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Exercise 3 - Solution

Problem 3.1. (Energy minimization problem for a rope)

Many models in mechanics originate in energy minimization principles. In this exercise, we will consider the problem of determining the postion of a rope under gravitational forces in a discretized setting – i.e., a number N of weights of mass m_i at positions (x_i, y_i), $i = 1, \dots, N$ that are connected by massless links. The links are assumed to be soft (but non-stretch) and the length of each link is at most ℓ.

(i) Model the problem for finding the unknown positions $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$ of the weights under gravitational load as a (finite-dimensional) potential energy minimization problem. At least one of the weights should be fixed at a certain position acting as supports, i.e.,

$$(x_i, y_i) = (\overline{x}_i, \overline{y}_i), \quad i \in L \subset \{1, \dots, N\},\ L \neq \emptyset.$$

Discuss the problems properties (linear/nonlinear, differentiable/non-smooth, convex/non-convex, etc.).

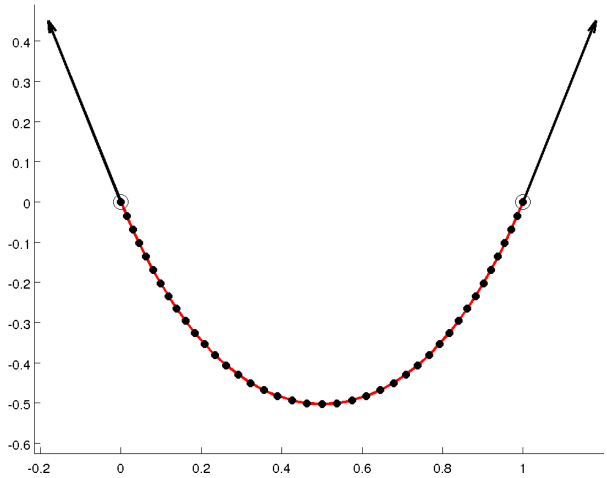
(ii) Can you expect LICQ to be satisfied at a minimizer (x^*, y^*)?

(iii) Formulate the KKT conditions of the problem for the nonsmooth and the squared, smooth link constraint formulation. Try to interpret the Lagrange multipliers as physical quantities. Check what will happen to the multipliers when you take the limit $N \to \infty$ while keeping the total weight and length of the chain constant in either formulation.

(iv) Suppose that the positions of the weights are constrained (in addition to the constraints already present due to the links and supports) by a rigid obstacle. The space occupied by the obstacle is described by $\{(x, y) \in \mathbb{R}^2 : \phi(x, y) \geq 0\}$, leading to the non-penetration conditions $\phi(x_i, y_i) \leq 0$ for $i = 1, \dots, N$.

How do the KKT conditions change?

(v) Solve the obstacle-constrained problem for some provided test configurations and investigate the behavior.



Solution.

(i) The potential energy of the chain is

$$E(x, y) = g \sum_{i=1}^{N} m_i \, y_i = g \, m^\top y,$$

where g is the gravitational constant. (Without loss of generality, we will use $g = 1$ in the numerical computations, which will only re-scale the multipliers associated with the rope stress.)

The simpler of the constraints of the problems is that of the support nodes/weights form the nonempty index set L, which will simply fix to their given positions via

$$(x_i, y_i) - (\overline{x}_i, \overline{y}_i) = (0, 0), \quad i \in L$$

giving $2|L|$ constraints. Note that there are other formulations we could employ in theory, such as, e.g.,

$$\|(x_i, y_i) - (\overline{x}_i, \overline{y}_i)\| = 0$$

giving only $|L|$ constraints. The formulation we employ has the undoubtable advantage of being a linear and smooth constraint that is as simple as it gets, only depending on a single variable per constraint.

The link length constraints, i.e., that the link lengths cannot exceed the given value ℓ, can equivalently be formulated as

$$\|(x_i, y_i) - (x_{i+1}, y_{i+1})\| - \ell \le 0, \quad i = 1, \ldots, N-1$$

or in a componentwise squared way as

$$\|(x_i, y_i) - (x_{i+1}, y_{i+1})\|^2 - \ell^2 \le 0, \quad i = 1, \ldots, N-1.$$

While the second (squared) version has the advantage of being smooth, we will see that it is actually inconvenient for the physical interpretation of the multipliers, because the multipliers associated with the rope stresses do not scale correctly and when the discretization is refined in a limiting process to recover the original infinite dimensional problem.

