Programming Languages: Imperative Program Construction Midterm

Shin-Cheng Mu

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1. (10 points) Prove (3.63) $p \Rightarrow (q \equiv r) \equiv p \Rightarrow q \equiv p \Rightarrow r$, using properties that appear before (3.63).

> **Solution:** There are many possible proofs. For example:
>
> $$\begin{array}{ll} & p \Rightarrow (q \equiv r) \\ = & \quad \{\ (3.59)\ \text{defn. of implication}\ \} \\ & \neg\, p \vee (q \equiv r) \\ = & \quad \{\ (3.27)\ \text{distributivity}\ \} \\ & \neg\, p \vee q \equiv \neg\, p \vee r \\ = & \quad \{\ (3.59)\ \text{defn. of implication}\ \} \\ & p \Rightarrow q \equiv p \Rightarrow r \ \ . \end{array}$$
>
> Alternatively,
>
> $$\begin{array}{ll} & p \Rightarrow (q \equiv r) \\ = & \quad \{\ (3.57)\ \text{defn. of implication}\ \} \\ & p \vee (q \equiv r) \equiv q \equiv r \\ = & \quad \{\ (3.27)\ \text{distributivity}\ \} \\ & p \vee q \equiv p \vee r \equiv q \equiv r \\ = & \quad \{\ (3.24)\ \text{and}\ (3.25)\ \} \\ & p \vee q \equiv q \equiv p \vee r \equiv r \\ = & \quad \{\ (3.57)\ \text{defn. of implication}\ \} \\ & p \Rightarrow q \equiv p \Rightarrow r \ \ . \end{array}$$
>
> Yet another interesting one:
>
> $$\begin{array}{ll} & p \Rightarrow (q \equiv r) \\ = & \quad \{\ (3.62)\ \} \\ & p \wedge q \equiv p \wedge r \\ = & \quad \{\ (3.60)\ \text{defn. of implication}\ \} \\ & (p \Rightarrow q \equiv p) \equiv (p \Rightarrow r \equiv p) \\ = & \quad \{\ (3.1)\ \text{and}\ (3.3)\ \} \\ & p \Rightarrow q \equiv p \Rightarrow r \ \ . \end{array}$$

2. (10 points) Prove that $\neg\, p \Rightarrow (p \Rightarrow q)$. **Hint**: there are many possible proofs. In some proofs you might try to reduce the entire expression to *True*.

Solution:

$$
\begin{aligned}
& \neg p \Rightarrow (p \Rightarrow q) \\
= \quad & \{ \text{(3.57) defn. of implication} \} \\
& \neg p \Rightarrow (p \vee q \equiv q) \\
= \quad & \{ \text{(3.59) defn. of implication, (3.12) double negation} \} \\
& p \vee (p \vee q \equiv q) \\
= \quad & \{ \text{(3.27) distributivity} \} \\
& p \vee p \vee q \equiv p \vee q \\
= \quad & \{ \text{(3.26) idempotency of } (\vee) \} \\
& p \vee q \equiv p \vee q \\
= \quad & \{ \text{(3.3)} \} \\
& \mathit{True} \ .
\end{aligned}
$$

Alternatively,

$$
\begin{aligned}
& \neg p \Rightarrow (p \Rightarrow q) \\
= \quad & \{ \text{(3.57) defn. of implication} \} \\
& \neg p \Rightarrow (p \vee q \equiv q) \\
= \quad & \{ \text{(3.62)} \} \\
& \neg p \wedge (p \vee q) \equiv \neg p \wedge q \\
= \quad & \{ \text{(3.44) absorption} \} \\
& \neg p \wedge q \equiv \neg p \wedge q \\
= \quad & \{ \text{(3.3)} \} \\
& \mathit{True} \ .
\end{aligned}
$$

3. (a) (5 points) Let N be an *Int* (integer) such that $N \geqslant 0$, and A an array of *Int* containing N elements, indexed by $A\,[0]$, $A\,[1]$... $A\,[N-1]$ (if these elements exist).

   For i, j such that $0 \leqslant i \leqslant j \leqslant N$, we denote by $A[i..j)$ a consecutive segment of an array that includes $A\,[i]$ but does not include $A\,[j]$. For example, if $N \geqslant 10$, by $A[3..10)$ we denote the segment $A\,[2], A\,[3] \mathinner{..} A\,[9]$. If $i = j$, the segment is empty.

