Programming Languages: Imperative Program Construction 1. Hoare Logic and Weakest Precondition: Non-Looping Constructs

Shin-Cheng Mu

Sep. 2021

National Taiwan University and Academia Sinica

Hoare Logic

In this course we will talk about program construction using Dijkstra's calculus. Most of the materials are from Kaldewaij.

- A program computing the greatest common divisor:

$$
\begin{array}{l}
\textbf{con } A, B : \mathit{Int} \\
\textbf{var } x, y : \mathit{Int} \\
x, y := A, B \\
\textbf{do } y < x \rightarrow x := x - y \\
\mid \quad x < y \rightarrow y := y - x \\
\textbf{od}
\end{array}
$$

.

- **do** denotes loops with guarded bodies.

In this course we will talk about program construction using Dijkstra's calculus. Most of the materials are from Kaldewaij.

- A program computing the greatest common divisor:

$$
\begin{array}{l}
\textbf{con } A, B : \mathit{Int}\ \{0 < A \land 0 < B\} \\
\textbf{var }\ x, y : \mathit{Int} \\
x, y := A, B \\
\textbf{do } y < x \to x := x - y \\
\mid\ \ x < y \to y := y - x \\
\textbf{od} \\
\{x = y = \mathit{gcd}\ (A, B)\}\ \ .
\end{array}
$$

- **do** denotes loops with guarded bodies.
- Assertions delimited in curly brackets.

- Given a program statement S and predicates P and Q, the *Hoare triple* $\{P\}\,S\,\{Q\}$ is a Boolean value.
- Operationally, $\{P\}\,S\,\{Q\}$ is $True$ iff. the statement S, when executed in a state satisfying P, *terminates* in a state satisfying Q.

Examples

- $\{x \geqslant 0 \wedge y \geqslant 0\}\, S\, \{r = x \times y\}$ is *True* iff. S is a program that, given non-negative x and y, terminates and stores $x \times y$ in r.

- $\{x \geqslant 0 \wedge y \geqslant 0\}\, S\, \{r = x \times y\}$ is *True* iff. S is a program that, given non-negative x and y, terminates and stores $x \times y$ in r.
    - Nothing is said about values of x and y upon termination.
    - When $x \geqslant 0 \wedge y \geqslant 0$ does not hold, S may do anything — including looping forever.

EXAMPLES

- $\{x \geqslant 0 \land y \geqslant 0\}\, S\, \{r = x \times y\}$ is *True* iff. S is a program that, given non-negative x and y, terminates and stores $x \times y$ in r.
  - Nothing is said about values of x and y upon termination.
  - When $x \geqslant 0 \land y \geqslant 0$ does not hold, S may do anything — including looping forever.
- $\{z \geqslant 0\}\, S\, \{x \times y = z\}$ is *True* iff. S, given non-negative z, computes a factorization of z, and terminates.

Examples

- $\{x \geqslant 0 \wedge y \geqslant 0\}\, S\, \{r = x \times y\}$ is *True* iff. S is a program that, given non-negative x and y, terminates and stores $x \times y$ in r.
  - Nothing is said about values of x and y upon termination.
  - When $x \geqslant 0 \wedge y \geqslant 0$ does not hold, S may do anything — including looping forever.
- $\{z \geqslant 0\}\, S\, \{x \times y = z\}$ is *True* iff. S, given non-negative z, computes a factorization of z, and terminates.
- $\{x > 0\}\, S\, \{\mathit{True}\}$ is *True* iff. S is any program that terminates, provided that $x > 0$.

