

# Programming Languages: Imperative Program Construction

## Practicals 0: Non-Looping Constructs and Weakest Precondition

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### Guarded Command Language Basics

1. Which of the following Hoare triples hold?

- (a)  $\{x = 7\} \text{skip} \{ \text{odd } x \};$
- (b)  $\{x > 60\} x := x \times 2 \{x > 100\};$
- (c)  $\{x > 40\} x := x \times 2 \{x > 100\};$
- (d)  $\{ \text{true} \} \text{if } x \leq y \rightarrow y := y - x \mid x \geq y \rightarrow x := x - y \text{ fi} \{x \geq 0 \wedge y \geq 0\};$
- (e)  $\{ \text{even } x \wedge \text{even } y \} \text{if } x \leq y \rightarrow y := y - x \mid x \geq y \rightarrow x := x - y \text{ fi} \{ \text{even } x \wedge \text{even } y \}.$

2. Is it always true that  $\{ \text{True} \} x := E \{x = E\}$ ? If you think the answer is yes, explain why. If your answer is no, give a counter example.

3. Verify:

```
{x = X ∧ y = Y}
x := x ≠ y
y := x ≠ y
x := x ≠ y
{x = Y ∧ y = X}
```

where  $x$  and  $y$  are boolean and  $(\neq)$  is the “not equal” or “exclusive or” operator. In fact, the code above works for any  $(\otimes)$  that satisfies the properties that for all  $a, b$ , and  $c$ :

associative :  $a \otimes (b \otimes c) = (a \otimes b) \otimes c$  ,  
unipotent :  $a \otimes a = 1$  ,

where 1 is the unit of  $(\otimes)$ , that is,  $1 \otimes b = b = b \otimes 1$ .

4. Verify the following program:

```
var r, b : Int
{0 ≤ r < 2 × b}
if b ≤ r → r := r - b
| r < b → skip
fi
{0 ≤ r < b}
```

5. Verify:

```

var  $x, y : Int$ 
{  $True$  }
 $x, y := x \times x, y \times y$ 
if  $x \geq y \rightarrow x := x - y$ 
  |  $y \geq x \rightarrow y := y - x$ 
fi
{  $x \geq 0 \wedge y \geq 0$  } .

```

6. Verify:

```

var  $a, b : Bool$ 
{  $True$  }
if  $\neg a \vee b \rightarrow a := \neg a$ 
  |  $a \vee \neg b \rightarrow b := \neg b$ 
fi
{  $a \vee b$  } .

```

7. Assuming that  $x, y$ , and  $z$  are integers, prove the following

- (a)  $\{True\} \text{ if } x \geq 1 \rightarrow x := x + 1 \mid x \leq 1 \rightarrow x := x - 1 \text{ fi } \{x \neq 1\}$ .
- (b)  $\{True\} \text{ if } x \geq y \rightarrow \text{skip} \mid y \geq x \rightarrow x, y := y, x \text{ fi } \{x \geq y\}$ .
- (c)  $\{x = 0\} \text{ if } True \rightarrow x := 1 \mid True \rightarrow x := -1 \{x = 1 \vee x = -1\}$ .
- (d)  $\{A = x \times y + z\} \text{ if even } x \rightarrow x, y := x / 2, y \times 2 \mid True \rightarrow y, z := y - 1, z + x \{A = x \times y + z\}$ .
- (e)  $\{x \times y = 0 \wedge y \leq x\} \text{ if } y < 0 \rightarrow y := -y \mid y = 0 \rightarrow x := -1 \{x < y\}$ .

### Weakest Precondition of Simple Statements

8. Given below is a list of statements and predicates. What are the weakest precondition for the predicates to be true after the statement?

- (a)  $x := x \times 2, x > 100$ ;
- (b)  $x := x \times 2, \text{even } x$ ;
- (c)  $x := x \times 2, x > 100 \wedge \text{even } x$ ;
- (d)  $x := x \times 2, \text{odd } x$ .
- (e)  $\text{skip}, \text{odd } x$ .

