Yoneda Lemma Notes

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1 Warm-Up: Cayley's Theorem

Before talking about the Yoneda Lemma, we have to talk about Cayley's Theorem, which we can't do without talking about groups.

1.1 Groups

Definition 1.1.1. A group $(G, \cdot, e)$ consists of a set G, together with a binary operation $\cdot : G \times G \to G$ and an element $e \in G$ satisfying the following properties:

- For all $x, y, z \in G$, $x(yz) = (xy)z$.
- For all $x \in G$, $ex = xe = x$.
- For each $x \in G$, there exists $y \in G$ so that $xy = yx = e$.

when the operation is clear from context, we refer to $(G, \cdot, e)$ as simply G.

Example 1.1.2. The group $GL(n)$ of invertible $n \times n$ matrices is a group under matrix multiplication.

Example 1.1.3. $GL(n)$ has a subgroup $SO(n)$ which consists of those matrices which also have determinant 1.

Example 1.1.4. For any set X, the set $S(X)$ of bijections $X \to X$ is a group under composition.

Example 1.1.5. The **free group on two generators**, F_2, is the set of reduced strings over the alphabet $\{a, a^{-1}, b, b^{-1}\}$, where a string w is **reduced** if no two adjacent symbols of w are inverses. The group operation is given by concatenation, followed by iterated removal of forbidden substrings.

Exercise 1.1.6. *Prove that each of the above examples satisfies the group axioms.*

1.2 The Category of Groups

Definition 1.2.1. Let G and H be groups. A function $\phi : G \to H$ is a homomorphism if $\phi(x)\phi(y) = \phi(xy)$ for all $x, y \in G$.

Exercise 1.2.2. *Let* $\textsc{Grp}$ *be the class of all groups. Prove that taking* $\textsc{Grp}(G, H)$ *to be the set of homomorphisms from* G *to* H *defines a category with composition given by function composition, which we will call the category of groups. Prove that a morphism* $\phi : G \to H$ *is an isomorphism iff the underlying function of sets is a bijection.* ***(Hint: show that the function*** ϕ^{-1} ***is a homomorphism.)***

1.3 Cayley's Theorem

Most of the examples were given as a set of symmetries, in a sense made precise by the following definition:

Definition 1.3.1. A permutation group P is a group which is isomorphic to a subgroup of $S(X)$ for some set X.

Remark 1.3.2. The examples of groups that we discussed were mostly of the form "the set of bijections $X \to X$ which satisfy property P," and so are permutation groups. However, Example 1.1.5 may appear to buck this trend; it is not immediately obvious that F_2 is a permutation group.

Theorem 1.3.3 (Cayley's Theorem)**.** *Let* G *be a group. Then* G *is isomorphic to a subgroup of* $S(G)$*. In particular,* G *is a permutation group.*

Proof. Let $H = \{f \in S(G) \mid \exists g \in G \,.\, \forall x \in G \,.\, f(x) = gx\}$. We define $\phi : G \to H$ by $\phi(g)(x) = gx$. ϕ is obviously surjective, and is injective since $\phi(g)(e) = ge = g$ for all $g \in G$. Finally, ϕ is an isomorphism, since if $g, g' \in G$, then $(\phi(g) \circ \phi(g'))(x) = \phi(g)(g'x) = gg'x = \phi(gg')(x)$, and in particular, $\phi(gg') = \phi(g) \circ \phi(g')$. $\square$

The upshot here is that every group, even F_2, is a symmetry group of **something**. The most noticeable consequence of Cayley's Theorem is that group theorists don't bother to distinguish symmetry groups from general groups; they just call them all, well, groups!

2 The Yoneda Lemma

2.1 Functors and Equivalences

Definition 2.1.1. Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and let $F : \mathcal{C} \to \mathcal{D}$ be a functor. We say that F is **full** if the induced function $f^* : \mathcal{C}(X, Y) \to \mathcal{D}(FX, FY)$ is always surjective, and we call F **faithful** if f^* is always injective. If F is both full and faithful, we say that F is **fully faithful**, or a **fully faithful embedding** when we want to think of $\mathcal{C}$ as a subcategory of $\mathcal{D}$.