- $\{P\}\, S\, \{Q\}$ and $P_0 \Rightarrow P$ implies $\{P_0\}\, S\, \{Q\}$.
- $\{P\}\, S\, \{Q\}$ and $Q \Rightarrow Q_0$ implies $\{P\}\, S\, \{Q_0\}$.
- $\{P\}\, S\, \{Q\}$ and $\{P\}\, S\, \{R\}$ equivales $\{P\}\, S\, \{Q \wedge R\}$.
- $\{P\}\, S\, \{Q\}$ and $\{R\}\, S\, \{Q\}$ equivales $\{P \vee R\}\, S\, \{Q\}$.
- **Note**: "A equivales B" is another way to say "A if and only if B", also denoted by $A \equiv B$.

- Perhaps the simplest statement: $\{P\}\, \mathit{skip}\, \{Q\}$ iff. $P \Rightarrow Q$.
  - E.g. $\{x > 0 \wedge y > 0\}\, \mathit{skip}\, \{x \geqslant 0\}$.
  - Note that the annotations need not be "exact."
- Operationally, *skip* is a statement that does nothing.
  - Why do we need a program that does nothing?
  - It is like why we need a number 0 that represents "nothing". It can be very useful sometimes.

Assignments

- $P[x \backslash E]$: substituting *free* occurrences of x in P for E.
- We do so in mathematics all the time. A formal definition of substitution, however, is rather tedious.
- For this lecture we will only appeal to "common sense":
  - E.g. $(x \leqslant 3)[x \backslash x - 1] \equiv x - 1 \leqslant 3 \equiv x \leqslant 4$.
  - $$\begin{aligned} & (\langle \exists y : y \in Nat : x < y \rangle \wedge y < x)[y \backslash y + 1] \\ \equiv\ & \langle \exists y : y \in Nat : x < y \rangle \wedge y + 1 < x. \end{aligned}$$
  - $$\begin{aligned} & \langle \exists y : y \in Nat : x < y \rangle [x \backslash y] \\ \equiv\ & \langle \exists z : z \in Nat : y < z \rangle. \end{aligned}$$

- The notation $[x \backslash E]$ hints at "divide by x and multiply by E."
  - We have $x[x \backslash E] = E$. Nice!
- Just in case you may see different notations in other papers…
  - Many papers use the notation $[E/x]$. Either way, x is the denominator.
  - Kaldewaij actually wrote $[x := E]$, since substitution is closely related to assignments.
  - Some papers write P^x_E for $P[x \backslash E]$.

- Which is correct:
    1. $\{P\}\, x := E\, \{P[x \backslash E]\}$, or
    2. $\{P[x \backslash E]\}\, x := E\, \{P\}$?

- Which is correct:
  1. $\{P\}\, x := E\, \{P[x \backslash E]\}$, or
  2. $\{P[x \backslash E]\}\, x := E\, \{P\}$?
- Answer: 2! For example:

$$
\begin{aligned}
& \{(x \leqslant 3)[x \backslash x+1]\}\, x := x+1\, \{x \leqslant 3\} \\
\equiv\ & \{x+1 \leqslant 3\}\, x := x+1\, \{x \leqslant 3\} \\
\equiv\ & \{x \leqslant 2\}\, x := x+1\, \{x \leqslant 3\}.
\end{aligned}
$$

Sequencing

- $\{P\}\, S; T\, \{Q\}$ equivals that there exists R such that $\{P\}\, S\, \{R\}$ and $\{R\}\, T\, \{Q\}$.
- Verify:

$$\mathbf{var}\ x, y : \mathit{Int}$$
$$\{x = A \wedge y = B\}$$
$$x := x - y$$

$$y := x + y$$

$$x := y - x$$
$$\{x = B \wedge y = A\}$$

- $\{P\}\, S; T\, \{Q\}$ equivals that there exists R such that $\{P\}\, S\, \{R\}$ and $\{R\}\, T\, \{Q\}$.
- Verify:

$$
\begin{aligned}
&\textbf{var}\ x, y : Int \\
&\{x = A \wedge y = B\} \\
&x := x - y \\
& \\
&y := x + y \\
&\{y - x = B \wedge y = A\} \\
&x := y - x \\
&\{x = B \wedge y = A\}
\end{aligned}
$$

