Programming Languages: Imperative Program Construction 6. Loop Construction II: Strengthening the Invariant

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Maximum Segment Sum

A classical problem: given an array of integers, find largest possible sum of a consecutive segment.

```
**con** $N : Int$ $\{0 \leqslant N\}$
**con** $f$ : **array** $[0..N)$ **of** $Int$
$S$
$\{r = \langle \uparrow p\ q : 0 \leqslant p \leqslant q \leqslant N : sum\ p\ q \rangle\}$
```

where $sum\ p\ q = \langle \Sigma i : p \leqslant i < q : f[i] \rangle$.

Details That Matter

- Note the use of $\leqslant$ and $<$ in the specification.
- The range in $sum\ p\ q$ is $p \leqslant i < q$. It computes the sum of $f\ [p..q)$ — not including $f[q]$!
- Therefore when $p = q$, $sum\ p\ q$ computes the sum of an empty segment.
- In the postcondition we have $p \leqslant q$ — we allow empty segments in our solution!
- We must have $q \leqslant N$ instead of $q < N$. Otherwise segments containing the rightmost element would not be valid solutions.

Previously Introduced Techniques

- Replace N by n. Use $P \wedge Q$ as the invariant, where

$$\begin{aligned} P &\equiv r = \langle \uparrow p\ q : 0 \leqslant p \leqslant q \leqslant n : sum\ p\ q \rangle \quad , \\ Q &\equiv 0 \leqslant n \leqslant N \quad . \end{aligned}$$

- Use $\neg\,(n = N)$ as guard. This way we immediately have that $P \wedge Q \wedge n = N$ imply the desired postcondition.
- How do we know we want $0 \leqslant n \leqslant N$? It can be forced by our development later. But let's expedite the pace.
- Initialisation: $n, r := 0, 0$.
- Use $N - n$ as the bound.
- To decrease the bound, let $n := n + 1$ be the last statement of the loop.

We get this program.

```
con $N : Int$ $\{0 \leqslant N\}$
con $f$ : array $[0..N)$ of $Int$
var $r, n : Int$
$r, n := 0, 0$
$\{P \wedge Q, bnd : N - n\}$
do $n \neq N \rightarrow$ ??? ; $n := n + 1$ od
$\{r = \langle \uparrow p\ q : 0 \leqslant p \leqslant q \leqslant N : sum\ p\ q \rangle\}$
```

Now we need to construct the ??? part.

Constructing the Loop Body

How to construct the ??? part?

$\{P \wedge Q \wedge n \neq N\}$

???

$\{(P \wedge Q)[n \backslash n+1]\}$

$n := n+1$

$\{P \wedge Q\}$

Examining the Expression

To reason about $P[n\backslash n+1]$, we calculate (assuming $P \wedge Q \wedge n \neq N$):

$$\begin{aligned}
&\langle\uparrow p\ q : 0 \leqslant p \leqslant q \leqslant n : sum\ p\ q\rangle[n\backslash n+1] \\
=\ &\langle\uparrow p\ q : 0 \leqslant p \leqslant q \leqslant n+1 : sum\ p\ q\rangle \\
=\ &\quad \{\ \text{split off } q = n+1\text{, see next slide}\ \} \\
&\langle\uparrow p\ q : 0 \leqslant p \leqslant q \leqslant n : sum\ p\ q\rangle \uparrow \\
&\quad \langle\uparrow p : 0 \leqslant p \leqslant (n+1) : sum\ p\ (n+1)\rangle \\
=\ &\quad \{\ P_0\ \} \\
&r \uparrow \langle\uparrow p : 0 \leqslant p \leqslant (n+1) : sum\ p\ (n+1)\rangle \quad .
\end{aligned}$$

Therefore we wish to update r by:

$$r := r \uparrow \langle \uparrow p : 0 \leqslant p \leqslant (n+1) : sum\ p\ (n+1) \rangle \quad .$$

But $\langle \uparrow p : 0 \leqslant p \leqslant (n+1) : sum\ p\ (n+1) \rangle$ cannot be computed in one step!

We could compute $\langle \uparrow p : 0 \leqslant p \leqslant (n+1) : sum\ p\ (n+1) \rangle$ in a loop...or can we store it in another variable?

Splitting Off?