Definition 2.1.2. A **full subcategory** $\mathcal{C}'$ of a category $\mathcal{C}$ is a subcategory such that the inclusion $i : \mathcal{C}' \hookrightarrow \mathcal{C}$ is a full functor.

Exercise 2.1.3. *Prove that the inclusion of a subcategory is always a faithful functor.*

Definition 2.1.4. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A functor $F : \mathcal{C} \to \mathcal{D}$ is **essentially surjective** if it is surjective on isomorphism classes of objects. Explicitly, F is essentially surjective if for every object $Y \in \mathcal{D}$, there exists an isomorphism $\phi : F(X) \xrightarrow{\sim} Y$ for some object $X \in \mathcal{C}$. We say that F is a **weak equivalence of categories**, or just an equivalence if F is fully faithful and essentially surjective.

Exercise 2.1.5. *Let $F : \mathcal{C} \to \mathcal{D}$ be a fully faithful functor. Prove that there is a full subcategory $\mathcal{D}' \subseteq \mathcal{D}$ and an equivalence $\widetilde{F} : \mathcal{C} \to \mathcal{D}'$ so that $F \cong i \circ \widetilde{F}$, where $i : \mathcal{D}' \to \mathcal{D}$ is the inclusion functor.*

Consider a full subcategory $\mathcal{C}$ of SET which contains the terminal object $1 = \{*\}$. The functor $h^1 = \mathcal{C}(1, -) : \text{SET} \to \text{SET}$ is then naturally isomorphic to the inclusion functior $i : \mathcal{C} \hookrightarrow \text{SET}$, by the transformation $\alpha : h^1 \Rightarrow i$ given by $\alpha_X(f) = f(*)$.

Exercise 2.1.6. *Suppose that $\mathcal{C}'$ is a full subcategory of $\mathcal{C}$, and that the terminal object $1 \in \mathcal{C}$ is also in $\mathcal{C}'$. Prove that 1 is a terminal object of $\mathcal{C}'$.*

Now let's stop and think about how nice this situations is. Suppose I give you a category $\mathcal{D}$, and I tell you that $\mathcal{D}$ is equivalent to a full subcategory $\mathcal{C}$ of SET which contains the terminal object, but I don't tell you which subcategory it is, or what the inclusion functor is. Then you can recover the inclusion functor up to a unique natural isomorphism by taking the hom-functor h^1, where 1 is the terminal object of $\mathcal{D}$. In particular, given an object $X \in \mathcal{D}$, you can figure out which set X is! Given an arbitrary category $\mathcal{C}$, even if there is a terminal object $1 \in \mathcal{C}$, we've no right to expect that $\mathcal{C}$ be equivalent to a full subcategory of SET.

Exercise 2.1.7. *Let $\mathcal{FD}$ be the small category whose set of objects $\{\mathbb{R}^n \mid 0 \leq n\}$ consists of one vector space of each finite dimension, and whose morphisms $\mathbb{R}^n \to \mathbb{R}^m$ are given by linear transformations, composed normally. Prove that $1 = \mathbb{R}^0$ is a terminal object, and that $h^1 : \mathcal{FD} \to \text{SET}$ is not faithful. Conclude that $\mathcal{FD}$ is not equivalent to any full subcategory of SET which contains the terminal object of SET.*

Exercise 2.1.8. *Suppose that $i : \mathcal{C} \to \text{SET}$ is fully faithful, and that X and Y are objects of $\mathcal{C}$ with two distinct morphisms $f \neq g : X \to Y$. Prove that any terminal object $\mathcal{C}$ must be mapped to a terminal object of SET under i. Conclude that $\mathcal{FD}$ is not equivalent to a full subcategory of SET.*

Exercise 2.1.9. *Let $F : \mathcal{C} \to \mathcal{D}$ be a fully faithful functor, and let $F(g) \in \mathcal{D}(F(A), F(B))$ be an isomorphism. Prove that g is an isomorphism.*