- $\{P\}\, S; T\, \{Q\}$ equivals that there exists R such that $\{P\}\, S\, \{R\}$ and $\{R\}\, T\, \{Q\}$.
- Verify:

$$
\begin{array}{ll}
\textbf{var}\ x, y : Int & \\
\{x = A \land y = B\} & \\
x := x - y & \\
 & \{x + y - x = B \land x + y = A\} \\
y := x + y & \\
\{y - x = B \land y = A\} & \\
x := y - x & \\
\{x = B \land y = A\} &
\end{array}
$$

- $\{P\}\, S; T\, \{Q\}$ equivals that there exists R such that $\{P\}\, S\, \{R\}$ and $\{R\}\, T\, \{Q\}$.
- Verify:

$$
\begin{array}{l}
\textbf{var}\ x, y : Int \\
\{x = A \wedge y = B\} \\
x := x - y \\
\{y = B \wedge x + y = A\} \;\; \Rightarrow \{x + y - x = B \wedge x + y = A\} \\
y := x + y \\
\{y - x = B \wedge y = A\} \\
x := y - x \\
\{x = B \wedge y = A\}
\end{array}
$$

- $\{P\}\, S; T\, \{Q\}$ equivals that there exists R such that $\{P\}\, S\, \{R\}$ and $\{R\}\, T\, \{Q\}$.
- Verify:

$$
\begin{array}{l}
\textbf{var}\ x, y : Int \\
\{x = A \wedge y = B\} \Rightarrow \{y = B \wedge x - y + y = A\} \\
x := x - y \\
\{y = B \wedge x + y = A\} \\
y := x + y \\
\{y - x = B \wedge y = A\} \\
x := y - x \\
\{x = B \wedge y = A\}
\end{array}
$$

- $\{P\}\ S; T\ \{Q\}$ equivals that there exists R such that $\{P\}\ S\ \{R\}$ and $\{R\}\ T\ \{Q\}$.
- Verify:

$$
\begin{array}{l}
\mathbf{var}\ x, y : Int \\
\{x = A \land y = B\} \\
x := x - y \\
\{y = B \land x + y = A\} \\
y := x + y \\
\{y - x = B \land y = A\} \\
x := y - x \\
\{x = B \land y = A\}
\end{array}
$$

Selection

- Selection takes the form **if** $B_0 \rightarrow S_0 \mid ... \mid Bn \rightarrow Sn$ **fi**.
- Each B_i is called a *guard*; $B_i \rightarrow S_i$ is a *guarded command*.
- If none of the guards $B_0 \ldots B_n$ evaluate to true, the program aborts. Otherwise, one of the command with a true guard is chosen *non-deterministically* and executed.

To annotate an **if** statement:

$$\begin{array}{l} \{P\} \\ \textbf{if } B_0 \to \{P \wedge B_0\}\, S_0\, \{Q, \mathrm{Pf}_0\} \\ \mid B_1 \to \{P \wedge B_1\}\, S_1\, \{Q, \mathrm{Pf}_1\} \\ \textbf{fi} \\ \{Q, \mathrm{Pf}_2\} \end{array} \quad ,$$

where Pf_0, Pf_1, Pf_2 are labels referring to proofs.

- Pf_0 refers to a proof of $\{P \wedge B_0\}\, S_0\, \{Q\}$;
- Pf_1 refers to a proof of $\{P \wedge B_1\}\, S_1\, \{Q\}$;
- Pf_2 refers to a proof of $P \Rightarrow B_0 \vee B_1$.
- The proofs and labels are sometimes omitted if they are trivial.

