Programming Languages: Imperative Program Construction
Practicals 2. Propositional Logic

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Prove each of the following properties using only axioms or theorems established before it (for example, prove (3.11) using only (1.?) and (3.1) - (3.10)).

Note that there are more than one ways to prove a property. You may discover a proof that is better than the one given in the solution.

1. Prove (3.9): $\neg(p \equiv q) \equiv \neg p \equiv q$.

Solution:

$$
\begin{aligned}
& \neg(p \equiv q) \\
= \quad & \{ \text{ definition of } \textit{False} \text{ (3.15) } \} \\
& p \equiv q \equiv \textit{False} \\
= \quad & \{ \text{ symmetry (3.2) and associativity (3.1) of } \equiv \} \\
& p \equiv \textit{False} \equiv q \\
= \quad & \{ \text{ definition of } \textit{False} \text{ (3.15) } \} \\
& \neg p \equiv q
\end{aligned}
$$

2. Prove (3.12): $\neg\neg p \equiv p$.

Solution:

$$
\begin{aligned}
& \neg\neg p \\
= \quad & \{ \text{ definition of } \textit{False} \text{ (3.5), associativity (3.1) and symmetry (3.2) of } \equiv \} \\
& \neg p \equiv \textit{False} \\
= \quad & \{ \text{ definition of } \textit{False} \text{ (3.5) } \} \\
& p
\end{aligned}
$$

3. Prove (3.13): $\neg \textit{False} \equiv \textit{True}$.

Solution:

$$
\neg \textit{False}
$$

$=$ { definition of *False* (3.5) }

   False ≡ *False*

$=$ { identity of ≡ (3.3) }

   True

4. Prove (3.29): $p \lor True \equiv True$.

Solution:

   $p \lor True$

$=$ { identity of ≡ (3.3) }

   $p \lor (p \equiv p)$

$=$ { distributivity (3.27) }

   $p \lor p \equiv p \lor p$

$=$ { identity of ≡ (3.3) }

   True

5. Prove (3.32): $p \lor q \equiv p \lor \neg q \equiv p$.

Solution:

   $p \lor q \equiv p \lor \neg q$

$=$ { distributivity (3.27) }

   $p \lor (q \equiv \neg q)$

$=$ { definition of *False* (3.15) }

   $p \lor False$

$=$ { identity of ∨ (3.30) }

   p

6. Prove (3.42): $p \land \neg p \equiv False$.

Solution:

   $p \land \neg p$

$=$ { golden rule (3.35) }

   $p \equiv \neg p \equiv p \lor \neg p$

$=$ { excluded middle (3.28) }

   $p \equiv \neg p \equiv True$

$=$ { identity of *True* (3.3) }

$p \equiv \neg p$
= { definition of *False* (3.15) }
  False

Another proof:

$p \wedge \neg p \equiv False$
= { definition of *False* (3.15) }
  $p \wedge \neg p \equiv p \equiv \neg p$
= { golden rule (3.35) }
  $p \vee \neg p$
= { excluded middle (3.28) }
  True

7. Prove (3.43a): $p \wedge (p \vee q) \equiv p$.

Solution:

$p \wedge (p \vee q)$
= { golden rule (3.35) }
  $p \equiv p \vee q \equiv p \vee p \vee q$
= { idempotency of $\vee$ (3.26) }
  $p \equiv p \vee q \equiv p \vee q$
= { identity of $\equiv$ (3.3) }
  $p \equiv True$
= { identity of $\equiv$ (3.3) }
  p

8. Prove (3.44a). $p \wedge (\neg p \vee q) \equiv p \wedge q$.

Solution:

$p \wedge (\neg p \vee q)$
= { golden rule (3.35) }
  $p \equiv \neg p \vee q \equiv p \vee \neg p \vee q$
= { excluded middle (3.28) }
  $p \equiv \neg p \vee q \equiv True \vee q$
= { zero of $\vee$ (3.29) and identity of $\equiv$ (3.3) }
  $p \equiv \neg p \vee q$
= { (3.32), with $p, q := q, p$ }
  $p \equiv q \equiv p \vee q$

