Programming Languages: Imperative Program Construction
Practicals 3. Quantifications

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1. An integer array $X[0..N)$ is given, where $N \geqslant 1$. Explain, in words, what each of the following expressions mean.

    1. $b \equiv \langle \forall i : 0 \leqslant i < N : X[i] \geqslant 0 \rangle$.
    2. $r = \langle \# k : 0 \leqslant k < N : \langle \forall i : 0 \leqslant i < k : X[i] < X[k] \rangle \rangle$.
    3. $r = \langle \uparrow p, q : 0 \leqslant p \leqslant q \leqslant N \ \wedge \ \langle \forall i : p \leqslant i < q : X[i] > 0 \rangle : q - p \rangle$.
    4. $r = \langle \# p, q : 0 \leqslant p < q < N : X[p] = 0 \ \wedge \ X[q] = 1 \rangle$.
    5. $s = \langle \uparrow p, q : 0 \leqslant p < q < N : X[p] + X[q] \rangle$.
    6. $b \equiv \langle \forall p, q : 0 \leqslant p \ \wedge \ 0 \leqslant q \ \wedge \ p + q = N - 1 : X[p] = X[q] \rangle$.

2. An integer array $X[0..N)$ is given, where $N \geqslant 1$. Express the following sentences in a formal way:

    1. r is the sum of the elements of X.
    2. X is increasing.
    3. all values of X are distinct.
    4. r is the length of a longest constant segment of X.
    5. r is the maximum of the sums of the segments of X.

Solution:

1. $r = \langle \Sigma i : 0 \leqslant i < N : X[i] \rangle$.
2. $\langle \forall i : 0 \leqslant i < N - 1 : X[i] < X[i + 1] \rangle$.
3. $\langle \forall i, j : 0 \leqslant i < j < N : X[i] \neq X[j] \rangle$.
4. $r = \langle \uparrow p, q : 0 \leqslant p \leqslant q \leqslant N \wedge \langle \forall i, j : p \leqslant i < j < q : X[i] = X[j] \rangle : q - p \rangle$.
5. $r = \langle \uparrow p, q : 0 \leqslant p \leqslant q \leqslant N : \langle \Sigma i : p \leqslant i < q : X[i] \rangle \rangle$

3. Expand the following textual substitutions. If necessary, change the dummy, according to Dummy Renaming (8.21).

    1. $\langle \star x : 0 \leqslant x + r < n : x + v \rangle[v \backslash 3]$
    2. $\langle \star x : 0 \leqslant x + r < n : x + v \rangle[x \backslash 3]$
    3. $\langle \star x : 0 \leqslant x + r < n : x + v \rangle[n \backslash n + x]$
    4. $\langle \star x : 0 \leqslant x < r : \langle \star y : 0 \leqslant y : x + y + n \rangle \rangle[n \backslash x + y]$
    5. $\langle \star x : 0 \leqslant x < r : \langle \star y : 0 \leqslant y : x + y + n \rangle \rangle[r \backslash y]$

Solution:

1. $\langle \star x : 0 \leqslant x + r < n : x + 3 \rangle$
2. $\langle \star x : 0 \leqslant x + r < n : x + v \rangle$ (it is also okay to answer $\langle \star y : 0 \leqslant y + r < n : y + v \rangle$).
3. $\langle \star y : 0 \leqslant y + r < n + x : y + v \rangle$ (it is *not* okay to answer $\langle \star x : 0 \leqslant x + r < n + x : x + v \rangle$).
4. $\langle \star z : 0 \leqslant z < r : \langle \star w : 0 \leqslant w : z + w + x + y \rangle \rangle$
5. $\langle \star x : 0 \leqslant x < y : \langle \star y : 0 \leqslant y : x + y + n \rangle \rangle$ (renaming is not necessary in this case).