   Assuming that $0 \leqslant i \leqslant j \leqslant N$, write down an expression stating that "s is the sum of $A[i..j)$."

   Solution:

   $$s = \langle \Sigma k : i \leqslant k < j : A\,[k] \rangle \ .$$

   (b) (10 points) A consecutive segment of an array of *Int* is called "steep" (陡 in Chinese) if each of its elements is larger than the sum of all elements to its lefthand side. For example, in the array below,

   6, 3, 4, 8, 10, 19, 38, 2, 7,

   the segment 3, 4, 8 is steep (since $0 < 3$, $3 < 4$ and $3 + 4 < 8$), the segments 8, 10, 19, 38 and 2, 7 are also steep (since $8 < 10$, $8 + 10 < 19$, $8 + 10 + 19 < 38$, etc). An empty segment is steep. A singleton segment containing one negative element, for example, -1, is *not* steep, since -1 is not larger than the sum of all elements to its lefthand side, which is 0.

   Assuming that $0 \leqslant i \leqslant j \leqslant N$, write down an expression stating that "b is true if and only if $A[i..j)$ is steep."

Solution:

$$b = \langle \forall k : i \leqslant k < j : \langle \Sigma m : i \leqslant m < k : A\,[m] \rangle < A\,[k] \rangle \ .$$

(c) (10 points) Write down an expression stating that "r is the length of the longest steep segment of the array A."

Solution: For ease of explanation we define:

$$\begin{array}{l} \mathit{sum}\ \ i\ j = \langle \Sigma k : i \leqslant k < j : A\,[k] \rangle\ , \\ \mathit{steep}\ i\ j = \langle \forall k : i \leqslant k < j : \mathit{sum}\ i\ k < A\,[k] \rangle\ . \end{array}$$

The expression is

$$r = \langle \uparrow p\ q : 0 \leqslant p \leqslant q \leqslant N \wedge \mathit{steep}\ p\ q : q - p \rangle\ ,$$

which can be expanded to:

$$\begin{array}{l} r = \langle \uparrow p\ q : 0 \leqslant p \leqslant q \leqslant N \wedge \\ \qquad \langle \forall k : p \leqslant k < q : \langle \Sigma m : p \leqslant m < k : A\,[m] \rangle < A\,[k] \rangle : \\ \qquad\qquad q - p \rangle\ . \end{array}$$

4. Consider the following program

$$\begin{array}{l} \textbf{if}\ x > 3 \rightarrow \mathit{skip} \\ \mid\ x < 0 \rightarrow x := -2 \times x \\ \textbf{fi} \end{array}$$

Denote this program by *PROG*.

(a) (10 points) Write down *wp PROG q*.

Solution:

$$\begin{array}{ll} & \mathit{wp}\ \mathit{PROG}\ q \\ = & \quad \{ \text{ def. of } \mathit{wp}\ \textbf{if} \ \} \\ & (x > 3 \Rightarrow \mathit{wp}\ \mathit{skip}\ q) \wedge (x < 0 \Rightarrow \mathit{wp}\ (x := -2 \times x)\ q) \wedge (x > 3 \vee x < 0) \\ = & \quad \{ \text{ def. of } \mathit{wp}\ \mathit{skip} \text{ and } \mathit{wp}\ (x := -2 \times x) \ \} \\ & (x > 3 \Rightarrow q) \wedge (x < 0 \Rightarrow q[x \backslash -2 \times x]) \wedge (x > 3 \vee x < 0)\ . \end{array}$$

(b) (10 points) What is the weakest precondition for *PROG* to terminate?

Solution:

$$\begin{array}{ll} & (x > 3 \Rightarrow \mathit{True}) \wedge (x < 0 \Rightarrow \mathit{True}[x \backslash -2 \times x]) \wedge (x > 3 \vee x < 0) \\ = & \mathit{True} \wedge \mathit{True} \wedge (x > 3 \vee x < 0) \\ = & x > 3 \vee x < 0\ . \end{array}$$

(c) (10 points) What is $\mathit{wp}\ \mathit{PROG}\ (x > 4)$? (You may use your knowledge about arithmetics to simplify the ranges.)

