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Exercise 5

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Homework Problem 5.1. (examples of operators and their norms)

Decide which of the following operators is a linear and bounded operator, and, if applicable, compute their respective operator norm. Here, Ω is assumed to be an open, bounded subset of $\mathbb{R}^n$ for an $n \in N$ and $[a, b]$ is a non-degenerated interval in $\mathbb{R}$ for $a, b \in \mathbb{R}$.

(a) $(L^2(\Omega), \|\cdot\|_{L^2}) \ni f \mapsto f \in (L^2(\Omega), \|\cdot\|_{L^2})$
(b) $(L^2(\Omega), \|\cdot\|_{L^2}) \ni f \mapsto f \in (L^1(\Omega), \|\cdot\|_{L^1})$
(c) $(L^6(\Omega), \|\cdot\|_{L^6}) \ni f \mapsto f^3 \in (L^2(\Omega), \|\cdot\|_{L^2})$
(d) $(C([a, b]), \|\cdot\|_\infty) \ni f \mapsto f \cdot g \in (C([a, b]), \|\cdot\|_\infty)$ for a fixed $g \in C([a, b])$
(e) $(C([a, b]), \|\cdot\|_\infty) \ni f \mapsto f - \int_a^b f(t)\,\mathrm{d}t \in (C([a, b]), \|\cdot\|_\infty)$
(f) $(W^{1,2}(a, b), \|\cdot\|_{W^{1,2}}) \ni f \mapsto f' \in (L^2(a, b), \|\cdot\|_{L^2})$ (mapping every function to its weak derivative)

Homework Problem 5.2. (convergence in the operator norm implies pointwise convergence)

Suppose that X and Y are normed linear spaces and $\big(A^{(k)}\big)$ is a sequence of bounded linear operators $X \to Y$. Show Lemma 4.6 of the lecture notes, i. e., that if $A^{(k)}$ converges to $A \in \mathcal{L}(X, Y)$ in the operator norm, then $A^{(k)}(x)$ converges to $A(x)$ for all $x \in X$.

Homework Problem 5.3. (boundedness is continuity)

Suppose that X and Y are normed linear spaces and $A\colon X \to Y$ is a linear operator. Prove Lemma 4.5 of the lecture notes, i. e., the equivalence of the following statements:

(a) A is continuous at 0.

(b) A is continuous on X.

(c) A is Lipschitz continuous.

(d) A is bounded.

Homework Problem 5.4. (composition of bounded linear operators)

Suppose that X, Y and Z are normed linear spaces and $B\colon Y \to Z$ as well as $A\colon X \to Y$ are bounded linear operators. Show that $B \circ A$ is a bounded linear operator from $X \to Z$ and show that

$$\|B \circ A\|_{\mathcal{L}(X,Z)} \leqslant \|B\|_{\mathcal{L}(Y,Z)} \|A\|_{\mathcal{L}(X,Y)}.$$

Give an example each for when this estimate holds with equality or strict inequality.

You are not expected to turn in your solutions.