Programming Languages: Imperative Program Construction
Practicals 9: Array Manipulation

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1. Given a : **array** $[0..10)$ **of** *Int*, compute $wp\ (a[i] := 0)\ (a[2] \neq 0)$.

Solution:

$$
\begin{aligned}
& wp\ (a[i] := 0)\ (a[2] \neq 0) \\
\equiv\ & 0 \leqslant i < 10 \land (a{:}i{\mapsto}0)[2] \neq 0 \\
\equiv\ & \quad \{ \text{ function alteration } \} \\
& 0 \leqslant i < 10 \land (i = 2 \Rightarrow 0 \neq 0) \land (i \neq 2 \Rightarrow a[2] \neq 0) \\
\equiv\ & \quad \{\ 0 \neq 0 \equiv \mathit{False}\ \} \\
& 0 \leqslant i < 10 \land (i = 2 \Rightarrow \mathit{False}) \land (i \neq 2 \Rightarrow a[2] \neq 0) \\
\equiv\ & \quad \{\ P \Rightarrow \mathit{False} \equiv \neg P\ \} \\
& 0 \leqslant i < 10 \land i \neq 2 \land (i \neq 2 \Rightarrow a[2] \neq 0) \\
\equiv\ & \quad \{ \text{ proposition logic } \} \\
& 0 \leqslant i < 10 \land i \neq 2 \land a[2] \neq 0 \quad .
\end{aligned}
$$

2. Given constant $N, Y : \mathit{Int}$ with $0 \leqslant N$, and variables b : **array** $[0..N)$ **of** *Int*, $x, i : \mathit{Int}$,

(a) compute $wp\ (b[i-1] := x + 1)\ \langle \forall j : i \leqslant j < N : b[j] = Y \rangle$.

Solution:

$$
\begin{aligned}
& wp\ (b[i-1] := x+1)\ \langle \forall j : i \leqslant j < N : b[j] = Y \rangle \\
\equiv\ & 0 \leqslant i - 1 < N \land \langle \forall j : i \leqslant j < N : (b{:}i-1{\mapsto}x+1)[j] = Y \rangle \\
\equiv\ & \quad \{ \text{ since } i - 1 < j \text{, function alteration } \} \\
& 1 \leqslant i \leqslant N \land \langle \forall j : i \leqslant j < N : b[j] = Y \rangle \quad .
\end{aligned}
$$

(b) Compute $wp\ (b[i-1] := x + 1; i := i - 1)\ \langle \forall j : i \leqslant j < N : b[j] = Y \rangle$.

Solution:

$$
\begin{aligned}
& wp\ (b[i-1] := x+1; i := i-1)\ \langle \forall j : i \leqslant j < N : b[j] = Y \rangle \\
\equiv\ & wp\ (b[i-1] := x+1)\ \langle \forall j : i - 1 \leqslant j < N : b[j] = Y \rangle \\
\equiv\ & 0 \leqslant i - 1 < N \land \langle \forall j : i - 1 \leqslant j < N : (b{:}i-1{\mapsto}x+1)[j] = Y \rangle \\
\equiv\ & \quad \{ \text{ split off } j = i - 1 \ \} \\
& 1 \leqslant i \leqslant N \land (b{:}i-1{\mapsto}x+1)[i-1] = Y \land {} \\
& \langle \forall j : i \leqslant j < N : (b{:}i-1{\mapsto}x+1)[j] = Y \rangle \\
\equiv\ & \quad \{ \text{ function alteration } \} \\
& 1 \leqslant i \leqslant N \land x + 1 = Y \land \langle \forall j : i \leqslant j < N : b[j] = Y \rangle \quad .
\end{aligned}
$$

3. Derive

$\textbf{con}\ N : Int\ \{1 \leqslant N\}$
$\textbf{con}\ F : \textbf{array}\ [0..N)\ \textbf{of}\ Int$
$\textbf{var}\ \ h : \textbf{array}\ [0..N)\ \textbf{of}\ Int$
running_sum
$\{\langle \forall k : 0 \leqslant k < N : h[k] = \langle \Sigma i : 0 \leqslant i \leqslant k : F[i]\rangle\rangle\}$.

