Problem Set 1 Solutions

Due: Monday, September 12

Problem 1. [20 points]

(a) [12 pts] Use a truth table to verify that $(P \oplus (P \oplus Q))$ is equivalent to Q.

Solution.

P	$\oplus$	$(P$	$\oplus$	$Q)$
true	true	true	false	true
true	false	true	true	false
false	true	false	true	true
false	false	false	false	false

Compare the (P XOR (P XOR Q)) column with the Q column to see that they are equivalent. ■

(b) [8 pts] Find a predicate $P(x)$ that is a counterexample to the following:

$$\exists x. P(x) \text{ IMPLIES } \forall x. P(x)$$

Solution. Define $P(x) ::=$ "$x = 1$" and let the domain of discourse be the two element set $\{1, 2\}$. Then the left hand side of the implication is true since there is an x, namely $x = 1$, for which $P(x)$ is true. The right hand side is false because it is not true that every element in $\{1, 2\}$ is equal to 1. Therefore the implication does not hold and the statement is not a validity. ■

Problem 2. [20 points] A student is trying to prove that propositions p, q, and r are all true. She proceeds as follows. First, she proves three facts: $p \to q$, $q \to r$, and $r \to p$. Then she concludes, "thus obviously p, q, and r are all true." Let's first formalize her deduction as a logical statement and then evaluate whether or not it is correct.

(a) [6 pts] Using logic notation and the symbols p, q, and r, write down the logical implication that she uses in her final step.

Solution.

$$((p \to q) \land (q \to r) \land (r \to p)) \to (p \land q \land r)$$

■

(b) [8 pts] Use a truth table to determine whether this logical implication is a tautology (i.e., a universal truth in logic).

Solution.

p	q	r	$((p \to q) \wedge (q \to r) \wedge (r \to p))$	$(p \wedge q \wedge r)$	complete expression
T	T	T	T	T	T
T	T	F	F	F	T
T	F	T	F	F	T
T	F	F	F	F	T
F	T	T	F	F	T
F	T	F	F	F	T
F	F	T	F	F	T
F	F	F	T	F	$\to F \leftarrow$

The truth table indicates that the implication she uses is *not* a tautology.

■

(c) [6 pts] Is her proof that propositions p, q, and r are all true correct? Briefly explain.

Solution. Her proof is incorrect; she makes use of a false proposition. (However, the first part of her argument is sufficient to show that either all three statements are true *or* all three statements are false.) ■

Problem 3. [24 points] Translate the following statements into predicate logic. For each, specify the domain. In addition to logic symbols, you may build predicates using arithmetic, relational symbols, and constants. For example, the statement "n is an odd number" could be translated into $\exists m.(2m + 1 = n)$, where the domain is $\mathbb{Z}$, the set of integers. Another example, "p is a prime number," could be translated to

$$(p > 1) \text{ AND NOT } (\exists m.\exists n.(m > 1 \text{ AND } n > 1 \text{ AND } mn = p))$$

Let prime(p) be an abbreviation that you could use to denote the above formula in this problem.

(a) [4 pts] (Lagrange's Four-Square Theorem) Every nonnegative integer is expressible as the sum of four perfect squares.

Solution. The domain is $\mathbb{N}$, the natural numbers.

$$\forall n.\exists w \exists x \exists y \exists z.(n = w^2 + x^2 + y^2 + z^2)$$

■

(b) [4 pts] (Goldbach Conjecture) Every even integer greater than two is the sum of two primes.

Solution. The domain is $\mathbb{N}$, the natural numbers. The statement could be translated to

$$\forall n\ (((n > 2) \text{ AND } \exists m(n = 2m)) \text{ IMPLIES } \exists p \exists q(\textsc{Prime}(p) \text{ AND } \textsc{Prime}(q) \text{ AND } (n = p + q)))$$

■

(c) [4 pts] Every finite integer set has a maximum element.

Solution. The statement could be translated to

$$\forall S \subset \mathbb{Z}\ (\exists n \in \mathbb{N}\ \ (|S| = n)) \text{ IMPLIES } \forall y \in S, \exists x \in S\ \ (y \leq x)$$

■

(d) [4 pts] (Fermat's Last Theorem) There are no nontrivial solutions to the equation:

$$x^n + y^n = z^n$$

over the nonnegative integers when $n > 2$.

