Programming Languages: Imperative Program Construction 4. Hoare Logic and Weakest Precondition: Loop

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1 Loop and loop invariants

Loops

- Repetition takes the form $\mathbf{do}\ B_0 \to S_0 \mid ... \mid Bn \to Sn\ \mathbf{od}$.
- If none of the guards $B_0 \ldots B_n$ evaluate to true, the loop terminates. Otherwise one of the commands is chosen non-deterministically, before the next iteration.
- To annotate a loop (for partial correctness):

$$\begin{array}{l}\{P\}\\ \mathbf{do}\ B_0 \to \{P \wedge B_0\}\ S_0\ \{P\}\\ \mid\ \ B_1 \to \{P \wedge B_1\}\ S_1\ \{P\}\\ \mathbf{od}\\ \{Q, Pf\}\ ,\end{array}$$

- where Pf refers to a proof of $P \wedge \neg B_0 \wedge \neg B_1 \Rightarrow Q$.
- P is called the *loop invariant*. Every loop should be constructed with an invariant in mind!

Linear-Time Exponentiation

$$\begin{array}{l}\mathbf{con}\ N\ \{0 \leqslant N\};\ \mathbf{var}\ x, n : Int\\ \\ x, n := 1, 0\\ \{x = 2^n\}\\ \mathbf{do}\ n \neq N \to\\ \quad \{x = 2^n \wedge n \neq N\}\\ \quad x, n := x + x, n + 1\\ \quad \{x = 2^n, Pf1\}\\ \mathbf{od}\\ \{x = 2^N, Pf2\}\end{array}$$

Pf1:

$$\begin{array}{ll} & (x = 2^n)[x, n \backslash x + x, n + 1]\\ \equiv & x + x = 2^{n+1}\\ \Leftarrow & x = 2^n \wedge n \neq N\end{array}$$

Pf2:

$$\begin{array}{l}x = 2^n \wedge n \leqslant N \wedge \neg(n \neq N)\\ \qquad \Rightarrow x = 2^N\end{array}$$

Greatest Common Divisor

- Known: $gcd(x, x) = x$; $gcd(x, y) = gcd(y, x - y)$ if $x > y$.
- $$\begin{array}{l}\mathbf{con}\ A, B : int\ \{0 < A \wedge 0 < B\}\\ \mathbf{var}\ x, y : int\\ \\ x, y := A, B\\ \{0 < x \wedge 0 < y \wedge gcd(x, y) = gcd(A, B)\}\\ \mathbf{do}\ y < x \to x := x - y\\ \quad \mid\ x < y \to y := y - x\\ \mathbf{od}\\ \{x = gcd(A, B) \wedge y = gcd(A, B)\}\end{array}$$
- $$\begin{array}{ll} & (0 < x \wedge 0 < y \wedge gcd(x, y) = gcd(A, B))[x \backslash x - y]\\ \equiv & 0 < x - y \wedge 0 < y \wedge gcd(x - y, y) = gcd(A, B)\\ \Leftarrow & 0 < x \wedge 0 < y \wedge gcd(x, y) = gcd(A, B) \wedge y < x\end{array}$$

A Weird Equilibrium

- Consider the following program:

$$
\begin{array}{l}
\textbf{var } x, y, z : int \\
\\
\{true, bnd : 3 \times (x \uparrow y \uparrow z) - (x + y + z)\} \\
\textbf{do } x < y \to x := x + 1 \\
\quad \mid \; y < z \to y := y + 1 \\
\quad \mid \; z < x \to z := z + 1 \\
\textbf{od} \\
\{x = y = z\}.
\end{array}
$$

- If it terminates at all, we do have $x = y = z$. But why does it terminate?
  1. $bnd \geqslant 0$, and $bnd = 0$ implies none of the guards are true.
  2. $\{x < y \wedge bnd = t\}\, x := x + 1\, \{bnd < t\}$.

Repetition

To annotate a loop for *total correctness*:

$$
\begin{array}{l}
\{P, bnd : t\} \\
\textbf{do } B_0 \to \{P \wedge B_0\}\, S_0\, \{P\} \\
\mid \quad B_1 \to \{P \wedge B_1\}\, S_1\, \{P\} \\
\textbf{od} \\
\{Q\} \;\; ,
\end{array}
$$

we have got a list of things to prove:

1. $P \wedge \neg B_0 \wedge \neg B_1 \Rightarrow Q$,
2. for all i, $\{P \wedge B_i\}\, S_i\, \{P\}$,
3. $P \wedge (B_0 \vee B_1) \Rightarrow t \geqslant 0$,
4. for all i, $\{P \wedge B_i \wedge t = C\}\, S_i\, \{t < C\}$.

