Programming Languages: Imperative Program Construction
Practicals 0: Non-Looping Constructs and Weakest Precondition

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Guarded Command Language Basics

1. Which of the following Hoare triples hold?
   (a) $\{x = 7\} skip \{odd\ x\}$;
   (b) $\{x > 60\} x := x \times 2 \{x > 100\}$;
   (c) $\{x > 40\} x := x \times 2 \{x > 100\}$;
   (d) $\{true\}\mathbf{if}\ x \leqslant y \rightarrow y := y - x \mid x \geqslant y \rightarrow x := x - y\ \mathbf{fi}\{x \geqslant 0 \wedge y \geqslant 0\}$;
   (e) $\{even\ x \wedge even\ y\}\mathbf{if}\ x \leqslant y \rightarrow y := y - x \mid x \geqslant y \rightarrow x := x - y\ \mathbf{fi}\{even\ x \wedge even\ y\}$.

2. Is it always true that $\{True\}\ x := E\ \{x = E\}$? If you think the answer is yes, explain why. If your answer is no, give a counter example.

3. Verify:

$$
\begin{array}{l}
\{x = X \wedge y = Y\} \\
x := x \not\Leftrightarrow y \\
y := x \not\Leftrightarrow y \\
x := x \not\Leftrightarrow y \\
\{x = Y \wedge y = X\}
\end{array}
$$

   where x and y are boolean and $(\not\Leftrightarrow)$ is the "not equal" or "exclusive or" operator. In fact, the code above works for any $(\otimes)$ that satisfies the properties that for all a, b, and c:

$$
\begin{array}{ll}
\text{associative}: & a \otimes (b \otimes c) = (a \otimes b) \otimes c\ , \\
\text{unipotent}: & a \otimes a = 1\ ,
\end{array}
$$

   where 1 is the unit of $(\otimes)$, that is, $1 \otimes b = b = b \otimes 1$.

4. Verify the following program:

$$
\begin{array}{l}
\mathbf{var}\ r, b : Int \\
\{0 \leqslant r < 2 \times b\} \\
\mathbf{if}\ b \leqslant r \rightarrow r := r - b \\
\mid\ r < b \rightarrow skip \\
\mathbf{fi} \\
\{0 \leqslant r < b\}
\end{array}
$$

5. Verify:

$$
\begin{array}{l}
\mathbf{var}\ x, y : \mathit{Int} \\
\{\mathit{True}\} \\
x, y := x \times x, y \times y \\
\mathbf{if}\ x \geqslant y \rightarrow x := x - y \\
\mid\ y \geqslant x \rightarrow y := y - x \\
\mathbf{fi} \\
\{x \geqslant 0 \wedge y \geqslant 0\}\ \ .
\end{array}
$$

6. Verify:

$$
\begin{array}{l}
\mathbf{var}\ a, b : \mathit{Bool} \\
\{\mathit{True}\} \\
\mathbf{if}\ \neg\, a \vee b \rightarrow a := \neg\, a \\
\mid\ a \vee \neg\, b \rightarrow b := \neg\, b \\
\mathbf{fi} \\
\{a \vee b\}\ \ .
\end{array}
$$

7. Assuming that x, y, and z are integers, prove the following
   (a) $\{\mathit{True}\}\ \mathbf{if}\ x \geqslant 1 \rightarrow x := x + 1 \mid x \leqslant 1 \rightarrow x := x - 1\ \mathbf{fi}\ \{x \neq 1\}$.
   (b) $\{\mathit{True}\}\ \mathbf{if}\ x \geqslant y \rightarrow \mathit{skip} \mid y \geqslant x \rightarrow x, y := y, x\ \mathbf{fi}\ \{x \geqslant y\}$.
   (c) $\{x = 0\}\ \mathbf{if}\ \mathit{True} \rightarrow x := 1 \mid \mathit{True} \rightarrow x := -1\ \{x = 1 \vee x = -1\}$.
   (d) $\{A = x \times y + z\}\ \mathbf{if}\ \mathit{even}\ x \rightarrow x, y := x\ /\ 2, y \times 2 \mid \mathit{True} \rightarrow y, z := y - 1, z + x\ \{A = x \times y + z\}$.
   (e) $\{x \times y = 0 \wedge y \leqslant x\}\ \mathbf{if}\ y < 0 \rightarrow y := -y \mid y = 0 \rightarrow x := -1\ \{x < y\}$.

