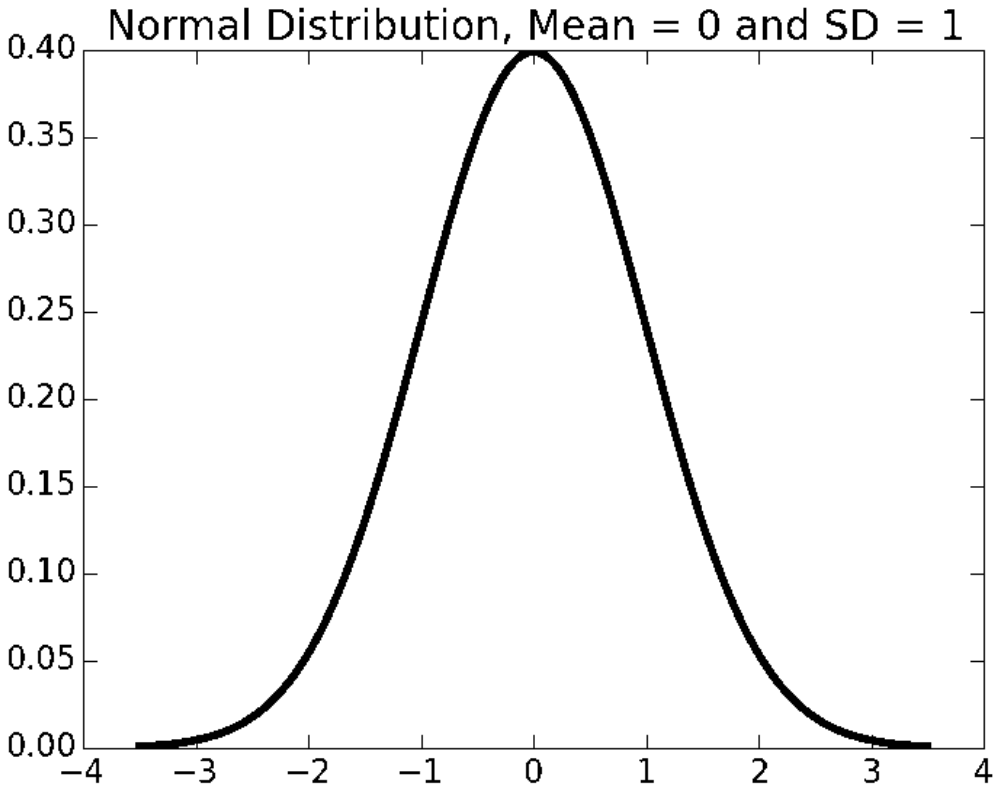
Lecture 7: Confidence Intervals

Assumptions Underlying Empirical Rule

- The mean estimation error is zero
- The distribution of the errors in the estimates is normal (Gaussian)

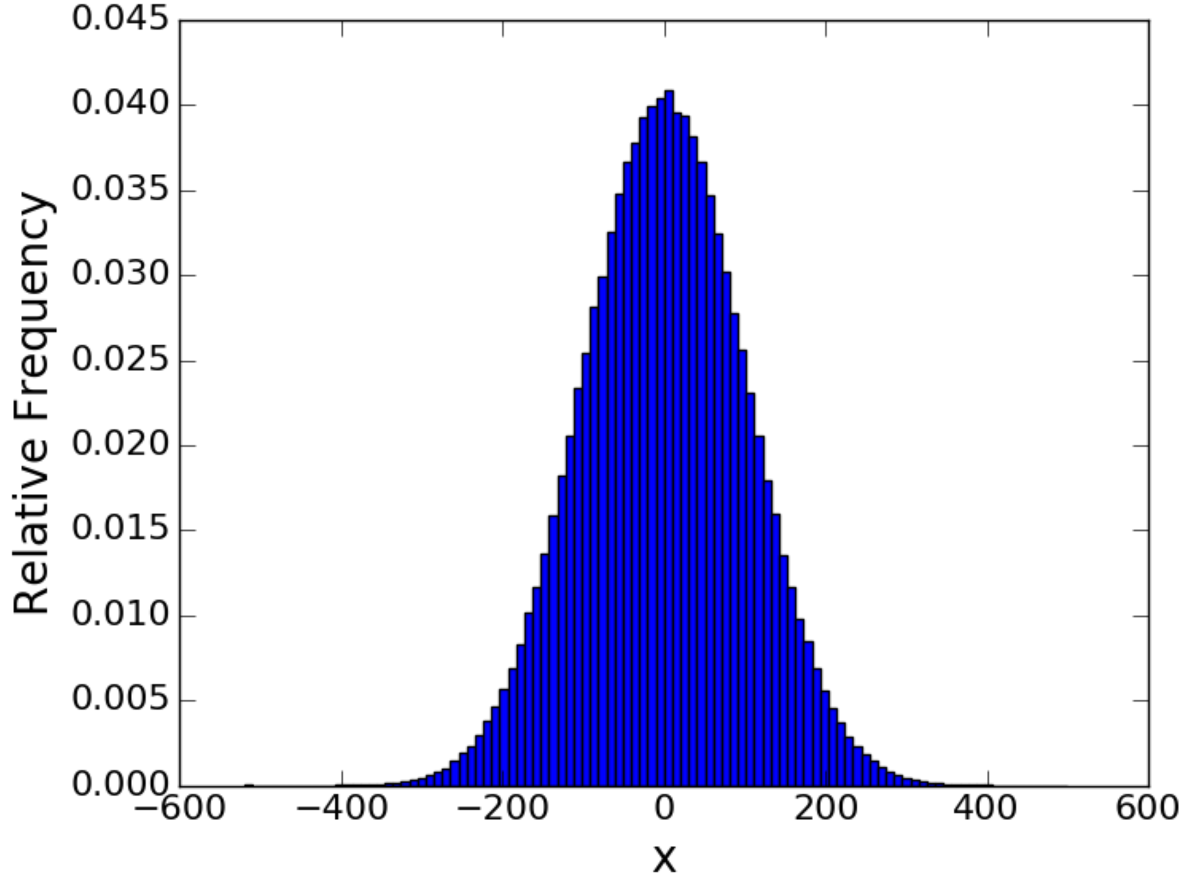


Generating Normally Distributed Data

```
dist, numSamples = [], 1000000

for i in range(numSamples):
    dist.append(random.gauss(0, 100))

weights = [1/numSamples]*len(dist)
v = pylab.hist(dist, bins = 100,
               weights = [1/numSamples]*len(dist))
pylab.xlabel('x')
pylab.ylabel('Relative Frequency')

print('Fraction within ~200 of mean =',
      sum(v[0][30:70]))
```