Exercise 2.1.10. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories, let $F, G : \mathcal{C} \to \mathcal{D}$ be functors, and $\alpha : F \Rightarrow G$ a natural transformation. Further suppose that α_X is an isomorphism for each $X \in \mathcal{C}$. Prove that α is an isomorphism in the category $[\mathcal{C}, \mathcal{D}]$ of functors from $\mathcal{C}$ to $\mathcal{D}$. (**Hint: Once you figure out what you're trying to prove, the answer is immediate**)*

2.2 Presheaves

Let $\mathcal{C}$ be a locally small category. Then for each object $A \in \mathcal{C}$, there is a representable contravariant functor $h_A = \mathcal{C}(-, A) : \mathcal{C}^{op} \to \textsc{Set}$. h_A acts on morphisms by sending each arrow $g : X \to Y$ to the function $h_A(g) = (- \circ g) : \mathcal{C}(Y, A) \to \mathcal{C}(X, A)$. Now suppose that $f : A \to B$ is a morphism in $\mathcal{C}$. Then f induces a natural transformation $h_f : h_A \Rightarrow h_B$. For each object $X \in \mathcal{C}$, $(h_f)_X : \mathcal{C}(X, A) \to \mathcal{C}(X, B)$ is defined by $(h_f)_X(t) = f \circ t$. Checking naturality amounts to checking that for any $g : X \to Y$, the diagram commutes:

$$\begin{tikzcd}
\mathcal{C}(Y,A) \arrow[r, "h_A(g)=(-\circ g)"] \arrow[d, "(h_f)_Y=(f\circ -)"'] & \mathcal{C}(X,A) \arrow[d, "(h_f)_X=(f\circ -)"] \\
\mathcal{C}(Y,B) \arrow[r, "h_B(g)=(-\circ g)"'] & \mathcal{C}(X,B)
\end{tikzcd}$$

or in other words, that for each map $m \in \mathcal{C}(Y, A)$, $(f \circ m) \circ g = f \circ (m \circ g)$. Furthermore, given $f : A \to B$ and $g : B \to C$, $h_g \circ f = h_{g \circ f}$, since precomposing a map with $f \circ g$ is the same as precomposing it with g and then precomposing the result with f. Finally, $h_{\mathrm{Id}} = \mathrm{Id}$, since composing with the identity gives you back what you started with.

Definition 2.2.1. Let $\mathcal{C}$ be a category. A **presheaf** on $\mathcal{C}$ is a contravariant functor from $\mathcal{C}$ to $\textsc{Set}$. If F and G are presheaves on $\mathcal{C}$, a morphism $F \to G$ of presheaves is a natural transformation. We denote the category of presheaves on $\mathcal{C}$ as $\mathrm{PSh}(\mathcal{C})$.

Remark 2.2.2. The previous discussion amounts to $h_{(-)}$ being a (covariant) functor from $\mathcal{C}$ to the category $\mathrm{PSh}(\mathcal{C})$ of **presheaves** on $\mathcal{C}$.

2.3 The Yoneda Lemma

Theorem 2.3.1 (The Yoneda Lemma)**.** *Let F be a presheaf on $\mathcal{C}$. Then there is bijection $\mathrm{PSh}(\mathcal{C})(h_X, F) \cong F(X)$ which is natural in X.*

Exercise 2.3.2 (The Yoneda Embedding)**.** *Assume the Yoneda lemma, and prove that the functor $h_{(-)} : \mathcal{C} \to \mathrm{PSh}(\mathcal{C})$ is a fully faithful embedding.*

Remark 2.3.3. In light of Exercise 2.3.2, we can now justify the choice to define $\mathrm{PSh}(\mathcal{C})$ as the category of *contravariant* functors from $\mathcal{C}$ to $\textsc{Set}$. We really care about the Yoneda embedding $h_{(-)}$, which allows us to freely interchange between objects of $X \in \mathcal{C}$ and representable presheaves $h_X \in \mathrm{PSh}(\mathcal{C})$. Indeed, we began with a bifunctor $\mathcal{C}(-, -) : \mathcal{C}^{op} \times \mathcal{C} \to \textsc{Set}$. We can give an alternate description of $h_{(-)}$ as the currying of $\mathcal{C}(-, -)$ into the form $\mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{op}, \textsc{Set})$.

