem (Bollobás, 1974)

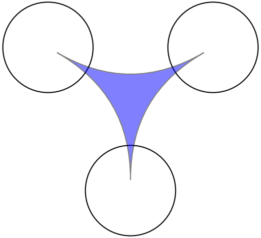
The maximum number of edges in a cancellative 3-graph on n vertices is

$$\left\lfloor \frac{n}{3} \right\rfloor \cdot \left\lfloor \frac{n+1}{3} \right\rfloor \cdot \left\lfloor \frac{n+2}{3} \right\rfloor .$$

Application 2: Almost congruent unit triangles

Theorem (Bollobás, 1974)

$$\mathrm{ex}(n, \{K_4^{3-}, F_5\}) = \left\lfloor \frac{n}{3} \right\rfloor \cdot \left\lfloor \frac{n+1}{3} \right\rfloor \cdot \left\lfloor \frac{n+2}{3} \right\rfloor .$$

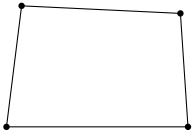


Application 2: Almost congruent unit triangles

Observation

Let P be a set of 4 points in the plane where the minimum pairwise distance is at least 1. Then diam$(P) \geq \sqrt{2}$.

convex 4-gon



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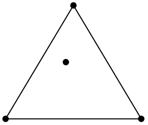

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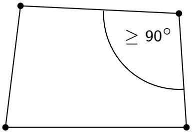

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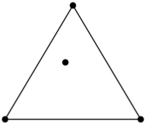

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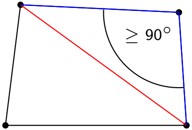

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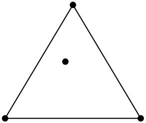

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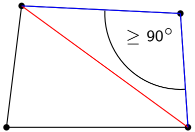

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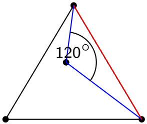
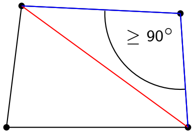


Application 2: Almost congruent unit triangles

Observation

Let a be a positive real number and P be a set of 4 points in the plane where the minimum pairwise distance is at least a. Then $\mathrm{diam}(P) \geq \sqrt{2}a$.

convex 4-gon



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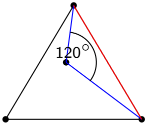

Proof of UB on $g(n)$:

Application 2: Almost congruent unit triangles

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- Let $\varepsilon > 0$ sufficiently small and P a planar point set of size n.

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- The shadowgraph of $H(P, \varepsilon)$ does not contain a copy of K_4, otherwise there were 4 points with pairwise distance in $(1 - \varepsilon, 1 + \varepsilon)$.

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- $H(P, \varepsilon)$ is K_4^{3-} and F_5-free, because they both contain a K_4 in the shadowgraph.

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- $H(P, \varepsilon)$ is K_4^{3-} and F_5-free, because they both contain a K_4 in the shadowgraph.

$$g(n) \leq \mathrm{ex}(n, \{K_4^{3-}, F_5\}) = \left\lfloor \frac{n}{3} \right\rfloor \cdot \left\lfloor \frac{n+1}{3} \right\rfloor \cdot \left\lfloor \frac{n+2}{3} \right\rfloor. \qquad \square$$

Application 2: Almost congruent unit triangles

Combinatorial Geometry

Extremal Graph Theory

4 points cannot have pairwise distance close to 1 $\rightarrow$ Shadowgraph of auxiliary 3-graph is K_4-free

$\downarrow$

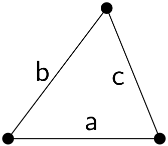
K_4^{3-}, F_5-free

$\downarrow$

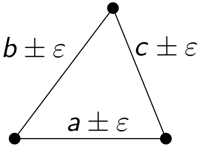
UB on $g(n)$ $\leftarrow$ UB on $\mathrm{ex}(n, \{K_4^{3-}, F_5\})$

Application 3: Almost congruent general triangles

triangle T



ε-congruent to T



Application 3: Almost congruent general triangles

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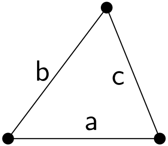
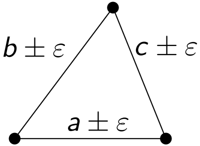


ε-congruent to T



Definition

$h_c(n, T, \varepsilon) =$ The maximum number of triangles ε-*congruent* to T in a point set $P \subseteq \mathbb{R}^2$ of size n.