Output

Discrete Approximation to PDF




```
Fraction within ~200 of mean = 0.957147
```

PDF’s (recapping)

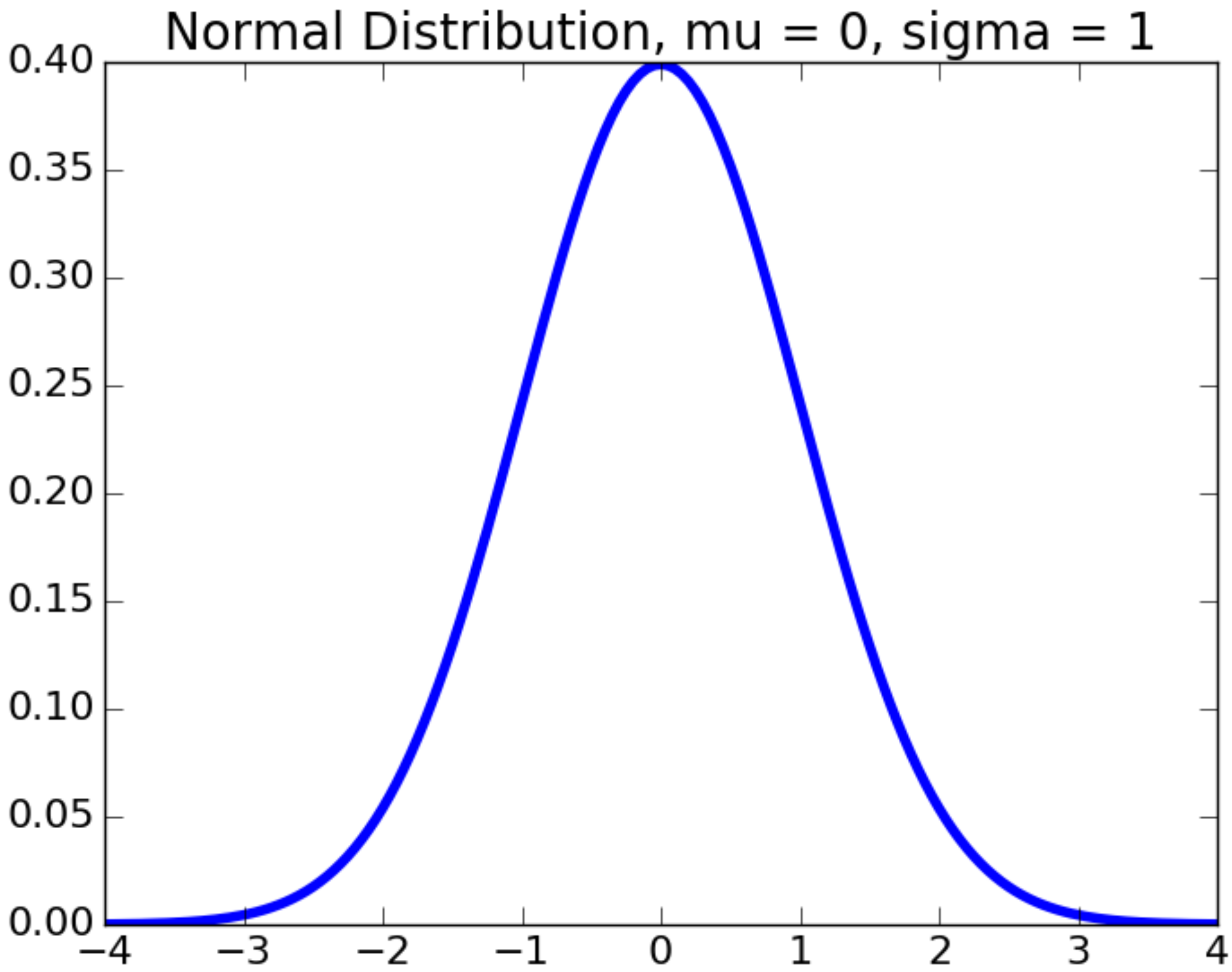
- Distributions defined by *probability density functions* (PDFs)
- Probability of a random variable lying between two values
- Defines a curve where the values on the x-axis lie between minimum and maximum value of the variable
- Area under curve between two points, is probability of example falling within that range

PDF for Normal Distribution

```
def gaussian(x, mu, sigma):
  factor1 = (1.0/(sigma*((2*pylab.pi)**0.5)))
  factor2 = pylab.e**-(((x-mu)**2)/(2*sigma**2))
  return factor1*factor2

xVals, yVals = [], []
mu, sigma = 0, 1
x = -4
while x <= 4:
    xVals.append(x)
    yVals.append(gaussian(x, mu, sigma))
    x += 0.05
pylab.plot(xVals, yVals)
pylab.title('Normal Distribution, mu = ' + str(mu)\
            + ', sigma = ' + str(sigma))
```

$$\boxed{P(x) = \frac{1}{\sigma\sqrt{2\pi}} * e^{-\frac{(x-\mu)^2}{2\sigma^2}}}$$

Output



Are values on y-axis probabilities?

They are densities. I.e., derivative of cumulative distribution function.

Hence we use integration to interpret a PDF

A Digression

- SciPy library contains my useful mathematical functions used by scientists and engineers
- scipy.integrate.quad has up to four arguments
  - a function or method to be integrated
  - a number representing the lower limit of the integration,
  - a number representing the upper limit of the integration, and
  - an optional tuple supplying values for all arguments, except the first, of the function to be integrated
- scipy.integrate.quad returns a tuple
  - Approximation to result
  - Estimate of absolute error

