Programming Languages: Imperative Program Construction
Practicals 3. Quantifications

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1. An integer array $X[0..N)$ is given, where $N \geqslant 1$. Explain, in words, what each of the following expressions mean.
   1. $b \equiv \langle\, \forall i : 0 \leqslant i < N : X[i] \geqslant 0 \,\rangle$.
   2. $r = \langle\, \#k : 0 \leqslant k < N : \langle\, \forall i : 0 \leqslant i < k : X[i] < X[k] \,\rangle \,\rangle$.
   3. $r = \langle\, \uparrow p, q : 0 \leqslant p \leqslant q \leqslant N \;\wedge\; \langle\, \forall i : p \leqslant i < q : X[i] > 0 \,\rangle : q - p \,\rangle$.
   4. $r = \langle\, \#p, q : 0 \leqslant p < q < N : X[p] = 0 \;\wedge\; X[q] = 1 \,\rangle$.
   5. $s = \langle\, \uparrow p, q : 0 \leqslant p < q < N : X[p] + X[q] \,\rangle$.
   6. $b \equiv \langle\, \forall p, q : 0 \leqslant p \;\wedge\; 0 \leqslant q \;\wedge\; p + q = N - 1 : X[p] = X[q] \,\rangle$.
2. An integer array $X[0..N)$ is given, where $N \geqslant 1$. Express the following sentences in a formal way:
   1. r is the sum of the elements of X.
   2. X is increasing.
   3. all values of X are distinct.
   4. r is the length of a longest constant segment of X.
   5. r is the maximum of the sums of the segments of X.
3. Expand the following textual substitutions. If necessary, change the dummy, according to Dummy Renaming (8.21).
   1. $\langle\, \star x : 0 \leqslant x + r < n : x + v \,\rangle [v \backslash 3]$
   2. $\langle\, \star x : 0 \leqslant x + r < n : x + v \,\rangle [x \backslash 3]$
   3. $\langle\, \star x : 0 \leqslant x + r < n : x + v \,\rangle [n \backslash n + x]$
   4. $\langle\, \star x : 0 \leqslant x < r : \langle\, \star y : 0 \leqslant y : x + y + n \,\rangle \,\rangle [n \backslash x + y]$
   5. $\langle\, \star x : 0 \leqslant x < r : \langle\, \star y : 0 \leqslant y : x + y + n \,\rangle \,\rangle [r \backslash y]$
4. Prove the following theorems. Provided $0 \leqslant n$,
   (a) $\langle\, \Sigma i : 0 \leqslant i < n + 1 : b[i] \,\rangle = b[0] + \langle\, \Sigma i : 1 \leqslant i < n + 1 : b[i] \,\rangle$
   (b) $\langle\, \exists i : 0 \leqslant i < n + 1 : b[i] = 0 \,\rangle = \langle\, \exists i : 0 \leqslant i < n : b[i] = 0 \,\rangle \vee b[n] = 0$
5. Prove that $\langle\, \forall x : R : P \,\rangle \equiv P \vee \langle\, \forall x :: \neg R \,\rangle$, provided $\neg \mathit{occurs}(x, P)$.
6. Prove the *range weakening* rule: $\langle\, \forall x : Q \vee R : P \,\rangle \Rightarrow \langle\, \forall x : Q : P \,\rangle$.
7. Prove the *body weakening* rule: $\langle\, \forall x : R : P \wedge Q \,\rangle \Rightarrow \langle\, \forall x : R : P \,\rangle$.