4. Prove the following theorems. Provided $0 \leqslant n$,

(a) $\langle \Sigma i : 0 \leqslant i < n+1 : b[i] \rangle = b[0] + \langle \Sigma i : 1 \leqslant i < n+1 : b[i] \rangle$

Solution:

$$
\begin{array}{cl}
 & \langle \Sigma i : 0 \leqslant i < n+1 : b[i] \rangle \\
= & \{\ 0 \leqslant i < n+1 \ \equiv\ i = 0 \ \vee\ 1 \leqslant i < n+1 \ \} \\
 & \langle \Sigma i : i = 0 \ \vee\ 1 \leqslant i < n+1 : b[i] \rangle \\
= & \{\ \text{range split (8.16), since } i = 0 \ \wedge\ 1 \leqslant i < n+1 \ \equiv\ \mathit{False} \ \} \\
 & \langle \Sigma i : i = 0 : b[i] = 0 \rangle + \langle \Sigma i : 1 \leqslant i < n+1 : b[i] \rangle \\
= & \{\ \text{one-point rule (8.14)} \ \} \\
 & b[0] + \langle \Sigma i : 1 \leqslant i < n+1 : b[i] \rangle
\end{array}
$$

(b) $\langle \exists i : 0 \leqslant i < n+1 : b[i] = 0 \rangle = \langle \exists i : 0 \leqslant i < n : b[i] = 0 \rangle \vee b[n] = 0$

Solution:

$$
\begin{array}{cl}
 & \langle \exists i : 0 \leqslant i < n+1 : b[i] = 0 \rangle \\
= & \{\ 0 \leqslant i < n+1 \ \equiv\ 0 \leqslant i < n \ \vee\ i = n \ \} \\
 & \langle \exists i : 0 \leqslant i < n \ \vee\ i = n : b[i] = 0 \rangle \\
= & \{\ \text{range split (8.16), since } 0 \leqslant i < n \ \wedge\ i = n \ \equiv\ \mathit{False} \ \} \\
 & \langle \exists i : 0 \leqslant i < n : b[i] = 0 \rangle \vee \langle \exists i : i = n : b[i] = 0 \rangle \\
= & \{\ \text{one-point rule (8.14)} \ \} \\
 & \langle \exists i : 0 \leqslant i < n : P \rangle \vee b[n] = 0
\end{array}
$$

5. Prove that $\langle \forall x : R : P \rangle \equiv P \vee \langle \forall x :: \neg R \rangle$, provided $\neg \mathit{occurs}(x, P)$.

Solution:

$$
\begin{array}{cl}
 & P \vee \langle \forall x :: \neg R \rangle \\
= & \{\ \text{distributivity, since } \neg \mathit{occurs}(x, P) \ \} \\
 & \langle \forall x :: P \vee \neg R \rangle \\
= & \{\ P \vee \neg R \ \equiv\ R \Rightarrow P, \text{trading} \ \}
\end{array}
$$

$$\langle \forall x : R : P \rangle$$

6. Prove the *range weakening* rule: $\langle \forall x : Q \vee R : P \rangle \Rightarrow \langle \forall x : Q : P \rangle$.

Solution:

$$\begin{aligned}
& \langle \forall x : Q \vee R : P \rangle \\
= \quad & \{ \text{ range split (8.18), since } \wedge \text{ idempotent } \} \\
& \langle \forall x : Q : P \rangle \wedge \langle \forall x : R : P \rangle \\
\Rightarrow \quad & \{ \text{ weakening (3.76b) } \} \\
& \langle \forall x : Q : P \rangle
\end{aligned}$$

7. Prove the *body weakening* rule: $\langle \forall x : R : P \wedge Q \rangle \Rightarrow \langle \forall x : R : P \rangle$.

Solution:

$$\begin{aligned}
& \langle \forall x : R : P \wedge Q \rangle \\
= \quad & \{ \text{ distributivity, since } P, Q : \mathit{Bool} \text{ } \} \\
& \langle \forall x : R : P \rangle \wedge \langle \forall x : R : Q \rangle \\
\Rightarrow \quad & \{ \text{ weakening (3.76b) } \} \\
& \langle \forall x : R : P \rangle
\end{aligned}$$