Programming Languages
0. Imperative Programming and Hoare Logic
Exercises

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Guarded Command Language Basics

1. Which of the following Hoare triples hold?
   (a) $\{x = 7\} skip \{odd\ x\}$;
   (b) $\{x > 60\} x := x \times 2 \{x > 100\}$;
   (c) $\{x > 40\} x := x \times 2 \{x > 100\}$;
   (d) $\{true\} \mathbf{if}\ x \leqslant y \to y := y - x \mid x \geqslant y \to x := x - y\ \mathbf{fi} \{x \geqslant 0 \land y \geqslant 0\}$;
   (e) $\{even\ x \land even\ y\} \mathbf{if}\ x \leqslant y \to y := y - x \mid x \geqslant y \to x := x - y\ \mathbf{fi} \{even\ x \land even\ y\}$.

Solution: As the first exercise I expect merely that you answer by informal reasoning. What follows is the more formal approach which you will learn later.

(a) The Hoare triple holds because:

$$\begin{aligned} & wp\ skip\ (odd\ x) \\ \equiv\ & \quad \{ \text{definition of } wp \} \\ & odd\ x \\ \Leftarrow\ & x = 7 \ . \end{aligned}$$

(b) The Hoare triple holds because:

$$\begin{aligned} & wp\ (x := x \times 2)\ (x > 100) \\ \equiv\ & \quad \{ \text{definition of } wp \} \\ & x \times 2 > 100 \\ \Leftarrow\ & x > 60 \ . \end{aligned}$$

(c) The Hoare triple does not hold because:

$$\begin{aligned} & wp\ (x := x \times 2)\ (x > 100) \\ \equiv\ & x \times 2 > 100 \\ \not\Leftarrow\ & x > 40 \ . \end{aligned}$$

(d) The annotated **if** statement is

$$\begin{array}{l}\{True\}\\ \textbf{if } x \leqslant y \rightarrow \{x \leqslant y\}\ y := y - x\ \{x \geqslant 0 \wedge y \geqslant 0\}\\ \quad x \geqslant y \rightarrow \{x \geqslant y\}\ x := x - y\ \{x \geqslant 0 \wedge y \geqslant 0\}\\ \textbf{fi}\\ \{x \geqslant 0 \wedge y \geqslant 0\}\ .\end{array}$$

That $x \leqslant y \vee x \geqslant y$ certainly holds. For the Hoare triple in the first branch we reason:

$$\begin{array}{ll} & (x \geqslant 0 \wedge y \geqslant 0)[y \backslash y - x]\\ \equiv & x \geqslant 0 \wedge y - x \geqslant 0\\ \equiv & x \geqslant 0 \wedge x \geqslant y\\ \not\Leftarrow & x \leqslant y\ .\end{array}$$

The situation with the other branch is similar. The bottom line is that the initial Hoare triple does not hold.

The initial Hoare triple would be true if the precondition were $x \geqslant 0 \wedge y \geqslant 0$.

(e) The annotated **if** statement is

$$\begin{array}{l}\{even\ x \wedge even\ y\}\\ \textbf{if } x \leqslant y \rightarrow \{even\ x \wedge even\ y \wedge x \leqslant y\}\ y := y - x\ \{even\ x \wedge even\ y\}\\ \quad x \geqslant y \rightarrow \{even\ x \wedge even\ y \wedge x \geqslant y\}\ x := x - y\ \{even\ x \wedge even\ y\}\\ \textbf{fi}\\ \{even\ x \wedge even\ y\}\ .\end{array}$$

That $x \leqslant y \vee x \geqslant y$ certainly holds. For the Hoare triple in the first branch we reason:

$$\begin{array}{ll} & (even\ x \wedge even\ y)[y \backslash y - x]\\ \equiv & even\ x \wedge even\ (y - x)\\ \equiv & even\ x \wedge even\ y\\ \Leftarrow & even\ x \wedge even\ y \wedge x \leqslant y\ .\end{array}$$

The situation with the other branch is similar. The bottom line is that the initial Hoare triple does hold.

