Problem 1

Theorem 1 (Tower of Hanoi)**.** *For every positive integer N EVERY sequence of legal movements that moves N discs from one peg to another has AT LEAST $2^N - 1$ moves.*

Proof (long form).

1. We can prove this by induction on N. Consider the base case N=1.
2. To move one disc to any other peg at minimum will take 1 move.
3. That is, when $N = 1$, the theorem holds.
4. Assume for induction that the theorem holds for moving some positive k discs.

4.1. That is, moving k discs takes at least $2^k - 1$ moves

4.2. Call the starting post we're moving from P_0, call the goal post P_1 and the other post, P_2.

4.3. Consider any legal sequence for moving $k + 1$ discs from P_0 to P_1.

4.3.1. At some point during this sequence, we must move the bottom disc (the $k + 1$ disc) to P_1.

4.3.2. By the definition of the problem, we can only move the top disc of any stack, and we cannot place it on any smaller disc.

4.3.3. Thus to move the $k + 1$ disc to P_1, the other k discs cannot be on top of it or on P_1.

4.3.4. That is, to move the $k + 1$ disc to P_1, we must first stack the other k discs on a post that is not P_1 or the post the $k + 1$ disc is on.

4.3.5. By 4.1, moving the k discs to another post will take at least $2^k - 1$ moves.

4.3.6. Moving the $k + 1$ disc to the target post will take at least 1 move.

4.3.7. Moving the k discs to the target post will take at least $2^k - 1$ moves.

4.3.8. Together, this sequence will take at least $2^k - 1 + 2^k - 1 + 1 = 2^{k+1} - 1$ moves.

4.4. By block 4.3, any legal sequence for moving $k+1$ discs from P_0 to P_1 will take at least $2^{k+1} - 1$.

4.5. Thus, the theorem holds for $k + 1$ discs.

5. By block 4 if the theorem holds for k discs, then it also holds for $k + 1$ discs for any positive k.
6. By 3 and 5, the theorem holds.

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