Programming Languages: Imperative Program Construction
Practicals 9: Array Manipulation

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1. Given $a : \mathbf{array}\ [0..10)\ \mathbf{of}\ \mathit{Int}$, compute $\mathit{wp}\ (a[i] := 0)\ (a[2] \neq 0)$.

Solution:

$$
\begin{aligned}
& \mathit{wp}\ (a[i] := 0)\ (a[2] \neq 0) \\
\equiv\ & 0 \leqslant i < 10 \land (a : i \mapsto 0)[2] \neq 0 \\
\equiv\ & \quad \{ \text{ function alteration } \} \\
& 0 \leqslant i < 10 \land (i = 2 \Rightarrow 0 \neq 0) \land (i \neq 2 \Rightarrow a[2] \neq 0) \\
\equiv\ & \quad \{ \ 0 \neq 0 \equiv \mathit{False} \ \} \\
& 0 \leqslant i < 10 \land (i = 2 \Rightarrow \mathit{False}) \land (i \neq 2 \Rightarrow a[2] \neq 0) \\
\equiv\ & \quad \{ \ P \Rightarrow \mathit{False} \equiv \neg\, P \ \} \\
& 0 \leqslant i < 10 \land i \neq 2 \land (i \neq 2 \Rightarrow a[2] \neq 0) \\
\equiv\ & \quad \{ \text{ proposition logic } \} \\
& 0 \leqslant i < 10 \land i \neq 2 \land a[2] \neq 0 \ \ .
\end{aligned}
$$

2. Given constant $N, Y : \mathit{Int}$ with $0 \leqslant N$, and variables $b : \mathbf{array}\ [0..N)\ \mathbf{of}\ \mathit{Int}$, $x, i : \mathit{Int}$,

   (a) compute $\mathit{wp}\ (b[i-1] := x + 1)\ \langle \forall j : i \leqslant j < N : b[j] = Y \rangle$.

Solution:

$$
\begin{aligned}
& \mathit{wp}\ (b[i-1] := x + 1)\ \langle \forall j : i \leqslant j < N : b[j] = Y \rangle \\
\equiv\ & 0 \leqslant i - 1 < N \land \langle \forall j : i \leqslant j < N : (b : i - 1 \mapsto x + 1)[j] = Y \rangle \\
\equiv\ & \quad \{ \text{ since } i - 1 < j \text{, function alteration } \} \\
& 1 \leqslant i \leqslant N \land \langle \forall j : i \leqslant j < N : b[j] = Y \rangle \ \ .
\end{aligned}
$$

   (b) Compute $\mathit{wp}\ (b[i-1] := x + 1; i := i - 1)\ \langle \forall j : i \leqslant j < N : b[j] = Y \rangle$.

Solution:

$$
\begin{aligned}
& \mathit{wp}\ (b[i-1] := x + 1; i := i - 1)\ \langle \forall j : i \leqslant j < N : b[j] = Y \rangle \\
\equiv\ & \mathit{wp}\ (b[i-1] := x + 1)\ \langle \forall j : i - 1 \leqslant j < N : b[j] = Y \rangle \\
\equiv\ & 0 \leqslant i - 1 < N \land \langle \forall j : i - 1 \leqslant j < N : (b : i - 1 \mapsto x + 1)[j] = Y \rangle \\
\equiv\ & \quad \{ \text{ split off } j = i - 1 \ \} \\
& 1 \leqslant i \leqslant N \land (b : i - 1 \mapsto x + 1)[i - 1] = Y \ \land \\
& \langle \forall j : i \leqslant j < N : (b : i - 1 \mapsto x + 1)[j] = Y \rangle \\
\equiv\ & \quad \{ \text{ function alteration } \} \\
& 1 \leqslant i \leqslant N \land x + 1 = Y \land \langle \forall j : i \leqslant j < N : b[j] = Y \rangle \ \ .
\end{aligned}
$$

3. Derive

$$\begin{array}{l}
\textbf{con } N : \mathit{Int}\ \{1 \leqslant N\} \\
\textbf{con } F : \textbf{array } [0..N) \textbf{ of } \mathit{Int} \\
\textbf{var } h : \textbf{array } [0..N) \textbf{ of } \mathit{Int} \\
\mathit{running_sum} \\
\{\langle \forall k : 0 \leqslant k < N : h[k] = \langle \Sigma i : 0 \leqslant i \leqslant k : F[i] \rangle \rangle\}\ .
\end{array}$$

Solution: This problem can be seen as a slightly varied instance of Simple Array Assignment mentioned in the handouts. We could have utilised the results. For practice, however, let's start from the basics.

