Programming Languages: Functional Programming
3. Definition and Proof by Induction

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- The next few lectures concerns inductive definitions and proofs of datatypes and programs.
- While Haskell provides allows one to define nonterminating functions, infinite data structures, for now we will only consider its total, finite fragment.
- That is, we temporarily
    - consider only finite data structures,
    - demand that functions terminate for all value in its input type, and
    - provide guidelines to construct such functions.
- Infinite datatypes and non-termination will be discussed later in this course.

Induction on Natural Numbers

The So-Called "Mathematical Induction"

- Let P be a predicate on natural numbers.
- We've all learnt this principle of proof by induction: to prove that P holds for all natural numbers, it is sufficient to show that
    - $P\,0$ holds;
    - $P\,(1+n)$ holds provided that $P\,n$ does.

- We can see the above inductive principle as a result of seeing natural numbers as defined by the datatype [1]

$$\mathbf{data}\ \mathit{Nat} = 0 \mid \mathbf{1}_+\ \mathit{Nat}\ .$$

- That is, any natural number is either 0, or $\mathbf{1}_+\ n$ where n is a natural number.
- In this lecture, $\mathbf{1}_+$ is written in bold font to emphasise that it is a data constructor (as opposed to the function $(+)$, to be defined later, applied to a number 1).

[1] Not a real Haskell definition.

A Proof Generator

Given $P\,0$ and $P\,n \Rightarrow P\,(1_+\,n)$, how does one prove, for example, $P\,3$?

$$
\begin{array}{ll}
 & P\,(1_+\,(1_+\,(1_+\,0))) \\
\Leftarrow & \{\; P\,(1_+\,n) \Leftarrow P\,n \;\} \\
 & P\,(1_+\,(1_+\,0)) \\
\Leftarrow & \{\; P\,(1_+\,n) \Leftarrow P\,n \;\} \\
 & P\,(1_+\,0) \\
\Leftarrow & \{\; P\,(1_+\,n) \Leftarrow P\,n \;\} \\
 & P\,0\;.
\end{array}
$$

Having done math. induction can be seen as having designed *a program that generates a proof* — given any $n :: Nat$ we can generate a proof of $P\,n$ in the manner above.

- Since the type *Nat* is defined by two cases, it is natural to define functions on *Nat* following the structure:

$$\begin{aligned} &exp && :: Nat \to Nat \to Nat \\ &exp\ b\ 0 && = 1 \\ &exp\ b\ (\mathbf{1}_+\ n) && = b \times exp\ b\ n \quad . \end{aligned}$$

- Even addition can be defined inductively

$$\begin{aligned} &(+) && :: Nat \to Nat \to Nat \\ &0 + n && = n \\ &(\mathbf{1}_+\ m) + n && = \mathbf{1}_+\ (m + n) \quad . \end{aligned}$$

- Exercise: define $(\times)$?

A Value Generator

Given the definition of exp, how does one compute $exp\ b\ 3$?

$$
\begin{array}{ll}
 & exp\ b\ (\mathbf{1}_+\ (\mathbf{1}_+\ (\mathbf{1}_+\ 0))) \\
= & \quad \{\ \text{definition of } exp\ \} \\
 & b \times exp\ b\ (\mathbf{1}_+\ (\mathbf{1}_+\ 0)) \\
= & \quad \{\ \text{definition of } exp\ \} \\
 & b \times b \times exp\ b\ (\mathbf{1}_+\ 0) \\
= & \quad \{\ \text{definition of } exp\ \} \\
 & b \times b \times b \times exp\ b\ 0 \\
= & \quad \{\ \text{definition of } exp\ \} \\
 & b \times b \times b \times 1 \quad .
\end{array}
$$

It is a program that generates a value, for any $n :: Nat$.
Compare with the proof of P above.

An inductive proof is a program that generates a proof for any given natural number.

An inductive program follows the same structure of an inductive proof.

Proving and programming are very similar activities.