$h_c(n, T) = \min_{\varepsilon > 0} h_c(n, T, \varepsilon)$.

Observation

If two triangles T, T' are similar to each other, then $h_c(n, T) = h_c(n, T')$

Application 3: Almost congruent general triangles

Theorem (Balogh, C., Dumitrescu, 2023+)

Let T be a triangle and n be a positive integer.

(a) *Let T be a right triangle. Then, $h_c(n, T) \leq \frac{n^3}{16}$, and if additionally n is divisible by 4, then $h_c(n, T) = \frac{n^3}{16}$.*

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(b) *Let T be of type $(120^\circ, 30^\circ, 30^\circ)$. Then, $h_c(n,T) \le \frac{4}{81}n^3$, and if additionally n is divisible by 9, then $h_c(n,T) = \frac{4}{81}n^3$.*

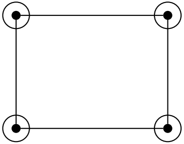
(c) *Let T be of type $\left(\frac{4\cdot 180}{7}^\circ, \frac{2\cdot 180}{7}^\circ, \frac{180}{7}^\circ\right)$. Then, $h_c(n,T) \le \frac{2}{49}n^3$, and if additionally n is divisible by 7, then $h_c(n,T) = \frac{2}{49}n^3$.*

(d) *Let T be of type $(108^\circ, 36^\circ, 36^\circ)$ or $(72^\circ, 72^\circ, 36^\circ)$. Then, $h_c(n,T) \le \frac{n^3}{25}$, and if additionally n is divisible by 5, then $h_c(n,T) = \frac{n^3}{25}$.*

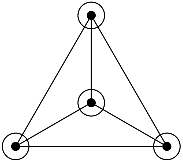
(e) *Let T be not of type (a)-(d). Then, $h_c(n,T) \le \frac{n^3}{27}$, and if additionally n is divisible by 3, then $h_c(n,T) = \frac{n^3}{27}$.*

Application 3: Almost congruent general triangles

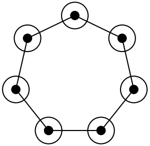
right triangle



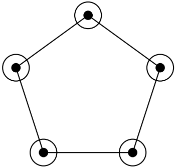
$(120^\circ, 30^\circ, 30^\circ)$



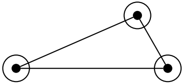
$\left(\frac{4\cdot 180}{7}^\circ, \frac{2\cdot 180}{7}^\circ, \frac{180}{7}^\circ\right)$



$(108^\circ, 36^\circ, 36^\circ)$ or $(72^\circ, 72^\circ, 36^\circ)$



arbitrary T



Application 3: Almost congruent general triangles

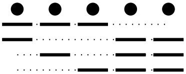
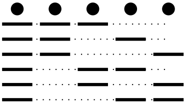
Definition

$H(P, T, \varepsilon) :=$ The 3-graph with vertex set P and edges the triples corresponding to triangles ε-congruent to T.

Combinatorial Geometry	Extremal Graph Theory
Some forbidden configuration on P for T $\longrightarrow$	$H(P, T, \varepsilon)$ is $\mathcal{F}(T)$-free for small ε
	$\downarrow$
UB on $h_c(n, T)$	$\longleftarrow$ UB on $\mathrm{ex}(n, \mathcal{F}(T))$

Theorem (Balogh, C., Dumitrescu)

(a) *Let T be a right triangle. Then,* $h_c(n, T) \leq \frac{n^3}{16}$.

