Programming Languages: Imperative Program Construction
Practicals 7: Loop Constuction III

Shin-Cheng Mu

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1. Solve:

   ```
   con $A, B : Int\{A \geqslant 0 \ \wedge B \geqslant 0\}$;
   var $r : Int$;
   $S$
   $\{r = A \times B\}$ ,
   ```

   using only (/2) (integral division by two), (×2), *even*, *odd*, addition, and subtraction.

2. The sum of all digits of a natural number can be computed by

   $$sd\ 0 = 0$$
   $$sd\ x = x \,\%\, 10 + sd\ (x \,/\, 10) \ ,$$

   where (/) is integral division and $a \,\%\, b$ computes the remainder of $a \,/\, b$. Solve

   ```
   $con\ N : Int\ \{0 \leqslant N\}$
   $var\ r : Int$
   ?
   $\{r = sd\ N\}$
   ```

3. Given a natural number N, derive a program that computes the number of factors 3 of N. For example, when $N = 945 = 3^3 \times 5 \times 7$ we output 3.

   ```
   $con\ N : Int\ \{0 \leqslant N\}$
   $var\ r : Int$
   ?
   $\{r = $ how do you write the post condition?$\}$
   ```

4. Solve:

   ```
   con $N, X : Int\ \{0 \leqslant N\}$
   con $f$ : array $[0..N)$ of $Int$
   var $r : Int$
   ?
   $\{r = \langle \Sigma i : 0 \leqslant i < N : f[i] \times X^i \rangle\}$
   ```

   We have seen this problem before but let us do it slightly differently this time. (This problem is not that much about associativity, but a practice constructing and using recursive function definition.)

   (a) Define $g\ n = \langle \Sigma i : n \leqslant i < N : f[i] \times X^{i-n} \rangle$ for $0 \leqslant n \leqslant N$, derive a recursive definition of g.

   (b) Use $r = g\ n$ as the main invariant, construct a program that solves the problem.

5. The function *fusc* is defined on natural numbers by:

$$\begin{array}{lcl} \mathit{fusc}\ 0 & = & 0 \\ \mathit{fusc}\ 1 & = & 1 \\ \mathit{fusc}\ (2 \times n) & = & \mathit{fusc}\ n \\ \mathit{fusc}\ (2 \times n + 1) & = & \mathit{fusc}\ n + \mathit{fusc}\ (n + 1). \end{array}$$

   Derive a program computing $\mathit{fusc}\ N$ for $N \geqslant 0$. Hint: try $\mathit{fusc}\ 78$.

6. Solve:

```
$\mathbf{con}\ N : \mathit{Int}\ \{0 \leqslant N\}$
$\mathbf{con}\ f : \mathbf{array}\ [0..N)\ \mathbf{of}\ \mathit{Int}$
$\mathbf{var}\ r : \mathit{Bool}$
 ?
$\{r = \langle \exists i : 0 \leqslant i < N : f[i] = 0 \rangle\}$
```

   (a) Define, for $0 \leqslant n \leqslant N$, $g\ n = \langle \exists i : n \leqslant i < N : f[i] = 0 \rangle$. Come up with a recursive definition of g.

   (b) Try come up with a program that, as soon as a zero is found in the array, terminates without having to scan the entire list. What invariant would you choose?