Without the $n + k$ Pattern

- Unfortunately, newer versions of Haskell abandoned the "$n + k$ pattern" used in the previous slides:

$$\begin{aligned} &exp && :: Int \to Int \to Int \\ &exp\ b\ 0 && = 1 \\ &exp\ b\ n && = b \times exp\ b\ (n - 1) \ . \end{aligned}$$

- Nat is defined to be Int in `MiniPrelude.hs`. Without `MiniPrelude.hs` you should use Int.
- For the purpose of this course, the pattern $1 + n$ reveals the correspondence between Nat and lists, and matches our proof style. Thus we will use it in the lecture.
- Remember to remove them in your code.

Proof by Induction

- To prove properties about Nat, we follow the structure as well.
- E.g. to prove that $exp\ b\ (m + n) = exp\ b\ m \times exp\ b\ n$.
- One possibility is to preform induction on m. That is, prove $P\,m$ for all $m :: Nat$, where $P\,m \equiv (\forall n :: exp\ b\ (m + n) = exp\ b\ m \times exp\ b\ n)$.

Proof by Induction

Recall $P\,m \equiv (\forall n :: exp\ b\ (m + n) = exp\ b\ m \times exp\ b\ n)$.

Case $m := 0$. For all n, we reason:

$$exp\ b\ (0 + n)$$

Recall $P\,m \equiv (\forall n :: \mathit{exp}\ b\ (m + n) = \mathit{exp}\ b\ m \times \mathit{exp}\ b\ n)$.

Case $m := 0$. For all n, we reason:

$$
\begin{aligned}
& \mathit{exp}\ b\ (0 + n) \\
= \quad & \quad \{\ \text{defn. of } (+)\ \} \\
& \mathit{exp}\ b\ n
\end{aligned}
$$

Recall $P\,m \equiv (\forall n :: exp\ b\ (m + n) = exp\ b\ m \times exp\ b\ n)$.

Case $m := 0$. For all n, we reason:

$$
\begin{array}{ll}
 & exp\ b\ (0 + n) \\
= & \quad \{\ \text{defn. of } (+)\ \} \\
 & exp\ b\ n \\
= & \quad \{\ \text{defn. of } (\times)\ \} \\
 & 1 \times exp\ b\ n
\end{array}
$$

Recall $P\,m \equiv (\forall n :: \mathit{exp}\ b\ (m+n) = \mathit{exp}\ b\ m \times \mathit{exp}\ b\ n)$.

Case $m := 0$. For all n, we reason:

$$
\begin{array}{ll}
 & \mathit{exp}\ b\ (0+n) \\
= & \quad \{\ \text{defn. of } (+)\ \} \\
 & \mathit{exp}\ b\ n \\
= & \quad \{\ \text{defn. of } (\times)\ \} \\
 & 1 \times \mathit{exp}\ b\ n \\
= & \quad \{\ \text{defn. of } \mathit{exp}\ \} \\
 & \mathit{exp}\ b\ 0 \times \mathit{exp}\ b\ n\ .
\end{array}
$$

We have thus proved $P\,0$.

Proof by Induction

Recall $P\, m \equiv (\forall n :: exp\ b\ (m + n) = exp\ b\ m \times exp\ b\ n)$.

Case $m := \mathbf{1}_+\ m$. For all n, we reason:

$$exp\ b\ ((\mathbf{1}_+\ m) + n)$$

Recall $P\, m \equiv (\forall n :: exp\ b\ (m+n) = exp\ b\ m \times exp\ b\ n)$.

Case $m := \mathbf{1}_+\ m$. For all n, we reason:

$$
\begin{array}{ll}
 & exp\ b\ ((\mathbf{1}_+\ m) + n) \\
= & \quad \{\ \text{defn. of } (+)\ \} \\
 & exp\ b\ (\mathbf{1}_+\ (m+n))
\end{array}
$$

Proof by Induction

Recall $P\ m \equiv (\forall n :: exp\ b\ (m + n) = exp\ b\ m \times exp\ b\ n)$.