Let $P\ n \equiv \langle \forall k : 0 \leqslant k < n : h[k] = \langle \Sigma i : 0 \leqslant i \leqslant k : F[i] \rangle \rangle$. Conjecture the following skeleton:

$$\begin{array}{l}
\textbf{con } N : \mathit{Int}\ \{1 \leqslant N\} \\
\textbf{con } F : \textbf{array } [0..N) \textbf{ of } \mathit{Int} \\
\textbf{var } h : \textbf{array } [0..N) \textbf{ of } \mathit{Int} \\
\textbf{var } n : \mathit{Int} \\
\mathit{initialise} \\
\{P\ 1\} \\
n := 1 \\
\{P\ n \wedge 1 \leqslant n \leqslant N, \mathit{bnd} : N - n\} \\
\textbf{do } n \neq N \rightarrow \mathit{step} \\
\qquad\qquad n := n + 1 \\
\textbf{od} \\
\{\langle \forall k : 0 \leqslant k < N : h[k] = \langle \Sigma i : 0 \leqslant i \leqslant k : F[i] \rangle \rangle\}\ .
\end{array}$$

Note that $1 \leqslant N$, and we decided to start the loop with $n = 1$. The *initialise* statement thus has to be $h[0] := F[0]$. (Proof omitted – do it if it is not yet familiar to you!) The reason we start with $n = 1$ will be evident later.

We conjecture that *step* can be performed by a single array assignment $h[I] := E$. We then have to find I and E such that

$$P\ n \wedge 1 \leqslant n < N \Rightarrow (P\ (n+1))[h \backslash (h{:}I{\mapsto}E)]\ .$$

Let us inspect $P\ (n+1)$, assuming $1 \leqslant n < N$:

$$\begin{array}{ll}
& \langle \forall k : 0 \leqslant k < n+1 : h[k] = \langle \Sigma i : 0 \leqslant i \leqslant k : F[i] \rangle \rangle \\
\equiv & \{\ 1 \leqslant n < N, \text{ split off } k = n\ \} \\
& \langle \forall k : 0 \leqslant k < n : h[k] = \langle \Sigma i : 0 \leqslant i \leqslant k : F[i] \rangle \rangle \wedge \\
& h[n] = \langle \Sigma i : 0 \leqslant i \leqslant n : F[i] \rangle\ .
\end{array}$$

(One could start with expanding $(P\ (n+1))[h \backslash (h{:}I{\mapsto}E)]$ directly. I find it easier to take it slower.)

Now consider $(P\ (n+1))[h \backslash (h{:}I{\mapsto}E)]$, assuming $P\ n \wedge 1 \leqslant n < N$

$$\begin{array}{ll}
& \langle \forall k : 0 \leqslant k < n : (h{:}I{\mapsto}E)[k] = \langle \Sigma i : 0 \leqslant i \leqslant k : F[i] \rangle \rangle \wedge \\
& (h{:}I{\mapsto}E)[n] = \langle \Sigma i : 0 \leqslant i \leqslant n : F[i] \rangle \\
\equiv & \{\ 1 \leqslant n < N, \text{ split off } i = n.\ ((*) \text{ see the } \textbf{Think} \text{ remark in the end})\ \} \\
& \langle \forall k : 0 \leqslant k < n : (h{:}I{\mapsto}E)[k] = \langle \Sigma i : 0 \leqslant i \leqslant k : F[i] \rangle \rangle \wedge \\
& (h{:}I{\mapsto}E)[n] = \langle \Sigma i : 0 \leqslant i \leqslant n-1 : F[i] \rangle + F[n] \\
\equiv & \{\ P\ n\ \} \\
& \langle \forall k : 0 \leqslant k < n : (h{:}I{\mapsto}E)[k] = h[i] \rangle \wedge \\
& (h{:}I{\mapsto}E)[n] = h[n-1] + F[n] \\
\equiv & \{\ \text{choose } I = n, E = h[n-1] + F[n]\ \} \\
& \langle \forall k : 0 \leqslant k < n : h[k] = h[i] \rangle \wedge \\
& h[n-1] + F[n] = h[n-1] + F[n] \\
\equiv & \{\ P\ n\ \} \\
& \mathit{True}\ .
\end{array}$$

