Problem 1

Theorem 1 (Problem 1)**.** *(Handshakes after the party) $N+1$ couples attend a dinner. At the end of the dinner, oen person (Alice) asks each other person to write down the number of distinct people with whom he or she shook hands at the dinner. Surprisingly, all numbers, $0, 1, 2, ..., 2N$ are written down. Assuming that no person shook hands with their partner, how many people did Alice shake hands with at the dinner?*

Proof (long form).

1. Assume we have a party of $N+1$ couples where indiduals have shaken hands with others but not with their own partner.
2. Call the individuals of one such couple Alice and Bob.
3. Assume that besides Alice, the number of hands shaken by each indivual ($h(i)$ for some individual i) is a unique integer in the range $[0, 2N]$.
4. By 3, there must exist an indidual who's shaken hands with $2N$ other individuals.
5. Assume for contradiction that Bob is the person who's shaken hands with $2N$ individuals

5.1. Besides Alice and Bob, there are $2N$ guests.
5.2. By 1, Bob cannot shake hands with Alice.
5.3. By 5.1 and 5.2, Bob must've shaken hands with everyone but Alice.
5.4. Handshaking is associative, so everyone but Alice has shaken hands with Bob.
5.5. Contradiction by 3, someone (besides Alice) must've shaken hands with no one, but everyone shook hands with Bob.

6. By block 5, Bob did not shake hands with $2N$ people.

Lemma 1. *Given that a couple, x, y where neither x nor y are Alice and $h(x) = 2N$, it is implied $h(y) = 0$.*

Proof of lemma.

a. Assume for contradiction that $h(y) \neq 0$
a.a. since $h(x) = 2N$, x cannot shake hands with y and there are $2N$ people other than x and y at the party, we know x must've shaken hands with everyone but y.
a.b. Since hand shaking is associative, everyone but y has shaken hands with x.
a.c. $h(y) \neq 0$ by a, but no one else has shaken hands with 0 people either, which is a contradiction by 3.
b. Therefore, if $h(x) = 2N$, $h(y)$ must be 0.

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7. By 3 and Lemma 1, there is some couple x, y where $h(x) = 2N, h(y) = 0$.
8. By removing the x, y couple, we are left with couples whose handshake numbers range in: $[1, 2N-1]$
9. If we remove this couple from the party, everyone will have shaken hands with one less person, since x has shaken hands with everyone but y.
10. After all the handshake numbers are decremented we are left with people in the range $[0, 2N-2]$ or $[0, 2(N-1)]$ which is a problem that is smaller than the previous problem by one, but still conforms to 1-3.
11. By this process, we reduced the number of hands Alice had shaken by 1.

12. We can repeat this process N times before the only couple left is Alice and Bob.
13. Thus, Alice shook hands with N people, since we could reduce the problem N times and each time would reduce the number of hands Alice had shaken by 1.

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