MATH4404 Theorems

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The theorem about the existence of a completion of a normed space.

Theorem 1. If E is a normed linear space, then there exists a complete normed space $\tilde{E}$ such that

i) $E \subset \tilde{E}$
ii) $||x||_E = ||x||_{\tilde{E}}$
iii) E is dense in $\tilde{E}$

Proof. Construct $\tilde{E}$ as equivalence classes of Cauchy Sequences $(x_1, x_2, \dots,) \sim (y_1, y_2, \dots,)$ if $(x_1, y_1, x_2, y_2, \dots,)$ is also Cauchy. We embed each $x \in E$ as $(x, x, x, \dots,) \in \tilde{E}$. This is certainly a Cauchy sequence and as such, well defined.

Now we will show that our operations $(+, \cdot)$ are well defined. Take two sequences $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ in $\tilde{E}$ such that $(x_n)_{n=1}^{\infty} \in [x']$ and $(y_n)_{n=1}^{\infty} \in [y']$. Our operations is defined as below,

$$(x_1, x_2, \cdots,) + (y_1, y_2, \dots,) = (x_1 + y_1, x_2 + y_2, \dots,)$$

We wish to show that this sequence is an element of $[x' + y']$. Consider the sequence,

$$(z_n)_{n=1}^{\infty} = (x_1 + y_1, x' + y', x_2 + y_2, x' + y', \dots,)$$

We will show this is Cauchy.

Given $\varepsilon > 0$, there exists large enough $N \in \mathbb{N}$ such that (x_n) and (y_n) behave as expected. Now for all $n > m$, there are two cases,

$$\begin{aligned} ||z_n - z_m|| &= ||x_n + y_n - x_m - y_m|| \\ &\leq ||x_n - x_m|| + ||y_n - y_m|| \\ &< \varepsilon \end{aligned}$$

or

$$\begin{aligned} ||z_n - z_m|| &= ||x_n + y_n - x' - y'|| \\ &\leq ||x_n - x'|| + ||y_n - y'|| \\ &< \varepsilon \end{aligned}$$

So we have $(z_n)_{n=1}^{\infty} \in [x' + y']$.

Multiplication is defined as expected. The norm in $\tilde{E}$ is defined as follows,

$$||[(x_1, x_2, \dots,)]||_{\tilde{E}} = \lim_{n\to\infty} ||x_n||_E$$

We will now show that the properties of the norm hold. It is trivial that $||\cdot||_{\tilde{E}\geq 0}$ since $||\cdot||_E \geq 0$. We will now show that $x = 0 \iff ||x||_{\tilde{E}} = 0$

$\Longrightarrow$
Let $(x_n)_{n=1}^\infty$ be an element of $[0]$, so we know for every $\varepsilon > 0$ and large enough n, $||x_n - 0|| < \varepsilon$. Continuing,

$$\begin{aligned} ||[(x_1, x_2, \dots,)]||_{\tilde{E}} &= \lim_{n\to\infty} ||x_n||_E \\ &< \varepsilon \end{aligned}$$

This holds for every $\varepsilon > 0$ So we have $||[(x_1, x_2, \dots,)]|| = 0$.

$\Longleftarrow$
Assume $||[(x_1, x_2, \dots,)]|| = 0$, we wish to show $(x_1, x_2, \dots,) \in [0]$. By the continuity of the norm (Reverse Triangle Inequality),

$$\begin{aligned} 0 &= \lim_{n\to\infty} ||x_n|| \\ &= ||\lim_{n\to\infty} x_n|| \end{aligned}$$

We have that E is a normed space so it holds that $||x_n||_E = 0 \implies x_n = 0$. So it is quite clear that $(x_1, 0, x_2, 0, \dots,)$ is Cauchy. Thus, $(x_n)_{n=1}^\infty \in [0]$.

Now we will prove the Triangle Inequality, let $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ in $\tilde{E}$. Wish to show $||[(x_1, x_2, \dots)] + [(y_1, y_2, \dots,)]|| \leq ||[(x_1, x_2, \dots)]|| + ||[(y_1, y_2, \dots)]||$

$$\begin{aligned} ||[(x_1, x_2, \dots)] + [(y_1, y_2, \dots,)]|| &= \lim_{n\to\infty} ||x_n + y_n|| \\ &\leq \lim_{n\to\infty} ||x_n|| + ||y_n|| \\ &= \lim_{n\to\infty} ||x_n|| + \lim_{n\to\infty} ||y_n|| \end{aligned}$$

Since $(||x_n||)_{n=1}^\infty$ and $(||y_n||)_{n=1}^\infty$ are Cauchy $\mathbb{R}$ and $\mathbb{R}$ is complete. So $||\cdot||_{\tilde{E}}$ is a valid norm. We will now show that $\tilde{E}$ is indeed complete.

Take some Cauchy sequence $(\tilde{x}_n)_{n=1}^\infty \subset \tilde{E}$, will show there exists an $\tilde{x} \in \tilde{E}$ such that $\lim_{n\to\infty} \tilde{x}_n = \tilde{x}$. Since E is dense in $\tilde{E}$, for every $n \in \mathbb{N}$ there exists $x_n \in E$ such that

$$||(x_n, x_n, \dots) - \tilde{x}_n|| < \frac{1}{n}$$

$$\begin{aligned} (x_1, x_1, x_1, \dots,) - \tilde{x}_2 &< 1 \\ (x_2, x_2, x_2, \dots,) - \tilde{x}_2 &< \frac{1}{2} \end{aligned}$$

Define $\tilde{x} = [(x_1, x_2, x_3, \dots)] = [(x_n)_{n=1}^\infty]$. We now need to verify that $(x_n)_{n=1}^\infty$ is indeed Cauchy.

$$\begin{aligned} ||x_k - x_m||_E &= ||(x_k, x_k, \dots,) - (x_m, x_m, \dots,)||_{\tilde{E}} \\ &\leq ||(x_k, x_k, \dots,) - \tilde{x}_k|| + ||\tilde{x}_k - \tilde{x}_m||_{\tilde{E}} + ||\tilde{x}_m - (x_m, x_m, \dots,)||_{\tilde{E}} \\ &\leq \frac{1}{k} + \frac{1}{m} + ||\tilde{x}_k - \tilde{x}_m||_{\tilde{E}} \end{aligned}$$

Since $(\tilde{x}_n)_{n=1}^{\infty}$ is Cauchy this goes to 0 as $k, m \to \infty$. Thus $(x_n)_{n=1}^{\infty}$ is Cauchy.

$$\begin{aligned} ||\tilde{x}_n - \tilde{x}||_{\tilde{E}} &= ||\tilde{x}_n - (x_n, x_n, \dots,)||_{\tilde{E}} + ||(x_n, x_n, \dots,) - \tilde{x}||_{\tilde{E}} \\ &\leq \frac{1}{n} + \lim_{m\to\infty} ||x_n - x_m||_E \end{aligned}$$

Since $(x_n)_{n=1}^{\infty}$ we have that this goes to 0 as $n, m \to \infty$, and so $(\tilde{x}_n)_{n=1}^{\infty}$ converges to $\tilde{x}$. □

The Bunyakovsky–Cauchy–Schwarz inequality.

