Lecture Notes
Infinite Dimensional Optimization

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Material for approximately 25 lectures, 14 weeks.

In these lecture notes we use colored markup for **definitions** and alerts.

Expert Knowledge: topic

A block like this contains further information that are not subject to examination.

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Chapter 0 Introduction

We will consider in this class optimization problems of the following kind:

$$\begin{aligned} \text{Minimize} \quad & f(x), \quad \text{where } x \in X \\ \text{subject to} \quad & h(x) = 0. \end{aligned}$$

In this problem, $f: X \to \mathbb{R}$ is called the **objective function** and $h: X \to Y$ is the **equality constraint**. The **optimization variable** x is sought in some **optimization space** X.

Inequality constraints may be added to the above problem, either

- explicitly in the form $g(x) \leqslant 0$ or, more generally, in the form $g(x) \in K \subseteq Z$,
- or implicitly, by imposing $x \in C \subseteq X$ or allowing f to take values in $\mathbb{R} \cup \{\infty\}$.

Often, K is a cone and C is a convex set.

What are reasonable choices for the "spaces" X, Y, Z?

(1) To define the notion of global minimizers, no structure at all is required, so X, Y, Z can be general sets.

(2) To define the notion of local minimizers, the space X of optimization variables must carry a topology since we require the concept of neighborhoods.

(3) Statements about the existence of global minimizers build on notions of continuity and compactness.[1] Therefore, topological spaces are required for this purpose as well.

(4) To formulate first-order optimality conditions, we need to be able to differentiate. A convenient setting for this are normed linear spaces.

(5) For algorithmic purposes, derivatives need to be converted into directions, e. g., directions of largest/smallest directional derivatives over the unit sphere. For this purpose, normed linear spaces or even Hilbert spaces, are convenient.

Based on these considerations, we will consider only **normed linear spaces** over the field of real numbers $\mathbb{R}$ (§ 2).[2]

We may anticipate a couple of differences compared to optimization over finite-dimensional linear spaces, as well as a number of questions that we will may want to to answer throughout the course:

(1) Different norms on an infinite-dimensional linear space are, in general, not equivalent to each other.

(2) How do we differentiate functions defined on infinite-dimensional normed linear space?

[1] Compare, for instance, the Weierstrass extreme value theorem: a continuous function $f: X \to \mathbb{R}$ attains its minimum (and its maximum) on a compact set $C \subseteq X$.

[2] We use the term **linear space** instead of the synonymous **vector space**.

(3) Can we formulate optimization algorithms on infinite-dimensional spaces?

(4) If so, then when and how do we discretize in order to realize them numerically?

§ 1 Motivating Examples

Example 1.1 (Brachistochrone problem).
In a 1696 article, Johann Bernoulli posted the following problem:

> Given two points A and B in a vertical plane, what is the curve traced out by a point acted on only by gravity, which starts at A and reaches B in the shortest time?

This problem is known as the **Brachistochrone problem** (ancient Greek: *βράχιστος χρόνος*). In modern terms, it can be formulated as follows. Suppose that the points have coordinates $A = (0, 0)$ and $B = (b_1, b_2)$ with $b_2 \geqslant 0$. Let $g > 0$ denote the gravitational constant.

We are seeking a function $y\colon [0, a] \to \mathbb{R}$ whose graph defines the curve from A to B. Using the principle of conservation of (potential plus kinetic) energy, we may express the speed of the particle at horizontal position x in terms of its height $y(x)$. Skipping the details, this eventually leads to the following optimization problem:

$$
\begin{aligned}
\text{Minimize} \quad & f(y) \coloneqq \int_0^a \frac{\sqrt{1 + y'(x)^2}}{\sqrt{2\, g\, y(x)}} \, \mathrm{d}x, \quad \text{where } y \in X \\
\text{s.t.} \quad & y(0) = 0 \\
\text{and} \quad & y(b_1) = b_2 \\
\text{as well as} \quad & y \geqslant 0 \text{ on } [0, b_1].
\end{aligned}
\tag{1.1}
$$

Here X is a suitable vector space of functions $y\colon [0, b_1] \to \mathbb{R}$, e. g., $X = C^1(0, b_1) \cap C([0, b_1])$, the space of continuous functions on $[0, b_1]$ whose restriction to the open interval $(0, b_1)$ is continuously differentiable. An alternative is the **Sobolev space** $X = H^1(0, b_1)$ of square integrable functions with square integrable weak derivative on $(0, b_1)$.[3]

(**Quiz 1.1:** Does the gravitational constant impact optimal curves?) One can show that the (unique) minimizer of (1.1) satisfies a first-order necessary optimality condition, which comes in the form of a differential equation:

$$
\frac{1}{2} \sqrt{\frac{1 + y'(x)^2}{y(x)^3}} + \frac{\mathrm{d}}{\mathrm{d}x} \frac{y'(x)}{\sqrt{y(x)\,\left(1 + y'(x)^2\right)}} = 0.
$$

The solutions of this equation satisfy

$$
y(x)\,\left(1 + y'(x)^2\right) = C \quad \text{in } (0, b_1) \tag{1.2}
$$

for some $C > 0$, and it has infinite slope initially:

$$
\lim_{x \searrow 0} y'(x) = \infty.
$$

[3] We will introduce Sobolev spaces later.

The unique solution is given by the curve

$$t \mapsto \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C \begin{pmatrix} t - \sin(t) \\ 1 - \cos(t) \end{pmatrix} \quad \text{for } t \in [0, T], \tag{1.3}$$

where $C > 0$ and $T \in (0, 2\pi]$ are determined by the conditions $x(T) = b_1$ and $y(T) = b_2$.

