Problem 1

Theorem 1 (Tower of Hanoi). *For every positive integer N EVERY sequence of legal movements that moves N discs from one peg to another has AT LEAST $2^N - 1$ moves.*

Proof (long form).

1. We can prove this by induction on N. Consider the base case N=1.
2. To move one disc to any other peg at minimum will take 1 move.
3. That is, when $N = 1$, the theorem holds.
4. Assume for induction that the theorem holds for moving some positive k discs.
4.1. That is, moving k discs takes at least $2^k - 1$ moves
4.2. Call the starting post we're moving from P_0, call the goal post P_1 and the other post, P_2.
4.3. Consider any legal sequence for moving $k + 1$ discs from P_0 to P_1.
4.3.1. To accomplish this, the bottom disc (the $k + 1$ disc) must make its way from P_0 to P_1.
4.3.2. Consider the first time the $k + 1$ disc is moved.
4.3.3. By the definition of the problem, we can only move the top disc of any stack, and we cannot place it on any smaller disc.
4.3.4. Thus to move the $k+1$ disc, the other k discs cannot be on top of it or on a post it's moving to.
4.3.5. That is, to move the $k + 1$ disc, we must first stack the other k discs on a post that is not P_0.
4.3.6. By 4.1, moving the k discs to another post will take at least $2^k - 1$ moves.
4.3.7. Moving the $k + 1$ disc to any other post post will take at least 1 move.
4.3.8. Now consider the problem state when the $k + 1$ disk is set on P_1.
4.3.9. At minimum, this is the same move from step 4.3.7
4.3.10. By the definition of the problem, the other k discs must all be stacked on a post that is not P_1.
4.3.11. Moving the k discs to the target post will take at least $2^k - 1$ moves.
4.3.12. Together, this sequence will take at least $2^k - 1 + 2^k - 1 + 1 = 2^{k+1} - 1$ moves.
4.4. By block 4.3, any legal sequence for moving $k+1$ discs from P_0 to P_1 will take at least $2^{k+1} - 1$.
4.5. Thus, the theorem holds for $k + 1$ discs.
5. By block 4 if the theorem holds for k discs, then it also holds for $k + 1$ discs for any positive k.
6. By 3 and 5, the theorem holds.

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