Weakest Precondition of Simple Statements

8. Given below is a list of statements and predicates. What are the weakest precondition for the predicates to be true after the statement?
   (a) $x := x \times 2$, $x > 100$;
   (b) $x := x \times 2$, $\mathit{even}\ x$;
   (c) $x := x \times 2$, $x > 100 \wedge \mathit{even}\ x$;
   (d) $x := x \times 2$, $\mathit{odd}\ x$.
   (e) skip, $\mathit{odd}\ x$.
9. Determine the weakest P that satisfies
   (a) $\{P\}\ x := x + 1; x := x + 1\ \{x \geqslant 0\}$.
   (b) $\{P\}\ x := x + y; y := 2 \times x\ \{y \geqslant 0\}$.
   (c) $\{P\}\ x := y; y := x\ \{x = A \wedge y = B\}$.
   (d) $\{P\}\ x := E; x := E\ \{x = E\}$.
10. What is the weakest P such that the following holds?

$$
\begin{array}{l}
\mathbf{var}\ x : \mathit{Int} \\
\{P\} \\
x := x + 1 \\
\mathbf{if}\ x > 0 \rightarrow x := x + 1 \\
\mid\ x < 0 \rightarrow x := x + 2 \\
\mid\ x = 1 \rightarrow \mathit{skip} \\
\mathbf{fi} \\
\{x \geqslant 1\}\ \ .
\end{array}
$$

11. Two programs S_0 and S_1 are equivalent if, for all Q, $wp\ S_0\ Q = wp\ S_1\ Q$. Show that the two following programs are equivalent.

$$\begin{array}{l} \textbf{if } B_0 \rightarrow S_0 \mid B_1 \rightarrow S_1 \textbf{ fi}; S \\ \textbf{if } B_0 \rightarrow S_0; S \mid B_1 \rightarrow S_1; S \textbf{ fi} \end{array}$$

12. Consider the two programs:

$$\begin{array}{l} \mathsf{IF}_0 = \textbf{if } B_0 \rightarrow S_0 \mid B_1 \rightarrow S_1 \textbf{ fi} \ , \\ \mathsf{IF}_1 = \textbf{if } B_0 \rightarrow S_0 \mid B_1 \wedge \neg B_0 \rightarrow S_1 \textbf{ fi} \ . \end{array}$$

Show that for all Q, $wp\ \mathsf{IF}_0\ Q \Rightarrow wp\ \mathsf{IF}_1\ Q$.

Properties of Weakest Precondition

13. Prove that $(wp\ S\ Q_0 \vee wp\ S\ Q_1) \Rightarrow wp\ S\ (Q_0 \vee Q_1)$.

14. Recall the definition of Hoare triple in terms of wp:

$$\{P\}\, S\, \{Q\} \ = \ P \Rightarrow wp\ S\ Q \ .$$

Prove that

1. $(\{P\}\, S\, \{Q\} \wedge (P_0 \Rightarrow P)) \Rightarrow \{P_0\}\, S\, \{Q\}$.
2. $\{P\}\, S\, \{Q\} \wedge \{P\}\, S\, \{R\} \equiv \{P\}\, S\, \{Q \wedge R\}$.

15. Recall the weakest precondition of **if**:

$$wp\ (\textbf{if } B_0 \rightarrow S_0 \mid B_1 \rightarrow S_1 \textbf{ fi})\ Q \ = \ (B_0 \Rightarrow wp\ S_0\ Q) \wedge (B_1 \Rightarrow wp\ S_1\ Q) \wedge (B_0 \vee B_1) \ .$$

Prove that

$$\begin{array}{l} \{P\}\, \textbf{if } B_0 \rightarrow S_0 \mid B_1 \rightarrow S_1 \textbf{ fi}\, \{Q\} \ \equiv \\ \quad \{P \wedge B_0\}\, S\, \{Q\} \ \wedge \ \{P \wedge B_1\}\, S\, \{Q\} \ \wedge \ (P \Rightarrow (B_0 \vee B_1)) \ . \end{array}$$

Note: having proved so shows that the way we annotate **if** is correct:

$$\begin{array}{l} \{P\} \\ \textbf{if } B_0 \rightarrow \{P \wedge B_0\}\, S_0\, \{Q\} \\ \mid \ B_1 \rightarrow \{P \wedge B_1\}\, S_1\, \{Q\} \\ \textbf{fi} \\ \{Q\} \ . \end{array}$$

16. Recall that $wp\ S\ Q$ stands for "the weakest precondition for program S to terminate in a state satisfying Q". What programs S, if any, satisfy each of the following conditions?

1. $wp\ S\ True = True$.
2. $wp\ S\ True = False$.
3. $wp\ S\ False = True$.
4. $wp\ S\ False = False$.