Problem 1

Describe a polynomial-time reduction from Vertex Cover to Set Cover. Definition of Set Cover: The input is U, C, k where U is a set, $C = \{S1, S2, ..., Sm\}$ is a collection of subsets of U, and k is an integer. The output is yes if there is a set cover in C of size k, and no otherwise. A set cover is a collection C C of sets in C such that the union of the sets in C is U. The size of C is the number of sets in C.

Given an instance $\langle G = (V, E), k\rangle$ of VERTEX COVER, the reduction outputs the instance $\langle U, C, k\rangle$ of SET COVER, where U, C, and k are defined as follows:

Let $U = E$ be the set of edges in G;
let C_i be the number of edges touched by V_i, that is $C_i = \{e_{V_iV_j} | e_{V_iV_j} \in E\}$;
Let k be k from the Vertex Cover problem.

Here is why the reduction can be computed in polynomial time:

Constructing C will require at worst a loop over the edge set for each vertex, resulting in $O(VE)$ time, which is polynomial.

Here is a proof that the reduction is correct.

Lemma 1. *Given any instance $\langle G, k\rangle$ of* VERTEX COVER*, let $\langle U, C, k\rangle$ be the instance of* SET COVER *produced by the reduction. Then G has a vertex cover of size k if and only if there is a set $C' \subseteq C$ of size k where $\bigcup_{i=1}^{k} C'_i = U$.*

Proof (long form).

1. First we show the "only if" direction.
2. Assume that G has a vertex cover of size k.
2.1. Let I be the vertex cover of size k in G.
2.2. Take C' to be the set of sets in C corresponding to the vertices in I.
2.3. Then C' has size k, and the union of all sets in C' equal U because each set in C', C'_i corresponds to the edges touched by including vertex V_i.
2.4. So there is a subset of size k of C such that the union over them all will equal U.
3. Next we show the "if" direction.
4. Assume there is a set of at least i ingredients that have total discord at most p.
4.1. Let S be a set of at i ingredients that have total discord at most p. Note $i = k$ and $p = 0$.
4.2. Take I to be the vertices corresponding to ingredients in S.
4.3. Then I has size i, and is an independent set because each pair (i, j) of vertices in I corresponds to a pair of ingredients with zero discord ($D_{ij} \leq p = 0$), so there is no edge (i, j).
4.4. There is an independent set of size $k = i$ in G.
5. By blocks 2 and 4, G has an independent set of size k if and only if there is a set of i ingredients that have total discord at most p.

□