Yoneda Lemma Notes

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1 Warm-Up: Cayley's Theorem

Before talking about the Yoneda Lemma, we have to talk about Cayley's Theorem, which we can't do without talking about groups.

Definition 1.1. A group $(G, \cdot, e)$ consists of a set G, together with a binary operation $\cdot : G \times G \to G$ and an element $e \in G$ satisfying the following properties:

- For all $x, y, z \in G$, $x(yz) = (xy)z$.
- For all $x \in G$, $ex = xe = x$.
- For each $x \in G$, there exists $y \in G$ so that $xy = yx = e$.

when the operation is clear from context, we refer to $(G, \cdot, e)$ as simply G.

Example 1.2. The group $GL(n)$ of invertible $n \times n$ matrices is a group under matrix multiplication.

Example 1.3. $GL(n)$ has a subgroup $SO(n)$ which consists of those matrices which also have determinant 1.

Example 1.4. For any set X, the set $S(X)$ of bijections $X \to X$ is a group under composition.

Example 1.5. The **free group on two generators**, F_2, is the set of reduced strings over the alphabet $\{a, a^{-1}, b, b^{-1}\}$, where a string w is **reduced** if no two adjacent symbols of w are inverses. The group operation is given by concatenation, followed by iterated removal of forbidden substrings.

Exercise 1.6. *Prove that each of the above examples satisfies the group axioms.*

Before continuing, we define morphisms of groups:

Definition 1.7. Let G and H be groups. A function $\phi : G \to H$ is a homomorphism if $\phi(x)\phi(y) = \phi(xy)$ for all $x, y \in G$.

Exercise 1.8. *Let* Grp *be the class of all groups. Prove that taking* Grp(G, H) *to be the set of homomorphisms from G to H defines a category with composition given by function composition, which we will call the category of groups. Prove that a morphism $\phi : G \to H$ is an isomorphism iff the underlying function of sets is a bijection.* ***(Hint: show that the function ϕ^{-1} is a homomorphism.)***

Some of the example groups are given as a set of symmetries, in a sense made precise by the following definition:

Definition 1.9. A symmetry group P is a group which is isomorphic to a subgroup of $S(X)$ for some set X.

Examples 1.2-1.4 are obviously symmetry groups, but it's less obvious whether or not Example 1.5 is. For that question, we have Cayley's Theorem.

Theorem 1.10 (Cayley's Theorem)**.** *Let G be a group. Then G is isomorphic to a subgroup of $S(G)$. In particular, G is a symmetry group.*

Proof. Let $H = \{f \in S(G) \mid \exists g \in G\,.\,\forall x \in G\,.\, f(x) = gx\}$. We define $\phi : G \to H$ by $\phi(g)(x) = gx$. ϕ is obviously surjective, and is injective since $\phi(g)(e) = ge = g$ for all $g \in G$. Finally, ϕ is an isomorphism, since if $g, g' \in G$, then $(\phi(g) \circ \phi(g'))(x) = \phi(g)(g'x) = gg'x = \phi(gg')(x)$, and in particular, $\phi(gg') = \phi(g) \circ \phi(g')$. □

The upshot here is that every group, even F_2, is a symmetry group of **something**. The most noticeable consequence of Cayley's Theorem is that group theorists don't bother to distinguish symmetry groups from general groups; they just call them all, well, groups!

2 The Yoneda Lemma: Categorical Preliminaries

Definition 2.1. Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and let $F : \mathcal{C} \to \mathcal{D}$ be a functor. We say that F is **full** if the induced function $f^* : \mathcal{C}(X, Y) \to \mathcal{D}(FX, FY)$ is always surjective, and we call F **faithful** if f^* is always injective. If F is both full and faithful, we say that F is **fully faithful**.

Definition 2.2. A **full subcategory** $\mathcal{C}'$ of a category $\mathcal{C}$ is a subcategory such that the inclusion $i : \mathcal{C}' \hookrightarrow \mathcal{C}$ is a full functor.

Exercise 2.3. *Prove that the inclusion of a subcategory is always a faithful functor.*

Definition 2.4. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A functor $F : \mathcal{C} \to \mathcal{D}$ is **essentially surjective** if it is surjective on isomorphism classes of objects. Explicitly, F is essentially surjective if for every object $Y \in \mathcal{D}$, there exists an isomorphism $\phi : F(X) \xrightarrow{\sim} Y$ for some object $X \in \mathcal{C}$. We say that F is a **weak equivalence of categories**, or just an equivalence if F is fully faithful and essentially surjective.