Solution. The domain is $\mathbb{N}$.

$$\forall x \forall y \forall z \forall n. ((x > 0 \text{ AND } y > 0 \text{ AND } z > 0 \text{ AND } n > 2) \text{ IMPLIES NOT } (x^n + y^n = z^n))$$

■

(e) [4 pts] There are infinitely many primes.

Solution. The domain is $\mathbb{Z}$.

$$\text{NOT } (\exists p(\textsc{Prime}(p) \text{ AND } (\forall q(\textsc{Prime}(q) \text{ IMPLIES } p \geq q))))$$

■

(f) [4 pts] If integers a and b are coprime, then there exist integers x and y such that $ax + by = 1$.

Solution. The domain is $\mathbb{Z}$.

$$\forall a, b\ \ (\forall d\ \ ((d|a, d|b) \text{ IMPLIES } d = 1)) \text{ IMPLIES } \exists x, y\ \ (ax + by = 1)$$

■

Problem 4. [16 points] Translate the following sentences from English to predicate logic. Let S denote the set of all students and let T denote the set of all TAs. You may use the functions $R(x, y)$, meaning that "x is in y's recitation," $P(x)$, meaning that "x will pass 6.042," $H(x)$, meaning that "x does their homework regularly," and $U(x)$, meaning that "x is an undergraduate."

(a) [4 pts] All the students in at least one TA's recitation will pass 6.042.

Solution.

$$\exists t \in T.\ \forall s \in S.\ R(s,t) \to P(s)$$

■

(b) [4 pts] All the students in 6.042 who do their homework will pass 6.042.

Solution.

$$\forall s \in S.\ H(s) \to P(s)$$

■

(c) [4 pts] Every TA will have a student in their recitation pass 6.042.

Solution.

$$\forall t \in T.\ \exists s \in S.\ R(s,t) \to P(s)$$

■

(d) [4 pts] There are at least three undergraduate TAs.

Solution.

$$\exists a, b, c \in T. a \neq b \wedge b \neq c \wedge a \neq c \wedge U(a) \wedge U(b) \wedge U(c)$$

■

Problem 5. [20 points] Suppose that $w^2 + x^2 + y^2 = z^2$, where w, x, y, and z always denote positive integers. (*Hint:* It may be helpful to represent even integers as $2i$ and odd integers as $2j + 1$, where i and j are integers.)

Prove the proposition: z is even if and only if w, x, and y are even. Do this by considering all the cases of w, x, y being odd or even.

Solution. As the problem suggests, we will build a truth table to figure out what z is in each case of x, y, z being even.

- w, x, y are all even. In this case, we can write w, x, y as $2i, 2j, 2k$, and the sum of squares is $4i^2 + 4j^2 + 4k^2 = 4(i^2 + j^2 + k^2)$. In this case, z can be any even integer when $i^2 + j^2 + k^2 = l^2$ for some l.
- w, x, y are all odd. In this case, each of their squares are odd, and the sum of three odd numbers is odd. However, if z is even, then z^2 is also even. So it cannot be that $z^2 = w^2 + x^2 + y^2$
- One of w, x, y is odd. This case is the same as above. Assume x is odd. The x^2 is odd, but w^2 and y^2 are even. So the sum is once again odd, which cannot equal the square of an even integer. So z cannot be even in that case either. The same argument can be repeated for when w and y are odd as well.

- Two of w, x, y are odd. Assume that w, x are odd and y is even. Then we can write the sum as $w^2 + x^2 + y^2 = (2i+1)^2 + (2j+1)^2 + (2k)^2$. This can be rewritten as $4i^2 + 4i + 1 + 4j^2 + 4j + 1 + 4k^2 = 4(i^2 + i + j^2 + j) + 2$. However if z is even, then z^2 is a multiple of 4. So z cannot be an even integer in this case. The same argument can be used when w, y are odd or when x, z are odd.

w even	x even	y even	z even
T	T	T	T
T	T	F	F
T	F	T	F
T	F	F	F
F	T	T	F
F	T	F	F
F	F	T	F
F	F	F	F

■