Case $m := \mathbf{1}_+\ m$. For all n, we reason:

$$
\begin{aligned}
& exp\ b\ ((\mathbf{1}_+\ m) + n) \\
= \quad & \{\ \text{defn. of } (+)\ \} \\
& exp\ b\ (\mathbf{1}_+\ (m + n)) \\
= \quad & \{\ \text{defn. of } exp\ \} \\
& b \times exp\ b\ (m + n)
\end{aligned}
$$

Proof by Induction

Recall $P\, m \equiv (\forall n :: exp\ b\ (m + n) = exp\ b\ m \times exp\ b\ n)$.

Case $m := \mathbf{1}_{+}\ m$. For all n, we reason:

$$
\begin{aligned}
& exp\ b\ ((\mathbf{1}_{+}\ m) + n) \\
= \quad & \quad \{\ \text{defn. of } (+)\ \} \\
& exp\ b\ (\mathbf{1}_{+}\ (m + n)) \\
= \quad & \quad \{\ \text{defn. of } exp\ \} \\
& b \times exp\ b\ (m + n) \\
= \quad & \quad \{\ \text{induction}\ \} \\
& b \times (exp\ b\ m \times exp\ b\ n)
\end{aligned}
$$

Proof by Induction

Recall $P\,m \equiv (\forall n :: \mathit{exp}\ b\ (m + n) = \mathit{exp}\ b\ m \times \mathit{exp}\ b\ n)$.

Case $m := \mathbf{1}_+\ m$. For all n, we reason:

$$
\begin{array}{ll}
 & \mathit{exp}\ b\ ((\mathbf{1}_+\ m) + n) \\
= & \quad \{\ \text{defn. of } (+)\ \} \\
 & \mathit{exp}\ b\ (\mathbf{1}_+\ (m + n)) \\
= & \quad \{\ \text{defn. of } \mathit{exp}\ \} \\
 & b \times \mathit{exp}\ b\ (m + n) \\
= & \quad \{\ \text{induction}\ \} \\
 & b \times (\mathit{exp}\ b\ m \times \mathit{exp}\ b\ n) \\
= & \quad \{\ (\times)\ \text{associative}\ \} \\
 & (b \times \mathit{exp}\ b\ m) \times \mathit{exp}\ b\ n
\end{array}
$$

Proof by Induction

Recall $P\ m \equiv (\forall n :: exp\ b\ (m + n) = exp\ b\ m \times exp\ b\ n)$.

Case $m := \mathbf{1}_+\ m$. For all n, we reason:

$$
\begin{aligned}
& exp\ b\ ((\mathbf{1}_+\ m) + n) \\
= & \quad \{\ \text{defn. of } (+)\ \} \\
& exp\ b\ (\mathbf{1}_+\ (m + n)) \\
= & \quad \{\ \text{defn. of } exp\ \} \\
& b \times exp\ b\ (m + n) \\
= & \quad \{\ \text{induction}\ \} \\
& b \times (exp\ b\ m \times exp\ b\ n) \\
= & \quad \{\ (\times)\ \text{associative}\ \} \\
& (b \times exp\ b\ m) \times exp\ b\ n \\
= & \quad \{\ \text{defn. of } exp\ \} \\
& exp\ b\ (\mathbf{1}_+\ m) \times exp\ b\ n \quad .
\end{aligned}
$$

We have thus proved $P\ (\mathbf{1}_+\ m)$, given $P\ m$.

- The inductive proof could be carried out smoothly, because both $(+)$ and exp are defined inductively on its lefthand argument (of type Nat).
- The structure of the proof follows the structure of the program, which in turns follows the structure of the datatype the program is defined on.

Lists and Natural Numbers

- We have yet to prove that $(\times)$ is associative.
- The proof is quite similar to the proof for associativity of $(+\!\!+)$, which we will talk about later.
- In fact, *Nat* and lists are closely related in structure.
- Most of us are used to think of numbers as atomic and lists as structured data. Neither is necessarily true.
- For the rest of the course we will demonstrate induction using lists, while taking the properties for *Nat* as given.

An Inductively Defined Set?

- For a set to be “inductively defined”, we usually mean that it is the *smallest* fixed-point of some function.
- What does that maen?