Exercise 2.3.4. *Let $\mathcal{C}$ be a category, and define* $\mathrm{coPSh}(\mathcal{C})$ *as the category of* ***covariant*** *functors $\mathcal{C} \to \textsc{Set}$. Assuming Exercise* 2.3.2*, show that there is a fully faithful embedding $h^{(-)} : \mathcal{C}^{op} \to \mathrm{coPSh}(\mathcal{C})$. Then, assuming the Yoneda lemma, prove that for any copresheaf $F \in \mathrm{coPSh}(\mathcal{C})$, there is a bijection* $\mathrm{coPSh}(\mathcal{C})(h^X, F) \cong F(X)$ *which is natural in X.*

Corollary 2.3.5. *Suppose that $A, B \in \mathcal{C}$, and that $\alpha : h_A \Rightarrow h_B$ is a natural isomorphism. Then there is a unique isomorphism $\widetilde{\alpha} : A \to B$ so that $h_{\widetilde{\alpha}} = \alpha$.*

Proof. The result follows immediately from the Yoneda lemma and Exercise 2.1.9 $\square$

Exercise 2.3.6. *Prove that the collection of representables is a separating family of* $\mathrm{PSh}(\mathcal{C})$

Example 2.3.7. Let $\mathcal{C} = \{V \rightrightarrows E\}$ be the category with two distinct non-identity morphisms $s, t : V \to E$. Then a presheaf $F \in \mathrm{PSh}(\mathcal{C})$ amounts to sets $F(E), F(V)$ with source and target maps $s, t : F(E) \to F(V)$, or in other words, a directed reflexive multigraph. A graph homomorphism $\phi : F \to G$ is consists of maps $\phi_E : F(E) \to G(E)$ and $\phi_V : F(V) \to G(V)$ which commutes with the source and target maps. The representable presheaf h_V consists of a single vertex Id_V and no edges, and its source and target maps are thus trivial. The representable h_E consists of a single edge Id_E and two distinct vertices $s, t \in h_E(V) = \mathcal{C}(V, E)$. In other words, h_V is the graph with one vertex, and h_E is the graph with a single edge and distinct endpoints.

Let $G \in \mathrm{PSh}(\mathcal{C})$ be a graph. We will try to understand the maps from representables into G. A graph homomorphism from h_V to G amounts to a mapping of vertices and edges which preserves source and target relations. If the Yoneda lemma is true, we should expect these to correspond to vertices of G. Indeed, there are no edges in h_V, and thus no source or target relations to preserve, so everything comes free besides the map of vertices. There's only one vertex in V, so that map is just given by an element of $G(V)$, as expected.

Now we direct our attention toward morphisms $\alpha : h_E \to G$. Such a morphism is determined by maps

$$\alpha_E : h_E(E) \to G(E)$$
$$\alpha_V : h_E(V) \to G(V)$$

and must satisfy

$$\alpha_E(s(e)) = s(\alpha_E(e))$$
$$\alpha_E(t(e)) = t(\alpha_E(e))$$

for each edge $e \in h_E(E)$. But h_E only has one edge, so our choice of morphism is reduced to picking an edge $e' \in G(E)$, along with the function $\alpha_V : h_E(V) = \{s, t\} \to G(V)$. But the constraints above can then be rewritten as

$$\alpha_E(s) = s(e')$$
$$\alpha_E(t) = t(e')$$

which looks an awful lot like a *definition* of α_V. Thus, the natural transformations from h_E to G are in bijection with $G(E)$.

2.4 Limits, Revisited

We will defer the proof of the Yoneda lemma to section 2.5 on the following page. First, we will attempt to understand limits in a locally small category $\mathcal{C}$ via presheaves on $\mathcal{C}$.

Let D be a small category. We define $\mathrm{pt} : D^{op} \to \textsc{Set}$ as the functor which sends every object of D to $1 = \{*\}$, and every morphism to the unique function $1 \to 1$.

Exercise 2.4.1. *Prove that* pt *is the terminal object of* $\mathrm{PSh}(\mathcal{C})$.

