

Programming Languages: Imperative Program Construction

Practicals 3. Quantifications

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1. An integer array $X[0..N)$ is given, where $N \geq 1$. Explain, in words, what each of the following expressions mean.
 1. $b \equiv \langle \forall i : 0 \leq i < N : X[i] \geq 0 \rangle$.
 2. $r = \langle \#k : 0 \leq k < N : \langle \forall i : 0 \leq i < k : X[i] < X[k] \rangle \rangle$.
 3. $r = \langle \uparrow p, q : 0 \leq p \leq q \leq N \wedge \langle \forall i : p \leq i < q : X[i] > 0 \rangle : q - p \rangle$.
 4. $r = \langle \#p, q : 0 \leq p < q < N : X[p] = 0 \wedge X[q] = 1 \rangle$.
 5. $s = \langle \uparrow p, q : 0 \leq p < q < N : X[p] + X[q] \rangle$.
 6. $b \equiv \langle \forall p, q : 0 \leq p \wedge 0 \leq q \wedge p + q = N - 1 : X[p] = X[q] \rangle$.
2. An integer array $X[0..N)$ is given, where $N \geq 1$. Express the following sentences in a formal way:
 1. r is the sum of the elements of X .
 2. X is increasing.
 3. all values of X are distinct.
 4. r is the length of a longest constant segment of X .
 5. r is the maximum of the sums of the segments of X .

Solution:

1. $r = \langle \sum i : 0 \leq i < N : X[i] \rangle$.
2. $\langle \forall i : 0 \leq i < N - 1 : X[i] < X[i + 1] \rangle$.
3. $\langle \forall i, j : 0 \leq i < j < N : X[i] \neq X[j] \rangle$.
4. $r = \langle \uparrow p, q : 0 \leq p \leq q \leq N \wedge \langle \forall i, j : p \leq i < j < q : X[i] = X[j] \rangle : q - p \rangle$.
5. $r = \langle \uparrow p, q : 0 \leq p \leq q \leq N : \langle \sum i : p \leq i < q : X[i] \rangle \rangle$

3. Expand the following textual substitutions. If necessary, change the dummy, according to Dummy Renaming (8.21).
 1. $\langle \star x : 0 \leq x + r < n : x + v \rangle [v \setminus 3]$
 2. $\langle \star x : 0 \leq x + r < n : x + v \rangle [x \setminus 3]$
 3. $\langle \star x : 0 \leq x + r < n : x + v \rangle [n \setminus n + x]$
 4. $\langle \star x : 0 \leq x < r : \langle \star y : 0 \leq y : x + y + n \rangle \rangle [n \setminus x + y]$
 5. $\langle \star x : 0 \leq x < r : \langle \star y : 0 \leq y : x + y + n \rangle \rangle [r \setminus y]$

Solution:

1. $\langle \star x : 0 \leq x + r < n : x + 3 \rangle$
2. $\langle \star x : 0 \leq x + r < n : x + v \rangle$ (it is also okay to answer $\langle \star y : 0 \leq y + r < n : y + v \rangle$).
3. $\langle \star y : 0 \leq y + r < n + x : y + v \rangle$ (it is *not* okay to answer $\langle \star x : 0 \leq x + r < n + x : x + v \rangle$).
4. $\langle \star z : 0 \leq z < r : \langle \star w : 0 \leq w : z + w + x + y \rangle \rangle$
5. $\langle \star x : 0 \leq x < y : \langle \star y : 0 \leq y : x + y + n \rangle \rangle$ (renaming is not necessary in this case).

4. Prove the following theorems. Provided $0 \leq n$,

(a) $\langle \Sigma i : 0 \leq i < n + 1 : b[i] \rangle = b[0] + \langle \Sigma i : 1 \leq i < n + 1 : b[i] \rangle$

Solution:

$$\begin{aligned}
 & \langle \Sigma i : 0 \leq i < n + 1 : b[i] \rangle \\
 = & \{ 0 \leq i < n + 1 \equiv i = 0 \vee 1 \leq i < n + 1 \} \\
 & \langle \Sigma i : i = 0 \vee 1 \leq i < n + 1 : b[i] \rangle \\
 = & \{ \text{range split (8.16), since } i = 0 \wedge 1 \leq i < n + 1 \equiv \text{False} \} \\
 & \langle \Sigma i : i = 0 : b[i] = 0 \rangle + \langle \Sigma i : 1 \leq i < n + 1 : b[i] \rangle \\
 = & \{ \text{one-point rule (8.14)} \} \\
 & b[0] + \langle \Sigma i : 1 \leq i < n + 1 : b[i] \rangle
 \end{aligned}$$

(b) $\langle \exists i : 0 \leq i < n + 1 : b[i] = 0 \rangle = \langle \exists i : 0 \leq i < n : b[i] = 0 \rangle \vee b[n] = 0$

Solution:

$$\begin{aligned}
 & \langle \exists i : 0 \leq i < n + 1 : b[i] = 0 \rangle \\
 = & \{ 0 \leq i < n + 1 \equiv 0 \leq i < n \vee i = n \} \\
 & \langle \exists i : 0 \leq i < n \vee i = n : b[i] = 0 \rangle \\
 = & \{ \text{range split (8.16), since } 0 \leq i < n \wedge i = n \equiv \text{False} \} \\
 & \langle \exists i : 0 \leq i < n : b[i] = 0 \rangle \vee \langle \exists i : i = n : b[i] = 0 \rangle \\
 = & \{ \text{one-point rule (8.14)} \} \\
 & \langle \exists i : 0 \leq i < n : P \rangle \vee b[n] = 0
 \end{aligned}$$

5. Prove that $\langle \forall x : R : P \rangle \equiv P \vee \langle \forall x : \neg R \rangle$, provided $\neg \text{occurs}(x, P)$.

Solution:

$$\begin{aligned}
 & P \vee \langle \forall x : \neg R \rangle \\
 = & \{ \text{distributivity, since } \neg \text{occurs}(x, P) \} \\
 & \langle \forall x : P \vee \neg R \rangle \\
 = & \{ P \vee \neg R \equiv R \Rightarrow P, \text{ trading} \}
 \end{aligned}$$

$$\langle \forall x : R : P \rangle$$

6. Prove the *range weakening* rule: $\langle \forall x : Q \vee R : P \rangle \Rightarrow \langle \forall x : Q : P \rangle$.

Solution:

$$\begin{aligned} & \langle \forall x : Q \vee R : P \rangle \\ = & \{ \text{range split (8.18), since } \wedge \text{ idempotent } \} \\ & \langle \forall x : Q : P \rangle \wedge \langle \forall x : R : P \rangle \\ \Rightarrow & \{ \text{weakening (3.76b)} \} \\ & \langle \forall x : Q : P \rangle \end{aligned}$$

7. Prove the *body weakening* rule: $\langle \forall x : R : P \wedge Q \rangle \Rightarrow \langle \forall x : R : P \rangle$.

Solution:

$$\begin{aligned} & \langle \forall x : R : P \wedge Q \rangle \\ = & \{ \text{distributivity, since } P, Q : \text{Bool} \} \\ & \langle \forall x : R : P \rangle \wedge \langle \forall x : R : Q \rangle \\ \Rightarrow & \{ \text{weakening (3.76b)} \} \\ & \langle \forall x : R : P \rangle \end{aligned}$$