Regarding the step “split off $q = n + 1$”:

$$\begin{aligned}
& 0 \leqslant p \leqslant q \leqslant n+1 \\
= {} & 0 \leqslant p \leqslant q \wedge q \leqslant n+1 \\
= {} & 0 \leqslant p \leqslant q \wedge (q \leqslant n \vee q = n+1) \\
= {} & (0 \leqslant p \leqslant q \wedge q \leqslant n) \vee (0 \leqslant p \leqslant q \wedge q = n+1) \\
= {} & 0 \leqslant p \leqslant q \leqslant n \vee (0 \leqslant p \leqslant q \wedge q = n+1) \quad .
\end{aligned}$$

Note that for the second step to be valid, we need $-1 \leqslant n$ (which is implied by $0 \leqslant n \leqslant N$). Always remember to check that the range is non-empty before you split!

Therefore we have (abbreviating *sum*... to R):

$$
\begin{array}{ll}
& \langle \uparrow p\, q : 0 \leqslant p \leqslant j \leqslant n+1 : R \rangle \\
= & \quad \{ \text{ previous calculation } \} \\
& \langle \uparrow p\, q : 0 \leqslant p \leqslant q \leqslant n \vee (0 \leqslant p \leqslant q \wedge q = n+1) : R \rangle \\
= & \quad \{ \text{ range split (8.16) } \} \\
& \langle \uparrow p\, q : 0 \leqslant p \leqslant q \leqslant n : R \rangle \uparrow \\
& \quad \langle \uparrow p\, q : 0 \leqslant p \leqslant q \wedge q = n+1 : R \rangle \\
= & \quad \{ \text{ one-point rule } \} \\
& \langle \uparrow p\, q : 0 \leqslant p \leqslant q \leqslant n : R \rangle \uparrow \\
& \quad \langle \uparrow p : 0 \leqslant p \leqslant n+1 : R \rangle \quad .
\end{array}
$$

Things to note:

- Calculation for other patterns of ranges (e.g. $0 \leqslant p \leqslant q \leqslant n+1$) are slightly different. Watch out!
- In practice, the "splitting off" step is but one quick step. We do not do the reasoning above in such detail.
- We show you the details above for expository purpose.
- In other problems we may see slightly different ranges, such as $0 \leqslant p < q < n+1$. The result of splitting is different too. Take extra care!

Strengthening the Invariant

Knowing that we need to update r with $\langle \uparrow p : 0 \leqslant p \leqslant (n+1) : \mathit{sum}\ p\ (n+1) \rangle$, let us store it in some variable! Introduce a new variable s, and *strengthen* the invariant to $P_0 \wedge P_1 \wedge Q$, where

$$\begin{aligned}
P_0 &\equiv r = \langle \uparrow p\ q : 0 \leqslant p \leqslant q \leqslant n : \mathit{sum}\ p\ q \rangle \quad , \\
P_1 &\equiv s = \langle \uparrow p : 0 \leqslant p \leqslant n : \mathit{sum}\ p\ n \rangle \quad , \\
Q &\equiv 0 \leqslant n \leqslant N \quad .
\end{aligned}$$

Maximum Suffix Sum

- That is, while r is the maximum *segment* sum so far, s is the maximum *suffix* sum so far.
- We discover the need of this concept through symbolic calculation.
- This is a pattern for many "segment problems": *to solve a problem about segments, solve a suffix problem for all prefixes.*

Q: Why don't we let $s = \langle \uparrow p : 0 \leqslant p \leqslant n+1 : \mathit{sum}\ p\ (n+1) \rangle$?

A: You can do that too! Left as an exercise. We use n instead of $n+1$ to keep the invariant looking simple.

Constructing the Loop Body

Therefore, a possible strategy would be:

```
$\{P_0 \wedge P_1 \wedge 0 \leqslant n \leqslant N \wedge n \neq N\}$
$s :=$ ???
$\{P_0 \wedge P_1[n \backslash n+1] \wedge 0 \leqslant n+1 \leqslant N\}$
$r := r \uparrow s$
$\{(P_0 \wedge P_1 \wedge 0 \leqslant n \leqslant N)[n \backslash n+1]\}$
$n := n+1$
$\{P_0 \wedge P_1 \wedge 0 \leqslant n \leqslant N\}$
```

Updating the Prefix Sum

Recall $P_1 \equiv s = \langle\uparrow p : 0 \leqslant p \leqslant n : sum\ p\ n\rangle$.