Theorem 2. Let H be an inner product space, for all $x, y \in H$,

$$|(x, y)|^2 \leq (x, x)(y, y)$$

Proof. If $x = 0$ or $y = 0$ use linearity to prove. Assume $y \neq 0$. Let $\lambda = \frac{(x,y)}{(y,y)}$. Note that

$$\begin{aligned} (x - \lambda y, y) &= (x, y) - \lambda(y, y) \\ &= (x, y) - \frac{(x, y)}{(y, y)}(y, y) \\ &= 0 \end{aligned}$$

So,

$$\begin{aligned} 0 &\leq (x - \lambda y, x - \lambda y) \\ &= (x - \lambda y, x) - \overline{\lambda}(x - \lambda y, y) \\ &= (x - \lambda y, x) \\ &= (x, x) - \lambda(y, x) \\ &= (x, x) - \frac{|(x, y)|^2}{(y, y)} \end{aligned}$$

So, $|(x, y)|^2 \leq (x, x)(y, y)$. □

The theorem about the equivalence of all the norms in $\mathbb{R}^n$.

Theorem 3. Any two norms in $\mathbb{R}^n$ are equivalent. If $|| \cdot ||_1$ and $|| \cdot ||_2$ are norms in $\mathbb{R}^n$ then there exists $c_1, c_2 > 0$ such that

$$c_1||x||_1 \leq ||x||_2 \leq c_2||x||_1$$

Proof. We may assume that $|| \cdot ||_1$ is the standard Euclidian norm in $\mathbb{R}^n$. We know that the unit sphere (w.r.t the standard norm) $S_1(0)$ is compact in $\mathbb{R}^n$ since it is closed and bounded (Heine-Borel). We will now prove that the function $x \mapsto ||x||_2$ is a continuous function. Indeed, consider the sequence $(x_n)_{n=1}^{\infty}$ which converges to x, we wish to show that $||x_n||_2$ converges to $||x||_2$.

$$|||x_n|| - ||x||| \leq ||x_n - x||_2 \to 0$$

So indeed this function is continuous. Therefore by EVT it attains its maximum and minimum on $S_1(0)$. Therefore

$$\begin{aligned} ||x||_2 &= ||x||_1 \times \left|\left|\frac{x}{||x||_1}\right|\right|_2 \\ &\leq ||x||_1 \times \sup_{x\in S_1(0)} ||x||_2 \end{aligned}$$

Analogously for the other direction, inf non-zero since $x \in S_1(0)$. □

A functional (operator) is continuous at one point if and only if it is continuous everywhere.

Theorem 4. (Same for both) Let E_1, E_2 be normed vector spaces, if A, a linear operator between E_1 and E_2 is continuous at one point, then A is continuous everywhere.

Assume A is continuous at the point $x_0 \in E_1$. Given $x \neq x_0$ we wish to prove A is continuous at x. Take a sequence $(x_n)_{n=1}^{\infty} \subset E$ such that $\lim_{n\to\infty} x_n = x$. We will show $\lim_{n\to\infty} A(x_n) = A(x)$.

$$\begin{aligned}
\lim_{n\to\infty} A(x_n + x_0 - x_0 + x - x) &= \lim_{n\to\infty} A((x_n - x) + x_0) + A(x - x_0) \\
&\to A(x_0) + A(x) - A(x_0) \\
&= A(x)
\end{aligned}$$

So $A(x_n) \to A(x)$, so A is continuous.

A functional (operator) is continuous if and only if it is bounded.

Theorem 5. (Same for both) Let E be a normed vector space and l be a linear functional on E. l is continuous if and only if it is bounded.

Proof. $\implies$ Assume l is bounded, we wish to show that l is continuous at one point, and thus everywhere. Consider a sequence $(x_n)_{n=1}^{\infty} \subset E$ such that $\lim_{n\to\infty} x_n = 0$.

$$|l(x_n)| \leq c||x_n|| \to 0 \quad n \to \infty$$

So l is continuous at 0 and thus everywhere.

$\impliedby$ For the sake of contradiction, assume l is continuous but unbounded. There exists a sequence $(x_n)_{n=1}^{\infty} \subset E$ such that

$$|l(x_n)| \geq n||x_n||$$

Define $\hat{x}_n = \frac{x_n}{n||x_n||}$, we have that $||\hat{x}_n|| = \frac{1}{n} \to 0$ as $n \to \infty$. So by continuity, $l(\hat{x}_n)$ should go to 0. However,

$$|l(\hat{x}_n)| = \frac{|l(x_n)|}{n||x_n||} \geq 1$$

Here lies our contradiction. □

The theorem about the completeness of the dual space.

Theorem 6. Let E be a normed vector space, E', it's dual, is complete.

Proof. Take a Cauchy sequence $(l_n)_{n=1}^{\infty} \subset E'$. Let us construct its limit l. For $x \in E$, we have

$$|l_n(x) - l_m(x)| \leq ||l_n - l_m|| \cdot ||x|| \to 0$$

As $n, m \to \infty$. So let us define,

$$l(x) = \lim_{n\to\infty} l_n(x)$$

We will now show linearity and boundedness of l.

$$\begin{aligned}
l(\lambda x + \gamma y) &= \lim_{n\to\infty} (l_n(\lambda x + \gamma y)) \\
&= \lim_{n\to\infty} (\lambda l_n(x) + \gamma l_n(y)) \\
&= \lambda l(x) + \gamma l(y)
\end{aligned}$$

$$
\begin{aligned}
||l|| &= \sup_{||x||\le 1} ||l(x)|| \\
&= \sup_{||x||\le 1} || \lim_{n\to\infty} l_n(x)|| \\
&\le \sup_{||x||\le 1} \lim_{n\to\infty} ||l_n|| \cdot ||x|| \\
&\le C \cdot \sup_{||x||\le 1} ||x|| \\
&\le C
\end{aligned}
$$

For some $C > 0$ as $(||l_n||)_{n=1}^{\infty}$ is Cauchy, it is bounded. It remains to show $||ln - l|| \to 0$. Since $(l_n)_{n=1}^{\infty}$ is Cauchy, for large enough n and m,

$$|l_n - l_m| < \varepsilon||x||$$

Now pass the limit $m \to \infty$

$$
\begin{aligned}
\lim_{m\to\infty} |l_n - l_m| &= |l_n - l| \\
&\le \varepsilon||x||
\end{aligned}
$$

So finally,

$$||l_n - l|| \to 0$$

□

The Hahn–Banach theorem.