Nevertheless we end up with two possible problem formulations, specifically:

$$\begin{aligned} \text{Minimize} \quad & g\, m^\top y, \quad (x, y) \in \mathbb{R}^N \times \mathbb{R}^N \\ \text{s.t.} \quad & (x_i, y_i) - (\overline{x}_i, \overline{y}_i) = (0, 0), \quad i \in L \\ \text{and} \quad & \|(x_i, y_i) - (x_{i+1}, y_{i+1})\|^2 - \ell^2 \le 0, \quad i = 1, \ldots, N-1 \end{aligned} \tag{CHAIN–V1}$$

and

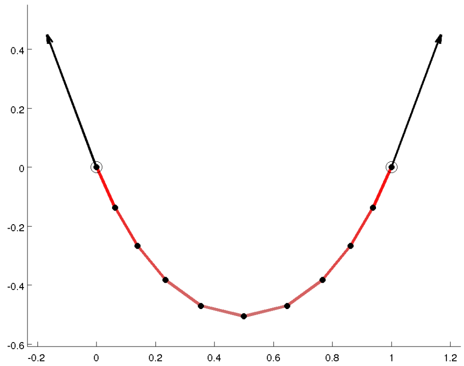
$$\begin{aligned} \text{Minimize} \quad & g\, m^\top y, \quad (x, y) \in \mathbb{R}^N \times \mathbb{R}^N \\ \text{s.t.} \quad & (x_i, y_i) - (\overline{x}_i, \overline{y}_i) = (0, 0), \quad i \in L \\ \text{and} \quad & \|(x_i, y_i) - (x_{i+1}, y_{i+1})\| - \ell \le 0, \quad i = 1, \ldots, N-1. \end{aligned} \tag{CHAIN–V2}$$

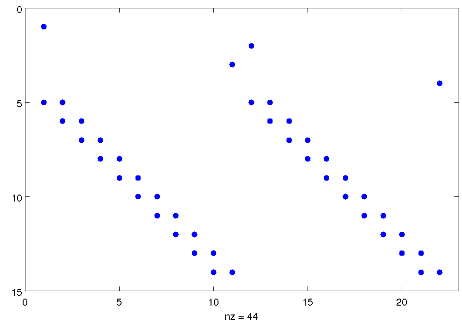
The objective is linear, the support constraints are linear equality constraints, while the link constraints are nonlinear (actually, quadratic in (CHAIN–V1) and fully nonlinear in (CHAIN–V2)) but convex. Hence the overall problem is convex.

We will not make use of the convexity in this example because we want to focus on the interpretation of lagrange multipliers in constrained problems in this exercise.

Also note that the link constraints in (CHAIN–V2) are also non-differentiable at points (x, y) where the positions of at least two of the weights coincide. This is, however, a rather pathological configuration that will seldomly appear in a solution, i.e., in the neighbourhood of a solution, this will not appear either, so an SQP-type solver should be expected to work regardless of this theoretical nonsmoothness.

(ii) In order get an idea on whether LICQ potentially holds, lets look at the constraint Jacobian. In the worst case (in the sense of most constraints' derivatives having to be checked for linear independence) all inequality constraints are active, i.e., all links are fully elongated. In this case, the sparsity structure of the Jacobian in both problem formulations has at most the nonzero entries shown below.





The first $2|L|$ rows correspond to the support nodes, the remaining $N-1$ rows correspond to the link constraints while the first N variables are the x_i and the remaining N variables are the y_i. Note that, when trying to find a linear combination of the rows that forms the $0 \in \mathbb{R}^{2N}$, fixing any coefficient of any of the lines concerning the link constraints will already predetermine all coefficients of all link constraint derivatives. Additionally, the staircase structure of nonzeros in the x and the y blocks are unlikely to contain the same numbers and the only additional dofs that come into play correspond to the support constraints, which can only change the result in the indices of dofs that are in L, i.e., it is highly likely that the Jacobian has full row rank in standard configurations.

