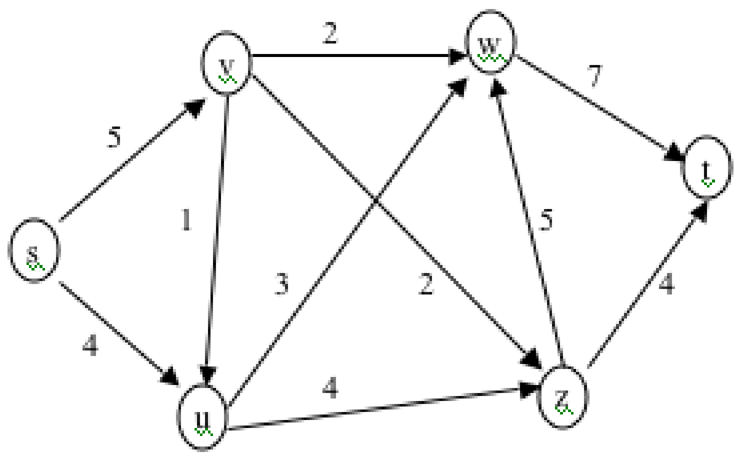
Problem 1



Maximum flows:

We can get a maximum flow of 9 by giving the edges in the graph the following flows:
$f(s, v) = 5$,
$f(s, u) = 4$,
$f(v, w) = 2$,
$f(v, z) = 2$,
$f(v, u) = 1$,
$f(u, w) = 1$,
$f(u, z) = 4$,
$f(z, t) = 4$,
$f(z, w) = 2$,
$f(w, t) = 5$

Minimum cuts:

There are a few minimum cuts for this problem. You could cut the set of edges $\{(s, v), (s, u)\}$ for a cost of 9, or you could cut $\{(s, u), (v, w), (v, u), (v, z)\}$ for a cost of 9.

Critical edges:

Critical edges here are all in the set $\{(s, v), (s, u), (v, w), (v, u), (v, z)\}$

Problem 2

Lemma 2. *An edge e is critical if and only if $f(e) = cap(e)$ in every maximum flow f.*

Proof (long form).

1. Consider any s-t flow network G and edge e.

1.1. First we will prove that, if $f(e) \neq \text{cap}(e)$ in some maximum flow f, then e is not critical.

1.2. Assume that there exists max-flow f such that $f(e) \neq \text{cap}(e)$.

1.2.1. Let $\epsilon = \text{cap}(e) - f(e) > 0$. (Using here that $f(e) \leq \text{cap}(e)$ and $f(e) \neq \text{cap}(e)$.)

1.2.2. Let G' be the graph obtained by reducing the capacity of e by ϵ.

1.2.3. cap(e) in G' is greater than or equal to $f(e)$ in G.

1.2.4. Therefore, the max flow, f in G is also feasible in G'.

1.2.5. So e is not critical.

1.3. By block 1.2, if $f(e) \neq \text{cap}(e)$ in some maximum flow f, then e is not critical.

1.4. Next we will prove that, if e is not critical, then $f(e) \neq \text{cap}(e)$ in some maximum flow.

1.5. Assume that e is not critical in G.

1.5.1. Let $\epsilon > 0$ be such that reducing the capacity of e by ϵ does not reduce the maximum flow value. (Such an ϵ exists by the definition of "critical").

1.5.2. Let G' be the graph obtained by reducing the capacity of e by ϵ.

1.5.3. By the definition of a critical edge, G' will have the same maximum flow value as G.

1.5.4. That is, the maximum flow in G' could be applied to G and $f(e)$ would be $\text{cap}(e) - \epsilon$.

1.5.5. There is a max flow f in G with $f(e) \neq \text{cap}(e)$.

1.6. By block 1.5, if e is not critical, then there exists a maximum flow f in G with $f(e) \neq \text{cap}(e)$.

1.7. By lines 1.3 and 1.6, e is critical if and only if $f(e) = \text{cap}(e)$ in every maximum flow f in G.

2. By block 1, for any edge e in any flow network, e is critical iff $f(e) = \text{cap}(e)$ in every max flow f in G.

$\square$