

Problem 2

1. For each pair (i, j) in $\langle (1, 1), (1, 2), (2, 1), (1, 3), (3, 1), (2, 2), (2, 3), (3, 2), (3, 3) \rangle$, do:
 - 1.1. Match every boy/girl pair where the boy is the girl's i th choice, the girl is the boy's j th choice, and the boy and girl are not yet matched.
 2. Return the resulting matching.
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While clever, this algorithm fails when those who get there third choices in two $(3, 1), (1, 3)$ couples are each other's second choices. Take for example the following boy/girl preferences:

$$\begin{array}{ll}
 \mathbf{B}_1 \rightarrow G_1, G_3, G_2 & \mathbf{G}_1 \rightarrow B_3, B_2, B_1 \\
 \mathbf{B}_2 \rightarrow G_3, G_1, G_2 & \mathbf{G}_2 \rightarrow B_2, B_1, B_3 \\
 \mathbf{B}_3 \rightarrow G_3, G_1, G_2 & \mathbf{G}_3 \rightarrow B_3, B_1, B_2
 \end{array}$$

The first run through the algorithm will attempt to match any pairs who are both each other's first choices. We get the match (B_3, G_3) .

The next few runs land no hits until the $(1, 3)$ round. Here we'll get the match (B_2, G_2) .

In the next round we look to match $(3, 1)$, so we'll get (B_1, G_1)

Overall we get the matching: $\{(B_1, G_1), (B_2, G_2), (B_3, G_3)\}$. However, there's an unstable pair, (B_2, G_1) , making this whole matching unstable.

Problem 3

Lemma 1. *In any execution of the stable-marriage algorithm (from the handout), there is a girl who is not proposed to until the last round.*

Proof (long form).

1. Consider any execution of the algorithm.
 - 1.1. Assume for contradiction that the lemma is false for this execution.
 - 1.1.1. That is, every girl has been proposed to when we begin the final iteration of the algorithm.
 - 1.1.2. Once a girl has been proposed to, she remains matched for the remainder of the algorithm.
 - 1.1.3. Since every girl has been proposed to, every girl is matched.
 - 1.1.4. Since the number of boys and girls is the same and each girl can match to no more than 1 boy, every boy is also matched.
 - 1.1.5. Contradiction. By the definition of the algorithm, it should've stopped when every boy was matched, but we're about to start the final iteration.
 - 1.2. By block 1.1 the lemma holds for this run of the algorithm.
2. By block 1 there is a girl who is not proposed to until the last round for any run of the algorithm. \square

Proof (short form). Take any run of the algorithm. Assume for contradiction you've had proposals to every girl when you are entering the last iteration. Once a girl has been proposed to, she will be matched for the rest of the run. If every girl is matched, every boy is also matched. Contradiction, the algorithm stops running when every boy is matched, so you cannot continue running after every girl has been proposed to. Thus, there is a girl who is not proposed to until the last round for any run of the algorithm. \square