Fixed-Point and Prefixed-Point

- A *fixed-point* of a function f is a value x such that $f\,x = x$.
- **Theorem**. f has fixed-point(s) if f is a *monotonic function* defined on a complete lattice.
    - In general, given f there may be more than one fixed-point.
- A *prefixed-point* of f is a value x such that $f\,x \leq x$.
    - Apparently, all fixed-points are also prefixed-points.
- **Theorem**. the smallest prefixed-point is also the smallest fixed-point.

EXAMPLE: *Nat*

- Recall the usual definition: Nat is defined by the following rules:
    1. 0 is in Nat;
    2. if n is in Nat, so is $\mathbf{1}_+\ n$;
    3. there is no other Nat.
- If we define a function F from sets to sets: $F\,X = \{0\} \cup \{\mathbf{1}_+\ n \mid n \in X\}$, 1. and 2. above means that $F\,Nat \subseteq Nat$. That is, Nat is a prefixed-point of F.
- 3. means that we want the *smallest* such prefixed-point.
- Thus Nat is also the least (smallest) fixed-point of F.

Least Prefixed-Point

Formally, let $F\,X = \{0\} \cup \{\mathbf{1}_+\ n \mid n \in X\}$, Nat is a set such that

$$F\,Nat \subseteq Nat \quad , \tag{1}$$

$$(\forall X : F\,X \subseteq X \;\Rightarrow\; Nat \subseteq X) \quad , \tag{2}$$

where (1) says that Nat is a prefixed-point of F, and (2) it is the least among all prefixed-points of F.

- Given property P, we also denote by P the set of elements that satisfy P.
- That $P\,0$ and $P\,n \Rightarrow P\,(\mathbf{1}_+ n)$ is equivalent to $\{0\} \subseteq P$ and $\{\mathbf{1}_+\, n \mid n \in P\} \subseteq P$,
- which is equivalent to $F\,P \subseteq P$. That is, P is a prefixed-point!
- By (2) we have $Nat \subseteq P$. That is, all Nat satisfy P!
- This is "why mathematical induction is correct."

Coinduction?

There is a dual technique called *coinduction* where, instead of least prefixed-points, we talk about *greatest postfixed points*. That is, largest x such that $x \leq f x$.

With such construction we can talk about infinite data structures.

Induction on Lists

- Recall that a (finite) list can be seen as a datatype defined by: [2]

$$\textbf{data}\ \mathit{List}\ a \ = \ [\,] \mid a : \mathit{List}\ a \ .$$

- Every list is built from the base case $[\,]$, with elements added by $(:)$ one by one: $[1, 2, 3] = 1 : (2 : (3 : [\,]))$.

[2] Not a real Haskell definition.

All Lists Today are Finite

But what about infinite lists?

- For now let's consider finite lists only, as having infinite lists make the *semantics* much more complicated. [3]
- In fact, all functions we talk about today are total functions. No $\perp$ involved.

[3] What does that mean? We will talk about it later.

Set-Theoretically Speaking...

The type *List a* is the *smallest* set such that

1. $[\,]$ is in *List a*;
2. if xs is in *List a* and x is in a, $x : xs$ is in *List a* as well.

- Many functions on lists can be defined according to how a list is defined:

$$
\begin{aligned}
&\mathit{sum} && :: \mathit{List}\ \mathit{Int} \rightarrow \mathit{Int} \\
&\mathit{sum}\ [\,] && = 0 \\
&\mathit{sum}\ (x : \mathit{xs}) && = x + \mathit{sum}\ \mathit{xs} \quad .
\end{aligned}
$$

$$
\begin{aligned}
&\mathit{map} && :: (a \rightarrow b) \rightarrow \mathit{List}\ a \rightarrow \mathit{List}\ b \\
&\mathit{map}\ f\ [\,] && = [\,] \\
&\mathit{map}\ f\ (x : \mathit{xs}) && = f\ x : \mathit{map}\ f\ \mathit{xs} \quad .
\end{aligned}
$$

- The function $(+\!\!+)$ appends two lists into one

$$
\begin{aligned}
(+\!\!+) &:: \mathit{List}\ a \to \mathit{List}\ a \to \mathit{List}\ a \\
[\,] +\!\!+ ys &= ys \\
(x : xs) +\!\!+ ys &= x : (xs +\!\!+ ys) \quad .
\end{aligned}
$$

- Compare the definition with that of $(+)$!