$$
\begin{aligned}
& \langle\uparrow p : 0 \leqslant p \leqslant n : sum\ p\ n\rangle[n\backslash n+1] \\
={}& \langle\uparrow p : 0 \leqslant p \leqslant n+1 : sum\ p\ (n+1)\rangle \\
={}& \quad \{\ \text{splitting off } p = n+1\ \} \\
& \langle\uparrow p : 0 \leqslant p \leqslant n : sum\ p\ (n+1)\rangle \uparrow \\
& \quad sum\ (n+1)\ (n+1) \\
={}& \quad \{\ [n+1..n+1) \text{ is an empty range}\ \} \\
& \langle\uparrow p : 0 \leqslant p \leqslant n : sum\ p\ (n+1)\rangle \uparrow 0 \\
={}& \quad \{\ \text{splitting off } i = n \text{ in } sum\ \} \\
& \langle\uparrow p : 0 \leqslant p \leqslant n : sum\ p\ n + f[n]\rangle) \uparrow 0 \\
={}& \quad \{\ \text{distributivity}\ \} \\
& (\langle\uparrow p : 0 \leqslant p \leqslant n : sum\ p\ n\rangle + f[n]) \uparrow 0 \quad .
\end{aligned}
$$

Thus, $\{P_1\}\ s :=\ ?\ \{P_1[n\backslash n+1]\}$ is satisfied by $s := (s + f[n]) \uparrow 0$.

A Key Property

- The last step labelled "distributivity" uses a rule mentioned before: provided that $\neg occurs(i, F)$ and R non-empty:

$$F + \langle \uparrow i : R : S \rangle = \langle \uparrow i : R : F + S \rangle$$
$$F + \langle \downarrow i : R : S \rangle = \langle \downarrow i : R : F + S \rangle \quad .$$

- The rules are valid because addition distributes into maximum/minimum:

$$x + (y \uparrow z) = (x + y) \uparrow (x + z) \quad ,$$
$$x + (y \downarrow z) = (x + y) \downarrow (x + z) \quad .$$

- That is the key property that allows us to have an efficient algorithm for the maximum segment sum problem!
- Through calculation, we not only have an algorithm, but also identified the key property that makes it work, which

Derived Program

```
con $N : Int$ $\{0 \leqslant N\}$
con $f$ : array $[0..N)$ of $Int$
var $r, n : Int$

$r, s, n := 0, 0, 0$
$\{P_0 \wedge P_1 \wedge Q, bnd : N - n\}$
do $n \neq N \rightarrow$
    $s := (s + f[n]) \uparrow 0$
    $r := r \uparrow s$
    $n := n + 1$
od
$\{r = \langle \uparrow p\ q : 0 \leqslant p \leqslant q \leqslant N : sum\ p\ q \rangle\}$
```

$$P_0 \equiv r = \langle \uparrow p\ q : 0 \leqslant p \leqslant q \leqslant n : sum\ p\ q \rangle) \quad ,$$
$$P_1 \equiv s = \langle \uparrow p : 0 \leqslant p \leqslant n : sum\ p\ n \rangle) \quad ,$$
$$Q \ \equiv 0 \leqslant n \leqslant N \quad .$$

“Strengthening”?

- We stay that the invariant $P_0 \wedge P_1 \wedge Q$ is “stronger” than $P \wedge Q$ because the former promises more.
- The resulting loop computes values for two variables rather than one.
- However, the program ends up being quicker because more results from the previous iteration of the loop can be utilised.
- It is a common phenomena: a generalised theorem is easier to prove.
- We will see another way to generalise the invariant in the rest of the course.

Lessons Learnt?

Let the symbols do the work!

- We discover how to strengthen the invariant by calculating and finding out what is missing.
- Expressions are your friend, and blind guessing can be minimised. We always get some clue from the expressions.
- Since we rely only on the symbols, the same calculation/algorithm can be generalised to other problems (e.g. as long as the same distributivity propery holds).

If we remove the pre/postconditions and the invariant, can you tell us what the program does?

- Without the assertions, programs mean nothing. The assertions are what matter about the program.
- Structured programming is not about making (the operational parts of) code easier to read/understand.
- Such efforts are bound to end in vain: even a simple three-line loop can be hard to understand if the assertions, encoding the intentions of the programmer, are stripped away.

- Instead, structured programming is about organising the code around the structure of the proofs.
- Once the pre/postconditions are given, and the invariants and bounds are determined, one can derive the code accordingly.
- It is pointless arguing, for example, “using a *break* here makes the code easier to read.”
- One shall not need to “understand“ the operational parts of the code, but to check whether it meets the specification.