Solution: This problem can be seen as a slightly varied instance of Simple Array Assignment mentioned in the handouts. We could have utilised the results. For practice, however, let's start from the basics.

Let $P\ n \equiv \langle \forall k : 0 \leqslant k < n : h[k] = \langle \Sigma i : 0 \leqslant i \leqslant k : F[i]\rangle\rangle$. Conjecture the following skeleton:

$\textbf{con}\ N : Int\ \{1 \leqslant N\}$
$\textbf{con}\ F : \textbf{array}\ [0..N)\ \textbf{of}\ Int$
$\textbf{var}\ \ h : \textbf{array}\ [0..N)\ \textbf{of}\ Int$
$\textbf{var}\ \ n : Int$
initialise
$\{P\ 1\}$
$n := 1$
$\{P\ n \wedge 1 \leqslant n \leqslant N, bnd : N - n\}$
$\textbf{do}\ n \neq N \rightarrow step$
$\qquad\qquad n := n + 1$
$\textbf{od}$
$\{\langle \forall k : 0 \leqslant k < N : h[k] = \langle \Sigma i : 0 \leqslant i \leqslant k : F[i]\rangle\rangle\}$.

Note that $1 \leqslant N$, and we decided to start the loop with $n = 1$. The *initialise* statement thus has to be $h[0] := F[0]$. (Proof omitted – do it if it is not yet familiar to you!) The reason we start with $n = 1$ will be evident later.

We conjecture that *step* can be performed by a single array assignment $h[I] := E$. We then have to find I and E such that

$$P\ n \wedge 1 \leqslant n < N \Rightarrow (P\ (n+1))[h \backslash (h{:}I{\mapsto}E)]\ .$$

Let us inspect $P\ (n+1)$, assuming $1 \leqslant n < N$:

$$\begin{aligned}
&\langle \forall k : 0 \leqslant k < n+1 : h[k] = \langle \Sigma i : 0 \leqslant i \leqslant k : F[i]\rangle\rangle \\
\equiv&\quad \{ 1 \leqslant n < N, \text{split off } k = n \} \\
&\langle \forall k : 0 \leqslant k < n : h[k] = \langle \Sigma i : 0 \leqslant i \leqslant k : F[i]\rangle\rangle \wedge \\
&h[n] = \langle \Sigma i : 0 \leqslant i \leqslant n : F[i]\rangle\ .
\end{aligned}$$

(One could start with expanding $(P\ (n+1))[h \backslash (h{:}I{\mapsto}E)]$ directly. I find it easier to take it slower.)

Now consider $(P\ (n+1))[h \backslash (h{:}I{\mapsto}E)]$, assuming $P\ n \wedge 1 \leqslant n < N$

$$\begin{aligned}
&\langle \forall k : 0 \leqslant k < n : (h{:}I{\mapsto}E)[k] = \langle \Sigma i : 0 \leqslant i \leqslant k : F[i]\rangle\rangle \wedge \\
&(h{:}I{\mapsto}E)[n] = \langle \Sigma i : 0 \leqslant i \leqslant n : F[i]\rangle \\
\equiv&\quad \{ 1 \leqslant n < N, \text{split off } i = n. \text{ ((*) see the } \textbf{Think} \text{ remark in the end)} \} \\
&\langle \forall k : 0 \leqslant k < n : (h{:}I{\mapsto}E)[k] = \langle \Sigma i : 0 \leqslant i \leqslant k : F[i]\rangle\rangle \wedge \\
&(h{:}I{\mapsto}E)[n] = \langle \Sigma i : 0 \leqslant i \leqslant n-1 : F[i]\rangle + F[n] \\
\equiv&\quad \{ P\ n \} \\
&\langle \forall k : 0 \leqslant k < n : (h{:}I{\mapsto}E)[k] = h[k]\rangle \wedge \\
&(h{:}I{\mapsto}E)[n] = h[n-1] + F[n] \\
\equiv&\quad \{ \text{choose } I = n, E = h[n-1] + F[n] \} \\
&\langle \forall k : 0 \leqslant k < n : h[k] = h[k]\rangle \wedge \\
&h[n-1] + F[n] = h[n-1] + F[n] \\
\equiv&\ True\ .
\end{aligned}$$