Proof by Structural Induction on Lists

- Recall that every finite list is built from the base case $[\,]$, with elements added by $(:)$ one by one.
- To prove that some property P holds for all finite lists, we show that
    1. $P\ [\,]$ holds;
    2. forall x and xs, $P\ (x : xs)$ holds provided that $P\ xs$ holds.

For a Particular List...

Given $P\,[\,]$ and $P\,xs \Rightarrow P\,(x : xs)$, for all x and xs, how does one prove, for example, $P\,[1, 2, 3]$?

$$
\begin{array}{ll}
 & P\,(1 : 2 : 3 : [\,]) \\
\Leftarrow & \{\ P\,(x : xs) \Leftarrow P\,xs\ \} \\
 & P\,(2 : 3 : [\,]) \\
\Leftarrow & \{\ P\,(x : xs) \Leftarrow P\,xs\ \} \\
 & P\,(3 : [\,]) \\
\Leftarrow & \{\ P\,(x : xs) \Leftarrow P\,xs\ \} \\
 & P\,[\,]\,.
\end{array}
$$

Appending is Associative

To prove that $xs +\!\!+ (ys +\!\!+ zs) = (xs +\!\!+ ys) +\!\!+ zs$.

Let $P\ xs = (\forall ys, zs :: xs +\!\!+ (ys +\!\!+ zs) = (xs +\!\!+ ys) +\!\!+ zs)$, we prove P by induction on xs.

Case $xs := [\,]$. For all ys and zs, we reason:

$$
\begin{array}{ll}
 & [\,] +\!\!+ (ys +\!\!+ zs) \\
= & \quad \{\ \text{defn. of } (+\!\!+)\ \} \\
 & ys +\!\!+ zs \\
= & \quad \{\ \text{defn. of } (+\!\!+)\ \} \\
 & ([\,] +\!\!+ ys) +\!\!+ zs \quad .
\end{array}
$$

We have thus proved $P\ [\,]$.

Case $xs := x : xs$. For all ys and zs, we reason:

$$
\begin{aligned}
& (x : xs) \mathbin{+\!\!+} (ys \mathbin{+\!\!+} zs) \\
= & \quad \{ \text{ defn. of } (\mathbin{+\!\!+}) \text{ } \} \\
& x : (xs \mathbin{+\!\!+} (ys \mathbin{+\!\!+} zs)) \\
= & \quad \{ \text{ induction } \} \\
& x : ((xs \mathbin{+\!\!+} ys) \mathbin{+\!\!+} zs) \\
= & \quad \{ \text{ defn. of } (\mathbin{+\!\!+}) \text{ } \} \\
& (x : (xs \mathbin{+\!\!+} ys)) \mathbin{+\!\!+} zs \\
= & \quad \{ \text{ defn. of } (\mathbin{+\!\!+}) \text{ } \} \\
& ((x : xs) \mathbin{+\!\!+} ys) \mathbin{+\!\!+} zs \quad .
\end{aligned}
$$

We have thus proved $P\ (x : xs)$, given $P\ xs$.

Do We Have To Be So Formal?

- In our style of proof, every step is given a reason. Do we need to be so pedantic?
- Being formal *helps* you to do the proof:
    - In the proof of $exp\ b\ (m + n) = exp\ b\ m \times exp\ b\ n$, we expect that we will use induction to somewhere. Therefore the first part of the proof is to generate $exp\ b\ (m + n)$.
    - In the proof of associativity, we were working toward generating $xs +\!\!+ (ys +\!\!+ zs)$.
- By being formal we can work on the *form*, not the *meaning*. Like how we solved the knight/knave problem
- Being formal actually makes the proof easier!
- *Make the symbols do the work.*

Length

- The function *length* defined inductively:

$$\begin{aligned} &\mathit{length} && :: \mathit{List}\ a \rightarrow \mathit{Nat} \\ &\mathit{length}\ [\,] && = 0 \\ &\mathit{length}\ (x : \mathit{xs}) && = \mathbf{1}_{+}\ (\mathit{length}\ \mathit{xs}) \quad . \end{aligned}$$