9. Determine the weakest  $P$  that satisfies

- (a)  $\{P\} x := x + 1; x := x + 1 \{x \geq 0\}$ .
- (b)  $\{P\} x := x + y; y := 2 \times x \{y \geq 0\}$ .
- (c)  $\{P\} x := y; y := x \{x = A \wedge y = B\}$ .
- (d)  $\{P\} x := E; x := E \{x = E\}$ .

10. What is the weakest  $P$  such that the following holds?

```

var  $x : Int$ 
{  $P$  }
 $x := x + 1$ 
if  $x > 0 \rightarrow x := x + 1$ 
  |  $x < 0 \rightarrow x := x + 2$ 
  |  $x = 1 \rightarrow \text{skip}$ 
fi
{  $x \geq 1$  } .

```

11. Two programs  $S_0$  and  $S_1$  are equivalent if, for all  $Q$ ,  $wp\ S_0\ Q = wp\ S_1\ Q$ . Show that the two following programs are equivalent.

**if**  $B_0 \rightarrow S_0 \mid B_1 \rightarrow S_1$  **fi**;  $S$   
**if**  $B_0 \rightarrow S_0; S \mid B_1 \rightarrow S_1; S$  **fi**

12. Consider the two programs:

$IF_0 = \text{if } B_0 \rightarrow S_0 \mid B_1 \rightarrow S_1 \text{ fi}$  ,  
 $IF_1 = \text{if } B_0 \rightarrow S_0 \mid B_1 \wedge \neg B_0 \rightarrow S_1 \text{ fi}$  .

Show that for all  $Q$ ,  $wp\ IF_0\ Q \Rightarrow wp\ IF_1\ Q$ .

### Properties of Weakest Precondition

13. Prove that  $(wp\ S\ Q_0 \vee wp\ S\ Q_1) \Rightarrow wp\ S\ (Q_0 \vee Q_1)$ .  
 14. Recall the definition of Hoare triple in terms of  $wp$ :

$$\{P\} S \{Q\} = P \Rightarrow wp\ S\ Q .$$

Prove that

1.  $(\{P\} S \{Q\} \wedge (P_0 \Rightarrow P)) \Rightarrow \{P_0\} S \{Q\}$ .
2.  $\{P\} S \{Q\} \wedge \{P\} S \{R\} \equiv \{P\} S \{Q \wedge R\}$ .

15. Recall the weakest precondition of **if**:

$$wp\ (\text{if } B_0 \rightarrow S_0 \mid B_1 \rightarrow S_1 \text{ fi})\ Q = (B_0 \Rightarrow wp\ S_0\ Q) \wedge (B_1 \Rightarrow wp\ S_1\ Q) \wedge (B_0 \vee B_1) .$$

Prove that

$$\{P\} \text{if } B_0 \rightarrow S_0 \mid B_1 \rightarrow S_1 \text{ fi } \{Q\} \equiv \{P \wedge B_0\} S_0 \{Q\} \wedge \{P \wedge B_1\} S_1 \{Q\} \wedge (P \Rightarrow (B_0 \vee B_1)) .$$

**Note:** having proved so shows that the way we annotate **if** is correct:

$\{P\}$   
**if**  $B_0 \rightarrow \{P \wedge B_0\} S_0 \{Q\}$   
 $\mid B_1 \rightarrow \{P \wedge B_1\} S_1 \{Q\}$   
**fi**  
 $\{Q\} .$

16. Recall that  $wp\ S\ Q$  stands for “the weakest precondition for program  $S$  to terminate in a state satisfying  $Q$ ”. What programs  $S$ , if any, satisfy each of the following conditions?

1.  $wp\ S\ \text{True} = \text{True}$ .
2.  $wp\ S\ \text{True} = \text{False}$ .
3.  $wp\ S\ \text{False} = \text{True}$ .
4.  $wp\ S\ \text{False} = \text{False}$ .