Theorem 2.4.2. *Let J be a small category, and $D \in \mathrm{PSh}(J)$ a J^{op}-shaped diagram of sets. Then the limit $\lim D$ can be identified with the set $\mathrm{PSh}(J)(\mathrm{pt}, D)$ of natural transformations from* pt *to D.*

Proof. Recall that the limit of D can be identified with the subset of $\prod_{j \in J} F(j)$ given by

$$\lim D = \left\{ (x_j)_{j \in J} \in \prod_{j \in J} D(j) \;\middle|\; \forall j, j' \in J, g \in J(j, j'),\ D(j)(x_{j'}) = x_j \right\}$$

which is a set since J is small. On the other hand, natural transformations from pt to D are given by those indexed collections

$$(\eta_j)_{j \in J} \in \prod_{j \in J} \{\mathrm{pt}(j) \to D(j)\} \cong \prod_{j \in J} D(j)$$

which satisfy

$$\eta_j \circ \mathrm{pt}(g) = D(g) \circ \eta_{j'}$$

for all $j, j' \in J$, $g \in J(j, j')$. But that's exactly what we wrote for $\lim D$. □

Remark 2.4.3. In light of this theorem, we can plainly see that the operation which sends a presheaf on D to its limit is just the functor $h^{\mathrm{pt}} : \mathrm{PSh}(D) \to \textsc{Set}$. In particular, $D \mapsto \lim D$ can naturally be made into a covariant functor.

Exercise 2.4.4 (Limits of Diagrams are Pointwise)**.** *[1, Proposition 8.8] Let $\mathcal{C}$ and $\mathcal{D}$ be (possibly large) categories. If D is complete, then so is the functor category $[\mathcal{C}, \mathcal{D}]$, and for every object $X \in \mathcal{C}$, the evaluation functor $\mathrm{ev}_X : [\mathcal{C}, \mathcal{D}] \to \mathcal{D}$ is continuous.*

Definition 2.4.5. Let $\mathcal{C}$ be a locally small category, J a small category, and $D : J^{op} \to \mathcal{C}$ a J^{op}-shaped diagram in $\mathcal{C}$. We define the **presheaf-valued limit**, $\widehat{\lim} D$ as the limit of the diagram $h_{D(-)} : J^{op} \to \mathrm{PSh}(\mathcal{C})$.

Remark 2.4.6. The definition of $\widehat{\lim} D$ is such that morphisms from h_X to D, which are the same as elements of $D(X)$, are just cones from X to D. In other words, if D really does have a limit L, then morphisms from h_X to F are just cones from X to F, which are themselves just morphisms from X to L

Definition 2.4.7 (Improved Definition of Limits)**.** Let $\mathcal{C}$ be a locally small category, and let $D : J \to \mathcal{C}$. We then define a limit of D as an object $L \in \mathcal{C}$, together with an isomorphism $h_L \cong \widehat{\lim} D$. In other words, the limit of D is the representing object of $\widehat{\lim} D$.

Exercise 2.4.8. *Prove that Definition* 2.4.7 *is equivalent to the standard definition. Conclode that the Yoneda embedding is continuous functor.*

Remark 2.4.9. Remember that our goal was to understand limits in an arbitrary (locally small) category $\mathcal{C}$. Unfortunately, a category $\mathcal{C}$ need not have small limits. However, by Exercise 2.4.4, that $\mathrm{PSh}(\mathcal{C})$ has all limits. Furthermore, by Exercise 2.4.8, if $D : J^{op} \to \mathcal{C}$ happens to have a limit L, there will be a unique isomorphism $L \cong \widehat{\lim} D$ which preserves the limit structure.

2.5 Proof of The Yoneda Lemma

Proof. Let $\mathcal{C}$ be a locally small category, let F be a presheaf on $\mathcal{C}$, and X an object of $\mathcal{C}$. We will define a family of maps $\alpha_X : \mathrm{PSh}(\mathcal{C})(h_X, F) \to F(X)$, and then show that they make up a natural isomorphism.