> **con** $N : Int$ $\{1 \leqslant N\}$
> **con** $F :$ **array** $[0..N)$ **of** Int
> **var** $h :$ **array** $[0..N)$ **of** Int
> **var** $n : Int$
> $h[0] := F[0]$
> $\{P\ 1\}$
> $n := 1$
> $\{P\ n \wedge 1 \leqslant n \leqslant N, bnd : N - n\}$
> **do** $n \neq N \rightarrow h[n] := h[n-1] + F[n]$
> $\qquad\qquad n := n + 1$
> **od**
> $\{\langle \forall k : 0 \leqslant k < N : h[k] = \langle \Sigma i : 0 \leqslant i \leqslant k : F[i]\rangle\rangle\}$.
>
> In retrospect, we need $1 \leqslant n < N$ to guarantee $def\ (h[n-1] + F[n])$ (that is, both array accesses are within bound). Therefore we have to start the loop with $n = 1$. Fortunately we can do so because $1 \leqslant N$.
>
> **Think**: in the step labelled (*) above, why could we not do the following instead of the splitting?
>
> $$\begin{array}{ll} & \ldots \wedge (h{:}I{\mapsto}E)[n] = \langle \Sigma i : 0 \leqslant i \leqslant n : F[i]\rangle \\ = & \quad \{\ P\ n\ \} \\ & \ldots \wedge (h{:}I{\mapsto}E)[n] = h[n] \ . \end{array}$$
>
> **Practice**: try solving this problem using Simple Array Assignment.

4. Derive

   con $N : Int$ $\{1 \leqslant N\}$
   var f : **array** $[0..N)$ **of** Int
   con $H :$ **array** $[0..N)$ **of** Int
   decompose
   $\{\langle \forall k : 0 \leqslant k < N : H[k] = \langle \Sigma i : 0 \leqslant i \leqslant k : f[i]\rangle\rangle\}$.

> **Solution:** Similar to the previous exercise, we let $P\ n \equiv \langle \forall k : 0 \leqslant k < n : H[k] = \langle \Sigma i : 0 \leqslant i \leqslant k : f[i]\rangle\rangle$, and conjecture the following skeleton:
>
> **con** $N : Int$ $\{1 \leqslant N\}$
> **con** $H :$ **array** $[0..N)$ **of** Int
> **var** f : **array** $[0..N)$ **of** Int
> **var** n : Int
> $f[0] := H[0]$
> $\{P\ 1\}$
> $n := 1$
> $\{P\ n \wedge 1 \leqslant n \leqslant N, bnd : N - n\}$
> **do** $n \neq N \rightarrow$ *step*
> $\qquad\qquad n := n + 1$
> **od**
> $\{\langle \forall k : 0 \leqslant k < N : H[k] = \langle \Sigma i : 0 \leqslant i \leqslant k : f[i]\rangle\rangle\}$.
>
> Conjecture that *step* can be performed by a single array assignment $f[I] := E$. We then have to find I and E such that

$$P\ n \wedge 1 \leqslant n < N \Rightarrow (P\ (n+1))[f \backslash (f : I \mapsto E)] \ .$$

Inspect $P\ (n+1)$, assuming $1 \leqslant n < N$:

$$\begin{array}{ll}
& \langle \forall k : 0 \leqslant k < n+1 : H[k] = \langle \Sigma i : 0 \leqslant i \leqslant k : f[i] \rangle \rangle \\
\equiv & \{ 1 \leqslant n < N, \text{split off } k = n \} \\
& \langle \forall k : 0 \leqslant k < n : H[k] = \langle \Sigma i : 0 \leqslant i \leqslant k : f[i] \rangle \rangle \wedge \\
& H[n] = \langle \Sigma i : 0 \leqslant i \leqslant n : f[i] \rangle \ .
\end{array}$$

