Programming Languages: Imperative Program Construction
Practicals 3. Quantifications

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Autumn Term, 2022

1. An integer array $X[0..N)$ is given, where $N \geqslant 1$. Explain, in words, what each of the following expressions mean.
    1. $b \equiv \langle \forall i : 0 \leqslant i < N : X[i] \geqslant 0 \rangle$.
    2. $r = \langle \# k : 0 \leqslant k < N : \langle \forall i : 0 \leqslant i < k : X[i] < X[k] \rangle \rangle$.
    3. $r = \langle \uparrow p, q : 0 \leqslant p \leqslant q \leqslant N \ \wedge \ \langle \forall i : p \leqslant i < q : X[i] > 0 \rangle : q - p \rangle$.
    4. $r = \langle \# p, q : 0 \leqslant p < q < N : X[p] = 0 \ \wedge \ X[q] = 1 \rangle$.
    5. $s = \langle \uparrow p, q : 0 \leqslant p < q < N : X[p] + X[q] \rangle$.
    6. $b \equiv \langle \forall p, q : 0 \leqslant p \ \wedge \ 0 \leqslant q \ \wedge \ p + q = N - 1 : X[p] = X[q] \rangle$.
2. An integer array $X[0..N)$ is given, where $N \geqslant 1$. Express the following sentences in a formal way:
    1. r is the sum of the elements of X.
    2. X is increasing.
    3. all values of X are distinct.
    4. r is the length of a longest constant segment of X.
    5. r is the maximum of the sums of the segments of X.
3. Expand the following textual substitutions. If necessary, change the dummy, according to Dummy Renaming (8.21).
    1. $\langle \star x : 0 \leqslant x + r < n : x + v \rangle [v \backslash 3]$
    2. $\langle \star x : 0 \leqslant x + r < n : x + v \rangle [x \backslash 3]$
    3. $\langle \star x : 0 \leqslant x + r < n : x + v \rangle [n \backslash n + x]$
    4. $\langle \star x : 0 \leqslant x < r : \langle \star y : 0 \leqslant y : x + y + n \rangle \rangle [n \backslash x + y]$
    5. $\langle \star x : 0 \leqslant x < r : \langle \star y : 0 \leqslant y : x + y + n \rangle \rangle [r \backslash y]$
4. Prove the following theorems. Provided $0 \leqslant n$,
    (a) $\langle \Sigma i : 0 \leqslant i < n + 1 : b[i] \rangle = b[0] + \langle \Sigma i : 1 \leqslant i < n + 1 : b[i] \rangle$
    (b) $\langle \exists i : 0 \leqslant i < n + 1 : b[i] = 0 \rangle = \langle \exists i : 0 \leqslant i < n : b[i] = 0 \rangle \vee b[n] = 0$
5. Prove that $\langle \forall x : R : P \rangle \equiv P \vee \langle \forall x :: \neg R \rangle$, provided $\neg occurs(x, P)$.
6. Prove the *range weakening* rule: $\langle \forall x : Q \vee R : P \rangle \Rightarrow \langle \forall x : Q : P \rangle$.
7. Prove the *body weakening* rule: $\langle \forall x : R : P \wedge Q \rangle \Rightarrow \langle \forall x : R : P \rangle$.