Binary Maximum

- Goal: to assign $x \uparrow y$ to z. By definition,
  $z = x \uparrow y \quad \equiv \quad (z = x \ \vee \ z = y) \ \wedge \ x \leqslant z \ \wedge \ y \leqslant z.$

- Goal: to assign $x \uparrow y$ to z. By definition,
  $z = x \uparrow y \quad \equiv \quad (z = x \ \lor \ z = y) \ \land \ x \leqslant z \ \land \ y \leqslant z$.
- Try $z := x$. We reason:

$$
\begin{aligned}
& ((z = x \ \lor \ z = y) \ \land \ x \leqslant z \ \land \ y \leqslant z)[z \backslash x] \\
\equiv \ & (x = x \ \lor \ x = y) \ \land \ x \leqslant x \ \land \ y \leqslant x \\
\equiv \ & y \leqslant x,
\end{aligned}
$$

  which hinted at using a guarded command: $y \leqslant x \to z := x$.

- Goal: to assign $x \uparrow y$ to z. By definition,
  $z = x \uparrow y \quad \equiv \quad (z = x \ \vee \ z = y) \ \wedge \ x \leqslant z \ \wedge \ y \leqslant z$.
- Try $z := x$. We reason:

$$
\begin{aligned}
& ((z = x \ \vee \ z = y) \ \wedge \ x \leqslant z \ \wedge \ y \leqslant z)[z \backslash x] \\
\equiv \ & (x = x \ \vee \ x = y) \ \wedge \ x \leqslant x \ \wedge \ y \leqslant x \\
\equiv \ & y \leqslant x,
\end{aligned}
$$

  which hinted at using a guarded command: $y \leqslant x \rightarrow z := x$.
- Indeed:

$$
\begin{array}{l}
\{True\} \\
\mathbf{if}\ y \leqslant x \rightarrow \{y \leqslant x\}\, z := x\, \{z = x \uparrow y\} \\
\ \mid x \leqslant y \rightarrow \{x \leqslant y\}\, z := y\, \{z = x \uparrow y\} \\
\mathbf{fi} \\
\{z = x \uparrow y\} \quad .
\end{array}
$$

- There are two ways to understand the program below:

$$\begin{array}{l} \textbf{if } B_{00} \to S_{00} \mid B_{01} \to S_{01} \textbf{ fi} \\ \textbf{if } B_{10} \to S_{10} \mid B_{11} \to S_{11} \textbf{ fi} \\ \qquad \vdots \\ \textbf{if } B_{n0} \to S_{n0} \mid B_{n1} \to S_{n1} \textbf{ fi}. \end{array}$$

- One takes effort exponential to n; the other is linear.
- Dijkstra: "…if we ever want to be able to compose really large programs reliably, we need a programming discipline such that the intellectual effort needed to understand a program does not grow more rapidly than in proportion to the program length."

Weakest Precondition

More precisely speaking...

- A *predicate* on A is a function having type $A \to Bool$.
  - E.g. $even :: Int \to Bool$ is a predicate on Int.

More precisely speaking...

- A *predicate* on A is a function having type $A \to Bool$.
    - E.g. $even :: Int \to Bool$ is a predicate on Int.
- The *state space* of a program is the states of all its variables.
    - E.g. state space for the GCD program, which has two variables x and y, is $(Int \times Int)$.

More precisely speaking...

- A *predicate* on A is a function having type $A \to Bool$.
  - E.g. $even :: Int \to Bool$ is a predicate on Int.
- The *state space* of a program is the states of all its variables.
  - E.g. state space for the GCD program, which has two variables x and y, is $(Int \times Int)$.
- An expression having free variables can be seen as a function.
  - E.g. $x \leqslant y$ is a predicate (a function) with type $(Int \times Int) \to Bool$ that yields $True$ for, e.g. $(x, y) = (3, 4)$ and $False$ for $(x, y) = (4, 3)$.

- In $\{P\}\ S\ \{Q\}$, P and Q shall be seen as *predicates* on the state space of the program S.

