Problem 1

Given an instance $\langle G = (V, E), k \rangle$ of INDEPENDENT SET, the reduction outputs the instance $\langle n, D, p, i \rangle$ of ECD, where n, D, p, and i are defined as follows:

Let $n = |V|$ be the number of vertices in G; let D_{ij} be one if there is an edge $(i, j) \in E$ and zero otherwise (for $i, j \in \{1, 2, \ldots, n\}$); Let $p = 0$ and $i = k$.

Here is why the reduction can be computed in polynomial time:

Given G and k, we can compute $n = |V|$ in linear time by counting the vertices. We can compute the matrix D in $O(n^2)$ time in a single pass over the edges. We can compute p and k in constant time.

Here is a proof that the reduction is correct.

Lemma 1. *Given any instance $\langle G, k \rangle$ of* INDEPENDENT SET*, let $\langle n, D, p, i \rangle$ be the instance of* ECD *produced by the reduction. Then G has an independent set of size k if and only if there is a set of i ingredients that have total discord at most p.*

Proof (long form).

1. First we show the "only if" direction.

2. Assume that G has an independent set of size k.

2.1. Let I be an independent set of size k in G.

2.2. Take S to be the set of ingredients corresponding to vertices in I.

2.3. Then S has size k, and has total discord zero because each pair (i, j) of ingredients in S corresponds to a pair of vertices with no edge ($D_{ij} = 0$).

2.4. So there is a set of $i = k$ ingredients that have total discord at most $p = 0$.

3. Next we show the "if" direction.

4. Assume there is a set of at least i ingredients that have total discord at most p.

4.1. Let S be a set of at i ingredients that have total discord at most p. Note $i = k$ and $p = 0$.

4.2. Take I to be the vertices corresponding to ingredients in S.

4.3. Then I has size i, and is an independent set because each pair (i, j) of vertices in I corresponds to a pair of ingredients with zero discord ($D_{ij} \leq p = 0$), so there is no edge (i, j).

4.4. There is an independent set of size $k = i$ in G.

5. By blocks 2 and 4, G has an independent set of size k if and only if there is a set of i ingredients that have total discord at most p.

□

Problem 2

Given an instance $\langle G = (V, E)\rangle$ of DIRECTED HAM CYCLE, the reduction outputs the instance $\langle G' = (V', E')\rangle$ of UNDIRECTED HAM CYCLE, where V' and E' are defined as follows:

For each vertex $v \in V$, add three *clone* vertices v^1, v^2, v^3 to V'; add edges (v^1, v^2) and (v^2, v^3) to E'. For each edge $(u, w) \in E$, add edge (w^3, v^1) to E'.

Here is why the reduction can be computed in polynomial time:

The reduction as described makes a single pass over the vertices and edges of G to build G'. So it takes linear time.

Here is a proof that the reduction is correct.

Lemma 2. *Given any instance $\langle G = (V, E)\rangle$ of* DIRECTED HAM CYCLE*, let $\langle G' = (V', E')\rangle$ be the instance of* UNDIRECTED HAM CYCLE *produced by the reduction. Then G has a directed Hamiltonian cycle if and only if G' has an undirected Hamiltonian cycle.*

Proof (long form).

1. First we show the "only if" direction.

2. Assume that G has a directed Hamiltonian cycle.

2.1. Let $C = (v_1, v_2, \ldots, v_n, v_1)$ be such a cycle.

2.2. Obtain C' from C by replacing each vertex v_i by its three clones v^1, v^2, v^3.

2.3. So $C' = (v_1^1, v_1^2 v_1^3, v_2^1, v_2^2, v_2^3, \ldots, v_n^1, v_n^2, v_n^3, v_1^1, v_1^2, v_1^3)$.

2.4. Since C is a Hamiltonian cycle, each edge (v_i, v_{i+1}) is present in G, and each vertex v in G is visited exactly once by C.

2.5. So each edge (v_i^3, v_{i+1}^1) is present in G' (by the reduction), and each vertex v_i^ℓ is visited exactly once by C'.

