**Problem 1**

Given an instance $\langle G = (V, E), k\rangle$ of Independent Set, the reduction outputs the instance $\langle n, D, p, i\rangle$ of ECD, where $n$, $D$, $p$, and $i$ are defined as follows:

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Let $D$ be the $VxV$ matrix constructed by the following steps. Fill the the matrix with zeroes. For every edge in $G$, $V_iV_j \in E$, change $D_{i,j} = .01$.

The reduced ECD will be of the form $\langle |V|, D, 0, k\rangle$

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Here is why the reduction can be computed in polynomial time:

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Given any Graph, to construct the reduced ECD only requires a loop through the edge set to setup the matrix. This should be linear which is easily polynomial.

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Here is a proof that the reduction is correct.

**Lemma 1.** *Given any instance $\langle G, k\rangle$ of* Independent Set*, let $\langle n, D, p, i\rangle$ be the instance of* ECD *produced by the reduction. Then $G$ has an independent set of size $k$ if and only if there is a set of $i$ ingredients that have total discord at most $p$.*

*Proof (long form).*

1. First we show the "only if" direction.

2. Assume that $G$ has an independent set of size $k$.

2.1. Let $I$ be an independent set of size $k$ in $G$.

2.2. That is, for any $V_j, V_k \in I$, $V_jV_k \notin E$.

2.3. Therefore, for any $V_k, V_k \in I$ in the Independent Set problem, $D_{j,k} = 0$ in the matrix for the reduced EDC problem.

2.4. There is a set of $i$ ingredients that have total discord at most $p$.

3. Next we show the "if" direction.

4. Assume there is a set of at least $i$ ingredients that have total discord at most $p$.

4.1. Let $S$ be a set of at $i$ ingredients that have total discord at most $p$.

4.2. That is, for every pair of ingredients, $I_j, I_k \in S$, in the matrix, $D_{j,k} = 0$.

4.3. This means in the independent set graph, there is no edge between $V_jV_k$.

4.4. Therefore there is an independent set of size $k$ in $G$.

5. By blocks 2 and 4, $G$ has an independent set of size $k$ if and only if there is a set of $i$ ingredients that have total discord at most $p$.

□

**Problem 2**

Given an instance $\langle G = (V, E)\rangle$ of Directed Ham Cycle, the reduction outputs the instance $\langle G' = (V', E')\rangle$ of Undirected Ham Cycle, where $V'$ and $E'$ are defined as follows:

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For each $v_i \in V$, construct a gizmo, ${v_i}^1, {v_i}^2, {v_i}^3$ to add to $V'$ and edges ${v_i}^1{v_i}^2$ and ${v_i}^2{v_i}^3$ to add to $E'$.

Then for each $v_iv_j \in E$, construct edge ${v_i}^3{v_j}^1$ to add to $E'$.

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Here is why the reduction can be computed in polynomial time:

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We need to create $3|V|$ nodes and $|E| + 2|V|$ edges to setup this graph. At worst this should be $O(|V| + |E|)$, which will be polynomial.

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Here is a proof that the reduction is correct.

**Lemma 2.** *Given any instance $\langle G = (V, E)\rangle$ of* Directed Ham Cycle*, let $\langle G' = (V', E')\rangle$ be the instance of* Undirected Ham Cycle *produced by the reduction. Then $G$ has a directed Hamiltonian cycle if and only if $G'$ has an undirected Hamiltonian cycle.*

*Proof (long form).*

1. First we show the "only if" direction.

2. Assume that $G$ has a directed Hamiltonian cycle.

2.1. Let $C$ be such a cycle.

2.2. That is, there is some order the vertices can be visited, $v_1, v_2, ..., v_n, v_1$ where all are unique except the first and last and all edge used are unique.

2.3. Then $G'$ will a cycle, ${v_1}^1, {v_1}^2, {v_1}^3, {v_2}^1, {v_2}^2, {v_2}^3, ..., {v_n}^1, {v_n}^2, {v_n}^3, {v_1}^1$.

2.4. This will be a Hamiltonian cycle since all of these nodes are unique except the first and last and all nodes in the graph are contained in the cycle.

3. Next we show the "if" direction.

4. Assume that $G'$ has an undirected Hamiltonian cycle.

4.1. Let $C'$ be such a cycle.

4.2. $C'$ must visit every node in the graph.

4.3. In any ${v_i}^1, {v_i}^2, {v_i}^3$ gizmo, ${v_i}^2$ must be visited either in the form ${v_i}^1 \to {v_i}^2 \to {v_i}^3$ or ${v_i}^3 \to {v_i}^2 \to {v_i}^1$ since it has two edges.

4.4. The only way to move directly from one gizmo, $v_i$ to another, $v_j$ will be via the edge ${v_i}^3{v_j}^1$ or ${v_i}^1{v_j}^3$.

4.5. Therefore, if one gizmo is visited $1 \to 2 \to 3$, they all will be and vice versa.

4.6. Without loss of generality, we can assume $C'$ will be of the form ${v_1}^1, {v_1}^2, {v_1}^3, {v_2}^1, ..., {v_n}^3, {v_1}^1$ where every node is unique except the first and last.

