## Problem 1

**Theorem 1** (All pairs have different sums)**.** *You are given a subset $S$ of $\{1, 2, ..., 100\}$ with the following property: for every quadruple $u, v, w, x$ of distinct numbers in $S$, the sum of $u$ and $v$ differs from the sum of $w$ and $x$. Must the size of $S$ be at most fifteen? Prove your answer.*

*Proof (long form).*

1. Consider any valid set $S$ in ascending order with $n$ items.
2. By definition, for any quadruple of distinct numbers, $u, v, w, x; u + v \neq w + x$.
3. Rearranging the equation, $u - w \neq x - v$.
4. If we look at the biggest item in $S$, item $n$, it will create $n-1$ positive differences with the other $n - 1$ numbers.
5. By step 3, the positive differences created by looking at the $(n-1)^{\text{th}}$ item must not be the same as those with the $n^{\text{th}}$ item (with the exception of the difference between the $n^{\text{th}}$ and $(n-1)^{\text{th}}$).
6. That is, the second biggest number will create $(n - 2) - 1$ new positive differences.
7. For a set $S$ of $n$ items, the lower bound for positive differences will be

$$(n-1) + (((n-2)-1) + ((n-3)-1) + ((n-4)-1) + \cdots + 1) = \frac{n \cdot (n-1)}{2} - n - 1$$

8. The differences for any $S$ must be a subset of $\{1, 2, ..., 99\}$.
9. That is, at most any set $S$ will contain 99 differences.
10. The greatest $n$ we can plug in to give us a value less than 99 is 15, $\frac{15 \cdot (15-1)}{2} - 15 - 1 = 91$.
11. Therefore, size 15 is the upper bound for any $S$.

$\square$