2. Is it always true that $\{True\}\ x := E\ \{x = E\}$? If you think the answer is yes, explain why. If your answer is no, give a counter example.

Solution: No. For a counterexample, let E be $x + 1$.

When do we do have the property that $\{True\}\ x := E\ \{x = E\}$? Since $(x = E)[x \backslash E] \equiv (E = E\ [x \backslash E])$, the Hoare triple holds if and only if $E = E\ [x \backslash E]$. Examples of such E include those that do not contain x, or those that are idempotent funtions on x, for example $E = 0 \uparrow x$.

The actual forward rule for assignment (due to Floyd) is:

$$\{P\}\ x := E\ \{(\exists\ x_0 :: x = E\ [x \backslash x_0] \wedge P\ [x \backslash x_0])\}\ ,$$

where x_0 is a fresh name.

3. Verify:

$$\begin{array}{l}
\{x = X \wedge y = Y\} \\
x := x \not\equiv y \\
y := x \not\equiv y \\
x := x \not\equiv y \\
\{x = Y \wedge y = X\}
\end{array}$$

where x and y are boolean and $(\not\equiv)$ is the "not equal" or "exclusive or" operator. In fact, the code above works for any $(\otimes)$ that satisfies the properties that for all a, b, and c:

$$\begin{array}{ll}
\text{associative}: & a \otimes (b \otimes c) = (a \otimes b) \otimes c\ , \\
\text{unipotent}: & a \otimes a = 1\ ,
\end{array}$$

where 1 is the unit of $(\otimes)$, that is, $1 \otimes b = b = b \otimes 1$.

Solution: The annotated program is:

$$\begin{array}{l}
\{x = X \wedge y = Y, \mathrm{Pf}_2\} \\
x := x \otimes y \\
\{y = Y \wedge x \otimes y = X, \mathrm{Pf}_1\} \\
y := x \otimes y \\
\{x \otimes y = Y \wedge y = X\} \\
x := x \otimes y \\
\{x = Y \wedge y = X\}\ \ .
\end{array}$$

Pf_1:

$$\begin{array}{ll}
& (x \otimes y = Y \wedge y = X)\ [x \otimes y\ /\ y] \\
\equiv & x \otimes (x \otimes y) = Y \wedge x \otimes y = X \\
\equiv & \quad \{\ (\otimes) \text{ associative }\} \\
& (x \otimes x) \otimes y = Y \wedge x \otimes y = X \\
\equiv & \quad \{\ \text{unipotence }\} \\
& 1 \otimes y = Y \wedge x \otimes y = X \\
\equiv & \quad \{\ \text{identity }\} \\
& y = Y \wedge x \otimes y = X\ \ .
\end{array}$$

Pf_2:

$$\begin{array}{ll}
& (y = Y \wedge x \otimes y = X)\ [x \otimes y\ /\ x] \\
\equiv & y = Y \wedge (x \otimes y) \otimes y = X \\
\equiv & \quad \{\ (\otimes) \text{ associative }\} \\
& y = Y \wedge x \otimes (y \otimes y) = X \\
\equiv & \quad \{\ \text{unipotence }\} \\
& y = Y \wedge x \otimes 1 = X \\
\equiv & \quad \{\ \text{identity }\} \\
& y = Y \wedge x = X\ \ .
\end{array}$$

4. Verify the following program:

$$
\begin{array}{l}
\textbf{var } r, b : \mathit{Int} \\
\{0 \leqslant r < 2 \times b\} \\
\textbf{if } b \leqslant r \to r := r - b \\
\mid\ r < b \to \mathit{skip} \\
\textbf{fi} \\
\{0 \leqslant r < b\}
\end{array}
$$

