Quiz 1 Solutions

- The quiz is **closed book**, but you may have one $8.5'' \times 11''$ sheet with notes (either printed or in your own handwriting) on both sides.
- Calculators and electronic devices (including cell phones) are not allowed.
- You may assume all of the results presented in class. This does **not** include results demonstrated in practice quiz material.
- Write your name on each page of the exam
- Please show your work. Partial credit cannot be given for a wrong answer if your work isn't shown.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Please keep your entire answer to a problem on that problem's page.
- Be neat and write legibly. You will be graded not only on the correctness of your answers, but also on the clarity with which you express them.
- If you get stuck on a problem, move on to others. The problems are not arranged in order of difficulty.

NAME: ______________________________

TA: ______________________________

Problem	**Value**	**Score**	**Grader**
1	10		
2	10		
3	10		
4	20		
5	10		
6	10		
7	10		
8	10		
9	10		
Total	100		

Name: ______________________

Problem 1. [10 points]

(a) [4 pts] Use a truth table to prove or disprove $\neg(A \wedge B) \Leftrightarrow (\neg A \vee \neg B)$.

Solution.

A	B	$A \wedge B$	$\neg(A \wedge B)$	$\neg A$	$\neg B$	$\neg A \vee \neg B$
`true`	`true`	`true`	`false`	`false`	`false`	`false`
`true`	`false`	`false`	`true`	`false`	`true`	`true`
`false`	`true`	`false`	`true`	`true`	`false`	`true`
`false`	`false`	`false`	`true`	`true`	`true`	`true`

Comparing the fourth and last colums, we see that the statement is true. *Note:* In fact, this is one of *DeMorgan's Laws.* ∎

(b) Translate the following statements from English into propositional logic or vice versa. You may use Prime(p) to denote that p is prime (i.e., Prime(p) is True if and only if p is a prime number) in the logical statements.

1. [2pts] If $n > 1$, then there is always at least one prime p such that $n < p < 2n$. n is an integer.

   Solution. The domain is $\mathbb{Z}$.

   $$\forall n.(n > 1) \Rightarrow (\exists p.\text{Prime}(p) \wedge (n < p) \wedge (p < 2n))$$

   Note: This is known as *Bertrand's Postulate* ∎

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2. [2pts] The domain is $\mathbb{N}$. $\forall m \ \exists \ p > m.\ \text{Prime}(p) \wedge \text{Prime}(p+2)$

   Solution. There are infinitely many primes p such that $p + 2$ is also prime. *Note:* This is known as the *Twin Prime Conjecture* ■

3. [2pts] Let T be the set of TA's, S be the set of students, and $G(x, y) := x$ grades y's exam

$$\exists \ t \in T \ \forall s \in S.\ G(t, s)$$

   Solution. One TA will grade all of the exams. *Note:* This is not a famous law, theorem, or conjecture, but would be quite impressive. ■

Name: ______________________

Problem 2. [10 points] Prove by induction that $\sum_{k=1}^{n} k \cdot k! = (n+1)! - 1$ for all integers $n \geq 1$

Solution. Hypothesis: $P(n) := \sum_{k=1}^{n} k \cdot k! = (n+1)! - 1$

Base case: $n = 1$.

$$\begin{aligned}\sum_{k=1}^{1} k \cdot k! &= (1+1)! - 1 \\ 1 \cdot 1! &= (2)! - 1 \\ 1 &= 1\end{aligned}$$

$\blacksquare$

Inductive step: Assume $P(n)$ holds. We show $P(n+1) := \sum_{k=1}^{n+1} k \cdot k! = (n+2)! - 1$

$$\begin{aligned}&\sum_{k=1}^{n+1} k \cdot k! \\ =&\sum_{k=1}^{n} k \cdot k! + (n+1) \cdot (n+1)! \\ =&(n+1)! - 1 + (n+1) \cdot (n+1)! \\ =&(n+1)!(1 + (n+1)) - 1 \\ =&(n+1)!(n+2) - 1 \\ =&(n+2)! - 1\end{aligned}$$

Therefore, by induction, $P(n)$ is true for all $n \geq 1$

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Problem 3. [10 points] Consider the matrix below.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix}$$

Suppose we are allowed to flip all of the signs of entries in any row or column. For example, flipping the signs of elements in column two will give the matrix:

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Prove that no matter how many operations you perform, there will always be at least one negative entry in the matrix. Do this by identifying an invariant and explain why you can use this invariant to show that at least one -1 will remain in the matrix.