However, there are configurations where LICQ in fact fails. To find an example, it is helpful to have the understanding of item *(iii)* that the Lagrange multipliers can be interpreted as forces. since we know that LICQ yields uniqueness of multipliers, it makes sense to try and find a configuration where the forces that result in the configuration are non-unique (a sufficient condition for LICQ not holding). To that end, consider, e. g., a situation with three mass points where the first and the last weight are fixed in vertical alignment with distance 2ℓ. There is only one feasible point in this scenario with the only free mass position (the middle one) being exactly in the middle between the two fixed ones. All constraints are active here and for any set of constraining forces (lagrangian multipliers) $f_i, i = 1, \ldots, 3$ acting on the three masses that would produce this setting, the forces $f_1 + \alpha e_y, f_2, f_3 - \alpha e_y$ for any $\alpha \in \mathbb{R}$ produce the same configuration. Here, all constraints are active and the Jacobian (e.g. for (CHAIN–V1)) can be easily computed to be

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

which clearly does not have full row rank (the kernel of $A^\top$ is spanned by $(0, \alpha, 0, -\alpha, -\alpha, -\alpha)^\top$). This example can easily be extended to more or fewer free mass points in between the two fixed ones.

(*iii*) The Lagrangian for (CHAIN–V1) reads

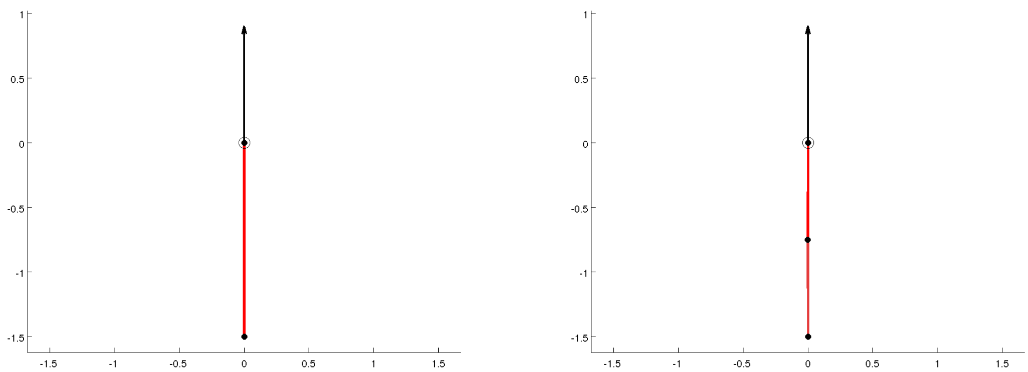
$$L(x, y, \lambda^{\text{support}}, \mu^{\text{rope}}) = \underbrace{g\, m^\top y}_{[\cdot]=\frac{N}{kg}\, kg\, m} + \sum_{i \in L} \underbrace{\left(\lambda_i^{\text{support}}\right)^\top}_{[\cdot]=N} \underbrace{\left[(x_i, y_i) - (\overline{x}_i, \overline{y}_i)\right]}_{[\cdot]=m}$$
$$+ \sum_{i=1}^{N-1} \underbrace{\mu_i^{\text{rope}}}_{[\cdot]=\frac{N}{m}} \underbrace{\left[\|(x_i, y_i) - (x_{i+1}, y_{i+1})\|^2 - \ell^2\right]}_{[\cdot]=m^2},$$

Its physical unit is $N\, m = J$, an energy. The Lagrange multipliers $\lambda_i^{\text{support}} \in \mathbb{R}^2$ represent the forces necessary to hold the support nodes in place. The components $\mu_i^{\text{rope}} \in \mathbb{R}$ seem to correspond to a force density (w.r.t. length) in the ropes. (We will come back to this below.)