Theorem 7. Let E be a normed space and G a linear subset. For every $l \in G'$, there exists some $L \in E'$ such that $L|_G = l$ and $||L|| = ||l||$.

Lemma 1 (Dense subset). Let E be a normed space, $G \subset E$ a dense subset and $l \in G'$. Then there exists a unique $L \in E'$ such that $L|_G = l$ and $||L|| = ||l||$.

Proof. Take $x \in E$. By density there exists a sequence $(g_n)_{n=1}6\infty \subset E$ such that $\lim_{n\to\infty} g_n = x$. Define

$$L(x) = \lim_{n\to\infty} l(g_n)$$

Why is this well defined?

$$|l(g_n) - l(g_m)| \le ||l|| \cdot ||g_n - g_m|| \to 0$$

as $n, m \to \infty$. Thus $(l(g_n))_{n=1}^{\infty}$ converges. Why is $L(x)$ independent of the sequence $(g_n)_{n=1}^{\infty}$. Indeed, consider $(g_n')_{n=1}^{\infty} \subset E$ converging to x.

$$|l(g_n) - l(g_n')| \le ||l|| \cdot (||g_n - x|| + ||g_n' - x||) \to 0$$

as $n \to \infty$. $L|_G = l$ since for every $x \in G$, take $g_n = x$ for every $n \in \mathbb{N}$. Let us show that the norm is preserved.

$$
\begin{aligned}
|L(x)| &= |\lim_{n\to\infty} l(g_n)| \\
&\le \lim_{n\to\infty} ||l|| \cdot ||g_n|| \\
&= ||l|| \cdot \lim_{n\to\infty} ||g_n|| \\
&= ||l|| \cdot ||x||, \quad \text{by continuity of the norm}
\end{aligned}
$$

Thus $||L|| \le ||l||$ and so $||L|| = ||l||$. Now we will show uniqueness, consider $L_1|_G = L_2|G = l$, by continuity,

$$L_1(x) = \lim_{n\to\infty} l(g_n) = L_2(x)$$

□

Lemma 2 (Extension of a functional by one dimension). Let E be a real normed space and $G \subset E$ a linear subspace. Choose $y \notin G$ and define $F = \text{span}(G \cup \{y\})$. For any linear bounded functional l on G. There is a bounded linear functional $L \in F'$ such that $L|_G = l$ and $||L|| = ||l||$.

Proof. Every $x \in F$ can be written as

$$x = g + \lambda y$$

where $g \in G$ and $\lambda \in \mathbb{R}$ in a unique way. Define $L : F \to \mathbb{R}$ by,

$$L(x) = l(y) + \lambda \cdot c$$

for some $c \in \mathbb{R}$. Let us now show that we can pick such a $c \in \mathbb{R}$ so $||L|| = ||l||$, it is sufficient to show that $||L|| \leq ||l||$, that is $||L||$ has not increased any larger than $||l||$.

The inequality $||L|| \leq ||l||$ is equivalent to

$$\begin{aligned} |L(g + \lambda y)| &= |l(y) + \lambda \cdot c| \\ &\leq ||l|| \cdot |g + \lambda y| \qquad (*) \end{aligned}$$

Sufficient to prove this for $\lambda > 0$, since $g + \lambda y = -(-g + (-\lambda)y)$. We can rewrite $(*)$ as follows,

$$-||l|| \cdot |g + y| - l(y) \leq \lambda c \leq ||l|| \cdot |g + y| - l(y)$$

Now let $h = \lambda^{-1} g$ and divide the inequality by λ

$$-||l|| \cdot |h + y| - l(h) \leq c \leq ||l|| \cdot |h + y| - l(h)$$

To find such a c, notice that for all $h_1, h_2 \in G$.

$$\begin{aligned} l(h_2) - l(h_1) &\leq ||l|| \cdot ||h_2 - h_1 + y - y|| \\ &\leq ||l|| \cdot ||(h_2 + y) + -(h_1 + y)|| \\ &\leq ||l|| \cdot |h_1 + y| + ||l|| \cdot |h_2 + y|| \end{aligned}$$

Therefore,

$$-||l|| \cdot |h_1 + y| - l(h_1) \leq ||l|| \cdot ||h_2 + y|| - l(h_2)$$

This implies that

$$\sup_{h_1 \in G} (-||l|| \cdot |h_1 + y| - l(h_1)) \leq \inf_{h_2 \in G} (||l|| \cdot ||h_2 + y|| - l(h_2))$$

Choosing c in between this sup and inf will obtain $(*)$. ☐

Lemma 3 (Real and separable). Let E be a real, linear, separable normed space and $G \subset E$ a linear subset. For every $l \in G'$ there exists an $L \in E'$ such that $L|_G = l$ and $||L|| = ||l||$.

Proof. Let

$$A = \{x_1, x_2, x_3, \dots\}$$

be a dense subset of E. Choose x_{n_1} to be the first element of A that does not lie in G. Define $G_1 = \text{span}(G \cup \{x_{n_1}\})$. Extend l to $l_1 : g_1 \to \mathbb{R}$ preserving the norm by Lemma 2. Repeating this procedure we obtain the subspaces,

$$G \subset G_1 \subset G_2 \subset \cdots$$

and similarly, functionals $l, l_1, l_2, \dots$ all with the same norm.

Define

$$M = \bigcup_{i=1}^{\infty} G_n$$

This is a linear set and is dense in E since it contains all of A. Now define $L_0 : M \to \mathbb{R}$ as follows, if $x \in M$, then $x \in G_n$ for some $n \in \mathbb{N}$, so define

$$L_0(x) = l_n(x)$$

This clearly also preserves the norm since $||l|| = ||l_n||$ for every $n \in \mathbb{N}$. By continuity we can extend L_0 to all of E. $\square$

Lemma 4 (Complex and separable). Let E be a complex, linear, separable normed space and $G \subset E$ a linear subset. For every $l \in G'$ there exists an $L \in E'$ such that $L|_G = l$ and $||L|| = ||l||$.

Proof. We can consider E as a vector space over the reals, denote it $E_\mathbb{R}$. Take $k \in E'$ and let $m = \Re(k)$ and $n = \Im(k)$, both of these lie in $E_\mathbb{R}$. Let us show they are bounded

$$|m(x)| \le |m(x) + in(x)| = |k(x)| \le ||k|| \cdot ||x||$$

Similar for $|n(x)|$. Thus $||m||, ||n|| \le ||k||$. Observe

$$\begin{aligned} m(ix) + in(ix) &= k(ix) \\ &= ik(x) \\ &= i(m(x) + in(x)) \\ &= -n(x) + im(x) \end{aligned}$$

So $m(ix) = -n(x)$ and $n(ix) = m(x)$.