$\mathbf{con}\ N : Int\ \{1 \leqslant N\}$
$\mathbf{con}\ F : \mathbf{array}\ [0..N)\ \mathbf{of}\ Int$
$\mathbf{var}\ h : \mathbf{array}\ [0..N)\ \mathbf{of}\ Int$
$\mathbf{var}\ n : Int$
$h[0] := F[0]$
$\{P\ 1\}$
$n := 1$
$\{P\ n \wedge 1 \leqslant n \leqslant N, bnd : N - n\}$
$\mathbf{do}\ n \neq N \rightarrow h[n] := h[n-1] + F[n]$
$\qquad\qquad n := n + 1$
$\mathbf{od}$
$\{\langle \forall k : 0 \leqslant k < N : h[k] = \langle \Sigma i : 0 \leqslant i \leqslant k : F[i] \rangle \rangle\}$.

In retrospect, we need $1 \leqslant n < N$ to guarantee $def\ (h[n-1] + F[n])$ (that is, both array accesses are within bound). Therefore we have to start the loop with $n = 1$. Fortunately we can do so because $1 \leqslant N$.

Think: in the step labelled (*) above, why could we not do the following instead of the splitting?

$$
\begin{array}{ll}
 & \ldots \wedge (h{:}I{\mapsto}E)[n] = \langle \Sigma i : 0 \leqslant i \leqslant n : F[i] \rangle \\
= & \quad \{\ P\ n\ \} \\
 & \ldots \wedge (h{:}I{\mapsto}E)[n] = h\ n\ .
\end{array}
$$

Practice: try solving this problem using Simple Array Assignment.

4. Derive

$\mathbf{con}\ N : Int\ \{1 \leqslant N\}$
$\mathbf{var}\ f : \mathbf{array}\ [0..N)\ \mathbf{of}\ Int$
$\mathbf{con}\ H : \mathbf{array}\ [0..N)\ \mathbf{of}\ Int$
decompose
$\{\langle \forall k : 0 \leqslant k < N : H[k] = \langle \Sigma i : 0 \leqslant i \leqslant k : f[i] \rangle \rangle\}$.

Solution: Similar to the previous exercise, we let $P\ n \equiv \langle \forall k : 0 \leqslant k < n : H[k] = \langle \Sigma i : 0 \leqslant i \leqslant k : f[i] \rangle \rangle$, and conjecture the following skeleton:

$\mathbf{con}\ N : Int\ \{1 \leqslant N\}$
$\mathbf{con}\ H : \mathbf{array}\ [0..N)\ \mathbf{of}\ Int$
$\mathbf{var}\ f : \mathbf{array}\ [0..N)\ \mathbf{of}\ Int$
$\mathbf{var}\ n : Int$
$f[0] := arrdix\ H\ 0$
$\{P\ 1\}$
$n := 1$
$\{P\ n \wedge 1 \leqslant n \leqslant N, bnd : N - n\}$
$\mathbf{do}\ n \neq N \rightarrow step$
$\qquad\qquad n := n + 1$
$\mathbf{od}$
$\{\langle \forall k : 0 \leqslant k < N : H[k] = \langle \Sigma i : 0 \leqslant i \leqslant k : f[i] \rangle \rangle\}$.