Based on the Lagrangian, we can evaluate the KKT conditions of (CHAIN–V1)

$$
\begin{aligned}
L_x(\cdot) &= \begin{bmatrix} \\ \lambda_{i,1}^{\text{support}} \\ \\ \end{bmatrix}_{i\in L} + 2 \begin{bmatrix} \mu_1^{\text{rope}}\,(x_1 - x_2) & & \\ \mu_2^{\text{rope}}\,(x_2 - x_3) & - & \mu_1^{\text{rope}}\,(x_1 - x_2) \\ \vdots \quad \vdots & & \vdots \quad \vdots \\ \mu_{N-1}^{\text{rope}}\,(x_{N-1} - x_N) & - & \mu_{N-2}^{\text{rope}}\,(x_{N-2} - x_{N-1}) \\ & - & \mu_{N-1}^{\text{rope}}\,(x_{N-1} - x_N) \end{bmatrix} \\
&= 0 \in \mathbb{R}^N \\
L_y(\cdot) &= \underbrace{g \begin{bmatrix} \\ m_i \\ \\ \end{bmatrix}_{i=1}^{N}}_{\text{gravity forces}} + \underbrace{\begin{bmatrix} \\ \lambda_{i,2}^{\text{support}} \\ \\ \end{bmatrix}_{i\in L}}_{\text{support forces}} + \underbrace{2 \begin{bmatrix} \mu_1^{\text{rope}}\,(y_1 - y_2) & & \\ \mu_2^{\text{rope}}\,(y_2 - y_3) & - & \mu_1^{\text{rope}}\,(y_1 - y_2) \\ \vdots \quad \vdots & & \vdots \quad \vdots \\ \mu_{N-1}^{\text{rope}}\,(y_{N-1} - y_N) & - & \mu_{N-2}^{\text{rope}}\,(y_{N-2} - y_{N-1}) \\ & - & \mu_{N-1}^{\text{rope}}\,(y_{N-1} - y_N) \end{bmatrix}}_{\text{rope forces}} \\
&= 0 \in \mathbb{R}^N \\
&(x_i, y_i) - (\overline{x}_i, \overline{y}_i) = (0, 0), \qquad i \in L \\
&0 \le \mu_i^{\text{rope}} \quad \perp \quad \|(x_i, y_i) - (x_{i+1}, y_{i+1})\|^2 - \ell^2 \le 0, \qquad i = 1, \ldots, N-1.
\end{aligned}
$$

We will now investigate a simple dangling chain problem to point out a problem with the scaling of the Lagrange multipliers in the formulation (CHAIN–V1), see below:



It is enough to compare the balance of forces in y direction for $N = 2$ and $N = 3$. Note that all rope segments are fully extended to length $\ell = L/(N-1)$ with (in this case) $L = 1.5$. The masses are $m_i = 1/N$, so the total mass of the chain is $M = 1$. In case $N = 2$ we get

$$
L_y(\cdot) = g \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} + \begin{bmatrix} \lambda_{1,2}^{\text{support}} \\ 0 \end{bmatrix} + 2 \begin{bmatrix} \mu_1^{\text{rope}}\,\ell & & \\ & - & \mu_1^{\text{rope}}\,\ell \end{bmatrix} = 0 \in \mathbb{R}^2,
$$

which gives the solution

$$
\begin{aligned}
\mu_1^{\text{rope}} &= \frac{1}{2\,\ell} \cdot \frac{g}{2} = \frac{2-1}{2\,L} \cdot \frac{g}{2} \\
\lambda_{1,2}^{\text{support}} &= -2\,\mu_1^{\text{rope}}\,\ell - \frac{g}{2}.
\end{aligned}
$$

In case $N = 3$ we get

$$
L_y(\cdot) = g \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} + \begin{bmatrix} \lambda_{1,2}^{\text{support}} \\ 0 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} \mu_1^{\text{rope}}\,\ell & & \\ \mu_2^{\text{rope}}\,\ell & - & \mu_1^{\text{rope}}\,\ell \\ & - & \mu_2^{\text{rope}}\,\ell \end{bmatrix} = 0 \in \mathbb{R}^3,
$$

which gives the solution

$$
\begin{aligned}
\mu_2^{\text{rope}} &= \frac{1}{2\,\ell} \cdot \frac{g}{3} = \frac{3-1}{2\,L} \cdot \frac{g}{3} \\
\mu_1^{\text{rope}} &= \mu_2^{\text{rope}} + \frac{1}{2\,\ell} \cdot \frac{g}{3} = 2\,\mu_2^{\text{rope}} \\
\lambda_{1,2}^{\text{support}} &= -2\,\mu_1^{\text{rope}}\,\ell - \frac{g}{3}.
\end{aligned}
$$