Solution. Consider the product of all the elements in this matrix. We claim that this is invariant under any of the operations provided. Suppose we flip all the signs of an element in one row. Then if there were previously b -1's in that row, then there are now $4 - b$ -1's in that row and b 1's in that row. However, $4 - b$ and b have the same parity for integer $b = 0, 1, 2, 3, 4$. Hence, the product remains invariant under a row operation. A similar argument can be used for the column operation. Hence as the product is initially -1, the product will always be -1 after any number of row or column operations and so there will always be at least one -1 in the matrix. ■

Name: ______________________

Problem 4. [20 points]

(a) [5 pts] Evaluate 2^{6042} mod 63.

Solution. We notice that $2^6 = 64 \equiv 1 \pmod{63}$. Now $6042 = 6 \cdot 1007$. Hence: $2^{6042} = (2^6)^{1007} \equiv 1 \pmod{63}$. ■

(b) [3 pts] What is 63^{6042} mod 6043? (Hint: 6043 is prime; don't do a messy calculation–it will just waste your time.)

Solution. By Fermat's little theorem, we can conclude that $63^{6042} \equiv 1 \pmod{6043}$. ■

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(c) [6 pts] Give a proof by contradiction that 33 does not have an inverse mod 121.

Solution. Suppose that x is the inverse of 33 mod 121. Then we have that $33x - 1 = 121y$ for some integer y. This means that $11 \mid 1$, which is a contradiction. ■

(d) [6 pts] Find the inverse of 32 mod 121 in the range $\{1, 2, \ldots 120\}$. (Hint: use the Pulverizer)

Solution.

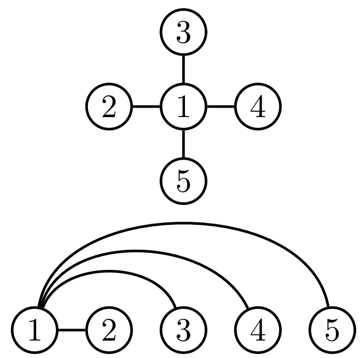
$$\begin{array}{ccrcl}
x & y & \text{rem}\,(x,y) & = & x - q \cdot y \\
\hline
121 & 32 & 25 & = & 121 - 3 \cdot 32 \\
32 & 25 & 7 & = & 32 - 25 \\
 & & & = & 32 - (121 - 3 \cdot 32) \\
 & & & = & 4 \cdot 32 - 121 \\
25 & 7 & 4 & = & 25 - 3 \cdot 7 \\
 & & & = & (121 - 3 \cdot 32) - 3 \cdot (4 \cdot 32 - 121) \\
 & & & = & 4 \cdot 121 - 15 \cdot 32 \\
7 & 4 & 3 & = & 7 - 4 \\
 & & & = & (4 \cdot 32 - 121) - (4 \cdot 121 - 15 \cdot 32) \\
 & & & = & 19 \cdot 32 - 5 \cdot 121 \\
4 & 3 & 1 & = & 4 - 3 \\
 & & & = & (4 \cdot 121 - 15 \cdot 32) - (19 \cdot 32 - 5 \cdot 121) \\
 & & & = & 9 \cdot 121 - 34 \cdot 32 \\
3 & 1 & 0 & &
\end{array}$$

Hence the inverse of 32 mod 121 is just $-34 \equiv 87 \pmod{121}$. ■

Name: ________________________

Problem 5. **[10 points]** A simple graph is said to have width k if you can order the nodes on a straight line so that each node is adjacent to at most k nodes to the left. Each node can be adjacent to any number of nodes to the right.

For example, the star graph with 5 nodes shown below has width 1 as can be seen from the ordering shown below the graph.



Prove by induction that any simple graph with width k can be colored in at most k+1 colors. (Hint: do not induct on k)

Solution. We induct on the number of vertices n.

Base Case: Suppose that a we have a graph with 1 node and width k. Then we only need one color to color our graph and so the graph can be colored in at most $k+1$ colors.

Inductive Hypothesis: Assume that a graph with $n-1$ nodes and width k can be colored in $k+1$ colors.