2.6. Also, for each i, each edge (v_i^1, v_i^2) and (v_i^2, v_i^3) is present in G' (by the reduction).

2.7. So C' is a Hamiltonian cycle in G'.

3. Next we show the "if" direction.

4. Assume that G' has a undirected Hamiltonian cycle.

4.1. Let C' be such a cycle.

4.2. For each $v \in V$, the vertex v^2 has edges only to v^1 and v^3, so edges (v^1, v^2) and (v^2, v^3) are present in C'. Further, each v^3 has edges only to vertices w^1 such that $(v, w) \in E$.

4.3. So (starting the cycle C' at an arbitrary vertex v^1, and orienting it so that v^2 is the second vertex on the cycle), by a simple induction, the $3n$ vertices in G' can be ordered so that

$$C' = (v_1^1, v_1^2 v_1^3, v_2^1, v_2^2, v_2^3, \ldots, v_n^1, v_n^2, v_n^3, v_1^1, v_1^2, v_1^3).$$

4.4. Let $C = (v_1, v_2, \ldots, v_n, v_1)$ be obtained by replacing each clone triple (v_i^1, v_i^2, v_i^3) in V' by its original vertex v_i in G. So each vertex v in G is visited exactly once by C.

4.5. Each edge (v_i^3, v_{i+1}^1) is present in G', so, by the reduction, each edge $(v_i, v_{i+1}$ is present in G.

4.6. So C is a Hamiltonian cycle in G.

5. By blocks 2 and 4, G has a directed Hamiltonian cycle if and only if G' has an undirected Hamiltonian cycle.

$\square$

Problem 3

Given an instance $\langle G = (V, E)\rangle$ of Directed Ham Cycle, the reduction outputs the instance $\langle \Phi \rangle$ of SAT, defined as follows:

Let $n = |V|$. The Boolean variables for the formula Φ are $X = \{x_{ij} : i \in V, j \in \{1, 2, \ldots, n\}\}$, with the interpretation that setting x_{ij} to be true should correspond to having a Hamiltonian cycle in G with vertex i at position j.

For any set of variables $S \subseteq X$, define $\mathsf{one}(S)$ to be the formula $(\bigvee_{x \in S} x) \wedge \neg(\bigvee_{x,x' \in S : x \neq x'} x \wedge x')$. So $\mathsf{one}(S)$ is satisfied if and only if exactly one of the variables in S is assigned true.

Take Φ to be logical and of the following expressions:

1. For each vertex i, $\mathsf{one}(\{x_{i1}, x_{i2}, \ldots, x_{in}\})$. (Each vertex should have one position.)
2. For each position j, $\mathsf{one}(\{x_{1j}, x_{2j}, \ldots, x_{nj}\})$. (Each position should have one vertex.)
3. For each position j, and pair of vertices i, i' such that $(i, i') \notin E$, add clause $\neg(x_{ij} \wedge x_{i',j+1})$ or, if $j = n$, then add clause $\neg(x_{ij} \wedge x_{i',1})$. (In other words, vertex i' cannot follow vertex i on the cycle if there is no edge (i, i')).

Here is why the reduction can be computed in polynomial time:

Φ can be built as described above in time $O(|V|^2 + |E|\,|V|)$ by appropriately enumerating the vertices $i \in V$ and positions $j \in \{1, \ldots, n\}$.

Here is a proof that the reduction is correct.

Lemma 3. *Given any instance $\langle G = (V, E)\rangle$ of* Directed Ham Cycle, *let Φ be the instance of* SAT *produced by the reduction. Then G has a directed Hamiltonian cycle if and only if Φ is satisfiable.*

Proof (long form).

1. First we show the "only if" direction.
2. Assume that G has a directed Hamiltonian cycle.

2.1. Let C be such a cycle.

. . . Somehow show that, since C exists in G, there must be an assignment A to the variables of Φ that makes Φ true. . .

2.2. Φ is satisfiable.

3. Next we show the "if" direction.
4. Assume that Φ is satisfiable.

4.1. Let A be an assignment to the variables of Φ that makes Φ true.

. . . Somehow show that, since A exists, there must be a directed Hamiltonian cycle C in G. . .

4.2. There is an directed Hamiltonian cycle in G.

5. By blocks 2 and 4, G has a directed Hamiltonian cycle if and only if Φ is satisfiable.

□