Checking the Empirical Rule

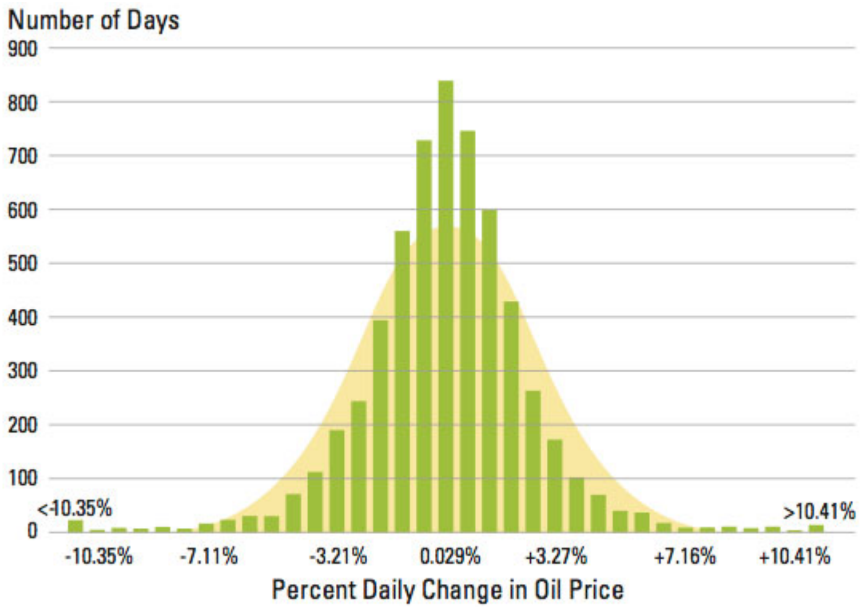
```
import scipy.integrate

def gaussian(x, mu, sigma)
   …
def checkEmpirical(numTrials):
  for t in range(numTrials):
     mu = random.randint(-10, 10)
     sigma = random.randint(1, 10)
     print('For mu =', mu, 'and sigma =', sigma)
     for numStd in (1, 1.96, 3):
        area = scipy.integrate.quad(gaussian,
                                    mu-numStd*sigma,
                                    mu+numStd*sigma,
                                    (mu, sigma))[0]
        print(' Fraction within', numStd,
              'std =', round(area, 4))
```

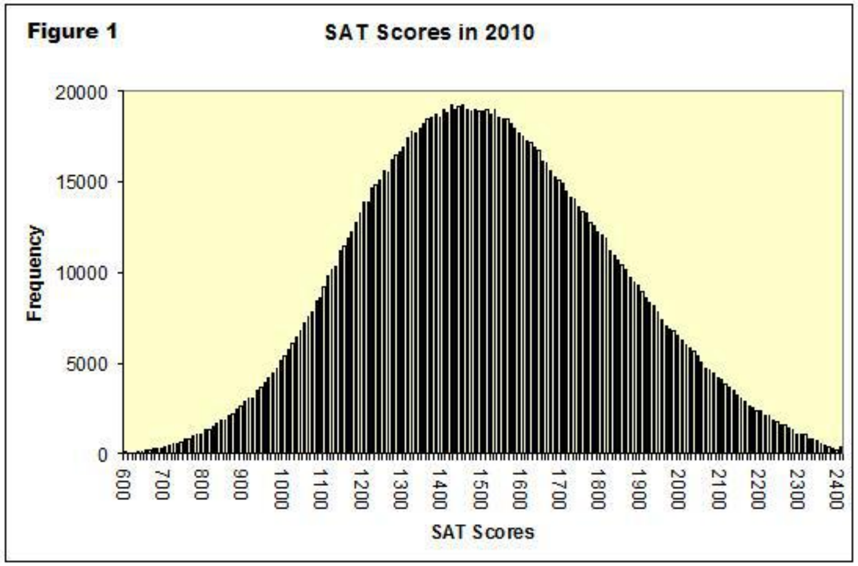
Results

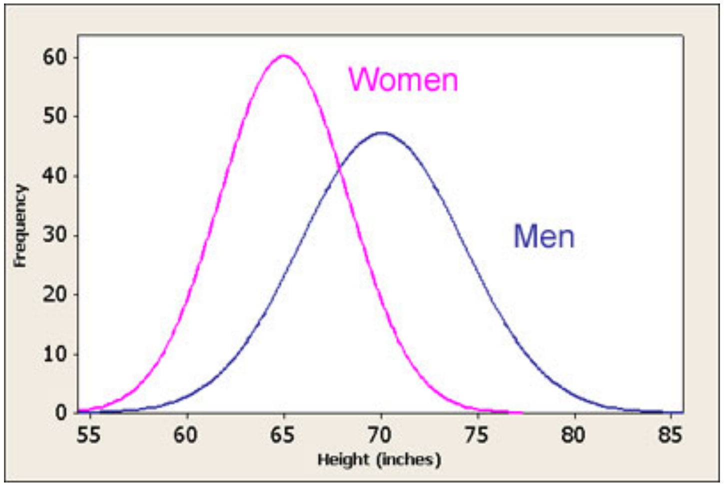
For mu = 9 and sigma = 6
  Fraction within 1 std = 0.6827
  Fraction within 1.96 std = 0.95
  Fraction within 3 std = 0.9973
For mu = -6 and sigma = 5
  Fraction within 1 std = 0.6827
  Fraction within 1.96 std = 0.95
  Fraction within 3 std = 0.9973
For mu = 2 and sigma = 6
  Fraction within 1 std = 0.6827
  Fraction within 1.96 std = 0.95
  Fraction within 3 std = 0.9973