$=$ { golden rule (3.35) }

     $p \land q$

9. Prove (3.65): $p \land q \Rightarrow r \equiv p \Rightarrow (q \Rightarrow r)$.

Solution:

     $p \Rightarrow (q \Rightarrow r)$
$=$ { definition of $\Rightarrow$ (3.60) }
     $p \land (q \Rightarrow r) \equiv p$
$=$ { definition of $\Rightarrow$ (3.60) }
     $p \land (q \land r \equiv q) \equiv p$
$=$ { (3.49) }
     $p \land q \land r \equiv p \land q \equiv p \equiv p$
$=$ { identity of $\equiv$ (3.3) }
     $p \land q \land r \equiv p \land q$
$=$ { definition of $\Rightarrow$ (3.60) }
     $(p \land q) \Rightarrow r$

10. Prove (3.66): $p \land (p \Rightarrow q) \equiv p \land q$.

Solution:

     $p \land (p \Rightarrow q)$
$=$ { definition of $\Rightarrow$ (3.60) }
     $p \land (p \land q \equiv p)$
$=$ { (3.49) and idempotency of $\land$ (3.38) }
     $p \land q \equiv p \equiv p$
$=$ { identity of $\equiv$ (3.3) }
     $p \land q$

11. Prove (3.67): $p \land (q \Rightarrow p) \equiv p$.

Solution:

     $p \land (q \Rightarrow p)$
$=$ { definition of $\Rightarrow$ (3.60) }
     $p \land (q \land p \equiv q)$
$=$ { (3.49) and idempotency of $\land$ (3.38) }

$$p \wedge q \equiv p \wedge q \equiv p$$
$$= \quad \{ \text{ identity of } \equiv (3.3) \}$$
$$p$$

12. Prove (3.68): $p \vee (p \Rightarrow q) \equiv \textit{True}$.

Solution:

$$p \vee (p \Rightarrow q)$$
$$= \quad \{ \text{ definition of } \Rightarrow (3.57) \}$$
$$p \vee (p \vee q \equiv q)$$
$$= \quad \{ \text{ distributvity (3.27) and idempotency (3.26) } \}$$
$$p \vee q \equiv p \vee q$$
$$= \quad \{ \text{ identity of } \equiv (3.3) \}$$
$$\textit{True}$$

Another proof:

$$p \vee (p \Rightarrow q)$$
$$= \quad \{ \text{ definition of } \Rightarrow (3.59) \}$$
$$p \vee \neg p \vee q$$
$$= \quad \{ \text{ excluded middle (3.28) and zero of } \vee (3.29) \}$$
$$\textit{True}$$

13. Prove (3.69): $p \vee (q \Rightarrow p) \equiv q \Rightarrow p$.

Solution:

$$p \vee (q \Rightarrow p)$$
$$= \quad \{ \text{ definition of } \Rightarrow (3.59) \}$$
$$p \vee \neg q \vee p$$
$$= \quad \{ \text{ idempotency of } \vee (3.26) \}$$
$$p \vee \neg q$$
$$= \quad \{ \text{ definition of } \Rightarrow (3.59) \}$$
$$q \Rightarrow p$$

14. Prove (3.78): $(p \Rightarrow r) \wedge (q \Rightarrow r) \equiv (p \vee q \Rightarrow r)$.

Solution:

$$p \vee q \Rightarrow r$$
$$= \quad \{ \text{ definition of } \Rightarrow (3.59) \}$$
$$\neg(p \vee q) \vee r$$
$$= \quad \{ \text{ de Morgan } (3.47) \}$$
$$(\neg p \wedge \neg q) \vee r$$
$$= \quad \{ \text{ distributivity } (3.45) \}$$
$$(\neg p \vee r) \wedge (\neg q \vee r)$$
$$= \quad \{ \text{ definition of } \Rightarrow (3.59) \}$$
$$(p \Rightarrow r) \wedge (q \Rightarrow r)$$

