

Programming Languages: Imperative Program Construction

Practicals 2. Propositional Logic

Shin-Cheng Mu

Autumn Term, 2021

Prove each of the following properties using only axioms or theorems established before it (for example, prove (3.11) using only (1.?) and (3.1) - (3.10)).

Note that there are more than one ways to prove a property. You may discover a proof that is better than the one given in the solution.

1. Prove (3.9): $\neg(p \equiv q) \equiv \neg p \equiv q$.

Solution:

$$\begin{aligned} & \neg(p \equiv q) \\ = & \{ \text{definition of } False \text{ (3.15)} \} \\ & p \equiv q \equiv False \\ = & \{ \text{symmetry (3.2) and associativity (3.1) of } \equiv \} \\ & p \equiv False \equiv q \\ = & \{ \text{definition of } False \text{ (3.15)} \} \\ & \neg p \equiv q \end{aligned}$$

2. Prove (3.12): $\neg\neg p \equiv p$.

Solution:

$$\begin{aligned} & \neg\neg p \\ = & \{ \text{definition of } False \text{ (3.5), associativity (3.1) and symmetry (3.2) of } \equiv \} \\ & \neg p \equiv False \\ = & \{ \text{definition of } False \text{ (3.5)} \} \\ & p \end{aligned}$$

3. Prove (3.13): $\neg False \equiv True$.

Solution:

$$\neg False$$

$$\begin{aligned}
&= \{ \text{definition of } \textit{False} \text{ (3.5)} \} \\
&\quad \textit{False} \equiv \textit{False} \\
&= \{ \text{identity of } \equiv \text{ (3.3)} \} \\
&\quad \textit{True}
\end{aligned}$$

4. Prove (3.29): $p \vee \textit{True} \equiv \textit{True}$.

Solution:

$$\begin{aligned}
&p \vee \textit{True} \\
&= \{ \text{identity of } \equiv \text{ (3.3)} \} \\
&\quad p \vee (p \equiv p) \\
&= \{ \text{distributivity (3.27)} \} \\
&\quad p \vee p \equiv p \vee p \\
&= \{ \text{identity of } \equiv \text{ (3.3)} \} \\
&\quad \textit{True}
\end{aligned}$$

5. Prove (3.32): $p \vee q \equiv p \vee \neg q \equiv p$.

Solution:

$$\begin{aligned}
&p \vee q \equiv p \vee \neg q \\
&= \{ \text{distributivity (3.27)} \} \\
&\quad p \vee (q \equiv \neg q) \\
&= \{ \text{definition of } \textit{False} \text{ (3.15)} \} \\
&\quad p \vee \textit{False} \\
&= \{ \text{identity of } \vee \text{ (3.30)} \} \\
&\quad p
\end{aligned}$$

6. Prove (3.42): $p \wedge \neg p \equiv \textit{False}$.

Solution:

$$\begin{aligned}
&p \wedge \neg p \\
&= \{ \text{golden rule (3.35)} \} \\
&\quad p \equiv \neg p \equiv p \vee \neg p \\
&= \{ \text{excluded middle (3.28)} \} \\
&\quad p \equiv \neg p \equiv \textit{True} \\
&= \{ \text{identity of } \textit{True} \text{ (3.3)} \}
\end{aligned}$$

$$\begin{aligned}
 & p \equiv \neg p \\
 = & \{ \text{definition of } False \text{ (3.15)} \} \\
 & False
 \end{aligned}$$

Another proof:

$$\begin{aligned}
 & p \wedge \neg p \equiv False \\
 = & \{ \text{definition of } False \text{ (3.15)} \} \\
 & p \wedge \neg p \equiv p \equiv \neg p \\
 = & \{ \text{golden rule (3.35)} \} \\
 & p \vee \neg p \\
 = & \{ \text{excluded middle (3.28)} \} \\
 & True
 \end{aligned}$$

7. Prove (3.43a): $p \wedge (p \vee q) \equiv p$.

Solution:

$$\begin{aligned}
 & p \wedge (p \vee q) \\
 = & \{ \text{golden rule (3.35)} \} \\
 & p \equiv p \vee q \equiv p \vee p \vee q \\
 = & \{ \text{idempotency of } \vee \text{ (3.26)} \} \\
 & p \equiv p \vee q \equiv p \vee q \\
 = & \{ \text{identity of } \equiv \text{ (3.3)} \} \\
 & p \equiv True \\
 = & \{ \text{identity of } \equiv \text{ (3.3)} \} \\
 & p
 \end{aligned}$$