4.7. Therefore, $G$ will have a Hamiltonian cycle of the form $v_1, v_2, ..., v_n, v_1$.

5. By blocks 2 and 4, $G$ has a directed Hamiltonian cycle if and only if $G'$ has an undirected Hamiltonian cycle.

$\square$

**Problem 3**

Given an instance $\langle G = (V, E) \rangle$ of Directed Ham Cycle, the reduction outputs the instance $\langle \Phi \rangle$ of SAT, defined as follows:

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*Oh boy...*

Create $|V|^2$ boolean variables and call them $x_{i,j}$ where $0 < i, j \leq |V|$. Conceptually we will use $x_{i,j}$ to represent whether or not the $i^th$ vertex is the $j^th$ node in the Hamiltonian cycle.

The boolean equation of all $|V|^2$ variables will be constructed by and'ing ($\wedge$) the following maxterms:

1. $x_{i,1} \vee x_{i,2} \vee \cdots \vee x_{i,|V|}$ for all $0 < i \leq |V|$.
2. $\neg x_{i,j} \vee \neg x_{i,k}$ for all $0 < i, j, k \leq |V|$ where $j \neq k$.
3. $x_{1,i} \vee x_{2,i} \vee \cdots \vee x_{|V|,i}$ for all $0 < i \leq |V|$.
4. $\neg x_{i,j} \vee \neg x_{k,j}$ for all $0 < i, j, k \leq |V|$ where $j \neq k$.
5. $\neg x_{i,k} \vee \neg x_{j,k+1}$ for all $V_i V_j \notin E$ for all $0 < k < |V|$.
6. $\neg x_{i,1} \vee \neg x_{j,|V|}$ for all $V_j V_i \notin E$.

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Here is why the reduction can be computed in polynomial time:

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Constructing maxterms 1-4 will always take $O(V)$ time, but 5 will take $O(V^3)$ time, so the whole thing will take $O(V^3)$ time, which is polynomial.

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Here is a proof that the reduction is correct.

**Lemma 3.** *Given any instance $\langle G = (V, E) \rangle$ of* Directed Ham Cycle*, let $\Phi$ be the instance of* SAT *produced by the reduction. Then $G$ has a directed Hamiltonian cycle if and only if $\Phi$ is satisfiable.*

*Proof (long form).*

1. First we show the "only if" direction.

2. Assume that $G$ has a directed Hamiltonian cycle.

2.1. Let $C$ be such a cycle.

2.2. That is, there is some order the vertices can be visited, $v_1, v_2, ..., v_n, v_1$ where all are unique except the first and last and all edge used are unique.

2.3. Then we can set variables $x_{i,j}$ to true where for every vertex in $C$, $i$ is the number of the vertex in $G$ and $j$ is the index it appears in $C$.

2.4. This will satisfy the first set of maxterms since there is some $x_{k,l}$ that is true for every $k$.

2.5. This will satisfy the second set of maxterms since there is only one $x_{k,l}$ that is true for each value of $k$.

2.6. This will satisfy the third set of maxterms since there is some $x_{k,l}$ that is true for every $l$.

2.7. This will satisfy the fourth set of maxterms since there is only one $x_{k,l}$ that is true for each value of $l$.

2.8. Since we know that every $x_{i,j}, x_{i,j+1}$ that are set to true are adjacent to each other in $G$, the fifth maxterms will be satisfied.

2.9. Since we know that $v_n v_1$ is an edge in $G$, the sixth maxterms will be satisfied.

2.10. $\Phi$ is satisfiable.

3. Next we show the "if" direction.
4. Assume that $\Phi$ is satisfiable.

4.1. Let $A$ be an assignment to the variables of $\Phi$ that makes $\Phi$ true.

4.2. Construct some cycle, $C$ by examining all true variables, $x_{i,j}$ and adding vertex $i$ to index $j$ in $C$.

4.3. Because of the first set of maxterms, for every $k$, at least one $x_{k,l}$ will be true.

4.4. That is, every node will be included in $C$.

4.5. Because of the second set of maxterms, for every $k$, only one $x_{k,l}$ can be true.

4.6. That is, every node will be included at most once in $C$.

4.7. By Step 4.4 and Step 4.6, every node will be included exactly once in $C$.

4.8. Because of the third set of maxterms, for every $l$, at least one $x_{k,l}$ will be true.

4.9. That is, every index of $C$ will have at least one node in it.

4.10. Because of the fourth set of maxterms, for every $l$, only one $x_{k,l}$ can be true.

4.11. That is, no index of $C$ will have more than one node in it.

4.12. By Step 4.9 and Step 4.11, every index will have exactly one node in $C$.

4.13. Because of the fifth set of maxterms, nodes in adjacent indexes of $C$ must have an edge between them in $G$.

4.14. By 4.7, 4.12, 4.13, $C$ is a Hamiltonian path.

4.15. Because of the sixth set of maxterms, the last and first nodes in $C$ must be ajacent in $G$.

4.16. That is, $C$ forms a cycle.

4.17. By steps 4.14, 4.16 there is a directed Hamiltonian cycle in $G$.

5. By blocks 2 and 4, $G$ has a directed Hamiltonian cycle if and only if $\Phi$ is satisfiable.

$\square$