Inductive Step: Suppose we have a graph with n nodes and width k. We now show this graph can be colored in at most $k+1$ colors. As the graph has width k, we can order the nodes from $1, 2, \dots n$ on a straight line so that each node is adjacent to at most k nodes to the left. Now if a graph has width k, then the graph without one vertex has width k. In particular, the subgraph using vertices $1, 2, \dots n-1$ has width k and so by our inductive hypothesis, it can be colored using at most $k+1$ colors. Now we consider the color of the last vertex n. We know that n is adjacent to at most k nodes in $1, 2, \dots n-1$. Hence, we have that we use at most k colors to color the neighbors of node n in the graph, meaning that we have at least one color left over to use for vertex n. Thus, we can color our graph using at most $k+1$ colors as desired.

As we have proved this for n vertices, by induction, we have that any graph with width k can be colored in $k+1$ colors. ■

Name: ________________________

Problem 6. [10 points] The following questions concern the following preferences.

girls	*boys*
Wendy : Stan, Kenny, Butters, Eric	Stan: Wendy, Bebe, Heidi, Annie
Bebe : Kenny, Eric, Butters, Stan	Kenny : Heidi, Wendy, Bebe, Annie
Heidi : Eric, Butters, Stan, Kenny	Butters : Bebe, Wendy, Heidi, Annie
Annie: Butters, Stan, Eric, Kenny	Eric : Bebe, Heidi, Wendy, Annie

(a) [3 pts] Is the following a stable marriage? If not, list a rogue couple.

Wendy – Stan, Bebe – Eric, Heidi – Butters, Annie – Kenny

Solution. The following is not a stable marriage. Heidi prefers Eric to Butters and Eric prefers Heidi to Bebe. ■

(b) [7 pts] Find the matching produced by the stable matching algorithm.

Solution.
Day 1
Wendy - Stan
Bebe - Eric, Butters
Heidi - Kenny
Annie - no one

Day 2
Wendy - Stan, Butters
Bebe - Eric
Heidi - Kenny
Annie - no one

Day 3
Wendy - Stan
Bebe - Eric
Heidi - Kenny, Butters
Annie - no one

Day 4
Wendy - Stan
Bebe - Eric, Kenny
Heidi - Butters

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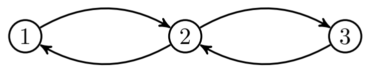
Annie - no one

Day 5 - Stable
Wendy - Stan
Bebe - Kenny
Heidi - Eric
Annie - Butters
■

Name: ______________________

Problem 7. [10 points] Consider the following graph:



Suppose that each node starts with a PageRank value of $\frac{1}{3}$.

(a) [3 pts] What weights will the nodes have after one iteration of the Pagerank algorithm?

Solution. The values will be $\frac{1}{6}, \frac{2}{3}, \frac{1}{6}$ on vertices 1, 2, 3 respectively. ■

(b) [7 pts] What will be the PageRank values of each node for the stationary distribution?

Solution. Suppose μ_1, μ_2, μ_3 are the stationary PageRank values on vertices 1, 2, 3 respectively. Then we must have the following equations:

$$\begin{aligned} \mu_1 &= \frac{\mu_2}{2} \\ \mu_2 &= \mu_1 + \mu_3 \\ \mu_3 &= \frac{\mu_2}{2} \\ \mu_1 + \mu_2 + \mu_3 &= 1 \end{aligned} \tag{1}$$

Hence by solving the system of equations, we have that $\mu_1 = \mu_3 = \frac{1}{4}$ and $\mu_2 = \frac{1}{2}$. ■

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Problem 8. [10 points] Consider the following relation:

$$R = \{(x, y) : x + y = 0, x, y \in \mathbb{R}\}.$$

Which of the following properties holds for R? If it has the property, prove it. If not, provide a counterexample.

(a) [2 pts] Symmetry.

Solution. Yes. $x + y = 0 \Leftrightarrow y + x = 0$ ■

(b) [2 pts] Antisymmetry.

Solution. No. Counterexample: $x = -1, y = 1$ gives us xRy and yRx but $x \neq y$ ■

(c) [2 pts] Reflexivity.

Solution. No. Counterexample: For any $x \neq 0$, $\neg xRx$ e.g. if $x = 1$, $x + x \neq 0$ ■

(d) [2 pts] Transitivity.

Solution. No. Counterexample: Let $x = 1, y = -1$ and $z = 1$. Then xRy and yRz but it is not true that xRz. ■

(e) [2 pts] The property of being an equivalence relation.

Solution. No. It is not reflexive or transitive. ■