8. Prove (3.44a). $p \wedge (\neg p \vee q) \equiv p \wedge q$.

Solution:

$$\begin{aligned}
 & p \wedge (\neg p \vee q) \\
 = & \{ \text{golden rule (3.35)} \} \\
 & p \equiv \neg p \vee q \equiv p \vee \neg p \vee q \\
 = & \{ \text{excluded middle (3.28)} \} \\
 & p \equiv \neg p \vee q \equiv True \vee q \\
 = & \{ \text{zero of } \vee \text{ (3.29) and identity of } \equiv \text{ (3.3)} \} \\
 & p \equiv \neg p \vee q \\
 = & \{ \text{(3.32), with } p, q := q, p \} \\
 & p \equiv q \equiv p \vee q
 \end{aligned}$$

$$= \{ \text{golden rule (3.35)} \}$$

$$p \wedge q$$

9. Prove (3.65): $p \wedge q \Rightarrow r \equiv p \Rightarrow (q \Rightarrow r)$.

Solution:

$$p \Rightarrow (q \Rightarrow r)$$

$$= \{ \text{definition of } \Rightarrow (3.60) \}$$

$$p \wedge (q \Rightarrow r) \equiv p$$

$$= \{ \text{definition of } \Rightarrow (3.60) \}$$

$$p \wedge (q \wedge r \equiv q) \equiv p$$

$$= \{ (3.49) \}$$

$$p \wedge q \wedge r \equiv p \wedge q \equiv p \equiv p$$

$$= \{ \text{identity of } \equiv (3.3) \}$$

$$p \wedge q \wedge r \equiv p \wedge q$$

$$= \{ \text{definition of } \Rightarrow (3.60) \}$$

$$(p \wedge q) \Rightarrow r$$

10. Prove (3.66): $p \wedge (p \Rightarrow q) \equiv p \wedge q$.

Solution:

$$p \wedge (p \Rightarrow q)$$

$$= \{ \text{definition of } \Rightarrow (3.60) \}$$

$$p \wedge (p \wedge q \equiv p)$$

$$= \{ (3.49) \text{ and idempotency of } \wedge (3.38) \}$$

$$p \wedge q \equiv p \equiv p$$

$$= \{ \text{identity of } \equiv (3.3) \}$$

$$p \wedge q$$

11. Prove (3.67): $p \wedge (q \Rightarrow p) \equiv p$.

Solution:

$$p \wedge (q \Rightarrow p)$$

$$= \{ \text{definition of } \Rightarrow (3.60) \}$$

$$p \wedge (q \wedge p \equiv q)$$

$$= \{ (3.49) \text{ and idempotency of } \wedge (3.38) \}$$

$$\begin{aligned}
 & p \wedge q \equiv p \wedge q \equiv p \\
 = & \{ \text{identity of } \equiv (3.3) \} \\
 & p
 \end{aligned}$$

12. Prove (3.68): $p \vee (p \Rightarrow q) \equiv \text{True}$.

Solution:

$$\begin{aligned}
 & p \vee (p \Rightarrow q) \\
 = & \{ \text{definition of } \Rightarrow (3.57) \} \\
 & p \vee (p \vee q \equiv q) \\
 = & \{ \text{distributivity (3.27) and idempotency (3.26)} \} \\
 & p \vee q \equiv p \vee q \\
 = & \{ \text{identity of } \equiv (3.3) \} \\
 & \text{True}
 \end{aligned}$$

Another proof:

$$\begin{aligned}
 & p \vee (p \Rightarrow q) \\
 = & \{ \text{definition of } \Rightarrow (3.59) \} \\
 & p \vee \neg p \vee q \\
 = & \{ \text{excluded middle (3.28) and zero of } \vee (3.29) \} \\
 & \text{True}
 \end{aligned}$$

13. Prove (3.69): $p \vee (q \Rightarrow p) \equiv q \Rightarrow p$.

Solution:

$$\begin{aligned}
 & p \vee (q \Rightarrow p) \\
 = & \{ \text{definition of } \Rightarrow (3.59) \} \\
 & p \vee \neg q \vee p \\
 = & \{ \text{idempotency of } \vee (3.26) \} \\
 & p \vee \neg q \\
 = & \{ \text{definition of } \Rightarrow (3.59) \} \\
 & q \Rightarrow p
 \end{aligned}$$