Solution:

$$
\begin{aligned}
& (x > 3 \Rightarrow x > 4) \wedge (x < 0 \Rightarrow (x > 4)[x \backslash -2 \times x]) \wedge (x > 3 \vee x < 0) \\
= \; & (x > 3 \Rightarrow x > 4) \wedge (x < 0 \Rightarrow -2 \times x > 4) \wedge (x > 3 \vee x < 0) \\
= \; & (x > 3 \Rightarrow x > 4) \wedge (x < 0 \Rightarrow x < -2) \wedge (x > 3 \vee x < 0) \\
= \; & (x \leqslant 3 \vee x > 4) \wedge (x \geqslant 0 \vee x < -2) \wedge (x > 3 \vee x < 0) \\
= \; & \quad \{ \text{ distributivity } \} \\
& (x \leqslant 3 \wedge x \geqslant 0 \wedge x > 3) \vee (x \leqslant 3 \wedge x \geqslant 0 \wedge x < 0) \vee \\
& (x \leqslant 3 \wedge x < -2 \wedge x > 3) \vee (x \leqslant 3 \wedge x < -2 \wedge x < 0) \vee \\
& (x > 4 \wedge x \geqslant 0 \wedge x > 3) \vee (x > 4 \wedge x \geqslant 0 \wedge x < 0) \vee \\
& (x > 4 \wedge x < -2 \wedge x > 3) \vee (x > 4 \wedge x < -2 \wedge x < 0) \\
= \; & \mathit{False} \vee \mathit{False} \vee \mathit{False} \vee x < -2 \vee x > 4 \vee \mathit{False} \vee \mathit{False} \vee \mathit{False} \\
= \; & x < -2 \vee x > 4 \;\; .
\end{aligned}
$$

(d) (10 points) What is *wp PROG False*?

Solution:

$$
\begin{aligned}
& (x > 3 \Rightarrow \mathit{False}) \wedge (x < 0 \Rightarrow \mathit{False}) \wedge (x > 3 \vee x < 0) \\
= \; & x \leqslant 3 \wedge x \geqslant 0 \wedge (x > 3 \vee x < 0) \\
= \; & \quad \{ \text{ distributivity } \} \\
& (x \leqslant 3 \wedge x \geqslant 0 \wedge x > 3) \vee (x \leqslant 3 \wedge x \geqslant 0 \wedge x < 0) \\
= \; & \mathit{False} \vee \mathit{False} \\
= \; & \mathit{False} \;\; .
\end{aligned}
$$

5. (15 points) Prove the following Hoare triple:

$$
\begin{aligned}
& \{3 \leqslant x \vee (-1 \leqslant x < 0)\} \\
& \textbf{if } 0 < x \rightarrow x := x - 1 \\
& \mid \; x < 0 \rightarrow x := x + 3 \\
& \textbf{fi} \\
& \{1 \leqslant x\} \;\; .
\end{aligned}
$$

Solution: Note that $(3 \leqslant x \vee (-1 \leqslant x < 0)) \wedge 0 < x$ simplifies to $3 \leqslant x$, and $(3 \leqslant x \vee (-1 \leqslant x < 0)) \wedge x < 0$ simplifies to $-1 \leqslant x < 0$. Therefore, the fully annotated program is:

$$
\begin{aligned}
& \{3 \leqslant x \vee (-1 \leqslant x < 0)\} \\
& \textbf{if } 0 < x \rightarrow \{3 \leqslant x\} \qquad\quad x := x - 1 \; \{1 \leqslant x, \mathit{pf0}\} \\
& \mid \; x < 0 \rightarrow \{-1 \leqslant x < 0\} \; x := x + 3 \;\; \{1 \leqslant x, \mathit{pf1}\} \\
& \textbf{fi} \\
& \{1 \leqslant x, \mathit{pf2}\} \;\; .
\end{aligned}
$$

pf0:

$$
\begin{aligned}
& (1 \leqslant x)[x \backslash x - 1] \\
\equiv \; & 1 \leqslant x - 1 \\
\equiv \; & 2 \leqslant x \\
\Leftarrow \; & 3 \leqslant x \;\; .
\end{aligned}
$$

pf1:

$$
\begin{aligned}
& (1 \leqslant x)[x \backslash x + 3] \\
\equiv\ & 1 \leqslant x + 3 \\
\equiv\ & -2 \leqslant x \\
\Leftarrow\ & -1 \leqslant x < 0 \ .
\end{aligned}
$$

pf2:

$$
\begin{aligned}
& 3 \leqslant x \vee (-1 \leqslant x < 0) \\
=\ & 3 \leqslant x \vee (-1 \leqslant x \wedge x < 0) \\
\Rightarrow\ & 0 < x \vee x < 0 \ .
\end{aligned}
$$