

Programming Languages: Imperative Program Construction

Practicals 3. Quantifications

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1. An integer array $X[0..N)$ is given, where $N \geq 1$. Explain, in words, what each of the following expressions mean.

1. $b \equiv \langle \forall i : 0 \leq i < N : X[i] \geq 0 \rangle$.
2. $r = \langle \#k : 0 \leq k < N : \langle \forall i : 0 \leq i < k : X[i] < X[k] \rangle \rangle$.
3. $r = \langle \uparrow p, q : 0 \leq p \leq q \leq N \wedge \langle \forall i : p \leq i < q : X[i] > 0 \rangle : p - q \rangle$.
4. $r = \langle \#p, q : 0 \leq p < q < N : X[p] = 0 \wedge X[q] = 1 \rangle$.
5. $s = \langle \uparrow p, q : 0 \leq p < q < N : X[p] + X[q] \rangle$.
6. $b \equiv \langle \forall p, q : 0 \leq p \wedge 0 \leq q \wedge p + q = N - 1 : X[p] = X[q] \rangle$.

2. An integer array $X[0..N)$ is given, where $N \geq 1$. Express the following sentences in a formal way:

1. r is the sum of the elements of X .
2. X is increasing.
3. all values of X are distinct.
4. r is the length of a longest constant segment of X .
5. r is the maximum of the sums of the segments of X .

3. Expand the following textual substitutions. If necessary, change the dummy, according to Dummy Renaming (8.21).

1. $\langle \star x : 0 \leq x + r < n : x + v \rangle [v \setminus 3]$
2. $\langle \star x : 0 \leq x + r < n : x + v \rangle [x \setminus 3]$
3. $\langle \star x : 0 \leq x + r < n : x + v \rangle [n \setminus n + x]$
4. $\langle \star x : 0 \leq x < r : \langle \star y : 0 \leq y : x + y + n \rangle \rangle [n \setminus x + y]$
5. $\langle \star x : 0 \leq x < r : \langle \star y : 0 \leq y : x + y + n \rangle \rangle [r \setminus y]$

4. Prove the following theorems. Provided $0 \leq n$,

- (a) $\langle \Sigma i : 0 \leq i < n + 1 : b[i] \rangle = b[0] + \langle \Sigma i : 1 \leq i < n + 1 : b[i] \rangle$
- (b) $\langle \exists i : 0 \leq i < n + 1 : b[i] = 0 \rangle = \langle \exists i : 0 \leq i < n : b[i] = 0 \rangle \vee b[n] = 0$

5. Prove that $\langle \forall x : R : P \rangle \equiv P \vee \langle \forall x :: \neg R \rangle$, provided $\neg \text{occurs}(x, P)$.

6. Prove the *range weakening* rule: $\langle \forall x : Q \vee R : P \rangle \Rightarrow \langle \forall x : Q : P \rangle$.

7. Prove the *body weakening* rule: $\langle \forall x : R : P \wedge Q \rangle \Rightarrow \langle \forall x : R : P \rangle$.