- Exercise: prove that *length* distributes into $(+\!\!+)$:

$$\mathit{length}\ (\mathit{xs} +\!\!+ \mathit{ys}) = \mathit{length}\ \mathit{xs} + \mathit{length}\ \mathit{ys}$$

Concatenation

- While $(+\!\!+)$ repeatedly applies $(:)$, the function *concat* repeatedly calls $(+\!\!+)$:

$$\begin{aligned} &\mathit{concat} &&:: \mathit{List}\ (\mathit{List}\ a) \to \mathit{List}\ a \\ &\mathit{concat}\ [\,] &&= [\,] \\ &\mathit{concat}\ (\mathit{xs} : \mathit{xss}) &&= \mathit{xs} +\!\!+ \mathit{concat}\ \mathit{xss} \quad . \end{aligned}$$

- Compare with *sum*.
- Exercise: prove $\mathit{sum} \cdot \mathit{concat} = \mathit{sum} \cdot \mathit{map}\ \mathit{sum}$.

Definition by Induction/Recursion

- Rather than giving commands, in functional programming we specify values; instead of performing repeated actions, we define values on inductively defined structures.
- Thus induction (or in general, recursion) is the only “control structure” we have. (We do identify and abstract over plenty of patterns of recursion, though.)
- To inductively define a function f on lists, we specify a value for the base case ($f\ [\,]$) and, assuming that $f\ xs$ has been computed, consider how to construct $f\ (x : xs)$ out of $f\ xs$.

Filter

- *filter p xs* keeps only those elements in *xs* that satisfy *p*.

$$
\begin{aligned}
&\mathit{filter} && :: (a \rightarrow \mathit{Bool}) \rightarrow \mathit{List}\ a \rightarrow \mathit{List}\ a\\
&\mathit{filter}\ p\ [\,] && = [\,]\\
&\mathit{filter}\ p\ (x : xs) && \mid p\ x = x : \mathit{filter}\ p\ xs\\
& && \mid \textbf{otherwise} = \mathit{filter}\ p\ xs \quad .
\end{aligned}
$$

Take and Drop

- Recall *take* and *drop*, which we used in the previous exercise.

$$\begin{array}{ll}
\mathit{take} & :: \mathit{Nat} \rightarrow \mathit{List}\ a \rightarrow \mathit{List}\ a \\
\mathit{take}\ 0\ \mathit{xs} & = [\,] \\
\mathit{take}\ (\mathbf{1}_+\ n)\ [\,] & = [\,] \\
\mathit{take}\ (\mathbf{1}_+\ n)\ (x : \mathit{xs}) & = x : \mathit{take}\ n\ \mathit{xs} \quad .
\end{array}$$

$$\begin{array}{ll}
\mathit{drop} & :: \mathit{Nat} \rightarrow \mathit{List}\ a \rightarrow \mathit{List}\ a \\
\mathit{drop}\ 0\ \mathit{xs} & = \mathit{xs} \\
\mathit{drop}\ (\mathbf{1}_+\ n)\ [\,] & = [\,] \\
\mathit{drop}\ (\mathbf{1}_+\ n)\ (x : \mathit{xs}) & = \mathit{drop}\ n\ \mathit{xs} \quad .
\end{array}$$

- Prove: $\mathit{take}\ n\ \mathit{xs} \mathbin{+\!\!+} \mathit{drop}\ n\ \mathit{xs} = \mathit{xs}$, for all n and xs.

TakeWhile and DropWhile

- *takeWhile p xs* yields the longest prefix of *xs* such that *p* holds for each element.

$$\begin{array}{lll} takeWhile & :: (a \to Bool) \to List\ a \to List\ a \\ takeWhile\ p\ [\,] & = [\,] \\ takeWhile\ p\ (x : xs) & \mid p\ x = x : takeWhile\ p\ xs \\ & \mid \mathbf{otherwise} = [\,] \quad . \end{array}$$

- *dropWhile p xs* drops the prefix from *xs*.

$$\begin{array}{lll} dropWhile & :: (a \to Bool) \to List\ a \to List\ a \\ dropWhile\ p\ [\,] & = [\,] \\ dropWhile\ p\ (x : xs) & \mid p\ x = dropWhile\ p\ xs \\ & \mid \mathbf{otherwise} = x : xs \quad . \end{array}$$

- Prove: $takeWhile\ p\ xs +\!\!+\ dropWhile\ p\ xs = xs$.