No. of Pairs in an Array

Consider constructing the following program:

```
con $N : Int \; \{0 \leqslant N\}; a : \mathbf{array}\ [0..N)\ \mathbf{of}\ Int$
var $r : Int$
$S$
$\{r = \langle \# i\ j : 0 \leqslant i < j < N : a[i] \leqslant 0 \wedge a[j] \geqslant 0 \rangle\}$
```

Previously Introduced Techniques

- Replace N by n. Use $P \wedge Q$ as the invariant, where

$$P \equiv \; r = \langle \# i, j : 0 \leqslant i < j < n : a[i] \leqslant 0 \wedge a[j] \geqslant 0 \rangle,$$
$$Q \equiv \; 0 \leqslant n \leqslant N.$$

- Use $\neg\,(n = N)$ as guard. This way we immediately have that $P \wedge Q \wedge n = N$ imply the desired postcondition.
- Initialisation: $n, r := 0, 0$.
- Use $N - n$ as the bound.
- To decrease the bound, let $n := n + 1$ be the last statement of the loop.

We get this program.

```
$\textbf{con}\ N : Int\ \{0 \leqslant N\};\ a : \textbf{array}\ [0..N)\ \textbf{of}\ Int$
$\textbf{var}\ r, n : Int$
$r, n := 0, 0$
$\{P \wedge Q, bnd : N - n\}$
$\textbf{do}\ n \neq N \rightarrow ...;\ n := n + 1\ \textbf{od}$
$\{r = \langle \# i\ j : 0 \leqslant i < j < N : a[i] \leqslant 0 \wedge a[j] \geqslant 0 \rangle\}$
```

Now we need to construct the ... part.

Constructing the Loop Body

How to construct the ... part?

$$
\begin{array}{l}
\{P \wedge Q \wedge n \neq N\} \\
\ldots \\
\{(P \wedge Q)[n \backslash n+1]\} \\
n := n + 1 \\
\{P \wedge Q\}
\end{array}
$$

No. of Pairs in an Array

To reason about $P[n\backslash n+1]$, we calculate (assuming $P \wedge Q \wedge n \neq N$):

$$\begin{aligned}
& \langle\, \#i,j : 0 \leqslant i < j < n+1 : a[i] \leqslant 0 \wedge a[j] \geqslant 0 \,\rangle \\
= & \quad \{ \text{ split off } j = n \text{, see the next slide } \} \\
& \langle\, \#i,j : 0 \leqslant i < j < n : a[i] \leqslant 0 \wedge a[j] \geqslant 0 \,\rangle + \\
& \langle\, \#i : 0 \leqslant i < n : a[i] \leqslant 0 \wedge a[n] \geqslant 0 \,\rangle \\
= & \quad \{ \; P \; \} \\
& r + \langle\, \#i : 0 \leqslant i < n : a[i] \leqslant 0 \wedge a[n] \geqslant 0 \,\rangle \\
= & \begin{cases} r, & \text{if } a[n] < 0; \\ r + \langle\, \#i : 0 \leqslant i < n : a[i] \leqslant 0 \,\rangle, & \text{if } a[n] \geqslant 0. \end{cases}
\end{aligned}$$

Let us try storing $\langle\, \#i : 0 \leqslant i < n : a[i] \leqslant 0 \,\rangle$ in another variable?

Splitting Off?

For expository purpose let us exam how the splitting was done:

$$\begin{aligned}
& 0 \leqslant i < j < n+1 \\
={}& 0 \leqslant i < j \wedge j < n+1 \\
={}& 0 \leqslant i < j \wedge (j < n \vee j = n) \\
={}& (0 \leqslant i < j \wedge j < n) \vee (0 \leqslant i < j \wedge j = n) \\
={}& 0 \leqslant i < j < n \vee (0 \leqslant i < j \wedge j = n) \quad .
\end{aligned}$$

The second step is valid if $0 \leqslant n$.

A Frequent Pattern

We may see this pattern often. For some $\star$, we need to calculate:

$$
\begin{aligned}
& \langle \star i\, j : 0 \leqslant i < j < n+1 : R \rangle \\
= & \quad \{ \text{ previous calculation } \} \\
& \langle \star i\, j : 0 \leqslant i < j < n \vee (0 \leqslant i < j \wedge j = n) : R \rangle \\
= & \langle \star i\, j : 0 \leqslant i < j < n : R \rangle \star \\
& \quad \langle \star i\, j : 0 \leqslant i < j \wedge j = n : R \rangle \\
= & \quad \{ \text{ one-point rule } \} \\
& \langle \star i\, j : 0 \leqslant i < j < n : R \rangle \star \\
& \quad \langle \star i : 0 \leqslant i < n : R \rangle \quad .
\end{aligned}
$$

Calculation for other ranges (e.g. $0 \leqslant i \leqslant j \leqslant n+1$) are slightly different. Watch out!