Everybody Likes Normal Distributions

- Occur a lot!
- Nice mathematical properties





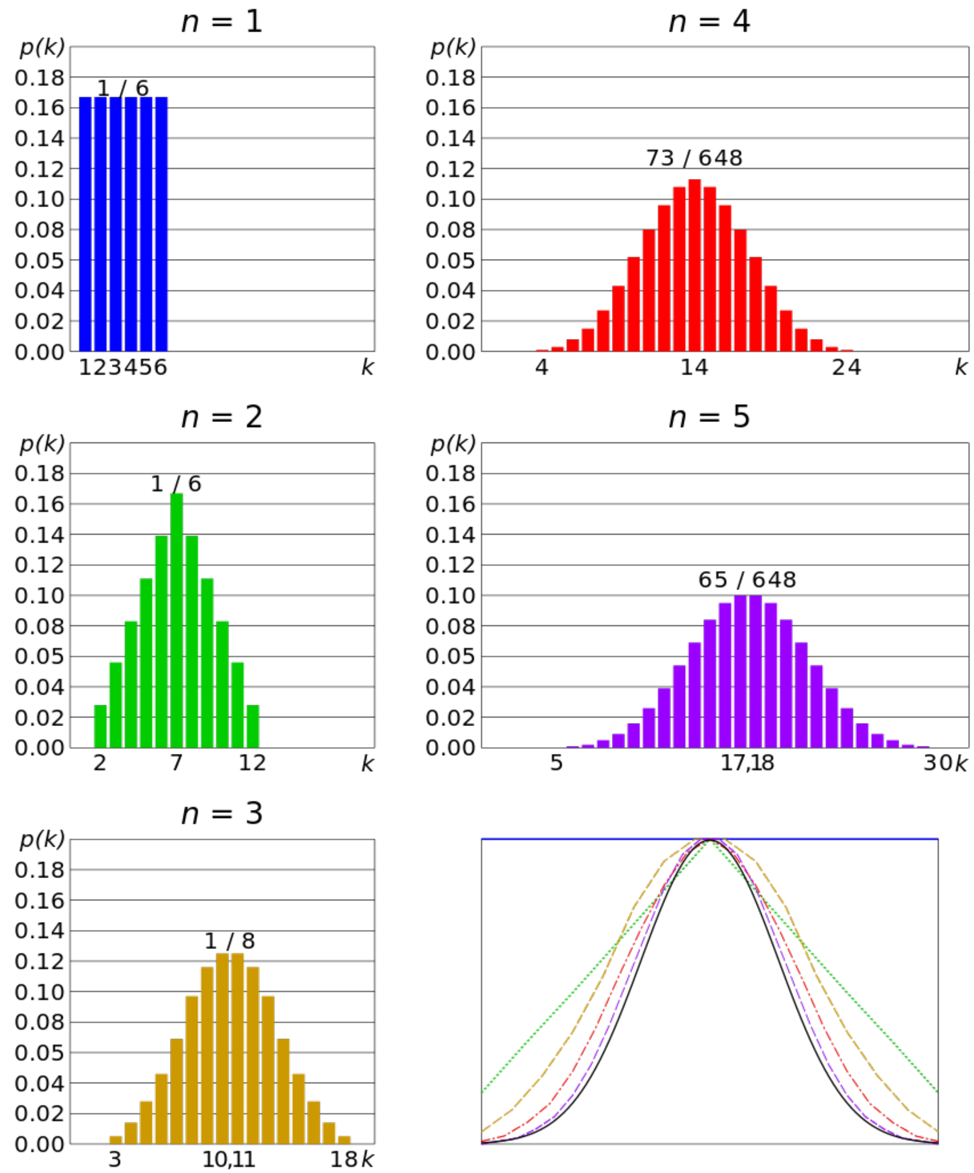


But Not All Distribution Are Normal

- Empirical works for normal distributions
- But are the outcomes of spins of a roulette wheel normally distributed?
- No, they are uniformly distributed
  - Each outcome is equally probable
- So, why does the empirical rule work here?

Why Did the Empirical Rule Work?

- Because we are reasoning not about a single spin, but about the mean of a set of spins
- And the central limit theorem applies



The Central Limit Theorem (CLT)

- Given a sufficiently large sample:

  1) The means of the samples in a set of samples (the sample means) will be approximately normally distributed,

  2) This normal distribution will have a mean close to the mean of the population, and

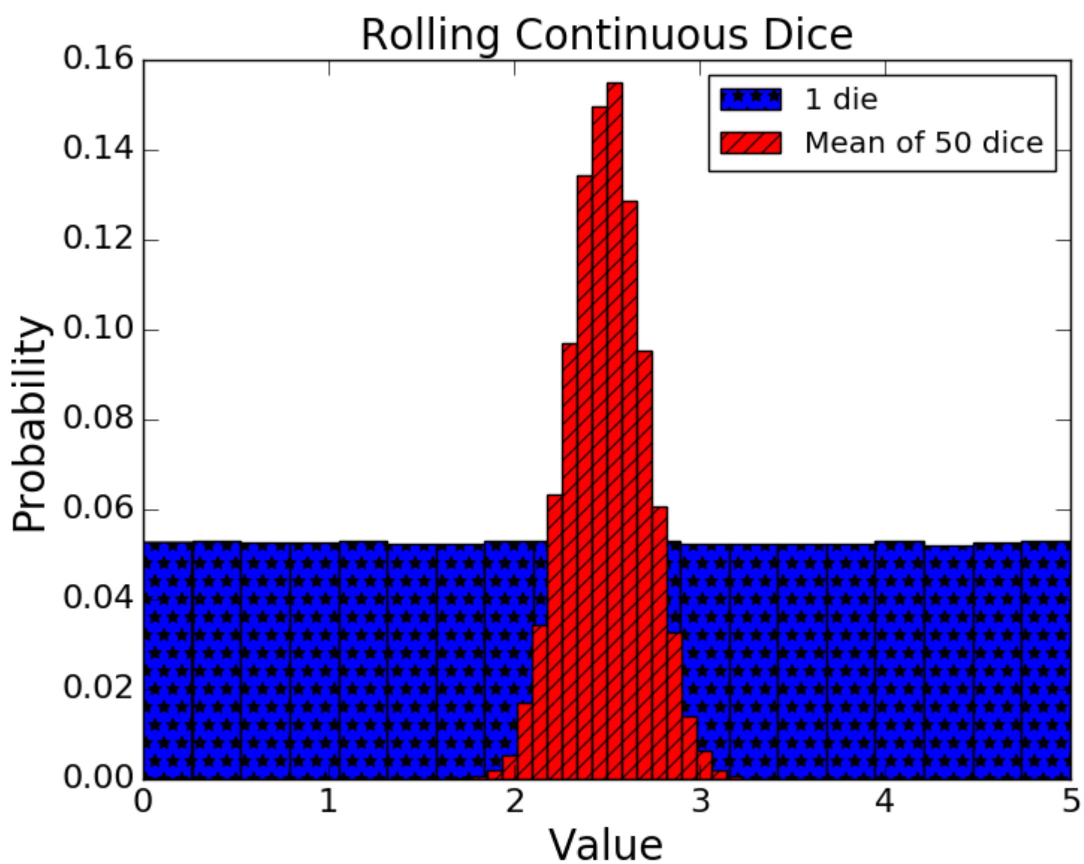
  3) The variance of the sample means will be close to the variance of the population divided by the sample size.

