Problem 1

Theorem 1 (Tower of Hanoi)**.** *You are given a subset S of $\{1, 2, ..., 100\}$ with the following property: for every quadruple u, v, w, x of distinct numbers in S, the sum of u and v differs from the sum of w and x. Must the size of S be at most fifteen? Prove your answer.*

Proof (long form).

1. Assume we have some valid set, S of size 15.

1.1. By definition, for any quadruple of distinct numbers $u, v, w, x \in S$, $u + v \neq w + x$.

1.2. In other words, $u - w \neq x - v$.

1.3. That is, given any two pairs, the differences between the pairs must differ.

1.4. Consider solely pairs made of consecutively numbers in S (when S is sorted in ascending order).

1.5. We can make 14 pairs of these consecutive numbers, each of which must have a unique difference.

1.6. At minimum, The differences between the pairs will be in the set $\{1, 2, ..., 14\}$.

1.7. However, $\sum_{n=1}^{14} n = 105$, which means that the largest number in S has a lower bound of 106 (since we started at 1 not 0).

1.8. This is a contradiciton since every number in S must be in the set $\{1, 2, ..., 100\}$.

2. By block 1, the size of S cannot be 15.

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