Lemma 1. *An edge e is critical if and only if there exists a minimum-capacity cut containing e.*

Proof (long form).

1. Consider any s-t flow network G and edge e.
1.1. Let $c_{\min}$ be the minimum capacity of any $s - t$ cut in G.
1.2. By the max-flow/min-cut theorem, the maximum $s - t$ flow value in G is $c_{\min}$.

1.3. First we will prove that, if no minimum cut contains e, then e is not critical.
1.4. Assume that no minimum-capacity cut contains e.
1.4.1. That is, the value of any cut that contains e will be a loose upper bound for the max $s - t$ flow value in G.
1.4.2. Therefore, there exists some $\epsilon > 0$ where e can be reduced by ϵ and all cuts with e will still not be minimum capacity cuts.
1.4.3. That is, the upper bound on the maximum $s - t$ flow in G will not change.
1.4.4. So by definition e is not critical.
1.5. By block 1.4, if no minimum-capacity cut contains e, then e is not critical.

1.6. Next we will prove that, if some minimum-capacity cut contains e, then e is critical.
1.7. Suppose that some minimum-capacity cut C (of capacity $c_{\min}$) contains e.
1.7.1. We will show that e is critical.

1.7.2. Consider any $\epsilon > 0$.
1.7.2.1. Let G' be the flow network obtained from G by reducing the capacity of e by ϵ.
1.7.2.2. Call the minimum cut C in G', C' of value $c'_{\min}$.
1.7.2.3. $c'_{\min} < c_{\min}$ since the value of e has been reduced.
1.7.2.4. By definition every cut is an upper bound on the max $s - t$ flow through some graph.
1.7.2.5. Since we have a cut in G' whose value is less than any minimum cut in G, the max $s - t$ flow of G' is strictly less than that of G.
1.7.2.6. That is, reducing e's capacity by ϵ reduces the max flow.

1.7.3. By block 1.7.2, for all $\epsilon > 0$, reducing e's capacity by ϵ reduces the max flow. That is, e is critical.
1.8. By block 1.7, if some minimum-capacity cut contains e, then e is critical.

1.9. By lines 1.5 and 1.8, e is critical if and only if e is on some minimum-capacity cut.
2. By block 1, for any edge e in any flow network, e is critical iff e is on some minimum-capacity cut.

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