Checking CLT for a Continuous Die

```
def plotMeans(numDice, numRolls, numBins, legend, color, style):
    means = []
    for i in range(numRolls//numDice):
        vals = 0
        for j in range(numDice):
            vals += 5*random.random()
        means.append(vals/float(numDice))
    pylab.hist(means, numBins, color = color, label = legend,
               weights = pylab.array(len(means)*[1])/len(means),
               hatch = style)
    return getMeanAndStd(means)

mean, std = plotMeans(1, 1000000, 19, '1 die', 'b', '*')
print('Mean of rolling 1 die =', str(mean) + ',', 'Std =', std)
mean, std = plotMeans(50, 1000000, 19, 'Mean of 50 dice', 'r', '//')
print('Mean of rolling 50 dice =', str(mean) + ',', 'Std =', std)
pylab.title('Rolling Continuous Dice')
pylab.xlabel('Value')
pylab.ylabel('Probability')
pylab.legend()
```

Output

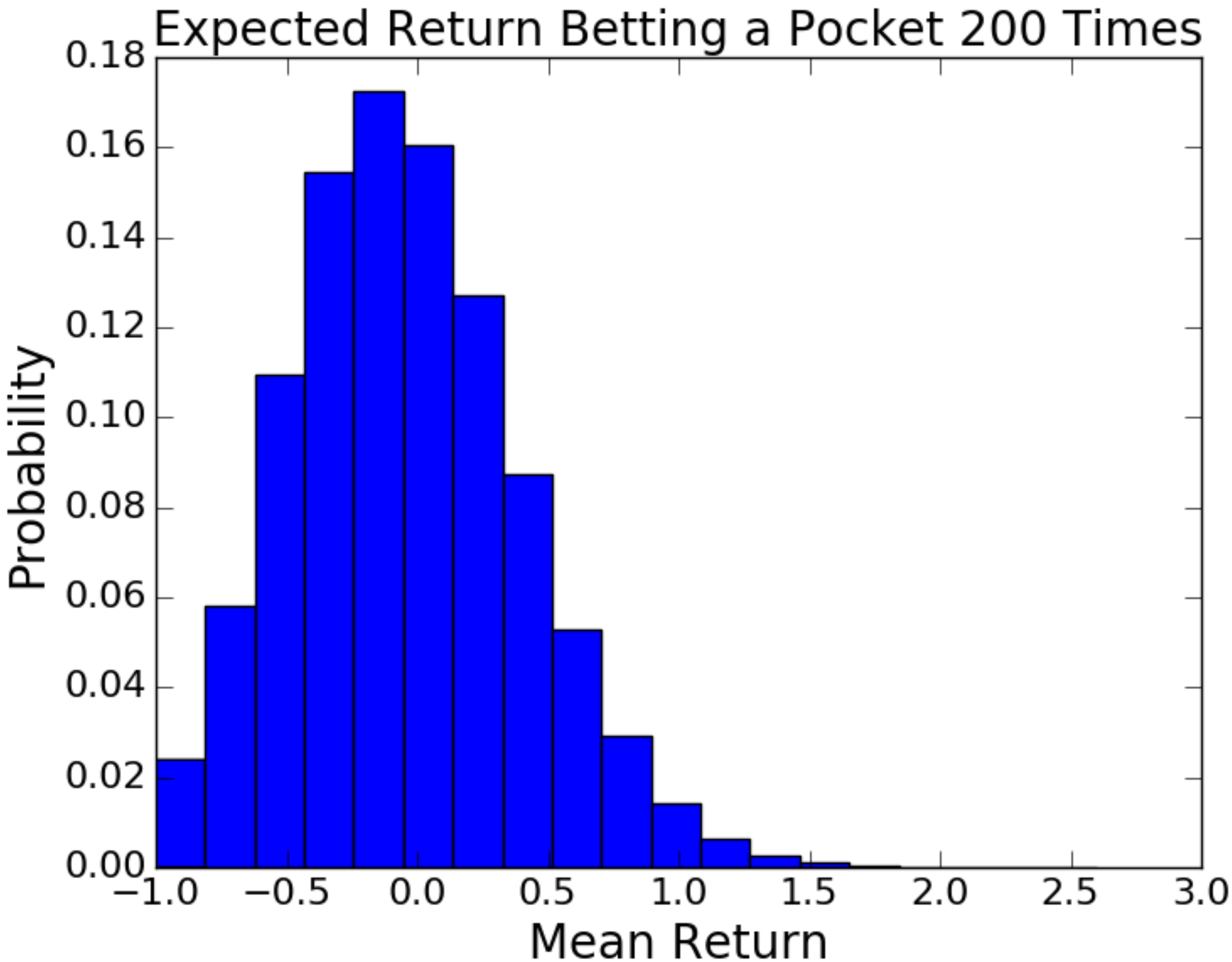
Mean of rolling 1 die = 2.49759575528, Std = 1.4439045633
Mean of rolling 50 dice = 2.49985051798, Std = 0.204887274645



Try It for Roulette

```
numTrials = 1000000
numSpins = 200
game = FairRoulette()

means = []
for i in range(numTrials):
    means.append(findPocketReturn(game, 1, numSpins,
                                  False)[0])

pylab.hist(means, bins = 19,
           weights = [1/len(means)]*len(means))
pylab.xlabel('Mean Return')
pylab.ylabel('Probability')
pylab.title('Expected Return Betting a Pocket 200 Times')
```

Betting a Pocket in Fair Roulette



Moral

- It doesn’t matter what the shape of the distribution of values happens to be
- If we are trying to estimate the mean of a population using sufficiently large samples
- The CLT allows us to use the empirical rule when computing confidence intervals