Solution: The annotated program is:

$$
\begin{array}{l}
\textbf{var } r, b : \mathit{Int} \\
\{0 \leqslant r < 2 \times b\} \\
\textbf{if } b \leqslant r \to \{0 \leqslant r < 2 \times b \wedge b \leqslant r\}\ r := r - b\ \{0 \leqslant r < b, \mathrm{Pf}_1\} \\
\mid\ r < b \to \{0 \leqslant r < 2 \times b \wedge r < b\}\ \mathit{skip}\ \{0 \leqslant r < b, \mathrm{Pf}_2\} \\
\textbf{fi} \\
\{0 \leqslant r < b, \mathrm{Pf}_3\}
\end{array}
$$

Pf_1. We reason:

$$
\begin{array}{ll}
 & (0 \leqslant r < b)\ [r \backslash r - b] \\
\equiv & 0 \leqslant r - b < b \\
\equiv & b \leqslant r < 2 \times b \\
\Leftarrow & 0 \leqslant r < 2 \times b \wedge b \leqslant r \ \ .
\end{array}
$$

Pf_2. Trivial.

Pf_3. Certainly any proposition implies $b \leqslant r \vee r < b$.

5. Verify:

$$
\begin{array}{l}
\textbf{var } x, y : \mathit{Int} \\
\{\mathit{True}\} \\
x, y := x \times x, y \times y \\
\textbf{if } x \geqslant y \to x := x - y \\
\mid\ y \geqslant x \to y := y - x \\
\textbf{fi} \\
\{x \geqslant 0 \wedge y \geqslant 0\} \ \ .
\end{array}
$$

Solution: For brevity we abbreviate $x \geqslant 0 \ \wedge \ y \geqslant 0$ to P. The fully annotated program could be:

$$
\begin{array}{l}
\{\mathit{True}\} \\
x, y := x \times x, y \times y \\
\{P, \mathrm{Pf}_4\} \\
\textbf{if } x \geqslant y \to \{x \geqslant y \wedge P\}\ x := x - y\ \{P, \mathrm{Pf}_1\} \\
\mid\ y \geqslant x \to \{y \geqslant x \wedge P\}\ y := y - x\ \{P, \mathrm{Pf}_2\} \\
\textbf{fi} \\
\{P, \mathrm{Pf}_3\} \ \ .
\end{array}
$$

To verify the **if** branching, we check that

Pf$_1$. $\{x \geqslant y \ \wedge \ P\}\, x := x - y\, \{P\}$. The Hoare triple is valid because

$$
\begin{aligned}
& (x \geqslant 0 \ \wedge \ y \geqslant 0)[x \backslash x - y] \\
\equiv \ & x - y \geqslant 0 \ \wedge \ y \geqslant 0 \\
\equiv \ & x \geqslant y \wedge \ y \geqslant 0 \\
\Leftarrow \ & x \geqslant y \ \wedge \ x \geqslant 0 \ \wedge \ y \geqslant 0.
\end{aligned}
$$

Pf$_2$. $\{y \geqslant x \ \wedge \ P\}\, y := y - x\, \{P\}$. Omitted.

Pf$_3$. And indeed $x \geqslant y \ \vee \ y \geqslant x$ always holds, thus $P \Rightarrow x \geqslant y \ \vee \ y \geqslant x$.

Do not forget that we have yet to verify $\{true\}\, x, y := x \times x, y \times y\, \{P\}$, which is not difficult either:

Pf$_4$.

$$
\begin{aligned}
& (x \geqslant 0 \ \wedge \ y \geqslant 0)[x, y \backslash x \times x, y \times y] \\
\equiv \ & x \times x \geqslant 0 \ \wedge \ y \times y \geqslant 0 \\
\equiv \ & true.
\end{aligned}
$$

6. Verify:

var $a, b : Bool$
$\{True\}$
if $\neg\, a \vee b \rightarrow a := \neg\, a$
$\mid\ a \vee \neg\, b \rightarrow b := \neg\, b$
fi
$\{a \vee b\}$.

Solution:

var $a, b : Bool$
$\{True\}$
if $\neg\, a \vee b \rightarrow \{\neg\, a \vee b\}\, a := \neg\, a\, \{a \vee b, \mathrm{Pf}_1\}$
$\mid\ a \vee \neg\, b \rightarrow \{a \vee \neg\, b\}\, b := \neg\, b\, \{a \vee b, \mathrm{Pf}_2\}$
fi
$\{a \vee b, \mathrm{Pf}_3\}$.