We have that l is defined on $G \subset E$. We can extend l from $G_\mathbb{R}$ to $E_\mathbb{R}$ and moreover $\Re(l) = m \in G'_\mathbb{R}$ can be extended to $M \in E'_\mathbb{R}$ by Lemma 3.

Define

$$L(x) = M(x) - iM(ix)$$

Clearly $L|_G = l$ and L is linear, indeed,

$$\begin{aligned} L((\alpha + i\beta)x) &= \alpha L(x) + \beta L(ix) \\ &= \alpha L(x) + \beta(M(ix) - iM(ix)) \\ &= \alpha L(x) + \beta i(M(x) - iM(-x)) \end{aligned}$$

Now we will show $||L|| = ||l||$, indeed take θ such that

$$L(x) = |L(x)|e^{i\theta}$$

Now,

$$\begin{aligned} |L(x)| &= L(x)e^{-i\theta} \\ &= L(xe^{-i\theta}) \\ &= M(xe^{-i\theta}) \\ &\le |M(xe^{-i\theta})| \\ &\le ||M|| \cdot ||xe^{-i\theta}|| \\ &= ||M|| \cdot |e^{-i\theta}| \cdot ||x|| \\ &= ||M|| \cdot ||x|| \\ &\le ||l|| \cdot ||x|| \end{aligned}$$

So $||L|| \le ||l||$. $\square$

Proof. (The field over E is $\mathbb{R}$) Let l_P be an extension of l to the subspace $P \subset E$ containing G. At least one such an extension exists by Lemma 2.

Let X be the set of all such extensions which preserve the norm. Define $l_p \leq l_q$ if $P \subset Q$ and $l_p = l_q|_P$. This is a partial ordering on X.

Let $Y = \{l_{P_\alpha} \mid \alpha \in A\}$ be a chain in X. This chain has an upper bound, set

$$\tilde{P} = \bigcup_{\alpha \in A} P_\alpha$$

The define $\tilde{l}$ on $\tilde{P}$ by,

$$\tilde{l}(x) = l_\alpha(x)$$

if $x \in P_\alpha$. Clearly well defined and $||\tilde{l}|| = ||l||$. We can then extend $\tilde{l}$ to the closure of $\tilde{P}$ by continuity, then $\tilde{l}$ is an upper bound for Y.

By Zorn's Lemma, X has a maximal element, denoted L. This is defined on all of E since if it weren't we could extend it and it wouldn't be maximal. By construction $\tilde{l}$ is a norm-preserving extension of l.

If E is complex, same argument as used in the separable case. □

The theorem about a monomorphism between a normed space and its second dual.

Theorem 8. The map $\Phi : E \to E''$ given by $x \mapsto \phi_x$ is an isometric mono-morphism.

Proof. Let us first show linearity,

$$\begin{aligned}\Phi(\alpha x + \beta y)(l) &= \phi_{\alpha x + \beta y}(l) \\ &= l(\alpha x + \beta y) \\ &= \alpha l(x) + \beta l(y) \\ &= \alpha \phi_x(l) + \beta \phi_y(l) \\ &= \alpha \Phi(x)(l) + \beta \Phi(y)(l)\end{aligned}$$

Now we will show the isomoprhism. Take $x \neq y$, we will show that $\phi_{x-y} \neq 0$. Indeed, by Hahn-Banach, there exists some $l \in E'$ such that $l(x-y) = ||x-y|| \neq 0$. Then,

$$\phi_{x-y}(l) = l(x-y) \neq 0$$

Now we will show the norm is preserved. Indeed

$$\begin{aligned}|\phi_x(l)| &= |l(x)| \\ &\leq ||l|| \cdot ||x||\end{aligned}$$

So $||\phi_x|| \leq ||x||$ Now we will show $||x|| \leq ||phi_x||$. If $x = 0$ this is trivial. Take $x \neq 0$. By Hahn Banach, there exists some $l \in E'$ such that $||l|| = 1$ and $l(x) = ||x||$.

$$\begin{aligned}||x|| &= l(x) \\ &= \phi_x(l) \\ &\leq ||\phi_x|| \cdot ||l|| \\ &= ||\phi_x||\end{aligned}$$

□

The Banach–Steinhaus theorem.

Theorem 9. Let E be a Banach space. Consider a sequence $(l_n)_{n=1}^{\infty} \subset E'$. Assume that for every $x \in E$, there exists $c_x > 0$ such that

$$|l_n(x)|| \leq c_x$$

for all $n \in \mathbb{N}$. The conclusion is that there exists $c > 0$ such that

$$||l_n|| \leq c$$

for all $n \in \mathbb{N}$.

Lemma 1 (Nested Ball Lemma). Consider a sequence of balls,

$$\overline{B_{r_1}(x_1)} \supset \overline{B_{r_1}(x_2)} \supset \overline{B_{r_3}(x_3)} \supset \cdots$$

Such that $r_i \to 0$ as $i \to \infty$. Then

$$\bigcap_{i=1}^{\infty} \overline{B_{r_i}(x_i)} \neq \emptyset$$

Proof. To begin $(x_i)_{i=1}^{\infty}$ is a Cauchy sequence. Denote

$$x^* = \lim_{i \to \infty} x_i$$

Given $m \in \mathbb{N}$, $x_k \in \overline{B_{r_m}(x_m)}$ for all $k \geq m$. By closedness we know the limit point $x^* \in \overline{B_{r_m}(x_m)}$. Thus x^* is in all of the balls and the intersection is non-empty. □

Lemma 2 (Baire Category Theorem). Let $(C_n) \subset E$ be a sequence of closed sets such that

$$\bigcup_{n=1}^{\infty} C_n = E$$

Then there exists $n \in \mathbb{N}$, $x \in E$ and $r > 0$ such that $B_r(x) \subset C_n$.

Proof. Assume the contrary, choose $x \in \underline{E \setminus C_1}$. Since C_1 is closed, the complement must be open. Thus there must exist some $r_1 > 0$ such that $\overline{B_{r_1}(x)} \subset E \setminus C_1$. Now by $\underline{\text{our assumption}}$, $B_{r_1}(x) \cap (E \setminus C_2) \neq \emptyset$. Take $x_2 \in B_{r_1}(x) \cap (E \setminus C_2)$. There exists some $r_2 < \frac{r_1}{2}$ such that $\overline{B_{r_2}(x_2)} \subset B_{r_1}(x) \cap (E \setminus C_2)$.

