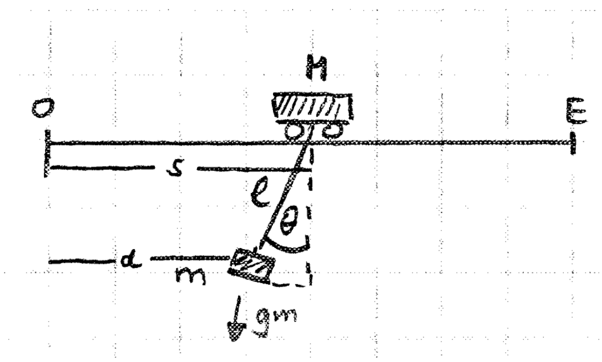
Exercise 4 - Solution

Problem 4.1. (Optimal Control of a hanging pendulum)

In this exercise, we want to investigate the linearized version of the control-constrained one dimensional problem of time/cost-optimally controlling a load on a hanging pendulum connected to a carriage system by a rod from an initial rest state to a final rest state. We consider being able to control the acceleration of the carriage (through forces). You can think of an application of a storage unit crane moving a load around.



The model is described by the following information.

(i) M is the mass of the carriage

(ii) m is the mass of the load hanging from the carriage

(iii) s is the x-displacement of the carriage

(iv) d is the x-displacement of the load

(v) Θ is the angle of the rod at the carriage relative to vertical

(vi) E is the final x-position that both the carriage and the load are supposed to end up in rest at.

(vii) The state variables of this problem are $x = (s, \dot{s}, z, \dot{z})$ where z is the x-position of the load relative to the carriage.

The corresponding optimal control problem reads as

$$
\begin{aligned}
\text{minimize} \int_0^T 1 + \frac{\gamma}{2} u(t)^2 \, \mathrm{d}t \qquad &\text{with respect to } (u, T) \in L^2(0, T) \times \mathbb{R} \\
\text{s.t. } \dot{x}(t) &= Ax(t) + Bu(t) \\
x(0) &= 0 \\
x(T) &= (E, 0, 0, 0) \\
u(t) &\in (-1, 1) \\
T &> 0
\end{aligned}
$$

for

$$
A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{g}{l} & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1/M \\ 0 \\ 1/M \end{bmatrix}
$$

and the regularization parameter $\gamma > 0$

Discretize the problem using piecewise constant controls, solve the problem numerically and investigate the behavior of the control and the influence of the problem parameters.

Solution.

A few words on how to get the dynamic model from the configuration. The nonlinear dynamical conditions in this setting read:

$$
\begin{aligned}
M\ddot{s} &= u \\
m\ddot{d} &= mg\sin(\Theta)
\end{aligned}
$$

We assume small angles Θ and linearize $\Theta \approx \sin(\Theta) = \frac{s-d}{l}$ to obtain the modified system

$$
\begin{aligned}
\ddot{s} &= \frac{1}{M} u \\
\ddot{d} &= \frac{g}{l}(s - d).
\end{aligned}
$$

Applying the notation $x \coloneqq (s, \dot{s}, z = s - d, \dot{z} = \dot{s} - \dot{d})$, we therefore obtain the system of ordinary differential equations

$$\dot{x} = (x_2, \frac{1}{M}u(t), x_4, -\frac{g}{l}x_3 + \frac{1}{M}u(t))^\intercal$$

which is precisely the system we have written down above.

The discrete form of this model is

$$\begin{aligned}
\text{minimize } & T + \mathrm{d}t\|u\|_2^2 \quad \text{with respect to } (u, T) \in \mathbb{R}^{N+1} \\
\text{s.t. } & x_0 = 0 \\
& x_{i+1} = x_i + f(\mathrm{d}t, x_i, u_i), \; i = 0, \dots, N \\
& x_N = (E, 0, 0, 0) \\
& u \in (-1, 1)^N
\end{aligned}$$

where $\mathrm{d}t = T/N$ is the step-length and $f(\mathrm{d}t, x_i, u_i)$ denotes the evaluation of an explicit one-step integrator such as the explicit Euler method.

We can employ an SQP method with finite difference approximations (or AD) to solve this problem.

In contrast to the Vandermonde-Oszillator, this problem is better behaved in the sense that it does admit a continuous solution and refining the discretization of the finite dimensional problem simply leads to a more accurate approximation of the continuous problem. The message of this exercise however is that it pays to think about the problem in a continuous manner to better interpret behavior of solutions that would be hard to expect in the discretized setting only.

Some effects you may want to observe in the results:

(i) The control cost parameter γ can be set to zero. The control will have Bang-Bang-structure, being active at its constraints essentially everywhere in the time interval.

(ii) Increasing the control cost parameter will favor controlling the system for longer time with less sharp forces (since the square of large numbers is unproportionally large and vice versa for small numbers).

(iii) Removing the control constraints on u and setting $\gamma = 0$, the problem does not have a solution in both the continuous and discrete level. Since the end time is free, one can produce a sequence of solutions with $T \longrightarrow 0$ by increasing the control indefinetely.

(iv) The problem will always have a solution regardless of E because the end time T is free, i. e., if we have to move the system far, we will simply take a long time. This changes if T were fixed.

In this case we can have situations where the problem is non-uniquely solvable or where no solutions exist.