Exercise 2.5. *Let $F : \mathcal{C} \to \mathcal{D}$ be a fully faithful functor. Prove that there is a full subcategory $\mathcal{D}' \subseteq \mathcal{D}$ and an equivalence $\widetilde{F} : \mathcal{C} \to \mathcal{D}'$ so that $F \cong i \circ \widetilde{F}$, where $i : \mathcal{D}' \to \mathcal{D}$ is the inclusion functor.*

Consider a full subcategory $\mathcal{C}$ of SET which contains the terminal object $1 = \{*\}$. The functor $h^1 = \mathcal{C}(1, -) : \text{SET} \to \text{SET}$ is then naturally isomorphic to the inclusion functior $i : \mathcal{C} \hookrightarrow \text{SET}$, by the transformation $\alpha : h^1 \Rightarrow i$ given by $\alpha_X(f) = f(*)$.

Exercise 2.6. *Suppose that $\mathcal{C}'$ is a full subcategory of $\mathcal{C}$, and that the terminal object $1 \in \mathcal{C}$ is also in $\mathcal{C}'$. Prove that 1 is a terminal object of $\mathcal{C}'$.*

Now let's stop and think about how nice this situations is. Suppose I give you a category $\mathcal{D}$, and I tell you that $\mathcal{D}$ is equivalent to a full subcategory $\mathcal{C}$ of SET which contains the terminal object, but I don't tell you which subcategory it is, or what the inclusion functor is. Then you can recover the inclusion functor up to a unique natural isomorphism by taking the hom-functor h^1, where 1 is the terminal object of $\mathcal{D}$. In particular, given an object $X \in \mathcal{D}$, you can figure out which set X is! Given an arbitrary category $\mathcal{C}$, even if there is a terminal object $1 \in \mathcal{C}$, we've no right to expect that $\mathcal{C}$ be equivalent to a full subcategory of SET.

Exercise 2.7. *Let $\mathcal{FD}$ be the small category whose set of objects $\{\mathbb{R}^n \mid 0 \leq n\}$ consists of one vector space of each finite dimension, and whose morphisms $\mathbb{R}^n \to \mathbb{R}^m$ are given by linear transformations, composed normally. Prove that $\mathbb{R}^0$ is a terminal object, and that $h_{\mathbb{R}^0} : \mathcal{FD} \to \text{SET}$ is not faithful. Conclude that $\mathcal{FD}$ is not equivalent to any full subcategory of* SET *which contains the terminal object of* SET.

Exercise 2.8. *Suppose that $i : \mathcal{C} \to \text{SET}$ is fully faithful, and that X and Y are objects of $\mathcal{C}$ with two distinct morphisms $f \neq g : X \to Y$. Prove that any terminal object $\mathcal{C}$ must be mapped to a terminal object of* SET *under i. Conclude that $\mathcal{FD}$ is not equivalent to a full subcategory of* SET.

Let $\mathcal{C}$ be a locally small category. Then for each object $A \in \mathcal{C}$, there is a representable contravariant functor $h_A = \mathcal{C}(-, A) : \mathcal{C}^{op} \to \text{SET}$. h_A acts on morphisms by sending each arrow $g : X \to Y$ to the function $h_A(g) = (- \circ g) : \mathcal{C}(Y, A) \to \mathcal{C}(X, A)$. Now suppose that $f : A \to B$ is a morphism in $\mathcal{C}$. Then f induces a natural transformation $h_f : h_A \Rightarrow h_B$. For each object $X \in \mathcal{C}$, $(h_f)_X : \mathcal{C}(X, A) \to \mathcal{C}(X, B)$ is defined by $(h_f)_X(t) = f \circ t$. Checking naturality amounts to checking that for any $g : X \to Y$, the diagram commutes:

$$\begin{tikzcd}
\mathcal{C}(Y, A) \arrow[r, "h_A(g) = (- \circ g)"] \arrow[d, "(h_f)_Y = (f \circ -)"'] & \mathcal{C}(X, A) \arrow[d, "(h_f)_X = (f \circ -)"] \\
\mathcal{C}(Y, B) \arrow[r, "h_B(g) = (- \circ g)"'] & \mathcal{C}(X, B)
\end{tikzcd}$$

or in other words, that for each $h \in \mathcal{C}(Y, A)$, $(f \circ h) \circ g = f \circ (h \circ g)$.