First, observe that $\mathrm{PSh}(\mathcal{C})(h_X, F) = h_F(h_X)$ as functors in X. For any morphism of presheaves $\beta : h_X \to F$, let $\alpha_X(\beta) = \beta_X(\mathrm{Id}_X)$. To show that α_X is natural in X, we want to show that for any $g : X \to Y$, the following diagram commutes,

$$\begin{tikzcd}
h_F(h_Y) \arrow[r, "h_F(h_g)"] \arrow[d, "\alpha_Y"'] & h_F(h_X) \arrow[d, "\alpha_X"] \\
F(Y) \arrow[r, "F(g)"'] & F(X)
\end{tikzcd}$$

which is to say that for any $\beta \in h_F(h_Y)$ (equivalently, $\beta : h_Y \to F$), we want to show that $F(g)(\alpha_Y(\beta)) = \alpha_X(\beta \circ h_g)$. Now, applying the definition of α gives us:

$$F(g)(\alpha_Y(\beta)) = F(g)(\beta_Y(\mathrm{Id}_Y))$$

$$\begin{aligned}
\alpha_X(\beta \circ h_g) &= (\beta \circ h_g)_X(\mathrm{Id}_X) \\
&= \beta_X(g)
\end{aligned}$$

from which point it suffices to show that $F(g)(\beta_Y(\mathrm{Id}_Y)) = \beta_X(g)$, which immediately follows from the naturality of β.

All that remains is to show that α_X is a bijection for each $X \in \mathcal{C}$. First we will show that α_X is surjective. Let $x \in F(x)$. We wish to construct a morphism $\eta : h_X \to F$ so that

$\alpha_X(\eta) = x$. For $Y \in \mathcal{C}$, we define $\eta_Y : h_X(Y) \to F(Y)$ as $\eta_Y(g) = F(g)(x)$. We verify:

$$\begin{aligned}
\alpha_X(\eta) &= \eta_X(\mathrm{Id}_X) \\
&= F(\mathrm{Id}_X)(x) \\
&= \mathrm{Id}_{F(X)}(x) \\
&= x
\end{aligned}$$

Next, we will check that η is a natural transformation. Indeed, if $g : Y \to Z$ is a morphism of $\mathcal{C}$, then for any $k \in h_X(Z)$, we have

$$\begin{aligned}
\eta_Y(h_X(g)(k)) &= \eta_Y(k \circ g) \\
&= F(k \circ g)(x) \\
&= F(g)(F(k)(x)) \\
&= F(g)(\eta_Z(k))
\end{aligned}$$

Finally, we will check that α_X is injective. Suppose that $\beta, \gamma : h_X \to F$ are morphisms of presheaves, such that $\alpha_X(\beta) = \alpha_X(\gamma)$, or in other words:

$$\beta_X(\mathrm{Id}_X) = \gamma_X(\mathrm{Id}_X)$$

We want to show, for an arbitrary $g : Y \to X$, that $\beta_Y(g) = \gamma_Y(g)$. We have the following commutative diagrams due to the naturality of β and γ,

$$\begin{tikzcd}
h_X X \arrow[r, "h_X(g)"] \arrow[d, "\beta_X"'] & h_X Y \arrow[d, "\beta_Y"] \\
F(X) \arrow[r, "F(g)"'] & F(Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
h_X X \arrow[r, "h_X(g)"] \arrow[d, "\gamma_X"'] & h_X Y \arrow[d, "\gamma_Y"] \\
F(X) \arrow[r, "F(g)"'] & F(Y)
\end{tikzcd}$$

which allow us to compute

$$\begin{aligned}
\beta_Y(g) &= \beta_Y(\mathrm{Id}_X \circ g) \\
&= \beta_Y(h_X(g)(\mathrm{Id}_X)) \\
&= F(g)(\beta_X(\mathrm{Id}_X)) \\
&= F(g)(\gamma_X(\mathrm{Id}_X)) \\
&= \gamma_Y(h_X(g)(\mathrm{Id}_X) \\
&= \gamma_Y(\mathrm{Id}_X \circ g) \\
&= \gamma_Y(g)
\end{aligned}$$

completing the proof that α_X is an isomorphism. $\square$

References

[1] Steve Awodey, *Basic Category Theory*, Oxford Logic Guides, second edition, 2010.