Problem 1

Lemma 2. *An edge e is critical iff $f(e) = cap(e)$ in every maximum flow f.*

Proof (long form).

1. Consider any s-t flow network G and edge e.

1.1. First we will prove that, if $f(e) \neq \mathrm{cap}(e)$ in some maximum flow f, then e is not critical.

1.2. Assume that there exists max-flow f such that $f(e) \neq \mathrm{cap}(e)$.

1.2.1. Let $\epsilon = \mathrm{cap}(e) - f(e) > 0$. (Using here that $f(e) \leq \mathrm{cap}(e)$ and $f(e) \neq \mathrm{cap}(e)$.)

1.2.2. Let G' be the graph obtained by reducing the capacity of e by ϵ.

1.2.3. Since the flow through e in G is less than or equal to the capacity of e in G', the flow through e in each graph can still be $f(e)$.

1.2.4. That is, the maximum flow in G' is the same as that in G.

1.2.5. So e is not critical.

1.3. By block 1.2, if $f(e) \neq \mathrm{cap}(e)$ in some maximum flow f, then e is not critical.

1.4. Next we will prove that, if $f(e) = \mathrm{cap}(e)$ in every maximum flow f, then e is critical.

1.5. Assume there exists some e such that in every max-flow f, $f(e) = \mathrm{cap}(e)$.

1.5.1. Let $\epsilon > 0$. (Reducing the capacity of e by any ϵ should reduce the max flow by the definition of "critical").

1.5.2. Since in every maximum flow, $f(e) = \mathrm{cap}(e)$, reducing the capacity of e by ϵ will reduce the maximum flow by ϵ

1.5.3. So e is critical.

1.6. By block 1.5, if $f(e) \neq \mathrm{cap}(e)$ in some maximum flow f, then e is not critical.

1.7. By lines 1.3 and 1.6, e is critical iff $f(e) = \mathrm{cap}(e)$ in every maximum flow f in G.

2. By block 1, for any edge e in any flow network, e is critical iff $f(e) = \mathrm{cap}(e)$ in every max flow f in G.

□