Pf$_1$. To verify the first branch:

$$
\begin{aligned}
& (a \vee b)[a \backslash \neg a] \\
\equiv \ & \neg a \vee b.
\end{aligned}
$$

Pf$_2$. The other branch is similar.

Pf$_3$. Certainly $true \Rightarrow \neg a \vee b \vee a \vee \neg b$.

Weakest Precondition

7. Given below is a list of statements and predicates. What are the weakest precondition for the predicates to be true after the statement?
   (a) $x := x \times 2$, $x > 100$;
   (b) $x := x \times 2$, $even\ x$;
   (c) $x := x \times 2$, $x > 100 \wedge even\ x$;
   (d) $x := x \times 2$, $odd\ x$.
   (e) $skip$, $odd\ x$.

Solution:

(a) $x \times 2 > 100$, that is, $x > 50$.

(b) $even\ (x \times 2)$, which simplifies to *True*.

(c) $x \times 2 > 100 \wedge even\ (x \times 2)$, that is, $x > 50$.

(d) $odd\ (x \times 2)$, that is, *False*.

(e) $odd\ x$.

8. Prove that $(wp\ S\ Q_0 \vee wp\ S\ Q_1) \Rightarrow wp\ S\ (Q_0 \vee Q_1)$.

Solution: Recall from propositional logic that $(P \vee Q) \Rightarrow R$ iff. $(P \Rightarrow R) \wedge (Q \Rightarrow R)$.

$$
\begin{array}{ll}
& (wp\ S\ Q_0 \vee wp\ S\ Q_1) \Rightarrow wp\ S\ (Q_0 \vee Q_1) \\
\equiv & \quad \{ \text{ said property above } \} \\
& (wp\ S\ Q_0 \Rightarrow wp\ S\ (Q_0 \vee Q_1)) \wedge \\
& (wp\ S\ Q_1 \Rightarrow wp\ S\ (Q_0 \vee Q_1)) \\
\Leftarrow & \quad \{ \text{ Monotonicity } \} \\
& (Q_0 \Rightarrow (Q_0 \vee Q_1)) \wedge (Q_1 \Rightarrow (Q_0 \vee Q_1)) \\
\equiv & True\ \ .
\end{array}
$$

9. Recall the definition of Hoare triple in terms of wp:

$$\{P\}\, S\, \{Q\} \;=\; P \Rightarrow wp\ S\ Q\ \ .$$

Prove that

1. $(\{P\}\, S\, \{Q\} \wedge (P_0 \Rightarrow P)) \Rightarrow \{P_0\}\, S\, \{Q\}$.
2. $\{P\}\, S\, \{Q\} \wedge \{P\}\, S\, \{R\} \equiv \{P\}\, S\, \{Q \wedge R\}$.

Solution:

1. We reason:

$$
\begin{array}{ll}
& \{P_0\}\, S\, \{Q\} \\
\equiv & \{ \text{definition of Hoare triple} \} \\
& P_0 \Rightarrow wp\ S\ Q \\
\Leftarrow & \{ \text{since } P_0 \Rightarrow P \} \\
& P \Rightarrow wp\ S\ Q \\
\equiv & \{ \text{definition of Hoare triple} \} \\
& \{P\}\, S\, \{Q\} \ .
\end{array}
$$

2. We reason:

$$
\begin{array}{ll}
& \{P\}\, S\, \{Q \wedge R\} \\
\equiv & \{ \text{definition of Hoare triple} \} \\
& P \Rightarrow wp\ S\ (Q \wedge R) \\
\equiv & \{ \text{distributivity over conjunction} \} \\
& P \Rightarrow (wp\ S\ Q \wedge wp\ S\ R) \\
\equiv & \{ \text{since } (P \Rightarrow (X \wedge Y)) \equiv (P \Rightarrow X) \wedge (P \Rightarrow Y) \} \\
& (P \Rightarrow wp\ S\ Q) \wedge (P \Rightarrow wp\ S\ R) \\
\equiv & \{ \text{definition of Hoare triple} \} \\
& \{P\}\, S\, \{Q\} \wedge \{P\}\, S\, \{R\} \ .
\end{array}
$$