Conjecture that *step* can be performed by a single array assignment $f[I] := E$. We then have to find I and E such that

$$P\ n \wedge 1 \leqslant n < N \Rightarrow (P\ (n+1))[f \backslash (f : I \mapsto E)] \ .$$

Inspect $P\ (n+1)$, assuming $1 \leqslant n < N$:

$$\begin{aligned}
& \langle \forall k : 0 \leqslant k < n+1 : H[k] = \langle \Sigma i : 0 \leqslant i \leqslant k : f[i] \rangle \rangle \\
\equiv \quad & \{\ 1 \leqslant n < N, \text{split off } k = n\ \} \\
& \langle \forall k : 0 \leqslant k < n : H[k] = \langle \Sigma i : 0 \leqslant i \leqslant k : f[i] \rangle \rangle \wedge \\
& H[n] = \langle \Sigma i : 0 \leqslant i \leqslant n : f[i] \rangle \ .
\end{aligned}$$

Now consider $(P\ (n+1))[f \backslash (f : I \mapsto E)]$, assuming $P\ n \wedge 1 \leqslant n < N$

$$\begin{aligned}
& \langle \forall k : 0 \leqslant k < n : H[k] = \langle \Sigma i : 0 \leqslant i \leqslant k : (f : I \mapsto E)[i] \rangle \rangle \wedge \\
& H[n] = \langle \Sigma i : 0 \leqslant i \leqslant n : (f : I \mapsto E)[i] \rangle \\
\equiv \quad & \{\ 1 \leqslant n < N, \text{split off } i = n\ \} \\
& \langle \forall k : 0 \leqslant k < n : H[k] = \langle \Sigma i : 0 \leqslant i \leqslant k : (f : I \mapsto E)[i] \rangle \rangle \wedge \\
& H[n] = \langle \Sigma i : 0 \leqslant i \leqslant n-1 : (f : I \mapsto E)[i] \rangle + (f : I \mapsto E)[n] \\
\equiv \quad & \{\ \text{choose } I = n, \text{see below (*)}\ \} \\
& \langle \forall k : 0 \leqslant k < n : H[k] = \langle \Sigma i : 0 \leqslant i \leqslant k : f[i] \rangle \rangle \wedge \\
& H[n] = \langle \Sigma i : 0 \leqslant i \leqslant n-1 : f[i] \rangle + E \\
\equiv \quad & \{\ P\ n\ \} \\
& \langle \forall k : 0 \leqslant k < n : H[k] = \langle \Sigma i : 0 \leqslant i \leqslant k : f[i] \rangle \rangle \wedge \\
& H[n] = H[n-1] + E \\
\equiv \quad & \{\ \text{choose } E = H[n] - H[n-1]\ \} \\
& \langle \forall k : 0 \leqslant k < n : H[k] = \langle \Sigma i : 0 \leqslant i \leqslant k : f[i] \rangle \rangle \wedge \\
& H[n] = H[n-1] + (H[n] - H[n-1]) \\
\equiv \quad & \{\ P\ n\ \} \\
& \mathit{True} \ .
\end{aligned}$$

In the step marked (*), since $0 \leqslant i \leqslant n-1$, by choosing $I = n$ both occurrences of $(f : I \mapsto E)[i]$ reduce to $f[i]$. Meanwhile, $(f : I \mapsto E)[n]$ reduces to E.

The derived program is

$$\begin{array}{l}
\mathbf{con}\ N : \mathit{Int}\ \{1 \leqslant N\} \\
\mathbf{con}\ H : \mathbf{array}\ [0..N)\ \mathbf{of}\ \mathit{Int} \\
\mathbf{var}\ f\ : \mathbf{array}\ [0..N)\ \mathbf{of}\ \mathit{Int} \\
\mathbf{var}\ n\ : \mathit{Int} \\
f[0] := H[0] \\
\{P\ 1\} \\
n := 1 \\
\{P\ n \wedge 1 \leqslant n \leqslant N, \mathit{bnd} : N - n\} \\
\mathbf{do}\ n \neq N \rightarrow f[n] = H[n] - H[n-1] \\
\qquad\qquad n := n + 1 \\
\mathbf{od} \\
\{\langle \forall k : 0 \leqslant k < N : H[k] = \langle \Sigma i : 0 \leqslant i \leqslant k : f[i] \rangle \rangle\} \ .
\end{array}$$