Continuing this procedure for larger N we find that (for general total mass M equidistributed among the N weights, and general length L also equidistributed among the $N-1$ links)

$$
\begin{aligned}
\mu_{N-1}^{\text{rope}} &= \frac{M}{2\,\ell} \cdot \frac{g}{N} = M\,\frac{N-1}{N}\,\frac{g}{2\,L} \qquad \text{(last = lowest) rope segment} \\
&\vdots \\
\mu_1^{\text{rope}} &= N\,\mu_{N-1}^{\text{rope}} = M\,(N-1)\,\frac{g}{2\,L} \qquad \text{(first = highest) rope segment}
\end{aligned}
$$

Thus we conclude that the multiplier $\mu_1^{\text{rope}} \to \infty$ when $N \to \infty$, hence it cannot be directly interpreted as the rope stress (force divided by length).

We resolve the issue when we consider the problem in the form (CHAIN–V2). Its Lagrangian is

$$
\begin{aligned}
L(x, y, \lambda^{\text{support}}, \mu^{\text{rope}}) = {} & \underbrace{g\,m^\top y}_{[\cdot]=\frac{N}{kg}\,kg\,m} + \sum_{i \in L} \underbrace{(\lambda_i^{\text{support}})^\top}_{[\cdot]=N} \underbrace{\big[(x_i, y_i) - (\overline{x}_i, \overline{y}_i)\big]}_{[\cdot]=m} \\
& + \sum_{i=1}^{N-1} \underbrace{\mu_i^{\text{rope}}}_{[\cdot]=N} \underbrace{\big[\|(x_i, y_i) - (x_{i+1}, y_{i+1})\| - \ell\big]}_{[\cdot]=m}.
\end{aligned}
$$

Note that now the multipliers μ_i^{rope} have the unit of a force (transmitted in a particular rope segment #i). The KKT conditions for problem (CHAIN–V2) become

$$
\begin{aligned}
L_x(\cdot) &= \begin{bmatrix} \\ \lambda_{i,1}^{\text{support}} \\ \\ \end{bmatrix}_{i \in L} + \begin{bmatrix} \mu_1^{\text{rope}}\, d_1^{-1}\,(x_1 - x_2) & & \\ \mu_2^{\text{rope}}\, d_2^{-1}\,(x_2 - x_3) & - & \mu_1^{\text{rope}}\, d_1^{-1}\,(x_1 - x_2) \\ \vdots \quad \vdots & & \vdots \quad \vdots \\ \mu_{N-1}^{\text{rope}}\, d_{N-1}^{-1}\,(x_{N-1} - x_N) & - & \mu_{N-2}^{\text{rope}}\, d_{N-2}^{-1}\,(x_{N-2} - x_{N-1}) \\ & - & \mu_{N-1}^{\text{rope}}\, d_{N-1}^{-1}\,(x_{N-1} - x_N) \end{bmatrix} \\
&= 0 \in \mathbb{R}^N \\
L_y(\cdot) &= \underbrace{g \begin{bmatrix} \\ m_i \\ \\ \end{bmatrix}_{i=1}^{N}}_{\text{gravity forces}} + \underbrace{\begin{bmatrix} \\ \lambda_{i,2}^{\text{support}} \\ \\ \end{bmatrix}_{i \in L}}_{\text{support forces}} \\
&\quad + \underbrace{\begin{bmatrix} \mu_1^{\text{rope}}\, d_1^{-1}\,(y_1 - y_2) & & \\ \mu_2^{\text{rope}}\, d_2^{-1}\,(y_2 - y_3) & - & \mu_1^{\text{rope}}\, d_1^{-1}\,(y_1 - y_2) \\ \vdots \quad \vdots & & \vdots \quad \vdots \\ \mu_{N-1}^{\text{rope}}\, d_{N-1}^{-1}\,(y_{N-1} - y_N) & - & \mu_{N-2}^{\text{rope}}\, d_{N-2}^{-1}\,(y_{N-2} - y_{N-1}) \\ & - & \mu_{N-1}^{\text{rope}}\, d_{N-1}^{-1}\,(y_{N-1} - y_N) \end{bmatrix}}_{\text{rope forces}} \\
&= 0 \in \mathbb{R}^N \\
& (x_i, y_i) - (\overline{x}_i, \overline{y}_i) = (0, 0), \quad i \in L \\
& 0 \le \mu_i^{\text{rope}} \quad \perp \quad \|(x_i, y_i) - (x_{i+1}, y_{i+1})\| - \ell \le 0, \quad i = 1, \ldots, N-1.
\end{aligned}
$$