15. Prove that $(p \Rightarrow q) \wedge (p \Rightarrow r) \equiv (p \Rightarrow q \wedge r)$.

Solution:

$$(p \Rightarrow q) \wedge (p \Rightarrow r)$$
$$= \quad \{ \text{ definition of } \Rightarrow (3.59) \}$$
$$(\neg p \vee q) \wedge (\neg p \vee r)$$
$$= \quad \{ \text{ distributivity } (3.45) \}$$
$$\neg p \vee (q \wedge r)$$
$$= \quad \{ \text{ definition of } \Rightarrow (3.59) \}$$
$$p \Rightarrow q \wedge r$$

16. Prove that $(r \Rightarrow)$ is monotonic with respect to implication. That is, $(p \Rightarrow q) \Rightarrow ((r \Rightarrow p) \Rightarrow (r \Rightarrow q))$.

Solution:

$$(p \Rightarrow q) \Rightarrow ((r \Rightarrow p) \Rightarrow (r \Rightarrow q))$$
$$= \quad \{ \text{ shunting } (3.65) \}$$
$$((p \Rightarrow q) \wedge (r \Rightarrow p)) \Rightarrow (r \Rightarrow q)$$
$$= \quad \{ \text{ shunting } (3.65) \}$$
$$((p \Rightarrow q) \wedge (r \Rightarrow p) \wedge r) \Rightarrow q$$
$$= \quad \{ (3.66) \}$$
$$((p \Rightarrow q) \wedge p \wedge r) \Rightarrow q$$
$$= \quad \{ (3.66) \}$$
$$(q \wedge p \wedge r) \Rightarrow q$$
$$= \quad \{ \text{ weakening } (3.76b) \}$$
$$\textit{True}$$

17. Prove that $(\Rightarrow r)$ is anti-monotonic with respect to implication. That is, $(p \Rightarrow q) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \Rightarrow r))$.

Solution:

$\qquad (p \Rightarrow q) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \Rightarrow r))$

$= \quad \{ \text{ shunting (3.65) } \}$

$\qquad ((p \Rightarrow q) \wedge (q \Rightarrow r)) \Rightarrow (p \Rightarrow r)$

$= \quad \{ \text{ shunting (3.65) } \}$

$\qquad ((p \Rightarrow q) \wedge (q \Rightarrow r) \wedge p) \Rightarrow r$

$= \quad \{ \text{ (3.66) } \}$

$\qquad (p \wedge q \wedge (q \Rightarrow r)) \Rightarrow r$

$= \quad \{ \text{ (3.66) } \}$

$\qquad (p \wedge q \wedge r) \Rightarrow r$

$= \quad \{ \text{ weakening (3.76b) } \}$

$\qquad \textit{True}$

18. Prove that conjunction is monotonic with respect to implication. That is, $(p \Rightarrow q) \Rightarrow ((p \wedge r) \Rightarrow (q \wedge r))$.

Solution:

$\qquad (p \Rightarrow q) \Rightarrow ((p \wedge r) \Rightarrow (q \wedge r))$

$= \quad \{ \text{ shunting (3.65) } \}$

$\qquad ((p \Rightarrow q) \wedge p \wedge r) \Rightarrow (q \wedge r)$

$= \quad \{ \text{ (3.66) } \}$

$\qquad (p \wedge q \wedge r) \Rightarrow (q \wedge r)$

$= \quad \{ \text{ weakening (3.76b) } \}$

$\qquad \textit{True}$

19. Prove (4.3) $(p \Rightarrow q) \Rightarrow (p \wedge r \Rightarrow q \wedge r)$. Start with the consequent, since it has more structure.