- In $\{P\}\, S\, \{Q\}$, P and Q shall be seen as *predicates* on the state space of the program S.
- E.g. In $\{z \geqslant 0\}\, S\, \{x \times y = z\}$, assuming that the program S uses only three variables x, y, and z.
    - The part $z \geqslant 0$ shall be understood as a predicate that takes x, y, and z, and returns *True* iff. $z \geqslant 0$.

- In $\{P\}\, S\, \{Q\}$, P and Q shall be seen as *predicates* on the state space of the program S.
- E.g. In $\{z \geqslant 0\}\, S\, \{x \times y = z\}$, assuming that the program S uses only three variables x, y, and z.
    - The part $z \geqslant 0$ shall be understood as a predicate that takes x, y, and z, and returns $True$ iff. $z \geqslant 0$.
    - The part $x \times y = z$ shall be understood as a predicate that takes x, y, and z, and returns $True$ iff. $x \times y = z$.

- In $\{P\}\, S\, \{Q\}$, P and Q shall be seen as *predicates* on the state space of the program S.
- E.g. In $\{z \geqslant 0\}\, S\, \{x \times y = z\}$, assuming that the program S uses only three variables x, y, and z.
  - The part $z \geqslant 0$ shall be understood as a predicate that takes x, y, and z, and returns *True* iff. $z \geqslant 0$.
  - The part $x \times y = z$ shall be understood as a predicate that takes x, y, and z, and returns *True* iff. $x \times y = z$.
- *True* in a Hoare triple can be understood as a predicate that returns *True* for any input; similarly with *False*.

- Let S be a program having variables x, y, z. That $\{P\}\, S\, \{Q\}$ being $True$ means that if S starts running in a state such that $P\,(x, y, z) = True$, it terminates and yields a state such that $Q\,(x, y, z) = True$.

- Given propositions P and Q, if $P \Rightarrow Q$, we say that Q is the *weaker* one, and P is the *stronger* one.

- Given propositions P and Q, if $P \Rightarrow Q$, we say that Q is the *weaker* one, and P is the *stronger* one.
- Precisely speaking, P is *no weaker than* Q and Q is *no stronger than* P. But let's be a bit sloppy to avoid confusion...

- The convention extends to predicates. If $P\, x \Rightarrow Q\, x$ for every x, Q is the *weaker* one, while P is the *stronger* one.

- The convention extends to predicates. If $P\,x \Rightarrow Q\,x$ for every x, Q is the *weaker* one, while P is the *stronger* one.
- Example: $0 \leqslant x < 4$ is weaker than $0 \leqslant x < 3$, which is in turn weaker than $1 \leqslant x < 3$.
    - Intuition: for first-order values, the set of values satisfying a weaker predicate is *larger* than that satisfying a stronger predicate.

- The convention extends to predicates. If $P\,x \Rightarrow Q\,x$ for every x, Q is the *weaker* one, while P is the *stronger* one.
- Example: $0 \leqslant x < 4$ is weaker than $0 \leqslant x < 3$, which is in turn weaker than $1 \leqslant x < 3$.
    - Intuition: for first-order values, the set of values satisfying a weaker predicate is *larger* than that satisfying a stronger predicate.
- Example: P can be weaker than $P \wedge Q$ (since $(P \wedge Q) \Rightarrow P$); $P \vee Q$ can be weaker than P (since $P \Rightarrow (P \vee Q)$).

- The convention extends to predicates. If $P\ x \Rightarrow Q\ x$ for every x, Q is the *weaker* one, while P is the *stronger* one.
- Example: $0 \leqslant x < 4$ is weaker than $0 \leqslant x < 3$, which is in turn weaker than $1 \leqslant x < 3$.
  - Intuition: for first-order values, the set of values satisfying a weaker predicate is *larger* than that satisfying a stronger predicate.
- Example: P can be weaker than $P \wedge Q$ (since $(P \wedge Q) \Rightarrow P$); $P \vee Q$ can be weaker than P (since $P \Rightarrow (P \vee Q)$).
- Intuition: a weaker predicate enforces less restriction, is more tolerant, and allows more inputs/states to be $True$.