$F_{3,2}$	$123, 145, 245, 345$	
J_4	$123, 124, 125, 134, 135, 145$	

If T is not $(90^\circ, 60^\circ, 30^\circ)$, then

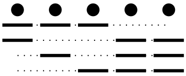
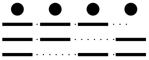
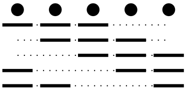
$$h_c(n, T) \leq \mathrm{ex}(n, \{F_{3,2}, J_4\}) = \frac{n^3}{16}(1 + o(1)),$$

by Falgas-Ravry and Vaughan. □

Application 3: Almost congruent general triangles

Theorem (Balogh, C., Dumitrescu)

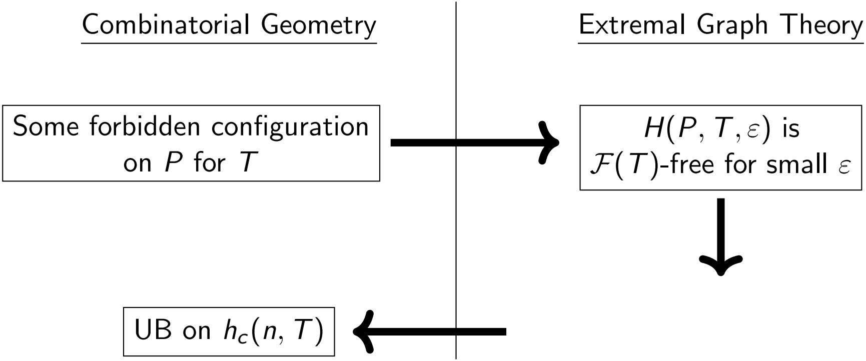
(c) *Let T be of type $\left(\frac{4\cdot180^\circ}{7}, \frac{2\cdot180^\circ}{7}, \frac{180^\circ}{7}\right)$. Then, $h_c(n,T) \le \frac{2}{49}n^3$.*

$F_{3,2}$	123, 145, 245, 345	
K_4^{3-}	123, 124, 134	
C_5^3	123, 234, 345, 145, 125	

$$h_c(n,T) \le \mathrm{ex}(n, \{K_4^{3-}, F_{3,2}, C_5^3\}) = \frac{2}{49}n^3(1+o(1)),$$

by Falgas-Ravry and Vaughan. $\square$

Application 3: A new strategy



Application 3: A new strategy

Combinatorial Geometry	Extremal Graph Theory
Some forbidden configuration on P for T $\longrightarrow$	$H(P, T, \varepsilon)$ is $\mathcal{F}(T)$-free for small ε
	$\downarrow$
UB on $h_c(n, T)$ $\longleftarrow$	Weighted hypergraph Turán problem

Application 3: Hypergraph Lagrangians

Let H be an n-vertex 3-graph. The *Lagrangian polynomial* of H is

$$\lambda_H(x_1, \ldots, x_n) := \sum_{ijk \in H} x_i x_j x_k,$$

and the *Lagrangian* of H is

$$\lambda(H) := \max\{\lambda_H(x_1, \ldots, x_n) : \ (x_1, x_2, \ldots, x_n) \in \Delta_n\},$$

where $\Delta_n = \{(x_1, x_2, \ldots, x_n) \in [0,1]^n : x_1 + x_2 + \ldots + x_n = 1\}$.

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<u>Example:</u> $V(K_4^{3-}) = \{1,2,3,4\}, \ E(K_4^{3-}) = \{123, 124, 134\}$

$$\lambda(K_4^{3-}) = \max_{x_i \geq 0, \ x_1+x_2+x_3+x_4=1} x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4$$

$$= \max_{0 \leq x \leq 1} 3x \left(\frac{1-x}{3}\right)^2 = \frac{4}{81}$$

Application 3: The key lemma

$H(P, T) :=$ The 3-graph with vertex set P and edges the triples corresponding to triangles congruent to T.

