Problem 1

Lemma 1. *For any boy-optimal stable matching M for any instance I, M is girl-pessimal.*

Proof (long form).

1. Consider any instance I and boy-optimal stable matching M.
1.1. Suppose for contradiction that M is not girl-pessimal.
1.1.1. Let $(b, g) \in M$ be a matched couple where b is not g's least-preferred possible partner.
1.1.2. Let b' be the least-preferred possible partner of g.
1.1.3. Let M' be a stable matching in which (b', g) are a couple.
1.1.4. b must prefer g to his partner in M', since g is his best possible partner.
1.1.5. Likewise, since b' is g's least preferred possible partner, g prefers b to b'.
1.1.6. Thus, (b, g) is an unstable pair in M'.
1.1.7. Contradiction by 1.1.3., M' is not a stable matching.
1.2. By block 1.1, M is girl-pessimal.
2. By block 1, for any boy-optimal stable matching M for any instance I, M is girl-pessimal.

$\square$

Proof (short form). Assume for contradiction that we have some stable boy optimal matching M on some instance that is not girl pessimal. That is, there is some $(b, g) \in M$ where g is paired with a less preferred boy, b' in some other stable matching, M'. However, g is b's most preferred possible choice, and g prefers b to b', so (b, g) is an unstable pair in M'. Therefore M' cannot be stable, contradiction. Thus any boy optimal matching M is also girl pessimal. $\square$

Problem 2

(a) First problem.

(b) Original:

$$\begin{aligned} \text{maximize } & x_1 + 2x_2 \\ \text{subject to } & 2x_1 + x_2 \leq 10 \\ & x_1 + 3x_2 \leq 9 \\ & x_1, x_2 \geq 0. \end{aligned}$$

Dual:

$$\begin{aligned} \text{minimize } & 10a + 9b \\ \text{subject to } & 2a + b \geq 1 \\ & a + 3b \geq 2 \\ & a, b \geq 0. \end{aligned}$$

(c) The optimal solution is $(a, b) = (1/5, 3/5)$, with value 37/5. (found graphically).

Problem 3

(a) Second Problem

(b) Original:

$$\begin{aligned} \text{minimize } & x_1 + 3x_2 + 5x_3 \\ \text{subject to } & x_1 + x_2 + x_3 \geq 1 \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

Dual:

$$\begin{aligned} \text{maximize } & a \\ \text{subject to } & a \leq 1 \\ & a \leq 3 \\ & a \leq 5 \\ & a \geq 0. \end{aligned}$$

These can be rewritten as simply:

$$\begin{aligned} \text{maximize } & a \\ \text{subject to } & 0 \leq a \leq 1 \end{aligned}$$

(c) Especially here in the dual, it becomes blindingly obvious that our optimal solution will be $a = 1$ with a value of 1.