Problem 1

Given an instance $\langle G = (V, E)\rangle$ of DIRECTED HAM CYCLE, the reduction outputs the instance $\langle G' = (V', E')\rangle$ of UNDIRECTED HAM CYCLE, where V' and E' are defined as follows:

For each $v_i \in V$, construct a gizmo, ${v_i}^1, {v_i}^2, {v_i}^3$ to add to V' and edges ${v_i}^1{v_i}^2$ and ${v_i}^2{v_i}^3$ to add to E'.

Then for each $v_iv_j \in E$, construct edge ${v_i}^3{v_j}^1$ to add to E'.

Here is why the reduction can be computed in polynomial time:

We need to create $3|V|$ nodes and $|E| + 2|V|$ edges to setup this graph. At worst this should be $O(|V| + |E|)$, which will be polynomial.

Here is a proof that the reduction is correct.

Lemma 1. *Given any instance $\langle G = (V, E)\rangle$ of* DIRECTED HAM CYCLE*, let $\langle G' = (V', E')\rangle$ be the instance of* UNDIRECTED HAM CYCLE *produced by the reduction. Then G has a directed Hamiltonian cycle if and only if G' has an undirected Hamiltonian cycle.*

Proof (long form).

1. First we show the "only if" direction.

2. Assume that G has a directed Hamiltonian cycle.

2.1. Let C be such a cycle.

2.2. That is, there is some order the vertices can be visited, $v_1, v_2, ..., v_n, v_1$ where all are unique except the first and last and all edge used are unique.

2.3. Then G' will have the Hamiltonian cycle, ${v_1}^1, {v_1}^2, {v_1}^3, {v_2}^1, {v_2}^2, {v_2}^3, ..., {v_n}^1, {v_n}^2, {v_n}^3, {v_1}^1$.

3. Next we show the "if" direction.

4. Assume that G' has an undirected Hamiltonian cycle.

4.1. Let C' be such a cycle.

4.2. Every ${v_i}^1, {v_i}^2, {v_i}^3$ gizmo must be visited either from ${v_i}^1 \to {v_i}^2 \to {v_i}^3$ or ${v_i}^3 \to {v_i}^2 \to {v_i}^1$.

4.3. The only way to move directly from one gizmo, v_i to another, v_j will be via the edge ${v_i}^3{v_j}^1$.

4.4. Therefore, if one gizmo is visited $1 \to 2 \to 3$, they all will be and vice versa.

4.5. Without loss of generality, we can assume C' will be of the form ${v_1}^1, {v_1}^2, {v_1}^3, {v_2}^1, ..., {v_n}^3, {v_1}^1$ where every node is unique except the first and last.

4.6. Therefore, G will have a Hamiltonian cycle of the form $v_1, v_2, ..., v_n, v_1$.

5. By blocks 2 and 4, G has a directed Hamiltonian cycle if and only if G' has an undirected Hamiltonian cycle.

□