Lemma 1. *An edge e is critical if and only if there exists a minimum-capacity cut containing e.*

Proof (long form).

1. Consider any s-t flow network G and edge e.
1.1. Let $c_{\min}$ be the minimum capacity of any $s-t$ cut in G.
1.2. By the max-flow/min-cut theorem, the maximum $s-t$ flow value in G is $c_{\min}$.

1.3. First we will prove that, if no minimum cut contains e, then e is not critical.
1.4. Assume that no minimum-capacity cut contains e.
1.4.1. Take some cut, c, that contains e.
1.4.1.1. By 1.4, c is not a minimum cut.
1.4.1.2. That is, the value of c is a loose upper bound for the max $s-t$ flow value in G.
1.4.1.3. Therefore, there exists some $\epsilon > 0$ where e can be reduced by ϵ and the value of c will still be greater than any minimum cut.
1.4.1.4. The amount of flow coming into the cut is unchanged after e is reduced.
1.4.1.5. The amount of flow leaving the cut is strictly greater than the max flow of the whole graph since this still isn't a minimum cut after e is reduced.
1.4.1.6. Therefore, the max flow of G is unchanged after e is reduced.
1.4.2. In every cut with e, there exists some ϵ where e can be reduced by ϵ and still not change the max flow in G.
1.4.3. So by definition e is not critical.
1.5. By block 1.4, if no minimum-capacity cut contains e, then e is not critical.

1.6. Next we will prove that, if some minimum-capacity cut contains e, then e is critical.
1.7. Suppose that some minimum-capacity cut C (of capacity $c_{\min}$) contains e.
1.7.1. We will show that e is critical.
1.7.2. Consider any $\epsilon > 0$.
1.7.2.1. Let G' be the flow network obtained from G by reducing the capacity of e by ϵ.
1.7.2.2. Consider the flow passing through C in G'.
1.7.2.3. The amount of flow coming into C is the same in G as in G'.
1.7.2.4. However, the capacity of C in G' is less than $c_{min} =$ max flow of G.
1.7.2.5. In other words, the flow leaving C in G' is strictly less than the maximum flow of the graph G.
1.7.2.6. Therefore, the max flow of G' must be less than that of G.
1.7.2.7. That is, reducing e's capacity by ϵ reduces the max flow.
1.7.3. By block 1.7.2, for all $\epsilon > 0$, reducing e's capacity by ϵ reduces the max flow. That is, e is critical.
1.8. By block 1.7, if some minimum-capacity cut contains e, then e is critical.

1.9. By lines 1.5 and 1.8, e is critical if and only if e is on some minimum-capacity cut.
2. By block 1, for any edge e in any flow network, e is critical iff e is on some minimum-capacity cut.

□