Now consider $(P\ (n+1))[f \backslash (f : I \mapsto E)]$, assuming $P\ n \wedge 1 \leqslant n < N$

$$\begin{array}{ll}
& \langle \forall k : 0 \leqslant k < n : H[k] = \langle \Sigma i : 0 \leqslant i \leqslant k : (f : I \mapsto E)[i] \rangle \rangle \wedge \\
& H[n] = \langle \Sigma i : 0 \leqslant i \leqslant n : (f : I \mapsto E)[i] \rangle \\
\equiv & \{ 1 \leqslant n < N, \text{split off } i = n \} \\
& \langle \forall k : 0 \leqslant k < n : H[k] = \langle \Sigma i : 0 \leqslant i \leqslant k : (f : I \mapsto E)[i] \rangle \rangle \wedge \\
& H[n] = \langle \Sigma i : 0 \leqslant i \leqslant n-1 : (f : I \mapsto E)[i] \rangle + (f : I \mapsto E)[n] \\
\equiv & \{ \text{choose } I = n, \text{see below (*)} \} \\
& \langle \forall k : 0 \leqslant k < n : H[k] = \langle \Sigma i : 0 \leqslant i \leqslant k : f[i] \rangle \rangle \wedge \\
& H[n] = \langle \Sigma i : 0 \leqslant i \leqslant n-1 : f[i] \rangle + E \\
\equiv & \{ P\ n \} \\
& \langle \forall k : 0 \leqslant k < n : H[k] = \langle \Sigma i : 0 \leqslant i \leqslant k : f[i] \rangle \rangle \wedge \\
& H[n] = H[n-1] + E \\
\equiv & \{ \text{choose } E = H[n] - H[n-1] \} \\
& \langle \forall k : 0 \leqslant k < n : H[k] = \langle \Sigma i : 0 \leqslant i \leqslant k : f[i] \rangle \rangle \wedge \\
& H[n] = H[n-1] + (H[n] - H[n-1]) \\
\equiv & \{ P\ n \} \\
& \mathit{True} \ .
\end{array}$$

In the step marked (*), since $0 \leqslant i \leqslant n-1$, by choosing $I = n$ both occurrences of $(f : I \mapsto E)[i]$ reduce to $f[i]$. Meanwhile, $(f : I \mapsto E)[n]$ reduces to E.

The derived program is

$$\begin{array}{l}
\textbf{con } N : \mathit{Int}\ \{1 \leqslant N\} \\
\textbf{con } H : \textbf{array } [0..N)\ \textbf{of } \mathit{Int} \\
\textbf{var } f\ : \textbf{array } [0..N)\ \textbf{of } \mathit{Int} \\
\textbf{var } n\ : \mathit{Int} \\
f[0] := H[0] \\
\{P\ 1\} \\
n := 1 \\
\{P\ n \wedge 1 \leqslant n \leqslant N, \mathit{bnd} : N - n\} \\
\textbf{do } n \neq N \rightarrow f[n] := H[n] - H[n-1] \\
\qquad\qquad\quad n := n+1 \\
\textbf{od} \\
\{\langle \forall k : 0 \leqslant k < N : H[k] = \langle \Sigma i : 0 \leqslant i \leqslant k : f[i] \rangle \rangle\} \ .
\end{array}$$

5. Prove:

$$\begin{array}{l}
\{h[0] = 0 \wedge h[1] = 1\} \\
\mathit{swap}\ h\ (h[0])\ (h[1]) \\
\{h[h[1]] = 1\}
\end{array}$$

Note that $h[0] = 0 \wedge h[1] = 1$ implies $h[h[0]] = 0$.