Problem 1

Theorem 1 (All pairs have different sums)**.** *Prove or disprove: It is possible to label each square of an 8 x 8 checkerboard with an integer, so that no two squares are labeled with the same integer and no two adjacent squares (up, down, left, right) have labels that differ by 8 or more.*

Proof (long form).

1. Assume you have filled some chessboard with integers to satisfy the theorem.

1.1. Construct a set S by doing the following...

1.2. Add the square with the smallest integer not in S into S, stopping when either every row has at least one square in S or when every column has at least one square in S.

1.3. Without loss of generality, we can assume that every row will have at least one square in S (if it's columns, just rotate the chessboard).

1.4. Call T the set of chessboard squares adjacent to those in S but not in S.

1.5. Consider the largest numbered square in S, s, where the number in s is $num(s)$.

1.6. Before adding s to S, there were no squares in S in this row.

1.7. Likewise, no row was full (if one had been, we would have already stopped adding to S since there was a square in every column).

1.8. Therefore, after the addition of s into S, no row will be completely contained in S.

1.9. Conversely, every row will have at least one square in S.

1.10. Therefore, there must be at least one square in T in every row.

1.11. That is, $|T| \geq 8$.

1.12. Since there are at least 8 squares in T and the numbers in the squares in T must be unique integers $> num(s)$, the lower bound on the largest numbered square in T will be $num(s) + 8$.

1.13. However, by the definition of the theorem, there is no square in S that the largest square in T can be adjacent to.

1.14. Contradiction.

2. By block 1 you cannot fill a chessboard with integers to satisfy the theorem.

3. That is, the theorem is false.

□