List Reversal

- $reverse\ [1, 2, 3, 4] = [4, 3, 2, 1]$.

$$\begin{aligned}
&reverse &&:: List\ a \rightarrow List\ a \\
&reverse\ [\,] &&= [\,] \\
&reverse\ (x : xs) &&= reverse\ xs +\!\!+ [x] \quad .
\end{aligned}$$

All Prefixes and Suffixes

- $inits\ [1,2,3] = [[\,],[1],[1,2],[1,2,3]]$

$$\begin{aligned}
&inits && :: List\ a \to List\ (List\ a)\\
&inits\ [\,] && = [[\,]]\\
&inits\ (x : xs) && = [\,] : map\ (x :)\ (inits\ xs) \quad .
\end{aligned}$$

- $tails\ [1,2,3] = [[1,2,3],[2,3],[3],[\,]]$

$$\begin{aligned}
&tails && :: List\ a \to List\ (List\ a)\\
&tails\ [\,] && = [[\,]]\\
&tails\ (x : xs) && = (x : xs) : tails\ xs \quad .
\end{aligned}$$

Totality

- Structure of our definitions so far:

$$\begin{aligned} f\,[\,] &= \ldots \\ f\,(x : xs) &= \ldots f\,xs \ldots \end{aligned}$$

  - Both the empty and the non-empty cases are covered, guaranteeing there is a matching clause for all inputs.
  - The recursive call is made on a “smaller” argument, guranteeing termination.
- Together they guarantee that every input is mapped to some output. Thus they define *total* functions on lists.

Variations with the Base Case

- Some functions discriminate between several base cases. E.g.

$$
\begin{aligned}
&\mathit{fib} && :: \mathit{Nat} \to \mathit{Nat} \\
&\mathit{fib}\ 0 && = 0 \\
&\mathit{fib}\ 1 && = 1 \\
&\mathit{fib}\ (2 + n) && = \mathit{fib}\ (\mathbf{1}_{+} n) + \mathit{fib}\ n \quad .
\end{aligned}
$$

- Some functions make more sense when it is defined only on non-empty lists:

$$\begin{aligned} f\,[x] &= \ldots \\ f\,(x : xs) &= \ldots \end{aligned}$$

- What about totality?
  - They are in fact functions defined on a different datatype:

    $$\mathbf{data}\ List^{+}\ a\ =\ Singleton\ a \mid a : List^{+}\ a\ .$$

  - We do not want to define *map*, *filter* again for $List^{+}\ a$. Thus we reuse *List a* and pretend that we were talking about $List^{+}\ a$.
  - It's the same with *Nat*. We embedded *Nat* into *Int*.
  - Ideally we'd like to have some form of *subtyping*. But that makes the type system more complex.

- It also occurs often that we perform *lexicographic induction* on multiple arguments: some arguments decrease in size, while others stay the same.
- E.g. the function *merge* merges two sorted lists into one sorted list:

$$
\begin{array}{ll}
\mathit{merge} & :: \mathit{List}\ \mathit{Int} \to \mathit{List}\ \mathit{Int} \to \mathit{List}\ \mathit{Int} \\
\mathit{merge}\ [\,]\ [\,] & = [\,] \\
\mathit{merge}\ [\,]\ (y : ys) & = y : ys \\
\mathit{merge}\ (x : xs)\ [\,] & = x : xs \\
\mathit{merge}\ (x : xs)\ (y : ys) & \mid x \leq y = x : \mathit{merge}\ xs\ (y : ys) \\
 & \mid \mathbf{otherwise} = y : \mathit{merge}\ (x : xs)\ ys \quad .
\end{array}
$$