Now there exists some $x_3 \in B_{r_2}(x_2) \cap (X \setminus C_3)$. Choose $r_3 < \frac{r_2}{2}$ such that $\overline{B_{r_3}(x_3)} \in B_{r_2}(x_2) \cap (E \setminus C_3)$. Continuing this procedure we obtain,

$$\overline{B_{r_1}(x_1)} \supset \overline{B_{r_2}(x_2)} \supset \cdots$$

With $r_n \to 0$ as $n \to \infty$. Clearly

$$\bigcap_{n=1}^{B} {}_{r_n}(x_n) \supset \{x\}$$

for some $x \in E$. By construction $x \in E \setminus C_n$ for all $n \in \mathbb{N}$. This means that

$$\bigcup_{n=1}^{\infty} C_n \neq E$$

Contradiction. □

Lemma 3. There exists $x \in E$ and $r > 0$ such that the set

$$S_{r,a} = \{l_n(x) \mid x \in \overline{B_r(a)}, n \in \mathbb{N}\}$$

is bounded.

Proof. Denote
$$C_m = \{x \in E \mid |l_n(x)| \leq m . \forall n \in \mathbb{N}\}$$
We have that C_m is closed. Indeed if $(x_k)_{k=1}^{\infty} \subset C_m$ converges to some $x \in E$. Then for every $n \in \mathbb{N}$
$$|l_n(x)| = \left| \lim_{k\to\infty} l_n(x_k) \right| \leq m$$
By assumption of the theorem,
$$E = \bigcup_{m=1}^{\infty} C_m$$
Thus, by Lemma 2, at least one C_m contains a ball. □

Proof. Recall that
$$||l_n|| = \sup_{x \in B_1(0)} |l_n(x)|$$
It suffices to show that $|l_n(x)| \leq c$ for all $x \in \overline{B_1(0)}$ and $n \in \mathbb{N}$.

Given $x \in \overline{B_1(0)}$, define $x' = rx + a$ where r and a are from Lemma 3. Thus $x = \frac{1}{r}(x' - a)$ and
$$\begin{aligned} |l_n(x)| &= \left| l_n(\frac{1}{r}(x' - a)) \right| \\ &\leq \frac{1}{r}(|l_n(x')| + |l_n(a)|) \end{aligned}$$
By Lemma 3 the two summands are bounded by some $c' > 0$ and so
$$|l_n(x)| \leq \frac{2c'}{r}$$
and is bounded on $B_1(0)$ uniformly in n. □

The theorem about the weak compactness of the unit sphere in the dual space.

Theorem 10. If E is separable then the unit sphere in E' is weakly compact, i.e. $(l_n)_{n=1}^{\infty} \subset S^1(0)$ has a weakly convergent subsequence.

Theorem "Previous Theorem". Let E be a Banach space. Assume that M is dense in E. The sequence $(l_n)_{n=-1}^{\infty} \subset E'$ converges weakly to $l \in E'$ iff
a) $l_n(x) \to l(x)$ on M.
(b) $||l_n|| \leq c$ for some c independent of M

Proof. Let $M = \{x_1, x_2, x_3, \dots\}$ be dense in E. Then,
$$|l_n(x_1)| \leq ||l|| \cdot ||x_1|| = ||x_1||$$
So there exists a subsequence $(l_{k_1})_{k=1}^{\infty}$ such that
$$(l_{k_1}(x_1))_{k=1}^{\infty}$$
converges. By the same token, there is a subsequence $(l_{k_2})_{k=1}^{\infty} \subset (l_{k_1})_{k=1}^{\infty}$ such that
$$(l_{k_2}(x_2))_{k=1}^{\infty}$$
converges. Continuing in this manner to obtain a subsequence $(l_{k_q})_k^{\infty} = 1$ such that
$$(l_{k_q}(x_q))_{k=1}^{\infty}$$

converges for every q. The sequence,

$$(l_{k_k}(x_k))_{k=1}^{\infty}$$

converges for every $k \in \mathbb{N}$. Define

$$l(x_i) = \lim_{k\to\infty} l_{k_k}(x_i)$$

for $i \in \mathbb{N}$. This is quite clearly linear, further more,

$$||l|| \le \sup ||l_{k_k}|| = 1$$

We can then extend l to all of E by continuity. The previous theorem implies that $l_{k_k} \to l$ weakly. ☐

The theorem about the existence and uniqueness of the orthogonal projection in a Hilbert space.

Theorem 11. Let $G \subset H$ be a subspace. Given $x \in H$ there exists unique $y \in G$ such that $(x-y) \perp G$, i.e. $(x-y) \in G^{\perp}$.

Proof. If $x \in G$ then $y = x$. Assume $x \notin G$. Then let

$$d := \mathrm{dist}(x, G) = \inf_{y\in G} ||x-y|| > 0$$

Choose $(y_n)_{n=1}^{\infty} \subset G$ s.t $||x - y_n|| \to d$ as $n \to \infty$. We will show that $(y_n)_{n=1}^{\infty}$ is a Cauchy sequence. Take $h \in G$ such that $h \neq 0$ and $\eta \in \mathbb{C}$. Then,

$$\begin{aligned} d^2 &\le ||x - (y_n + \eta h)||^2 \\ &= (x - (y_n + \eta h), x - (y_n + \eta h)) \\ &= ||x - y_n||^2 + |\eta|||h^2|| - \overline{\eta}(x - y_n, h) - \eta(h, x - y_n) \end{aligned}$$

Now let

$$\eta = ||h||^{-2}(x - y_n, h)$$

This simplifies the above to

$$d^2 \le ||x - y_n||^2 - \frac{|(x - y_n, h)|^2}{||h||^2}$$

That is,

$$||h||^2||x - y_n||^2 - |(x - y_n, h)|^2 - d^2||h||^2 \ge 0$$

or

$$|(x - y_n, h)|^2 \le ||h||^2(d_n^2 - d^2) \quad (*)$$

Where $d_n = ||x - y_n||$. Thus,

$$|(x - y_n, h)| \le ||h||\sqrt{d_n^2 - d^2}$$

This holds for $h \in G$. Set $h = y_n - y_m$, for $n, m \in \mathbb{N}$. Then we have

$$\begin{aligned} |(y_n - y_m \pm x, h)| &\le |(x - y_n, y)| + |(x - y_m, h)| \\ &\le ||h||(\sqrt{d_n^2 - d} + \sqrt{d_m^2 - d}) \end{aligned}$$

So

$$||y_n - y_m||^2 \le ||y_n - y_m||(\sqrt{d_n^2 - d} + \sqrt{d_m^2 - d}) \to 0$$

as $n, m \to \infty$. Thus $(y_n)_{n=1}^{\infty}$ is Cauchy. Denote $y = \lim_{n\to\infty} y_n$, certainly $y \in G$. Passing the limit through to $(*)$ we find that $(x - y, h) = 0$ for all $h \in G$. Thus $y = \mathrm{proj}_G(x)$. Why is y unique? If $(x - y) \perp G$ and $(x - y') \perp G$ then

$$(y - y' \pm x, h) = (y - x, h) + (x - y', h) = 0$$

for all $h \in G$ so set $h = y - y'$, we find

$$||y - y'||^2 = 0$$

So $y = y'$. ☐

The Riesz representation theorem.