Solution:

$\qquad p \wedge r \Rightarrow q \wedge r$

$= \quad \{ \text{ definition of } \Rightarrow \text{ (3.59) } \}$

$\qquad \neg p \vee \neg r \vee (q \wedge r)$

$= \quad \{ \text{ distributivity (3.45) } \}$

$\qquad \neg p \vee ((\neg r \vee q) \wedge (\neg r \vee r))$

$= \quad \{ \text{ excluded middle (3.28) and identity of } \wedge \text{ (3.39) } \}$

$\qquad \neg p \vee \neg r \vee q$

$\Leftarrow \quad \{ \text{ strengthening (3.76a) } \}$

$\qquad \neg p \vee q$

$$= \quad \{ \text{ definition of} \Rightarrow (3.59) \}$$
$$p \Rightarrow q$$

20. Prove (3.76d) $p \vee (q \wedge r) \Rightarrow p \vee q$ using inequality reasoning. Start with the antecedent, since it has more structure, and use distributivity.

Solution:

$$p \vee (q \wedge r)$$
$$= \quad \{ \text{ distributivity (3.45) } \}$$
$$(p \vee q) \wedge (p \vee r)$$
$$\Rightarrow \quad \{ \text{ weakening (3.76b) } \}$$
$$p \vee q$$

21. Prove $(p \Rightarrow q) \wedge (r \Rightarrow s) \Rightarrow (p \vee r \Rightarrow q \vee s)$ using inequality reasoning. **Hint**: first remove the implication in the antecedent, distribute as much as possible, and use (3.76d) and an absorption theorem.

Solution:

$$(p \Rightarrow q) \wedge (r \Rightarrow s)$$
$$= \quad \{ \text{ definition of} \Rightarrow (3.59) \}$$
$$(\neg p \vee q) \wedge (\neg r \vee s)$$
$$= \quad \{ \text{ distributivity (3.46) } \}$$
$$(\neg p \wedge \neg r) \vee (\neg p \wedge s) \vee (q \wedge \neg r) \vee (q \wedge s)$$
$$\Rightarrow \quad \{ (3.76d) \}$$
$$(\neg p \wedge \neg r) \vee s \vee (q \wedge \neg r) \vee q$$
$$= \quad \{ \text{ absorption (3.43b) } \}$$
$$(\neg p \wedge \neg r) \vee s \vee q$$
$$= \quad \{ \text{ de Morgan (3.47b) } \}$$
$$\neg(p \vee r) \vee s \vee q$$
$$= \quad \{ \text{ definition of} \Rightarrow (3.59) \}$$
$$p \vee r \Rightarrow q \vee s$$

22. Prove (4.1) $p \Rightarrow (q \Rightarrow p)$ by assuming the antecedent.

Solution:
Assume: p

$$q \Rightarrow p$$

$$= \quad \{ \text{ assumption } p \}$$
$$q \Rightarrow True$$
$$= \quad \{ \text{ zero of } \Rightarrow \text{ (3.76b) } \}$$
$$True$$

23. Prove $(\neg p \Rightarrow q) \Rightarrow ((p \Rightarrow q) \Rightarrow q)$ by assuming the antecedent.

Solution:
Assume: $\neg p \Rightarrow q$

$$p \Rightarrow q$$
$$= \quad \{ \text{ definition of } \Rightarrow \text{ (3.59) } \}$$
$$\neg p \lor q$$
$$\Rightarrow \quad \{ \text{ assumption, and monotonicity of } \lor \text{ (4.2) } \}$$
$$q \lor q$$
$$= \quad \{ \text{ idempotency of } \lor \text{ (3.26) } \}$$
$$q$$

24. Prove $(p \Rightarrow p') \land (q \Rightarrow q') \Rightarrow (p \lor q \Rightarrow p' \lor q')$ by assuming the antecedent.

Solution:
Assume: $(p \Rightarrow p')$ and $(q \Rightarrow q')$

$$p \lor q$$
$$\Rightarrow \quad \{ \text{ assumption, and monotonicity of } \lor \text{ (4.2) } \}$$
$$p' \lor q$$
$$\Rightarrow \quad \{ \text{ assumption, and monotonicity of } \lor \text{ (4.2) } \}$$
$$p' \lor q'$$