Strengthening the Invariant

New plan: define

$$P_0 \equiv r = \langle \#i,j : 0 \leqslant i < j < n : a[i] \leqslant 0 \wedge a[j] \geqslant 0 \rangle,$$
$$P_1 \equiv s = \langle \#i : 0 \leqslant i < n : a[i] \leqslant 0 \rangle,$$
$$Q \equiv 0 \leqslant n \leqslant N,$$

and try to derive

```
con $N : Int\ \{N \geqslant 0\}$; $a$ : array $[0..N)$ of $Int$
var $r, s : Int$

$n, r, s := 0, 0, 0$
$\{P_0 \wedge P_1 \wedge Q, bnd : N - n\}$
do $n \neq N \rightarrow \ldots n := n + 1$ od
$\{r = \langle \#i,j : 0 \leqslant i < j < N : a[i] \leqslant 0 \wedge a[j] \geqslant 0 \rangle\}$
```

Update the New Variable

$$
\begin{aligned}
& \langle \# i : 0 \leqslant i < n : a[i] \leqslant 0 \rangle [n \backslash n+1] \\
= \;& \langle \# i : 0 \leqslant i < n+1 : a[i] \leqslant 0 \rangle \\
= \;& \quad \{ \text{ split off } i = n \text{ (assuming } 0 \leqslant n) \; \} \\
& \langle \# i : 0 \leqslant i < n : a[i] \leqslant 0 \rangle + \#(a[n] \leqslant 0) \\
= \;& \quad \{ \; P_1 \; \} \\
& s + \#(a[n] \leqslant 0) \\
= \;& \begin{cases} s & \text{if } a[n] > 0, \\ s+1 & \text{if } a[n] \leqslant 0. \end{cases}
\end{aligned}
$$

Resulting Program

```
$n, r, s := 0, 0, 0$
$\{P_0 \wedge P_1 \wedge Q, bnd : N - n\}$
do $n \neq N \rightarrow \{P_0 \wedge P_1 \wedge Q \wedge n \neq N\}$
  if $a[n] < 0 \rightarrow skip$
   $| \; a[n] \geqslant 0 \rightarrow r := r + s$
  fi
  $\{P_0[n \backslash n + 1] \wedge P_1 \wedge Q \wedge n \neq N\}$
  if $a[n] > 0 \rightarrow skip$
   $| \; a[n] \leqslant 0 \rightarrow s := s + 1$
  fi
  $\{(P_0 \wedge P_1 \wedge Q)[n \backslash n + 1]\}$
  $n := n + 1$
od
$\{r = \langle \#i, j : 0 \leqslant i < j < N : a[i] \leqslant 0 \wedge a[j] \geqslant 0 \rangle\}$
```

Resulting Program

Since $P_0 \wedge P_1 \wedge Q \wedge n \neq N$ is a common precondition for the **if**'s (the second **if** does not use P_0), they can be combined:

```
$n, r, s := 0, 0, 0$
$\{P_0 \wedge P_1 \wedge Q, bnd : N - n\}$
do $n \neq N \rightarrow \{P_0 \wedge P_1 \wedge Q \wedge n \neq N\}$
    if $a[n] < 0 \rightarrow s := s + 1$
     | $a[n] = 0 \rightarrow r, s := r + s, s + 1$
     | $a[n] > 0 \rightarrow r := r + s$
    fi
    $\{(P_0 \wedge P_1 \wedge Q)[n \backslash n + 1]\}$
   $n := n + 1$
od
$\{r = \langle \#i, j : 0 \leqslant i < j < N : a[i] \leqslant 0 \wedge a[j] \geqslant 0 \rangle\}$
```

Isn't It Getting A Bit Too Complicated?

- Quantifier and indexes manipulation tend to get very long and tedious.
    - Expect to see even longer expressions later!
- To certain extent, it is a restriction of the data structure we are using. With arrays we have to manipulate the indexes.
- Is it possible to use higher-level data structures? Lists? Trees?
    - Heap-allocated data structure with pointers is a horrifying beast!
    - Trying to be more abstract lead to further developments in programming languages, e.g. algebraic datatypes.