Here we use the abbreviation

$$
d_i = \|(x_i, y_i) - (x_{i+1}, y_{i+1})\|, \quad i = 1, \ldots, N-1,
$$

all of which are assumed to be $\neq 0$ (no two weights are in the same place).

When we repeat the above computations to verify the scaling of the multipliers μ_i^{rope}, we can use $d_i \equiv \ell$ to see

$$
\begin{aligned}
\mu_{N-1}^{\text{rope}} &= M \, \frac{g}{N} && \text{(last = lowest) rope segment} \\
&\vdots \\
\mu_1^{\text{rope}} &= M\, N\, \mu_{N-1}^{\text{rope}} = M\, g && \text{(first = highest) rope segment,}
\end{aligned}
$$

which give the desired behavior as $N \to \infty$.

Before we continue, let us write the KKT conditions in a more compact form. To this end, we introduce the diagonal matrix operator $\operatorname{diag}(v)$ which converts a vector of length N into a diagonal matrix with the entries of v along the diagonal. We also set $\Delta v := (v_2 - v_1, \dots, v_N - v_{N-1})^\top \in \mathbb{R}^{N-1}$ for $v \in \mathbb{R}^N$. Then the KKT condtions for problem (CHAIN–V2) become

$$\begin{aligned}
L_x(\cdot) &= \left[\lambda_{i,1}^{\text{support}}\right]_{i\in L} - \begin{bmatrix} \operatorname{diag}(\mu^{\text{rope}})\ \operatorname{diag}(d)^{-1}\Delta x \\ 0 \end{bmatrix} \\
&\quad + \begin{bmatrix} 0 \\ \operatorname{diag}(\mu^{\text{rope}})\ \operatorname{diag}(d)^{-1}\Delta x \end{bmatrix} = 0 \in \mathbb{R}^N \\
L_y(\cdot) &= g\, m + \left[\lambda_{i,2}^{\text{support}}\right]_{i\in L} - \begin{bmatrix} \operatorname{diag}(\mu^{\text{rope}})\ \operatorname{diag}(d)^{-1}\Delta y \\ 0 \end{bmatrix} \\
&\quad + \begin{bmatrix} 0 \\ \operatorname{diag}(\mu^{\text{rope}})\ \operatorname{diag}(d)^{-1}\Delta y \end{bmatrix} = 0 \in \mathbb{R}^N \\
&(x_i, y_i) - (\overline{x}_i, \overline{y}_i) = (0,0), \quad i \in L \\
&0 \le \mu_i^{\text{rope}} \quad \perp \quad \|(x_i, y_i) - (x_{i+1}, y_{i+1})\| - \ell \le 0, \quad i = 1, \dots, N-1.
\end{aligned}$$

(And of course the KKT conditions of (CHAIN–V1) are obtained by omitting the terms $\operatorname{diag}(d)^{-1}$.)

Note that the complementarity between the multipliers μ_i^{rope} and the link length constraints $\|(x_i, y_i) - (x_{i+1}, y_{i+1})\| - \ell \le 0$ means that the rope can only transmit a force when it is fully extended.