Finding Pi

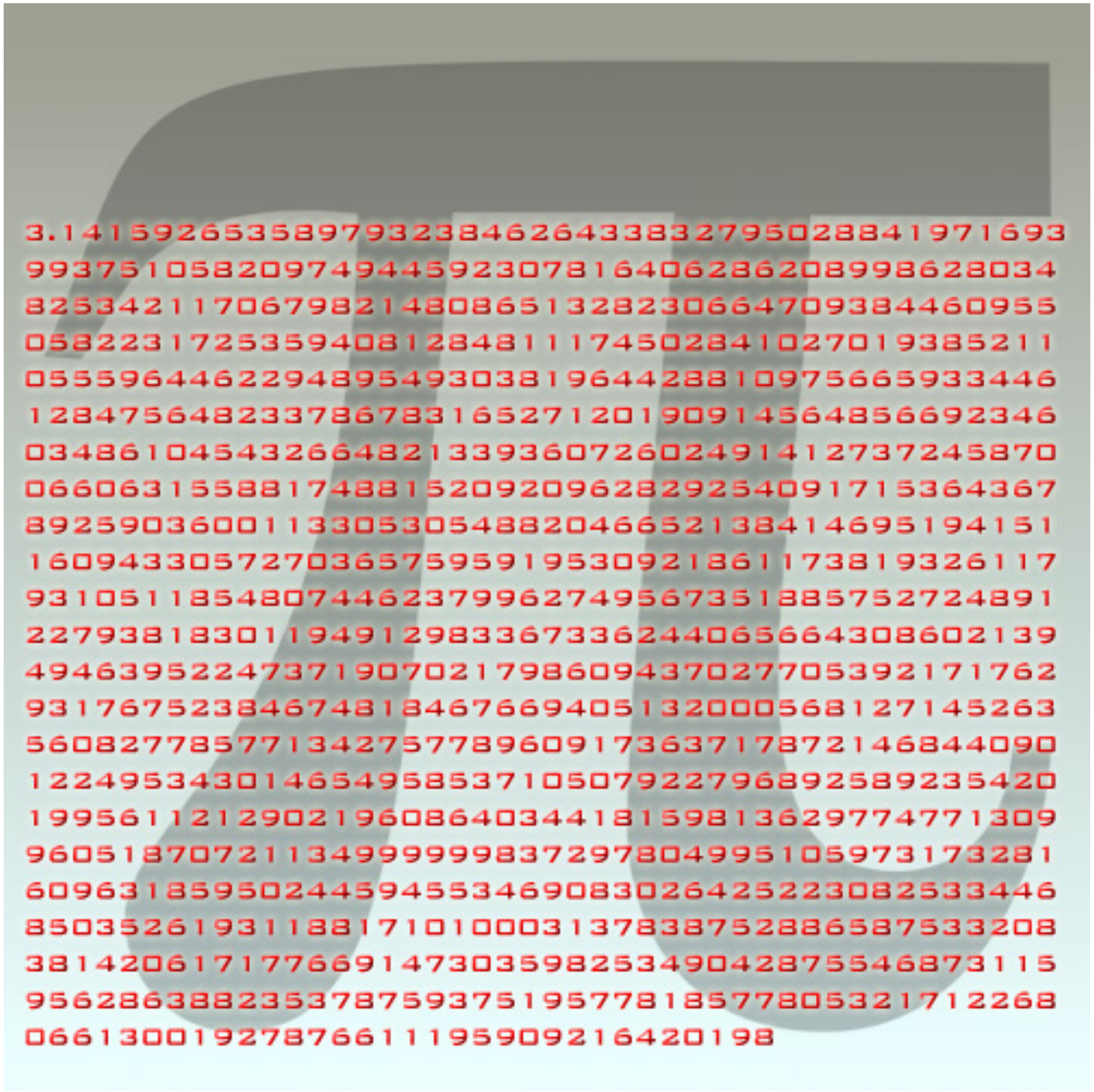
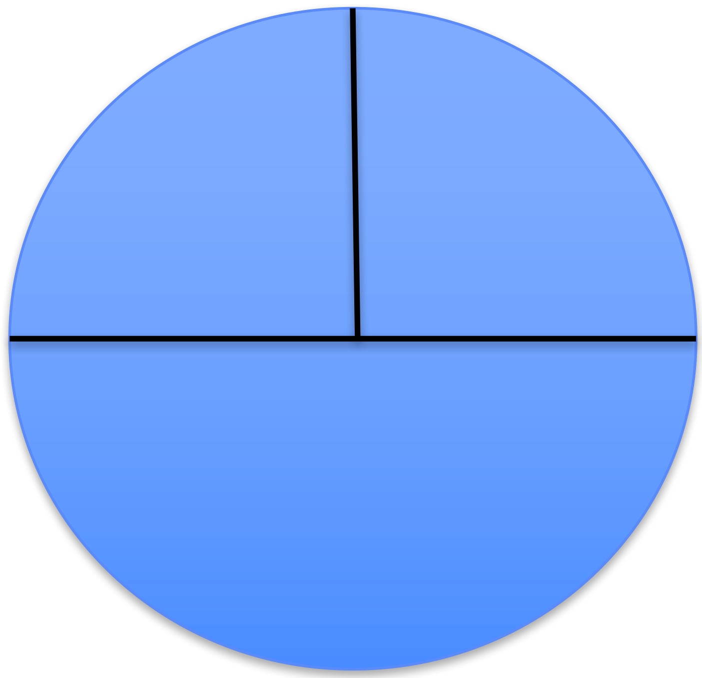


