Problem 1

Describe a polynomial-time reduction from Vertex Cover to Set Cover.

Given an instance $\langle G = (V, E), k\rangle$ of Vertex Cover, the reduction outputs the instance $\langle U, C, k'\rangle$ of Set Cover, where U, C, and k' are defined as follows:

Let $U = E$ be the set of edges in G;
let C_i be the edges touched by v_i, that is $C_i = \{e_{v_i v_j} | e_{v_i v_j} \in E\}$, unioned with v_i itself;
Let k' be k from the Vertex Cover problem.

Here is why the reduction can be computed in polynomial time:

Constructing C will require at worst a loop over the edge set for each vertex, resulting in $O(VE)$ time, which is polynomial.

Here is a proof that the reduction is correct.

Lemma 1. *Given any instance $\langle G, k\rangle$ of* Vertex Cover*, let $\langle U, C, k\rangle$ be the instance of* Set Cover *produced by the reduction. Then G has a vertex cover of size k if and only if there is a set cover, C' of size k'.*

Proof (long form).

1. First we show the “only if” direction.
2. Assume that G has a vertex cover of size k.
2.1. Let I be the vertex cover of size k in G.
2.2. Take C' to be the set of sets in C corresponding to the vertices in I.
2.3. Then C' has size k'.
2.4. Since every edge, $v_i v_j$ in E, has a vertex in I (that is, either v_i or v_j is in I by definition of vertex cover), all of the edges will be present in the union of the sets of C'.
2.5. That is, the union of the sets of C' will produce a set of all the edges in the graph, which is exactly U.
2.6. So there is a subset of size k of C such that the union over them all will equal U.
3. Next we show the “if” direction.
4. Assume that there are some k' sets of C such that the union over them all will equal U.
4.1. Let C' be the set of k' sets of C.
4.2. Take I to be the vertices corresponding to sets in C'.
4.3. Then I has size k.
4.4. The union of the edges touched by the nodes in I will be U, which is exactly E.
4.5. There is a vertex cover of size $k = k'$ in G.
5. By blocks 2 and 4, G has a vertex cover of size k if and only if there is a set cover, C' of size k'.

□

Problem 2

Describe a polynomial-time reduction from Subset Sum to Zero Path.

Definition of Zero Path: The input is $\langle G, s, t\rangle$, where $G = (V, E)$ is an acyclic directed graph with (possibly negative) integer edge weights, and s and t are vertices. The output is 'yes' if there is a path P from s to t in G such that the total weight of the edges in P is zero, and 'no' otherwise.

Definition of Subset Sum: The input is $\langle x, T\rangle$, where $x = (x_1, x_2, \ldots, x_n)$ is a sequence of n integers, and T is an integer. The output is 'yes' if there is a subset $S \subseteq \{1, 2, \ldots, n\}$ with $\sum_{i\in S} x_i = T$, and 'no' otherwise.

Create two vertices, v_s and $v_{s'}$ with an edge, $v_s v_{s'}$ of weight, $-T$.

Then create a widget for each x_i in x with two vertices, v_i and v_i' and two edges between them, $v_i v_i'$ of weights 0 and x_i.

String together any two widgets, i and j by creating an edge $v_i' v_j$ of weight 0.

Connect the first widget, k with the start by creating the edge $v_{s'} v_k$ of weight 0.

Connect the last widget, l to the the end node, v_t by adding the edge $v_l' v_t$ of weight 0.

Here is why the reduction can be computed in polynomial time:

The reduction as described makes a single pass over the set x to construct the graph, so it'll work in polynomial time.

Here is a proof that the reduction is correct.

Lemma 2. *Given any instance $\langle x, T\rangle$ of* Subset Sum, *let $\langle G = (V, E), s, t\rangle$ be the instance of* Zero Path *produced by the reduction. Then x has a Subset Sum if and only if G has a Zero Path.*

Proof (long form).

1. First we show the "only if" direction.
2. Assume that x has a Subset Sum of T.

2.1. For each $x_i \in x$, if x_i was used in the sum to get T, take the edge in the corresponding widget of weight x_i. Else, take the edge of weight 0.
2.2. Among the widgets traversed, the weight of the path will be T, but the first edge taken, edge $v_s v_{s'}$ was of weight $-T$, so the path weight will be 0.
2.3. So G has a path from v_s to v_t of weight 0.

3. Next we show the "if" direction.
4. Assume that G has a Zero Path from v_s to v_t.

4.1. Let P be such a path.
4.2. P must traverse the edge $v_s v_{s'}$ as it is the only edge out of v_s.
4.3. P must traverse every widget since there is no other way to reach v_t from v_s.
4.4. At every widget, the weight of the path either increased by some x_i, or remained unchanged.
4.5. For the weight of P to be Zero, some set x' of non-zero widget edges must've been traversed that sum to the weight of $v_s v_{s'}$.
4.6. Then x' is the set of widget weight that sum to $v_s v_{s'}$, which is exactly T.
4.7. Further, the non-zero widget weights, $x' \subseteq x$.
4.8. There is a subset, $x' \subseteq x$ whose values sum to T.

5. By blocks 2 and 4, x has a Subset Sum if and only if G has a Zero Path.

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