Zip

Another example:

$$
\begin{aligned}
&zip :: List\ a \to List\ b \to List\ (a, b) \\
&zip\ [\,]\ [\,] && = [\,] \\
&zip\ [\,]\ (y : ys) && = [\,] \\
&zip\ (x : xs)\ [\,] && = [\,] \\
&zip\ (x : xs)\ (y : ys) && = (x, y) : zip\ xs\ ys \quad .
\end{aligned}
$$

Non-Structural Induction

- In most of the programs we've seen so far, the recursive call are made on direct sub-components of the input (e.g. $f\ (x : xs) = ..f\ xs..$). This is called *structural induction*.
    - It is relatively easy for compilers to recognise structural induction and determine that a program terminates.
- In fact, we can be sure that a program terminates if the arguments get "smaller" under some (well-founded) ordering.

Mergesort

- In the implemenation of mergesort below, for example, the arguments always get smaller in size.

$$
\begin{array}{ll}
\mathit{msort} & :: \mathit{List\ Int} \rightarrow \mathit{List\ Int} \\
\mathit{msort}\ [\,] & = [\,] \\
\mathit{msort}\ [x] & = [x] \\
\mathit{msort}\ \mathit{xs} & = \mathit{merge}\ (\mathit{msort}\ \mathit{ys})\ (\mathit{msort}\ \mathit{zs}) \quad , \\
\quad \mathbf{where}\ n & = \mathit{length}\ \mathit{xs}\ \text{`}\mathit{div}\text{`}\ 2 \\
\qquad\quad \mathit{ys} & = \mathit{take}\ n\ \mathit{xs} \\
\qquad\quad \mathit{zs} & = \mathit{drop}\ n\ \mathit{xs} \quad .
\end{array}
$$

  - What if we omit the case for $[x]$?

- If all cases are covered, and all recursive calls are applied to smaller arguments, the program defines a total function.

A Non-Terminating Definition

- Example of a function, where the argument to the recursive does not reduce in size:

$$\begin{aligned} f &:: Int \to Int \\ f\,0 &= 0 \\ f\,n &= f\,n \quad . \end{aligned}$$

- Certainly f is not a total function. Do such definitions "mean" something? We will talk about these later.

User Defined Inductive Datatypes

- This is a possible definition of internally labelled binary trees:

$$\textbf{data}\ Tree\ a\ =\ \mathrm{Null} \mid \mathrm{Node}\ a\ (Tree\ a)\ (Tree\ a)\quad ,$$

- on which we may inductively define functions:

$$\begin{aligned}
&sumT && ::\ Tree\ Nat \rightarrow Nat \\
&sumT\ \mathrm{Null} && =\ 0 \\
&sumT\ (\mathrm{Node}\ x\ t\ u) && =\ x + sumT\ t + sumT\ u\quad .
\end{aligned}$$

Exercise: given $(\downarrow) :: Nat \to Nat \to Nat$, which yields the smaller one of its arguments, define the following functions

1. $minT :: Tree\ Nat \to Nat$, which computes the minimal element in a tree.
2. $mapT :: (a \to b) \to Tree\ a \to Tree\ b$, which applies the functional argument to each element in a tree.
3. Can you define $(\downarrow)$ inductively on Nat? [4]

[4] In the standard Haskell library, $(\downarrow)$ is called *min*.

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- Exercise: prove that for all n and t, $minT\ (mapT\ (n+)\ t) = n + minT\ t$. That is, $minT \cdot mapT\ (n+) = (n+) \cdot minT$.

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- Recall that **data** $Bool = False \mid True$. Do we have an induction principle for $Bool$?
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- Recall that **data** $Bool = False \mid True$. Do we have an induction principle for $Bool$?
- To prove a predicate P on $Bool$ holds for all booleans, it is sufficient to show that
  1. $P\ False$ holds, and
  2. $P\ True$ holds.
- Well, of course.

- What about $(A \times B)$? How to prove that a predicate P on $(A \times B)$ is always true?
- One may prove some property P_1 on A and some property P_2 on B, which together imply P.
- That does not say much. But the “induction principle” for products allows us to extract, from a proof of P, the proofs P_1 and P_2.

- *Every inductively defined datatype comes with its induction principle.*
- We will come back to this point later.