14. Prove (3.78): $(p \Rightarrow r) \wedge (q \Rightarrow r) \equiv (p \vee q \Rightarrow r)$.

Solution:

$$\begin{aligned} & p \vee q \Rightarrow r \\ = & \{ \text{definition of } \Rightarrow (3.60) \} \\ & \neg(p \vee q) \vee r \\ = & \{ \text{de Morgan (3.47)} \} \\ & (\neg p \wedge \neg q) \vee r \\ = & \{ \text{distributivity (3.45)} \} \\ & (\neg p \vee r) \wedge (\neg q \vee r) \\ = & \{ \text{definition of } \Rightarrow (3.60) \} \\ & (p \Rightarrow r) \wedge (q \Rightarrow r) \end{aligned}$$

15. Prove that $(p \Rightarrow q) \wedge (p \Rightarrow r) \equiv (p \Rightarrow q \wedge r)$.

Solution:

$$\begin{aligned} & (p \Rightarrow q) \wedge (p \Rightarrow r) \\ = & \{ \text{definition of } \Rightarrow (3.60) \} \\ & (\neg p \vee q) \wedge (\neg p \vee r) \\ = & \{ \text{distributivity (3.45)} \} \\ & \neg p \vee (q \wedge r) \\ = & \{ \text{definition of } \Rightarrow (3.60) \} \\ & p \Rightarrow q \wedge r \end{aligned}$$

16. Prove that $(r \Rightarrow)$ is monotonic with respect to implication. That is, $(p \Rightarrow q) \Rightarrow ((r \Rightarrow p) \Rightarrow (r \Rightarrow q))$.

Solution:

$$\begin{aligned} & (p \Rightarrow q) \Rightarrow ((r \Rightarrow p) \Rightarrow (r \Rightarrow q)) \\ = & \{ \text{shunting (3.65)} \} \\ & ((p \Rightarrow q) \wedge (r \Rightarrow p)) \Rightarrow (r \Rightarrow q) \\ = & \{ \text{shunting (3.65)} \} \\ & ((p \Rightarrow q) \wedge (r \Rightarrow p) \wedge r) \Rightarrow q \\ = & \{ (3.66) \} \\ & ((p \Rightarrow q) \wedge p \wedge r) \Rightarrow q \\ = & \{ (3.66) \} \\ & (q \wedge p \wedge r) \Rightarrow q \\ = & \{ \text{weakening (3.76b)} \} \\ & \text{True} \end{aligned}$$

17. Prove that $(\Rightarrow r)$ is anti-monotonic with respect to implication. That is, $(p \Rightarrow q) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \Rightarrow r))$.

Solution:

$$\begin{aligned}
 & (p \Rightarrow q) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \Rightarrow r)) \\
 = & \{ \text{shunting (3.65)} \} \\
 & ((p \Rightarrow q) \wedge (q \Rightarrow r)) \Rightarrow (p \Rightarrow r) \\
 = & \{ \text{shunting (3.65)} \} \\
 & ((p \Rightarrow q) \wedge (q \Rightarrow r) \wedge p) \Rightarrow r \\
 = & \{ (3.66) \} \\
 & (p \wedge q \wedge (q \Rightarrow r)) \Rightarrow r \\
 = & \{ (3.66) \} \\
 & (p \wedge q \wedge r) \Rightarrow r \\
 = & \{ \text{weakening (3.76b)} \} \\
 & \text{True}
 \end{aligned}$$

18. Prove that conjunction is monotonic with respect to implication. That is, $(p \Rightarrow q) \Rightarrow ((p \wedge r) \Rightarrow (q \wedge r))$.

Solution:

$$\begin{aligned}
 & (p \Rightarrow q) \Rightarrow ((p \wedge r) \Rightarrow (q \wedge r)) \\
 = & \{ \text{shunting (3.65)} \} \\
 & ((p \Rightarrow q) \wedge p \wedge r) \Rightarrow (q \wedge r) \\
 = & \{ (3.66) \} \\
 & (p \wedge q \wedge r) \Rightarrow (q \wedge r) \\
 = & \{ \text{weakening (3.76b)} \} \\
 & \text{True}
 \end{aligned}$$

19. Prove (4.3) $(p \Rightarrow q) \Rightarrow (p \wedge r \Rightarrow q \wedge r)$. Start with the consequent, since it has more structure.