- Functions can be hard to grasp.

- Functions can be hard to grasp.
- A predicate P is isomorphic to the set of values that satisfy the predicate — at least for first order values. Therefore I tend to equate them.

- Functions can be hard to grasp.
- A predicate P is isomorphic to the set of values that satisfy the predicate — at least for first order values. Therefore I tend to equate them.
- E.g. think of $x \leqslant 3$ as the set of values satisfying $x \leqslant 3$.

Predicate-Set Correspondence

- Functions can be hard to grasp.
- A predicate P is isomorphic to the set of values that satisfy the predicate — at least for first order values. Therefore I tend to equate them.
- E.g. think of $x \leqslant 3$ as the set of values satisfying $x \leqslant 3$.
- *False* is the empty set, *True* is the set of all values (of the right type).

Predicate-Set Correspondence

- Functions can be hard to grasp.
- A predicate P is isomorphic to the set of values that satisfy the predicate — at least for first order values. Therefore I tend to equate them.
- E.g. think of $x \leqslant 3$ as the set of values satisfying $x \leqslant 3$.
- *False* is the empty set, *True* is the set of all values (of the right type).
- $P \Rightarrow Q$ iff. $P \subseteq Q$.
  - A weaker predicate is a bigger set!

- Functions can be hard to grasp.
- A predicate P is isomorphic to the set of values that satisfy the predicate — at least for first order values. Therefore I tend to equate them.
- E.g. think of $x \leqslant 3$ as the set of values satisfying $x \leqslant 3$.
- *False* is the empty set, *True* is the set of all values (of the right type).
- $P \Rightarrow Q$ iff. $P \subseteq Q$.
    - A weaker predicate is a bigger set!
- $P \wedge Q$ corresponds to $P \cap Q$; $P \vee Q$ corresponds to $P \cup Q$.

- Recall that the predicates in a Hoare triple need not be exact.
  - $\{x \leqslant 2\}\, x := x + 1\, \{x \leqslant 3\}$ is a valid triple.
  - So is $\{0 < x \leqslant 2\}\, x := x + 1\, \{x \leqslant 3\}$. Note that $x \leqslant 2$ is weaker than $0 < x \leqslant 2$.
  - $x \leqslant 2$ is in fact the weakest (most tolerating) P such that $\{P\}\, x := x + 1\, \{x \leqslant 3\}$ holds.

- Defining weakest precondition in terms of Hoare triple….
- **Definition**: given a statement S, its *weakest precondition* with respect to Q, denoted $wp\ S\ Q$, is the weakest predicate such that $\{wp\ S\ Q\}\ S\ \{Q\}$ holds.

$wp\ S$ is a function from predicates to predicates.

- Also called a *predicate transformer*.
- I myself find it sometimes easier to think of a predicate transformer as a function from sets to sets.
- E.g. $wp\ S\ Q$ gives you the *largest* set P such that for all $x \in P$, running S starting from initial state x gives you a final state in Q.

Weakest Precondition: Skip and Assignment

- Weakest preconditions for *skip* and *assignment*:
- $wp\ skip\ P = P$.
- $wp\ (x := E)\ P = P[x \backslash E]$.

Hoare Triple, Revisited

- We can do it the other way round: specify *wp* for each program construct, and define Hoare triple in terms of *wp*.
- **Definition**: $\{P\}\ S\ \{Q\}$ if and only if $P \Rightarrow wp\ S\ Q$.

