

EXERCISE 9 (SOLUTION)

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Homework Problem 9.1. (Traces in L^p)

Let $\Omega := B_1^{\|\cdot\|_2}(0) \subseteq \mathbb{R}^2$. Show that there can not be an extension of the trace map $\tau: C(\overline{\Omega}) \rightarrow C(\partial\Omega)$ to a continuous map on $L^2(\Omega)$.

Solution.

Consider the family of functions $(f^{(k)}) \in C(\Omega)$, where $f^{(k)}: x \mapsto \|x\|^k$. Then

$$\|f^{(k)}\|_{L^2}^2 = \int_{\Omega} \|x\|^k \, dx = \int_0^{2\pi} \int_0^1 r^k \, dr \, d\varphi = 2\pi \frac{1}{k+1} \xrightarrow{k \rightarrow \infty} 0,$$

meaning that the $f^{(k)}$ converge to 0 in $L^2(\Omega)$, but as they are continuous, their boundary values on $\partial\Omega = S_2$ are 1 constantly, however the constant zero function is continuous as well with boundary values 0. This shows that the boundary trace operator can not be L^2 -continuous (even on $C(\Omega)$). It really is the L^2 topology that does not work well with the continuity of the extension.

Homework Problem 9.2. (The Lax-Milgram lemma)

- Let $n \in \mathbb{N}$, $b \in \mathbb{R}^n$ and a symmetric $A \in R^{n \times n}$ such that $x^\top Ax > c\|x\|_2^2$ for a $c \in \mathbb{R}_>$. Use the Lax-Milgram lemma to show that the linear system $Ax = b$ has a unique solution $x \in \mathbb{R}^n$.
- Let H be a Hilbert space and let $A: H \mapsto H$ be a bounded, linear operator such that $(Ax, x) \geq 0$ for every $x \in H$. Use the Lax-Milgram lemma to show that the operator $\text{id} + \alpha A: H \rightarrow H$ is bijective for every $\alpha \geq 0$. Show boundedness of $(\text{id} + \alpha A)^{-1}$.

Solution.

- (a) We define the bilinear form $a: (x, y) \mapsto x^T A y$ on the Hilbert space \mathbb{R}^2 with the euclidean inner product. By assumption, we have that

$$a(x, x) = x^T A x > c \|x\|_2^2,$$

i. e., ellipticity of the bilinear form. Additionally, we have that

$$|a(x, y)| = |x^T A y| = (x, A y)_2 \leq \|x\|_2 \|A y\|_2 \leq \|x\|_2 \|A\|_{2 \rightarrow 2} \|y\|_2$$

where $\|A\|_{2 \rightarrow 2}$ denotes the operator norm, so a is bounded. By the Lax-Milgram theorem, we obtain existence and uniqueness of the solution (and even the estimate $\|x\|_2 \leq \frac{1}{c} \|y\|_2$).

- (b) For injectivity, let $x, y \in H$ with $(\text{id} + \alpha A)x = (\text{id} + \alpha A)y$, then

$$\begin{aligned} 0 &= ((\text{id} + \alpha A)(x - y), x - y) \\ &= (\text{id}(x - y), x - y) + \alpha(A(x - y), x - y) \\ &= \|x - y\|^2 + \alpha(A(x - y), x - y) \\ &\geq 1\|x - y\|^2 \geq 0 \end{aligned}$$

meaning that $x = y$.

For surjectivity, let $v \in H$ and consider its image under the Riesz map, i. e., $(v, \cdot) \in H^*$. As id and A are linear and bounded, their (scaled) sum is as well. Accordingly, we have the boundedness of the bilinear form $(x, y) \mapsto ((\text{id} + \alpha A)x, y)$ because

$$((\text{id} + \alpha A)x, y) \leq \|(\text{id} + \alpha A)x\| \|y\| \leq \|\text{id} + \alpha A\| \|x\| \|y\|.$$

Additionally, by the same computation as for the injectivity, we obtain ellipticity with the constant 1. Lax-Milgram's lemma therefore yields a unique x such that $(Ax, \cdot) = (v, \cdot)$ and invertability of the Riesz map yields $\text{id} + \alpha A x = v$ uniquely.

That $(\text{id} + \alpha A)^{-1}$ is linear is clear, boundedness follows from

$$\|(\text{id} + \alpha A)^{-1}v\| = \|x\| \leq \frac{1}{1 + \alpha} \|v\|$$

by the a-priori estimate in Lax-Milgram's lemma.

You are not expected to turn in your solutions.