Solution:

$$\begin{aligned}
 & p \wedge r \Rightarrow q \wedge r \\
 = & \{ \text{definition of } \Rightarrow (3.59) \} \\
 & \neg p \vee \neg r \vee (q \wedge r) \\
 = & \{ \text{distributivity (3.45)} \} \\
 & \neg p \vee ((\neg r \vee q) \wedge (\neg r \vee r)) \\
 = & \{ \text{excluded middle (3.28) and identity of } \wedge (3.39) \} \\
 & \neg p \vee \neg r \vee q \\
 \Leftarrow & \{ \text{strengthening (3.76a)} \} \\
 & \neg p \vee q
 \end{aligned}$$

$$= \{ \text{definition of } \Rightarrow (3.59) \}$$

$$p \Rightarrow q$$

20. Prove (3.76d) $p \vee (q \wedge r) \Rightarrow p \vee q$ using inequality reasoning. Start with the antecedent, since it has more structure, and use distributivity.

Solution:

$$p \vee (q \wedge r)$$

$$= \{ \text{distributivity (3.45)} \}$$

$$(p \vee q) \wedge (p \vee r)$$

$$\Rightarrow \{ \text{weakening (3.76b)} \}$$

$$p \vee q$$

21. Prove $(p \Rightarrow q) \wedge (r \Rightarrow s) \Rightarrow (p \vee r \Rightarrow q \vee s)$ using inequality reasoning. **Hint:** first remove the implication in the antecedent, distribute as much as possible, and use (3.76d) and an absorption theorem.

Solution:

$$(p \Rightarrow q) \wedge (r \Rightarrow s)$$

$$= \{ \text{definition of } \Rightarrow (3.59) \}$$

$$(\neg p \vee q) \wedge (\neg r \vee s)$$

$$= \{ \text{distributivity (3.46)} \}$$

$$(\neg p \wedge \neg r) \vee (\neg p \wedge s) \vee (q \wedge \neg r) \vee (q \wedge s)$$

$$\Rightarrow \{ (3.76d) \}$$

$$(\neg p \wedge \neg r) \vee s \vee (q \wedge \neg r) \vee q$$

$$= \{ \text{absorption (3.43b)} \}$$

$$(\neg p \wedge \neg r) \vee s \vee q$$

$$= \{ \text{de Morgan (3.47b)} \}$$

$$\neg(p \vee r) \vee s \vee q$$

$$= \{ \text{definition of } \Rightarrow (3.59) \}$$

$$p \vee r \Rightarrow q \vee s$$

22. Prove (4.1) $p \Rightarrow (q \Rightarrow p)$ by assuming the antecedent.

Solution:

Assume: p

$$q \Rightarrow p$$

$$\begin{aligned}
&= \{ \text{assumption } p \} \\
&\quad q \Rightarrow \text{True} \\
&= \{ \text{zero of } \Rightarrow (3.76b) \} \\
&\quad \text{True}
\end{aligned}$$

23. Prove $(\neg p \Rightarrow q) \Rightarrow ((p \Rightarrow q) \Rightarrow q)$ by assuming the antecedent.

Solution:

Assume: $\neg p \Rightarrow q$

$$\begin{aligned}
&p \Rightarrow q \\
&= \{ \text{definition of } \Rightarrow (3.59) \} \\
&\quad \neg p \vee q \\
&\Rightarrow \{ \text{assumption, and monotonicity of } \vee (4.2) \} \\
&\quad q \vee q \\
&= \{ \text{idempotency of } \vee (3.26) \} \\
&\quad q
\end{aligned}$$

24. Prove $(p \Rightarrow p') \wedge (q \Rightarrow q') \Rightarrow (p \vee q \Rightarrow p' \vee q')$ by assuming the antecedent.

Solution:

Assume: $(p \Rightarrow p')$ and $(q \Rightarrow q')$

$$\begin{aligned}
&p \vee q \\
&\Rightarrow \{ \text{assumption, and monotonicity of } \vee (4.2) \} \\
&\quad p' \vee q \\
&\Rightarrow \{ \text{assumption, and monotonicity of } \vee (4.2) \} \\
&\quad p' \vee q'
\end{aligned}$$