Examples

- $\{x > 0\}\ \mathit{skip}\ \{x \geqslant 0\}$ is valid, because:

$$
\begin{aligned}
& \mathit{wp}\ \mathit{skip}\ (x \geqslant 0) \\
\equiv\quad & \{\ \text{definition of } \mathit{wp}\ \} \\
& x \geqslant 0 \\
\Leftarrow\ & x > 0 \quad .
\end{aligned}
$$

- $\{x > 0\}\ \mathit{skip}\ \{x \geqslant 0\}$ is valid, because:

$$
\begin{aligned}
& \mathit{wp}\ \mathit{skip}\ (x \geqslant 0) \\
\equiv\ & \quad \{\ \text{definition of } \mathit{wp}\ \} \\
& x \geqslant 0 \\
\Leftarrow\ & x > 0 \quad .
\end{aligned}
$$

- $\{0 < x < 2\}\ x := x + 1\ \{x \leqslant 3\}$ is valid, because

$$
\begin{aligned}
& \mathit{wp}\ (x := x + 1)\ (x \leqslant 3) \\
\equiv\ & \quad \{\ \text{definition of } \mathit{wp}\ \} \\
& (x \leqslant 3)[x \backslash x + 1] \\
\equiv\ & x + 1 \leqslant 3 \\
\Leftarrow\ & 0 < x < 2 \quad .
\end{aligned}
$$

- $wp\ (S;T)\ Q = wp\ S\ (wp\ T\ Q)$.
  - Or $wp\ (S;T) = wp\ S \cdot wp\ T$, where $(\cdot)$ denotes function composition.
- $wp\ (\mathbf{if}\ B_0 \to S_0 \mid B_1 \to S_1\ \mathbf{fi})\ Q =$ $(B_0 \Rightarrow wp\ S_0\ Q) \wedge (B_1 \Rightarrow wp\ S_1\ Q) \wedge (B_0 \vee B_1)$.

What does a program *mean*?

- **Denotational semantics**: what a program *is*. Mapping programs to mathematical objects.
- **Operational semantics**: what a program *does*. How one program term transforms to another.
- **Axiomatic semantics**: what a program *guarantees*.

- *Predicate transformer semantics* can be seen as a kind of denotational semantics, and axiomatic semantics.
- The meaning of a program is a *predicate transformer*: give it a post condition Q, it tells us what precondition is sufficient to guarantee Q.
- It is a “goal oriented” semantics that is more suitable for reasoning about and constructing imperative programs.

- wp must satisfy certain conditions.
- **Strictness**: $wp\ S\ False = False$.
- **Monotonicity**: $P \Rightarrow Q$ implies $wp\ S\ P \Rightarrow wp\ S\ Q$.
- **Distributivity over Conjunction**:
  $(wp\ S\ Q_0 \wedge wp\ S\ Q_1) \equiv wp\ S\ (Q_0 \wedge Q_1)$.

- wp must satisfy certain conditions.
- **Strictness**: $wp\ S\ False = False$.
- **Monotonicity**: $P \Rightarrow Q$ implies $wp\ S\ P \Rightarrow wp\ S\ Q$.
- **Distributivity over Conjunction**: $(wp\ S\ Q_0 \wedge wp\ S\ Q_1) \equiv wp\ S\ (Q_0 \wedge Q_1)$.
- One can prove that $(wp\ S\ Q_0 \vee wp\ S\ Q_1) \Rightarrow wp\ S\ (Q_0 \vee Q_1)$.

- *wp* must satisfy certain conditions.
- **Strictness**: $wp\ S\ False = False$.
- **Monotonicity**: $P \Rightarrow Q$ implies $wp\ S\ P \Rightarrow wp\ S\ Q$.
- **Distributivity over Conjunction**:
  $(wp\ S\ Q_0 \wedge wp\ S\ Q_1) \equiv wp\ S\ (Q_0 \wedge Q_1)$.
- One can prove that $(wp\ S\ Q_0 \vee wp\ S\ Q_1) \Rightarrow wp\ S\ (Q_0 \vee Q_1)$.
- $(wp\ S\ Q_0 \vee wp\ S\ Q_1) \equiv wp\ S\ (Q_0 \vee Q_1)$ holds only for *deterministic* programs.