10. Recall the weakest precondition of **if**:

$$wp\ (\textbf{if}\ B_0 \to S_0\ \lor\ B_1 \to S_1\ \textbf{fi})\ Q \ = \ (B_0 \Rightarrow wp\ S_0\ Q) \wedge (B_1 \Rightarrow wp\ S_1\ Q) \wedge (B_0 \vee B_1) \ .$$

Prove that

$$
\begin{array}{l}
\{P\}\ \textbf{if}\ B_0 \to S_0\ \lor\ B_1 \to S_1\ \textbf{fi}\ \{Q\} \ \equiv \\
\quad \{P \wedge B_0\}\, S\, \{Q\} \ \wedge \ \{P \wedge B_1\}\, S\, \{Q\} \ \wedge \ (P \Rightarrow (B_0 \vee B_1)) \ .
\end{array}
$$

Note: having proved so shows that the way we annotate **if** is correct:

$$
\begin{array}{l}
\{P\} \\
\textbf{if}\ B_0 \to \{P \wedge B_0\}\, S_0\, \{Q\} \\
\ |\ \ B_1 \to \{P \wedge B_1\}\, S_1\, \{Q\} \\
\textbf{fi} \\
\{Q\} \ .
\end{array}
$$

Solution: We reason:

$$
\begin{array}{ll}
& \{P\}\ \textbf{if}\ B_0 \to S_0\ \lor\ B_1 \to S_1\ \textbf{fi}\ \{Q\} \\
\equiv & \{ \text{definition of Hoare triple} \} \\
& P \Rightarrow wp\ (\textbf{if}\ B_0 \to S_0\ \lor\ B_1 \to S_1\ \textbf{fi})\ Q \\
\equiv & \{ \text{definition of } wp \} \\
& P \Rightarrow ((B_0 \Rightarrow wp\ S_0\ Q) \wedge (B_1 \Rightarrow wp\ S_1\ Q) \wedge (B_0 \vee B_1)) \\
\equiv & \{ \text{since } (P \Rightarrow (Q \wedge R)) \equiv (P \Rightarrow Q) \wedge (P \Rightarrow R) \} \\
& (P \Rightarrow (B_0 \Rightarrow wp\ S_0\ Q)) \wedge \\
& (P \Rightarrow (B_1 \Rightarrow wp\ S_1\ Q)) \wedge \\
& (P \Rightarrow (B_0 \vee B_1)) \\
\equiv & \{ \text{since } (P \Rightarrow (Q \Rightarrow R)) \equiv ((P \wedge Q) \Rightarrow R) \} \\
& ((P \wedge B_0) \Rightarrow wp\ S_0\ Q) \wedge \\
& ((P \wedge B_1) \Rightarrow wp\ S_1\ Q) \wedge \\
& (P \Rightarrow (B_0 \vee B_1)) \\
\equiv & \{ \text{definition of Hoare triple} \} \\
& \{P \wedge B_0\}\, S_0\, \{Q\} \wedge \\
& \{P \wedge B_1\}\, S_1\, \{Q\} \wedge \\
& (P \Rightarrow (B_0 \vee B_1)) \ .
\end{array}
$$

11. Recall that *wp S Q* stands for "the weakest precondition for program *S* to terminate in a state satisfying *Q*". What programs *S*, if any, satisfy each of the following conditions?

    1. *wp S True* = *True*.
    2. *wp S True* = *False*.
    3. *wp S False* = *True*.
    4. *wp S False* = *False*.

Solution:

1. *wp S True* = *True*: *S* is a program that always terminates.
2. *wp S True* = *False*: *S* is a program that never terminates.
3. *wp S False* = *True*: there is no such a program *S*.
4. *wp S False* = *False*: *S* can be any program.