Theorem 12. Let H be a Hilbert space, given $y \in H$ the formula

$$x \mapsto (x, y) \quad (*)$$

defines a functional in H'. Conversely for every $l \in H'$, there exists some $y \in H$ such that

$$l(x) = (x, y)$$

for all $x \in H$. Moreover $||l|| = ||y||$.

Proof. $(*)$, we will denote this as l from here on is clearly linear, why is it bounded? By Cauchy-Schwarz we have

$$|(x, y)| \leq ||x|| \cdot ||y||$$

so

$$||l|| \leq ||y||$$

Indeed, this is an equality, since $|(y, y)| = ||y||^2$.

Conversely, take $l \in H'$. Denote $G = \ker(l)$. G is a subspace and $H = G \oplus G^{\perp}$. Moreover $\dim(G^{\perp})$ has dimension 1. Fix $c \in G^{\perp}$ with $||c|| = 1$. For every $x \in H$ we have $x = g + \lambda c$ where $g \in \ker(l)$ and $\lambda \in \mathbb{C}$. Thus,

$$\begin{aligned} l(x) &= l(g + \lambda c) \\ &= l(g) + \lambda l(c) \\ &= \lambda l(c) \end{aligned}$$

and

$$\begin{aligned} (x, c) &= (g + \lambda c, c) \\ &= (g, c) + \lambda (c, c) \\ &= \lambda \end{aligned}$$

We conclude that

$$l(x) = (x, c) l(c) = (x, \overline{l(c)} c)$$

Now take $y = \overline{l(c)} c$ to get $(*)$. Now why is y unique? If $l(x) = (x, y) = (x, y')$. Then $(x, y - y') = l(x) - l(x) = 0$ for all $x \in H$ so choose $x = y - y'$ and $||y - y'||^2 = 0$ so $y = y'$. □

The theorem about the invertibility of $\mathbb{1} - A$ when $||A||$ is small.

Theorem 13. Let E be a Banach Space. Let $A \in \mathcal{L}(E)$, satisfy $||A|| = q < 1$. Then $\mathbb{1} - A$ is invertible and

$$(\mathbb{1} - A)^{-1} = \mathbb{1} + A + A^2 + A^3 + \cdots$$

Proof. Let

$$S_n = \mathbb{1} + A + A^2 + \cdots + A^n$$

We claim that S_n is Cauchy, we have

$$\begin{aligned} ||S_{n+p} - S_n|| &= ||A^{n+1}|| + ||A^{n+2}|| + \cdots + ||A^{n+p}|| \\ &\leq q^{n+1} + \cdots + q^{n+p} \\ &\leq q^n \frac{1}{1 - q} \\ &\to 0 \end{aligned}$$

as $n \to \infty$. Since E is Banach so is $\mathcal{L}(E)$, so $(S_n)_{n=1}^{\infty}$ converges to S in norm. We need to show $S(\mathbb{1} - A) = (\mathbb{1} - A)S = \mathbb{1}$. We will first show that $(\mathbb{1} - A)S_n \to (\mathbb{1} - A)S$. Indeed

$$||(\mathbb{1} - A)S_n - (\mathbb{1} - A)S|| \leq ||(\mathbb{1} - A)|| \cdot ||S_n - S|| \to 0$$

It suffices to show that $(\mathbb{1} - A)S_n \to \mathbb{1}$. We have that

$$\begin{aligned}
(\mathbb{1} - A)S_n &= S_n(\mathbb{1} - A) \\
&= \mathbb{1} + A + \cdots + A^n - A - A^2 - \cdots - A^n - A^{n+1} \\
&= \mathbb{1} - A^{n+1}
\end{aligned}$$

So

$$\begin{aligned}
||(\mathbb{1} - A)S_n - \mathbb{1}|| &= ||S_n(\mathbb{1} - A) - \mathbb{1}|| \\
&= ||\mathbb{1} - A^{n+1} - \mathbb{1}|| \\
&= ||A^{n+1}|| \\
&\leq ||A||^{n+1} \\
&\to 0
\end{aligned}$$

as $n \to \infty$. □

The open mapping theorem and the bounded inverse theorem.

Theorem 14.1 (Open Mapping Theorem). Let E_1 and E_2 be Banach spaces. The operator $A \in \mathcal{L}(E_1, E_2)$ is open iff it is surjective.

Lemma 1. There exists some $y_0 \in E_1$, $r_0 > 0$ and $\alpha \in (0, 1)$ such that

$$B_{\alpha r_0}(Ay_0) \subset \overline{A(B_{r_0}(y_0))}$$

Proof. The sets $\overline{B_n(0)}$ are closed an their union over $n \in \mathbb{N}$ is E_2. By the Baire Category Theorem. Atleast one of these sets contains a ball, i.e. there exists $r > 0, n \in \mathbb{N}$ and $x \in E_2$ s.t.

$$B_r(x) \subset \overline{A(B_n(0))}$$

Choose $y_0 \in E_1$ such that $x = Ay_0$ and choose $r_0 > 0$ such that $B_{r_0}(y_0) \supset B_n(0)$. Thus,

$$B_r(Ay_0) \subset \overline{A(B_n(0))} \subset \overline{A(B_{r_0}(y_0))}$$

It suffices to set $\alpha = \frac{r}{r_0}$. □

Lemma 2. For every point $y \in E_1$ and $r > 0$

$$B_{\alpha r}(Ay) \subset \overline{A(B_r(y))}$$

Proof. Take $z \in B_{\alpha r}(Ay)$ then,

$$A(y_0) + \frac{r_0}{r}(z - Ay) \in B_{\alpha r_0}(Ay_0)$$

So,

$$A(y_0) + \frac{r_0}{r}(z - Ay) \in \overline{A(B_{r_0}(y_0))}$$

By Lemma 1. Hence there exists some sequence $(z_n)_{n=1}^{\infty} \subset B_{r_0}(y_0)$ such that

$$Ay_0 + \frac{r_0}{r}(z - Ay) - Az_n \to 0$$

Rewriting this we get,

$$z - A(y + \frac{r}{r_0}(z_n - y_0)) \to 0$$

and we have

$$A(y + \frac{r}{r_0}(z_n - y_0)) \in B_r(y)$$

So $z \in \overline{A(B_r(y))}$ $\square$

Lemma 3. For every $y \in E_1$ and every $r > 0$

$$B_{\frac{\alpha r}{2}}(Ay) \subset A(B_r(y))$$

Proof. Take $z \in B_{\frac{\alpha}{2}r}(Ay)$ and $\varepsilon < 1$. By Lemma 2 there exists some $y_1 \in B_{\frac{r}{2}}(y)$ such that