E.g. Linear-Time Exponentiation

- What is the bound function?

$$
\begin{array}{l}
\textbf{con } N\ \{0 \leqslant N\};\ \textbf{var } x, n : Int \\
\\
x, n := 1, 0 \\
\{x = 2^n \wedge n \leqslant N, bnd : N - n\} \\
\textbf{do } n \neq N \to \\
\qquad x, n := x + x, n + 1 \\
\textbf{od} \\
\{x = 2^N\} \\
]|
\end{array}
$$

- $x = 2^n \wedge n \leqslant N \wedge n \neq N \Rightarrow N - n \geqslant 0$,
- $\{\ldots \wedge N - n = t\}\, x, n := x + x, n + 1\, \{N - n < t\}$.

E.g. Greatest Common Divisor

- What is the bound function?

$$
\begin{array}{l}
\textbf{con } A, B : Int\ \{0 < A \wedge 0 < B\} \\
\textbf{var } x, y : Int \\
\\
x, y := A, B \\
\{0 < x \wedge 0 < y \wedge gcd(x, y) = gcd(A, B), \\
\quad bnd : x + y\} \\
\textbf{do } y < x \to x := x - y \\
\quad \mid \; x < y \to y := y - x \\
\textbf{od} \\
\{x = gcd(A, B) \wedge y = gcd(A, B)\} \\
]|
\end{array}
$$

- $\ldots \Rightarrow x + y \geqslant 0$,
- $\{\ldots 0 < y \wedge y < x \wedge x + y = t\}\, x := x - y\, \{x + y < t\}$.

2 Weakest Precondition

- What about the weakest precondition?
- Denote the program $\textbf{do } B \to S \textbf{ od}$ by DO. It should behave the same as

$$\textbf{if } B \to S; DO \mid \neg\, B \to skip \textbf{ fi} \;\; .$$

- For any R, if $wp\ DO\ R = X$, it should satisfy

$$X = (B \Rightarrow wp\ S\ X) \wedge (\neg\, B \Rightarrow R) \;\; ,$$

- which is equivalent to

$$X = (B \wedge wp\ S\ X) \vee (\neg\, B \wedge R) \;\; .$$ (Why?)

- We let $wp\ DO\ R$ be the *strongest* X satifying the equation above.

Weakest Precondition for Loop

To be slightly more general,

- denote $\textbf{do } B_0 \to S_0 \mid B_1 \to S_1 \textbf{ od}$ by DO,
- denote $\textbf{if } B_0 \to S_0 \mid B_1 \to S_1 \textbf{ fi}$ by IF, and
- denote $B_0 \vee B_1$ by BB.
- For all R, $wp\ DO\ R$ is the strongest predicate satisfying

$$X \equiv wp\ IF\ X \vee (R \wedge \neg\, BB) \;\; .$$

A Bottom-Up Formulation

- Alternatively, let H_i denote "DO terminates, in at most i iterations, in a state satisfying R."
- $H_0 = R \wedge \neg\, BB$.
- $H_{n+1} = wp\ IF\ (H_n) \vee (R \wedge \neg\, BB)$.
- We may define

$$wp\ DO\ R = \langle \exists i : 0 \leqslant i : H_i \rangle\ .$$

- Theory on *fixed points* shows that the two definitions are equivalent.

Relationship to Hoare Logic

- However, how does $wp\ DO\ R$ relate to the way we annotate loops in the previous section?
- We had a theorem about IF which justified the way to annotate branches:

$$\begin{aligned} wp\ IF\ R = {} & (B_0 \Rightarrow wp\ S_0\ R) \\ & \wedge\ (B_1 \Rightarrow wp\ S_1\ R) \wedge (B_0 \vee B_1)\ . \end{aligned}$$

- Do we have a similar result about loops?

Fundamental Invariance Theorem

Theorem Let $(D, \leqslant)$ be a partially ordered set; let C be a subset of D such that $(C, <)$ is *well-founded*. Let t be a function on the state with value of type D. Then

$$\begin{aligned} & (P \wedge BB \Rightarrow t \in C) \wedge \\ & \langle \forall x :: P \wedge t = x \Rightarrow wp\ IF\ (P \wedge t < x) \rangle \\ & \quad \Rightarrow (P \Rightarrow wp\ DO\ (P \wedge \neg\, BB))\ . \end{aligned}$$

- Informally, $(C, <)$ being *well-founded* means that there is no infinite chain $c1 > c2 > c3 \ldots$ in C.
- The Fundamental Invariance Theorem was proved several times [Dij76, Bac81, Boo82, DvG86, Mor89]. Proving this theorem motivated developments in many related fields.

References

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[Mor89] J. M. Morris. Well-founded induction and the invariance theorem for loops. *Information Processing Letters*, 32(3):155–158, 1989.