Download this file (along with assignment-macros-1.tex) as a template for turn-in 6 or 7.

Problem 2

1. For each pair (i, j) in $\langle (1,1), (1,2), (2,1), (2,2), (2,3), (3,2), (3,3) \rangle$, do:

1.1. Match every boy/girl pair where the boy is the girl's ith choice, the girl is the boy's jth choice, and the boy and girl are not yet matched.

2. Return the resulting matching.

While clever, this algorithm fails to take into account any case where a stable matching requires a first choice to be matched with a last choice. Take for example the following boy/girl preferences:

$$\mathbf{B_1} \to G_1, G_2, G_3 \qquad \mathbf{G_1} \to B_1, B_2, B_3$$

$$\mathbf{B_2} \to G_1, G_2, G_3 \qquad \mathbf{G_2} \to B_1, B_2, B_3$$

$$\mathbf{B_3} \to G_3, G_1, G_2 \qquad \mathbf{G_3} \to B_1, B_2, B_3$$

The first run through the algorithm will attempt to match any pairs who are both each other's first choices. We get the match (B_1, G_1).

The next few runs land no hits until the $(2, 2)$ round. Here we'll get the match (B_2, G_2).

Even though we still have an unmatched boy and girl, the algorithm will run all the way through without matching them and return the matching of $\{(B_1, G_1), (B_2, G_2)\}$ which is clearly not stable as not even every boy and girl are matched.

(I know I'm only supposed to be turning in one problem here and I would've just done problem 3, but I wanted some feedback on this one. I found what I believe to be a counterexample, so do I still need a longform proof? Also is this a valid counterexample or am I misunderstanding how their algorithm works? Thanks!)

Problem 3

Lemma 1. *In any execution of the stable-marriage algorithm (from the handout), there is a girl who is not proposed to until the last round.*

Proof (long form).

1. Consider any execution of the algorithm.

1.1. Assume for contradiction that the lemma is false.

1.1.1. That is, every girl has been proposed to and the algorithm continues running.

1.1.2. Once a girl has been proposed to, she remains matched for the remainder of the algorithm.

1.1.3. Since every girl has been proposed to, every girl is matched.

1.1.4. Since the number of boys and girls is the same and each girl can match to no more than 1 boy, every boy is also matched.

1.1.5. Contradiction. By the definition of the algorithm, it will stop once every boy is matched.

1.2. By block 1.1 when every girl has been proposed to, the algorithm ends.

1.3. That is, there is a girl who is not proposed to until the last round.

2. By block 1 there is a girl who is not proposed to until the last round for any run of the algorithm. □

Proof (short form). Take any run of the algorithm. Assume for contradiction you've had proposals to every girl and continue running the algorithm. Once a girl has been proposed to, she will be

matched for the rest of the run. If every girl is matched, every boy is also matched. Contradiction, the algorithm stops running when every boy is matched, so you cannot continue running after every girl has been proposed to. Thus, there is a girl who is not proposed to until the last round for any run of the algorithm. □