$$||z - Ay_1|| < \alpha\varepsilon$$

so, $z \in B_{\alpha\varepsilon}(Ay_1) \subset \overline{A(B_\varepsilon(y_1))}$. Again there is a $y_2 \in B_\varepsilon(y_1)$ such that

$$||z - Ay_2|| < \alpha\varepsilon^2$$

, then $z \in B_{\alpha\varepsilon^2}(Ay_2) \subset \overline{A(B_{\varepsilon^2}(y_2))}$. Continue this process to obtain a sequence $(y_n)_{n=1}^\infty$ such that

$$||z - Ay_n|| < \alpha\varepsilon^n$$

and

$$||y_n - y_{n+1}|| < \varepsilon^n$$

So $(y_n)_{n=1}^\infty$ converges to some $y^* \in E_1$ and $Ay^* = z$. Now we will show that y^* is indeed in $B_r(y)$.

$$\begin{aligned}
||y - y^*|| &\leq ||y \pm y_n \pm y_{n-1} \pm \cdots - y^*|| \\
&\leq ||y - y_n|| + \sum_{k=2}^{n} ||y_k - y_{k-1}|| + ||y_1 - y^*|| \\
&\leq 0 + \sum_{k=2}^{n} \varepsilon^k + \frac{r}{2} \\
&\leq \frac{\varepsilon}{1-\varepsilon} + \frac{r}{2}
\end{aligned}$$

We can choose ε small enough such that this is less than r. $\square$

Proof. $\Longrightarrow$

Assume A is an open mapping, we will show A is surjective. Indeed take $B_1(0) \subset E_1$, we know that $A(B_1(0))$ is an open set in E_2 so there exists $\varepsilon > 0$ such that

$$B_\varepsilon(0) \subset A(B_1(0))$$

We can write every $y \in E_2$ as en element of this ball by setting $\tilde{y} = \frac{\varepsilon}{2||y||}y$. Thus there exists $x \in B_1(0)$ such that

$$\begin{aligned}
A(x) &= \tilde{y} \\
A(x) &= \frac{\varepsilon}{2||y||}y \\
A\left(\frac{2||y||}{\varepsilon}x\right) &= y
\end{aligned}$$

So we have that A is surjective.

$\Longleftarrow$

Assume that A is surjective. Take an open set $V \subset E_1$, we need to show $A(V)$ is open. Take $y \in A(V)$, let x be such that $Ax = y$. Since V is open, there exists some $r > 0$ such that

$$B_r(x) \subset V$$

By Lemma 3 we have

$$B_{\frac{\alpha}{2}} r(Ax) \subset A(B_r(x)) \subset AV$$

This holds for all y and x satisfying the above, thus AV is open. $\square$

Theorem 14.2 (Bounded Inverse Theorem). Let $A \in \mathcal{L}(E_1, E_2)$ be a bijection. Then A is invertible. That is, $A^{-1} : E_2 \to E_1$.

Proof. For A^{-1} the pre-image of an open set is open. $\square$

The closed graph theorem.

Theorem 15. If E_1 and E_2 are Banach spaces, then $A : E_1 \to E_2$ is a linear operator and $\Gamma(A)$ is closed. Then A is bounded.

Proof. Observe that $\Gamma(A)$ is a Banach space with the norm

$$||(x, Ax)||_{\Gamma(A)} = ||x||_{E_1} + ||Ax||_{E_2}$$

Now define $\pi_1 : \Gamma(A) \to E_1$ and $\pi_2 : \Gamma(A) \to E_2$ by

$$\pi_1((x, Ax)) = x$$

and

$$\pi_2((x, Ax)) = Ax$$

Clearly π_1 and π_2 are bounded. Moreover π_1 is invertible by the Bounded Inverse Theorem. This implies that

$$A = \pi_2 \pi_1^{-1}$$

is bounded. $\square$

The theorem about the decomposition of an operator into it's real and imaginary parts.

Theorem 16. Let H be a Hilbert space. For every operator $A \in \mathcal{L}(H)$ there exists $\Re(A), \Im(A)$ both in $\mathcal{L}(H)$ such that they are self adjoint and $A = \Re(A) + \Im(a)$.

Proof. Let $\Re(A) = \frac{A+A^*}{2}$ and $\Im(A) = \frac{A-A^*}{2i}$.

$$(\Re(A))^* = \frac{A^* + (A^*)^*}{2} = \frac{A * +A}{2} = \Re(A)$$

$$(\Im(A))^* = \frac{A^* - (A^*)^*}{-2i} = \frac{A - A^*}{2i} = \Im(A)$$

$\square$

The Hellinger-Toeplitz Theorem

Theorem 17. Let H be a Hilbert space. Assume $A : H \to H$ is a linear and $(Ax, y) = (x, Ay)$ for all $x, y \in H$. Then A is bounded, i.e. $A \in \mathcal{L}(H)$.

Proof. Let us prove that $\Gamma(A)$ is closed. Take a sequence (x_n, Ax_n) converging to $(x, y) \in H \times H$. We wish to show that $y = Ax$. Given $z \in H$, we can see that

$$\begin{aligned}(z, y) &= \lim_{n\to\infty}(z, Ax_n) \\ &= \lim_{n\to\infty}(Az, x_n) \\ &= (Az, y) \\ &= (z, Ay)\end{aligned}$$

Thus $(z, y - Ax) = 0$ for all $z \in H$, take $z = y - Ax$ and $||y - Ax||^2 = 0$ so $y = Ax$. $\square$

Every sesquilinear form is generated by an operator.

Theorem 18. Let H be a Hilbert space. Assume b is a sesquilinear form on H such that

$$|b(x, y)| \leq c||x|| \cdot ||y||$$

for some $c > 0$. Then there exists some $A \in \mathcal{L}(H)$ such that $b(x, y) = (Ax, y)$.

Proof. Fix $x \in H$ and set $l(y) = \overline{b(x, y)}$. This is a bounded linear functional on H.

By Riesz Representation Theorem, there exists $a_x \in H$ such that

$$\overline{b(x, y)} = l(y) = (y, a_x)$$

Now set $Ax = a_x$. We will show that A is linear and bounded.

$$\begin{aligned}(A(\alpha x_1 + \beta x_2), y) &= b(\alpha x_1 + \beta x_2, y) \\ &= b(\alpha x_1, y) + b(\beta x_2, y) \\ &= \alpha b(x_1, y) + \beta(x_2, y) \\ &= \alpha(Ax_1, y) + \beta(Ax_2, y)\end{aligned}$$

$$||Ax||^2 = b(x, Ax) \leq c||x|| \cdot ||Ax||$$

So $||Ax|| \leq c||x||$ $\square$

Every self adjoint operator that squares itself is a projector

Theorem 19. Let $A \in \mathcal{L}(H)$ be a self-adjoint operator with $A^2 = A$. Then $A = P_G$ for some $G \subset H$.

