Math 120 Optimization: Homework 4

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Problem 1: Consider the system of equations

$$\begin{bmatrix} 2 & -1 & 2 & -1 & 3 \\ 1 & 2 & 3 & 1 & 0 \\ 1 & 0 & -2 & 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 14 \\ 5 \\ -10 \end{bmatrix}.$$

Find one basic solution. State explicitly which basis you are using.

$det(\begin{bmatrix} 2 & -1 & 2 \\ 1 & 2 & 3 \\ 1 & 0 & -2 \end{bmatrix}) = -17 \neq 0$, so these three columns are linearly independent.

We will use these as the basis for our basic solution.

$\left[\begin{array}{ccc|c} 2 & -1 & 2 & 14 \\ 1 & 2 & 3 & 5 \\ 1 & 0 & -2 & -10 \end{array}\right]$ reduces to $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -4/17 \\ 0 & 1 & 0 & -80/17 \\ 0 & 0 & 1 & 83/17 \end{array}\right]$, so our basic solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -4/17 \\ -80/17 \\ 83/17 \\ 0 \\ 0 \end{bmatrix}$$

Problem 2: Consider the system of equations

$$\begin{bmatrix} 2 & 1 & -1 & 0 \\ 1 & 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$

Find all basic solutions. (*bruh.*)

All of the column vectors are linearly independent (*yay*) so we're going to have $\binom{4}{2} = 6$ possible basic solutions. In no particular order, these will be

1. $$\left[\begin{array}{cc|c} 2 & 1 & 3 \\ 1 & 3 & 5 \end{array}\right] \Longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 4/5 \\ 0 & 1 & 7/5 \end{array}\right] \Longrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 4/5 \\ 7/5 \\ 0 \\ 0 \end{bmatrix}$$

2. $$\left[\begin{array}{cc|c} 2 & -1 & 3 \\ 1 & 1 & 5 \end{array}\right] \Longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 8/3 \\ 0 & 1 & 7/3 \end{array}\right] \Longrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 8/3 \\ 0 \\ 7/3 \\ 0 \end{bmatrix}$$

3. $$\left[\begin{array}{cc|c} 2 & 0 & 3 \\ 1 & 3 & 5 \end{array}\right] \Longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 3/2 \\ 0 & 1 & 7/6 \end{array}\right] \Longrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 0 \\ 0 \\ 7/6 \end{bmatrix}$$

4. $$\left[\begin{array}{cc|c} 1 & -1 & 3 \\ 3 & 1 & 5 \end{array}\right] \Longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \end{array}\right] \Longrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -1 \\ 0 \end{bmatrix}$$

5. $$\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 3 & 3 & 5 \end{array}\right] \Longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -4/3 \end{array}\right] \Longrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 0 \\ -4/3 \end{bmatrix}$$

6. $$\left[\begin{array}{cc|c} -1 & 0 & 3 \\ 1 & 3 & 5 \end{array}\right] \Longrightarrow \left[\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 8/3 \end{array}\right] \Longrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -3 \\ 8/3 \end{bmatrix}$$

Problem 3: Solve the linear programming problem:

$$\begin{aligned} &\text{minimize} \quad x_1 - x_3 \\ &\text{subject to} \quad \begin{bmatrix} 2 & 1 & -1 & 0 \\ 1 & 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \\ &\qquad\qquad\quad \vec{x} \geq \vec{0}. \end{aligned}$$

You may use the result from the previous problem.

From the previous problem, solutions 4, 5, 6 are all infeasible, so we only need to check 1-3. Plugging our values from our basic solutions into the function we're attempting to minimize will yield:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 4/5 \\ 7/5 \\ 0 \\ 0 \end{bmatrix} \implies 4/5 - 0 = 4/5$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 8/3 \\ 0 \\ 7/3 \\ 0 \end{bmatrix} \implies 8/3 - 7/3 = 1/3$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 0 \\ 0 \\ 7/6 \end{bmatrix} \implies 3/2 - 0 = 3/2$$

Our optimal minimum value will be 1/3 when $[x_1, x_2, x_3, x_4, x_5] = [8/3, 0, 7/3, 0]$.

Problem 4: Prove that any subspace V of $\mathbb{R}^n$ is a convex set.

1. Assume we have some subspace, V.
2. By definition, V is closed under addition of two vectors within V and under scalar multiplication.
3. By definition, V is convex iff $\forall \vec{u}, \vec{v} \in V, \alpha\vec{u} + (1-\alpha)\vec{v} \in V$

4. Assume we have some vectors, $\vec{u}, \vec{v} \in V$.

5. Define $\vec{u'} = \alpha\vec{u}, \vec{v'} = (1-\alpha)\vec{v}$.

6. By 2, $\vec{u'}, \vec{v'} \in V$ since V is closed under scalar multiplication.

7. By 2, $\vec{u'} + \vec{v'} \in V$ since V is closed under vector addition.

8. That is, $\alpha\vec{u} + (1-\alpha)\vec{v} \in V$.

9. By 4-8, for any vectors $\vec{u}, \vec{v} \in V$, $\alpha\vec{u} + (1-\alpha)\vec{v} \in V$.

10. That is, V is convex.

11. By 1-10, for any subspace V, V is convex.

Problem 5: Consider the system of equations

$$\begin{bmatrix} 2 & -1 & 2 & -1 & 3 \\ 1 & 2 & 3 & 1 & 0 \\ 1 & 0 & -2 & 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 14 \\ 5 \\ -10 \end{bmatrix}.$$

Write down the augmented matrix of the system and convert it to the canonical form using elementary row operations. State explicitly which basis your canonical form is with respect to.

$$\text{Augmented matrix will be: } \left[\begin{array}{ccccc|c} 2 & -1 & 2 & -1 & 3 & 14 \\ 1 & 2 & 3 & 1 & 0 & 5 \\ 1 & 0 & -2 & 0 & -5 & -10 \end{array}\right]$$

Row reducing to put it into canonical form with respect to the basis of the first three column vectors will yield:

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & -2/17 & -23/17 & -4/17 \\ 0 & 1 & 0 & 11/17 & -35/17 & -80/17 \\ 0 & 0 & 1 & -1/17 & 31/17 & 83/17 \end{array}\right]$$

Problem 6: Consider the system of equations

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 0 & 2 & 0 & 0 & 2 \\ 3 & 4 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}.$$

Write down the augmented matrix of the system and convert it to the canonical form using elementary row operations(Hint: the first 4 columns are linearly independent).

$$\text{Augmented matrix will be: } \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 2 & 2 \\ 3 & 4 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{array}\right]$$

Row reducing to put it into canonical form with respect to the basis of the first three column vectors will yield:

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -2/3 & -4/3 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 2/3 & 1/3 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{array}\right]$$