

Problem 1

Theorem 1 (All pairs have different sums). *Prove or disprove: It is possible to label each square of an 8×8 checkerboard with an integer, so that no two squares are labeled with the same integer and no two adjacent squares (up, down, left, right) have labels that differ by 8 or more.*

Proof (long form).

1. Assume you have filled some chessboard with integers to satisfy the theorem.
 - 1.1. Construct a set S by doing the following...
 - 1.2. Add the square with the smallest integer not in S into S , stopping when either every row has at least one square in S or when every column has at least one square in S .
 - 1.3. Without loss of generality, we can assume that every row will have at least one square in S (if it's columns, just rotate the chessboard).
 - 1.4. Call T the set of chessboard squares adjacent to those in S but not in S .
 - 1.5. Since every value in the squares of T will be greater than those of S , we only need to show $|T| \geq 8$ to disprove the theorem.
 - 1.6. No row will be completely contained in S , since if it was, then there would be a square in every column in S .
 - 1.7. Conversely, every row will have at least one square in S .
 - 1.8. Therefore, there must be at least one square in T in every row.
 - 1.9. That is, $|T| \geq 8$.
 - 1.10. Call the largest numbered square in S , s , where the number in s is $num(s)$.
 - 1.11. Since there are at least 8 squares in T and the numbers in the squares in T must be unique integers $> num(s)$, the lower bound on the largest numbered square in T will be $num(s) + 8$.
 - 1.12. However, by the definition of the theorem, there is no square in S that the largest square in T can be adjacent to.
 - 1.13. Contradiction.
2. By block 1 you cannot fill a chessboard with integers to satisfy the theorem.
3. That is, the theorem is false.

□