Lemma (Balogh, C., Dumitrescu)

Let T be a triangle and n be a positive integer. Then there exists a point set Q of size $|Q| \leq 7$ such that in $H(Q, T)$ every pair of vertices is contained in an edge, and $h_c(n, T) \leq n^3 \lambda(H(Q, T))$.

Application 3: Sketch of proof of key lemma

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Sketch of proof:

$$\frac{h_c(n,T)}{n^3} = \frac{e(H(P,T,\varepsilon))}{n^3} = \lambda_{H(P,T,\varepsilon)}\left(\frac{1}{n},\ldots,\frac{1}{n}\right) \leq \lambda(H(P,T,\varepsilon)).$$

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- Let $\mathbf{x} \in \Delta_n$ be such that $\lambda(H(P, T, \varepsilon)) = \lambda_{H(P,T,\varepsilon)}(\mathbf{x})$ with the fewest non-zero entries.

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- If there are two vertices, not contained in an edge, with positive weights, we can move weights from one to the other.
- Compactness argument; the points with positive weight form a 3-distance set. Shinohara: At most 7 points have positive weight. $\square$

Application 3: Applying the key lemma

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Theorem (Balogh, C., Dumitrescu)

(d) *Let T be $(108^\circ, 36^\circ, 36^\circ)$ or $(72^\circ, 72^\circ, 36^\circ)$. Then, $h_c(n, T) \leq \frac{n^3}{25}$.*

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- Q is a 2-distance set $\rightarrow |Q| \leq 5$.
- If $|Q| = 5$, then because Q is a 2-distance set, Q is a regular pentagon. Then $H(Q,T) \cong C_5^3$ and $\lambda(C_5^3) = \frac{1}{25}$.

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- If $|Q| = 3$, then $H(Q, T)$ is just a single edge and

$$\lambda(H(Q, T)) = \max_{x_i \geq 0, x_1 + x_2 + x_3 = 1} x_1 x_2 x_3 = \frac{1}{27}.$$

$\square$

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- If $|Q| = 6, 7$, we use a characterization of 3-distance sets by Shinohara.

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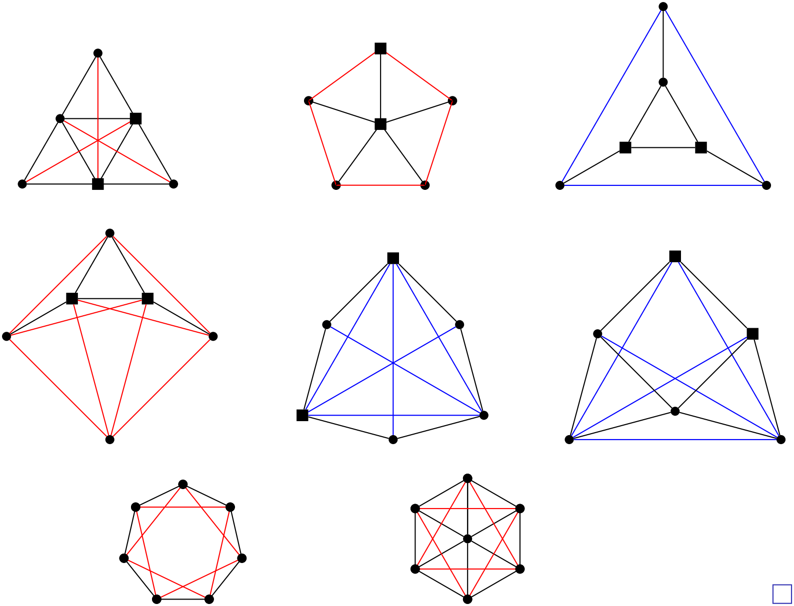

$\square$

Further questions

One of my favourite questions:

- Determine the maximum number of acute triangles in a planar point set of size n.

A Question for graduate students:

- Similar questions but $P \subseteq \mathbb{R}^d$ for $d \geq 3$.

Thank you for your attention!