Image from Tom Murphy

Pi



$$\frac{circumference}{diameter} = \Pi \qquad area = \Pi * radius^2$$

Rhind Papyrus

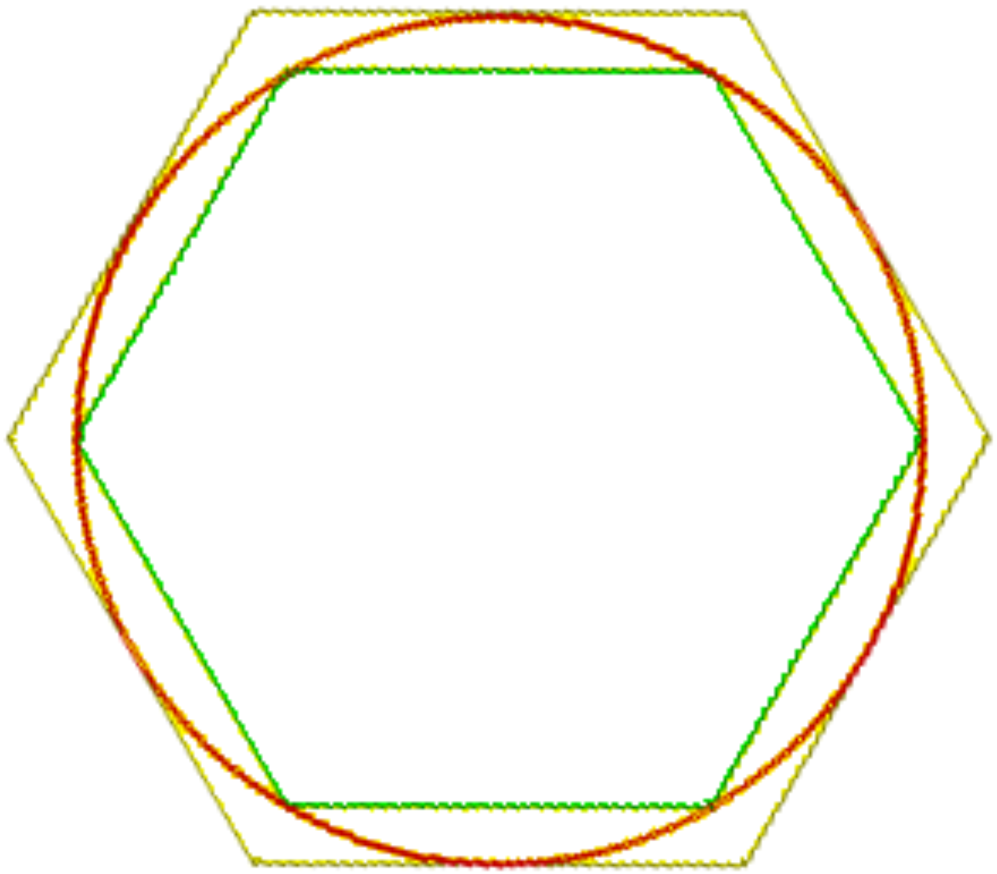
$4*(8/9)^2 = 3.16$

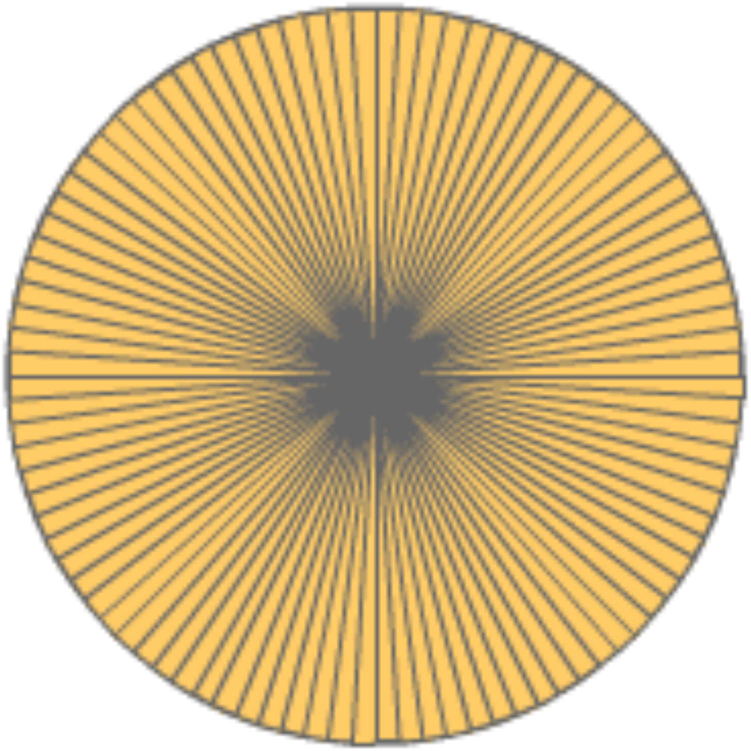
~1100 Years Later

"And he made a molten sea, ten cubits from the one brim to the other: it was round all about, and his height was five cubits: and a line of thirty cubits did compass it round about."

—1 Kings 7.23

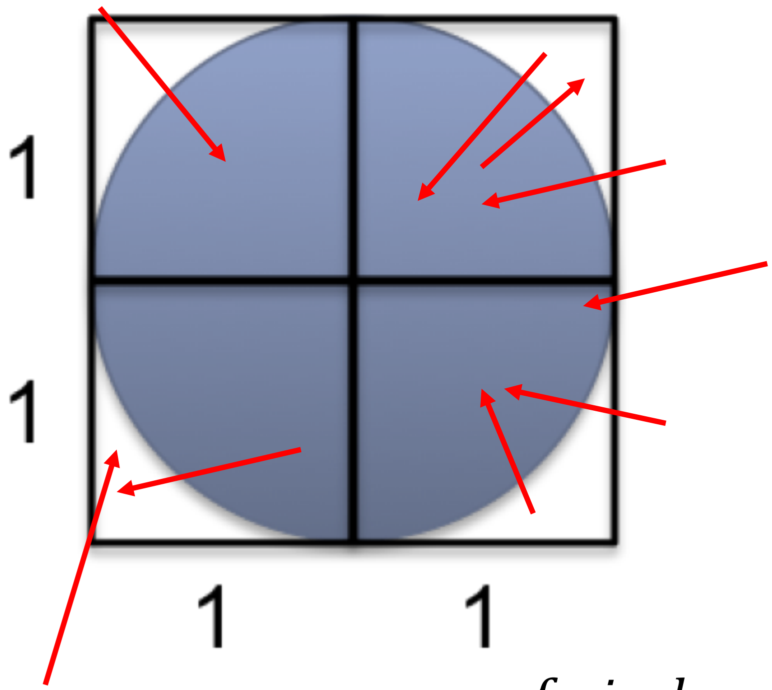
~300 Years Later (Archimedes)





223/71 < pi < 22/7

~2000 Years Later (Buffon-Laplace)



$A_s = 2*2 = 4$
$A_c = \pi r^2 = \pi$

$$\frac{needles\ in\ circle}{needles\ in\ square} = \frac{area\ of\ circle}{area\ of\ square}$$

$$area\ of\ circle = \frac{area\ of\ square * needles\ in\ circle}{needles\ in\ square}$$

$$area\ of\ circle = \frac{4 * needles\ in\ circle}{needles\ in\ square}$$

~200 Years Later

https://www.youtube.com/watch?v=oYM6MIjZ8IY

Very End of Video

Simulating Buffon-Laplace Method

```
def throwNeedles(numNeedles):
    inCircle = 0
    for Needles in range(1, numNeedles + 1, 1):
        x = random.random()
        y = random.random()
        if (x*x + y*y)**0.5 <= 1.0:
            inCircle += 1
    return 4*(inCircle/float(numNeedles))
```

Simulating Buffon-Laplace Method, cont.

```
def getEst(numNeedles, numTrials):
    estimates = []
    for t in range(numTrials):
        piGuess = throwNeedles(numNeedles)
        estimates.append(piGuess)
    sDev = stdDev(estimates)
    curEst = sum(estimates)/len(estimates)
    print('Est. = ' + str(curEst) +\
          ', Std. dev. = ' + str(round(sDev, 6))\
          + ', Needles = ' + str(numNeedles))
    return (curEst, sDev)
```