(*iv*) Finally, we add the constraints $\phi(x_i, y_i) \le 0$ for $i = 1, \dots, N$ to the problem (CHAIN–V2):

$$\begin{aligned}
\text{Minimize} \quad & g\, m^\top y, \quad (x, y) \in \mathbb{R}^N \times \mathbb{R}^N \\
\text{s.t.} \quad & (x_i, y_i) - (\overline{x}_i, \overline{y}_i) = (0,0), \quad i \in L \\
\text{and} \quad & \|(x_i, y_i) - (x_{i+1}, y_{i+1})\| - \ell \le 0, \quad i = 1, \dots, N-1 \\
\text{and} \quad & \phi(x_i, y_i) \le 0, \quad i = 1, \dots, N.
\end{aligned} \tag{CHAIN–V3}$$

I.e., we model that the weights cannot penetrate the obstacle described by the surface height ϕ (measured in meters). When the obstacle is convex, this does not necessarily carry over to the links though.

Accordingly, we have to add an appropriate term to the Lagrangian,

$$\begin{aligned} L(x, y, \lambda^{\text{support}}, \mu^{\text{rope}}) = {} & \underbrace{g\, m^\top y}_{[\cdot]=\frac{N}{kg}\,kg\,m} + \sum_{i \in L} \underbrace{(\lambda_i^{\text{support}})^\top}_{[\cdot]=N} \underbrace{\big[(x_i, y_i) - (\overline{x}_i, \overline{y}_i)\big]}_{[\cdot]=m} \\ & + \sum_{i=1}^{N-1} \underbrace{\mu_i^{\text{rope}}}_{[\cdot]=N} \underbrace{\big[\|(x_i, y_i) - (x_{i+1}, y_{i+1})\| - \ell\big]}_{[\cdot]=m} \\ & + \sum_{i=1}^{N} \underbrace{\mu_i^{\text{obstacle}} \phi(x_i, y_i)}_{[\cdot]=N\,m}. \end{aligned}$$

and to the KKT conditions:

$$\begin{aligned} L_x(\cdot) &= \Big[\lambda_{i,1}^{\text{support}}\Big]_{i \in L} - \begin{bmatrix} \operatorname{diag}(\mu^{\text{rope}})\ \operatorname{diag}(d)^{-1} \Delta x \\ 0 \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 \\ \operatorname{diag}(\mu^{\text{rope}})\ \operatorname{diag}(d)^{-1} \Delta x \end{bmatrix} + \Big[\mu_i^{\text{obstacle}} \frac{\partial}{\partial x} \phi(x_i, y_i)\Big]_{i \in N} = 0 \in \mathbb{R}^N \\ L_y(\cdot) &= g\, m + \Big[\lambda_{i,2}^{\text{support}}\Big]_{i \in L} - \begin{bmatrix} \operatorname{diag}(\mu^{\text{rope}})\ \operatorname{diag}(d)^{-1} \Delta y \\ 0 \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 \\ \operatorname{diag}(\mu^{\text{rope}})\ \operatorname{diag}(d)^{-1} \Delta y \end{bmatrix} + \Big[\mu_i^{\text{obstacle}} \frac{\partial}{\partial y} \phi(x_i, y_i)\Big]_{i \in N} = 0 \in \mathbb{R}^N \\ & (x_i, y_i) - (\overline{x}_i, \overline{y}_i) = (0, 0), \quad i \in L \\ & 0 \le \mu_i^{\text{rope}} \quad \perp \quad \|(x_i, y_i) - (x_{i+1}, y_{i+1})\| - \ell \le 0, \quad i = 1, \ldots, N-1 \\ & 0 \le \mu_i^{\text{obstacle}} \quad \perp \quad \phi(x_i, y_i) \le 0, \quad i = 1, \ldots, N. \end{aligned}$$

We can interpret the new multipliers μ_i^{obstacle} as (scaled) contact forces, which act in normal direction

$$n_i = \nabla \phi(x_i, y_i)$$

to the obstacle at the point (x_i, y_i). One can verify that

$$\mu_i^{\text{obstacle}}\, \nabla \phi(x_i, y_i), \quad i = 1, \ldots, N$$

is the vector representing the contact force. It is independent of the scaling of the obstacle function ϕ.

The complementarity condition means that the contact forces are zero in the absence of contact.

(v) Using Python, one can e.g. use scipy.optimize.minimize's SLSQP solver (which unfortunately does not return Lagrange multipliers). Matlab provides fmincon for constrained optimization with another SQP method.