Proof. Define $G = \ker(A - \mathbb{1})$. This is certainly a subspace of H. Since $A^2 = A$ for all $x \in H$ we have $Ax \in G$. Given $g \in G$,

$$\begin{aligned}(Ax, g) &= (x, Ag) \\ &= (x, g) \\ &= (x, P_G g) \\ &= (P_G x, g)\end{aligned}$$

So we have $(Ax - P_G x, g) = 0$ for all $g \in G$. Let $g = Ax - P_G x$ and we arrive at $||Ax - P_G x||^2 = 0$ so $Ax = P_G x$. $\square$

The theorem about the inverse of a unitary operator

Theorem 20. Let H be a Hilbert space and $A \in \mathcal{L}(H)$ a unitary operator. Then $A^* = A^{-1}$.

Proof. We know that A is invertible since

$$||Ax|| = ||x||$$

since A is unitary. Now given $x, y \in H$

$$\begin{aligned}(x, A^*y) &= (Ax, y) \\ &= (Ax, AA^{-1}y) \\ &= (x, A^{-1}y)\end{aligned}$$

So $A^*y = A^{-1}y$ for all $y \in H$. ☐

The Hilbert-Schmidt norm is basis independent

Theorem 21. If $(e_i)_{i=1}^{\infty}$ and $(f_i)_{i=1}^{\infty}$ are both orthonormal bases of H, then

$$\sum_{i=1}^{\infty} ||Ae_i||^2 = \sum_{i=1}^{\infty} ||Af_i||^2$$

Proof.

$$\begin{aligned}\sum_{i=1}^{\infty} ||Ae_i||^2 &= \sum_{i=1,j=1}^{\infty} |(Ae_i, f_j)|^2 \\ &= \sum_{i=1,j=1}^{\infty} |(e_i, A^*f_j)|^2 \\ &= \sum_{i=1,j=1}^{\infty} |(A^*f_j, e_i)|^2 \\ &= \sum_{i=1}^{\infty} ||A^*f_i||^2 \\ &= \sum_{k=1,j=1}^{\infty} |(A^*f_j, f_k)|^2 \\ &= \sum_{k=1,j=1}^{\infty} |(Af_k, f_j)|^2 \\ &= \sum_{k=1}^{\infty} ||Af_k||^2\end{aligned}$$

☐

The theorem about the closedness and boundedness of the spectrum.

Theorem 22.1. Let E be a normed space and consider an operator $A \in \mathcal{L}(E)$. Then

$$\operatorname{spec}(A) \subset \overline{B_{||A||}(0)}$$

Proof. Assume that $|z| > ||A||$, we will show that z is a regular point of A. That is, $(A - z\mathbb{1})$ is invertible. Indeed, $A - z\mathbb{1} = -z(\mathbb{1} - \frac{A}{z})$ and $\frac{||A||}{|z|} < 1$ so $(A - z\mathbb{1})$ is invertible and z is a regular point of A. ☐

Theorem 22.2. The spectrum of an operator A is closed.

Proof. Let z_0 be a regular point of A, if $z \in \mathbb{C}$ then

$$\begin{aligned} A - z\mathbb{1} &= A + z_0\mathbb{1} - z_0\mathbb{1} - z\mathbb{1} \\ &= A - z_0\mathbb{1} + (z_0 - z)\mathbb{1} \end{aligned}$$

Then choose z small enough so that $||(z_0 - z)\mathbb{1}|| < ||(A - z_0\mathbb{1})||^{-1}$. Then the sum is invertible. □

The spectrum is non-empty.

Theorem 23. Let A be an operator, $\mathrm{spec}(A) \neq \emptyset$.

Lemma 1. The function $z \mapsto ||R_z||$ is bounded on $\overline{B_{2||A||}(0)}$. Moreover,

$$\lim_{|z| \to \infty} ||R_z|| \to 0$$

Proof. Outside of $\overline{B_{2||A||}(0)}$,

$$\begin{aligned} ||R_z|| &= ||(-z)^{-1}(\mathbb{1} - A\frac{1}{z})^{-1}|| \\ &= |z|^{-1}||(\mathbb{1} - A\frac{1}{z})||^{-1} \\ &\leq \frac{1}{|z|} \cdot \frac{1}{1 - \frac{||A||}{|z|}} \\ &\leq \frac{1}{z} \cdot \frac{1}{1 - \frac{||A||}{2||A||}} \\ &= \frac{2}{z} \\ &\leq \frac{1}{||A||} \end{aligned}$$

□

Lemma 2. Fix $\alpha \in E$ and $l \in E'$. Define $f_{x,l} : \mathbb{C} \setminus \mathrm{spec}(A) \to \mathbb{C}$ by

$$f_{x,l}(z) = l(R_z(x))$$

Then $f_{x,l}$ is analytic.

Proof. For $z \in \mathbb{C} \setminus \mathrm{spec}(A)$,

$$\begin{aligned} \lim_{h \to 0} \frac{1}{h}(f_{x,l}(z+h) - f_{x,l}(z)) &= \lim_{h \to 0} \frac{1}{h} l((R_{z+h} - R_z)(x)) \\ &= l(\lim_{h \to 0} \frac{1}{h}(R_{z+h} - R_z)(x)) \\ &= l(\lim_{h \to 0} \frac{1}{h}(z + h - z)(R_{z+h}R_z)(x)) \\ &= l(R_z^2(x)) \end{aligned}$$

□

Proof. Assume $\mathrm{spec}(A) = \emptyset$. Then $f_{x,l}$ is analytic on $\mathbb{C}$. Moreover, it is bounded. Indeed $||R_z||$ is bounded on $\mathbb{C} \setminus \overline{B_{2||A||}(0)}$ by Lemma 1 and on $\overline{B_{2||A||}(0)}$ by continuity. Thus

$$\sup_{z \in \mathbb{C}} ||R_z|| = d < \infty$$

Therefore,

$$|f_{x,l}(z)| = |l(R_z(x))| \leq ||l|| \cdot ||x|| \cdot d$$

By Liousville's Theorem, $f_{x,l}$ is constant. Since $\lim_{|z|\to\infty} ||R_z|| = 0$, $f_{x,l}$ is identically 0. This means

$$l(R_z x) = 0$$

for all l and so $R_z x = 0$ for every x and so $R_z = 0$. This is impossible since R_z is invertible, here lies our contradiction. $\square$