Simulating Buffon-Laplace Method, cont.

```
def estPi(precision, numTrials):
    numNeedles = 1000
    sDev = precision
    while sDev >= precision/2:
        curEst, sDev = getEst(numNeedles,
                              numTrials)
        numNeedles *= 2
    return curEst

estPi(0.005, 100)
```

Output

```
Est. = 3.1484400000000012, Std. dev. = 0.047886, Needles = 1000
Est. = 3.1391799999999987, Std. dev. = 0.035495, Needles = 2000
Est. = 3.1410799999999997, Std. dev. = 0.02713, Needles = 4000
Est. = 3.141435, Std. dev. = 0.016805, Needles = 8000
Est. = 3.141355, Std. dev. = 0.0137, Needles = 16000
Est. = 3.1413137500000006, Std. dev. = 0.008476, Needles = 32000
Est. = 3.141171874999999, Std. dev. = 0.007028, Needles = 64000
Est. = 3.1415896874999993, Std. dev. = 0.004035, Needles = 128000
Est. = 3.1417414062499995, Std. dev. = 0.003536, Needles = 256000
Est. = 3.14155671875, Std. dev. = 0.002101, Needles = 512000
```

Being Right is Not Good Enough

- Not sufficient to produce a good answer
- Need to have reason to believe that it is close to right
- In this case, small standard deviation implies that we are close to the true value of π

Right?

Is it Correct to State

- 95% of the time we run this simulation, we will estimate that the value of pi is between 3.13743875875 and 3.14567467875?
- With a probability of 0.95 the actual value of π is between 3.13743875875 and 3.14567467875?
- Both are factually correct
- But only one of these statement can be inferred from our simulation
- $statiscally\ valid\ \neq true$

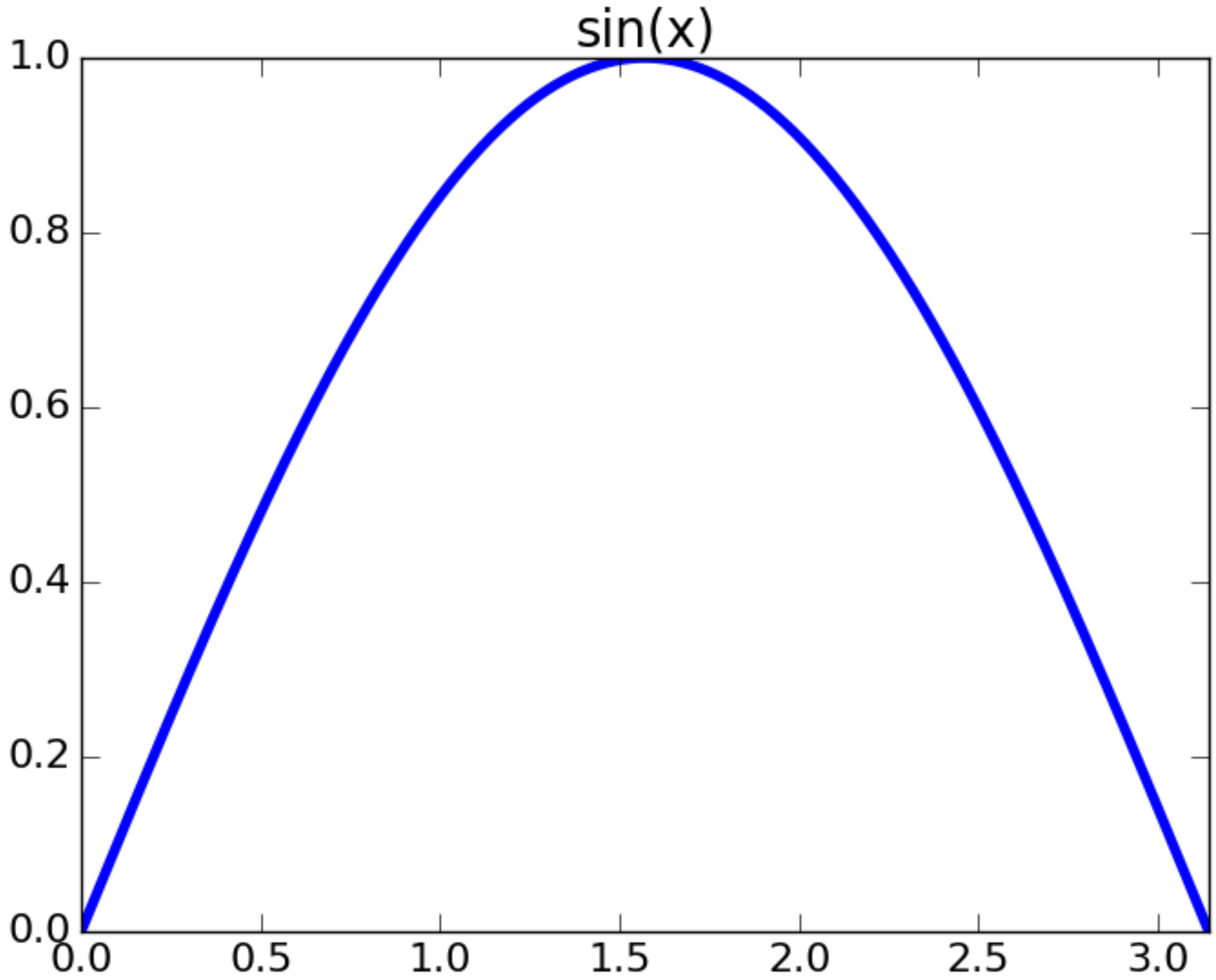
Introduce a Bug

```
def throwNeedles(numNeedles):
    inCircle = 0
    for Needles in range(1, numNeedles + 1, 1):
        x = random.random()
        y = random.random()
        if (x*x + y*y)**0.5 <= 1.0:
            inCircle += 1
    return 2*(inCircle/float(numNeedles))
```

Generally Useful Technique

- To estimate the area of some region, R
  - Pick an enclosing region, E, such that the area of E is easy to calculate and R lies completely within E
  - Pick a set of random points that lie within E
  - Let F be the fraction of the points that fall within R
  - Multiply the area of E by F
- Way to estimate integrals

Sin(x)



Random Points