This curve is a segment of a cycloid with radius C. △

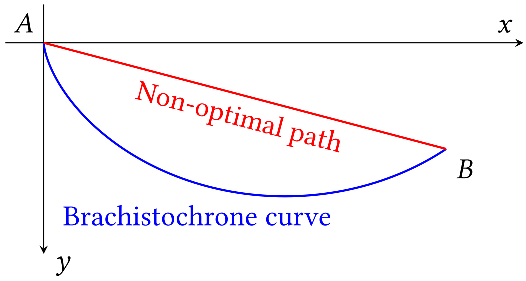


Figure 1.1: Some non-optimal curve $\gamma\colon [0, b_1] \to \mathbb{R}$ from A to B (left) as well as the unique global minimizer of the Brachistochrone problem (1.1), given by the segment of a cycloid (left). Image of an experimental device on display at Technoseum Mannheim (right), shot by Roland Herzog.

Remark 1.2 (on the Brachistochrone problem)**.**
The first-order optimality condition of the Brachistochrone problem come in the form of a differential equation (1.2). This is typical for optimization problems whose variables are functions and whose objectives involve derivatives of those functions. As a result, minimizers may be more regular than suggested by the optimization space X. This is indeed the case in the Brachistochrone problem (1.1), where the unique minimizer turns out to be a $C^\infty(0, b_1)$-function. △

Expert Knowledge: The origins of the calculus of variations

The Brachistochrone problem belongs to a class of problems referred to as calculus of variations, where optimization variables are functions and objectives are typically integrals involving values of the function and its derivative(s). This term was coined in 1766 by Leonhard Euler. The first-order optimality conditions for calculus of variations problems are referred to as **Euler-Lagrange equations**.

Newton's problem of minimal resistance from 1687 is considered the first problem of this type, and the Brachistochrone problem (1696) is second. That problem attracted the attention of Johann Bernoulli's brother Jakob, as well as of Isaac Newton, Gottfried Leibniz, Ehrenfried Walther von Tschirnhaus and Guillaume de l'Hôpital, who all turned in solutions.

Example 1.3 (Fermat's principle in optics)**.**

Suppose that $n\colon \mathbb{R}^2 \longrightarrow \mathbb{R}_{>0}$ is the material dependent refractive index of an optical material. Let $\gamma\colon [0, b_1] \longrightarrow \mathbb{R}$ denote a function whose graph defines a curve through this material. Then the optical length of this curve is defined by

$$\int_0^{b_1} n(x, \gamma(x)) \sqrt{1 + \gamma'(x)^2} \, \mathrm{d}x.$$

Fermat's principle stipulates that the path a ray of light will take minimizes the optical length. Suppose that the end points of that path are $A = (0, 0)$ and $B = (b_1, b_2)$. Then we obtain the following optimization problem:

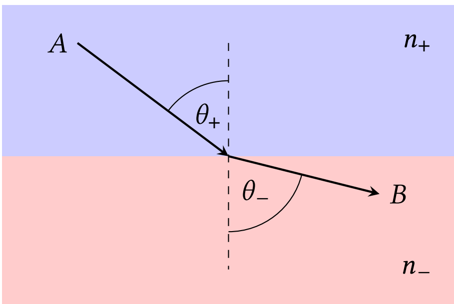
$$\begin{aligned} \text{Minimize} \quad & f(\gamma) \coloneqq \int_0^{b_1} n(x, \gamma(x)) \sqrt{1 + \gamma'(x)^2} \, \mathrm{d}x, \quad \text{where } \gamma \in X \\ \text{s. t.} \quad & \gamma(0) = 0 \\ \text{and} \quad & \gamma(b_1) = b_2. \end{aligned} \tag{1.4}$$

In the particular case where the refractive index is piecewise constant on slabs, the unique global minimizer of (1.4) satisfies **Snell's law**, which states that the incident angles θ_+, θ_- (measured against the normal) of two neighboring slabs satisfy the relation $n_+ \sin(\theta_+) = n_- \sin(\theta_-)$, see Figure 1.2.

Similar as in Example 1.1, every minimizer satisfies a first-order optimality condition that amounts to a differential equation:

$$-\frac{n(x, \gamma(x))\, \gamma'(x)}{\sqrt{1 + \gamma'(x)^2}} + n_y(x, \gamma(x)) \sqrt{1 + \gamma'(x)^2} = 0.$$

In this case, however, the discontinuous coefficient n may limit the regularity of an optimal path. Again, for piecewise constant refractive index, an optimal curve will be piecewise linear with discontinuous derivative at optical interfaces. △



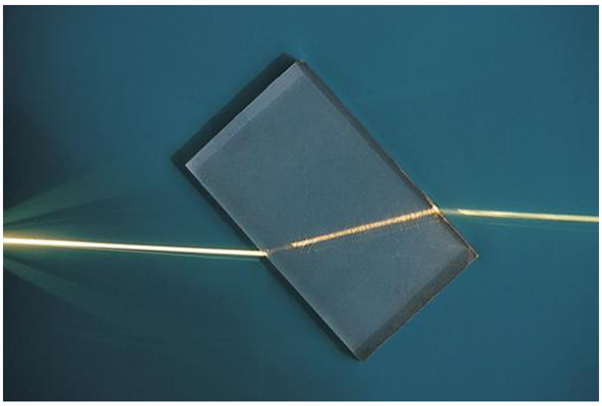


Figure 1.2: Illustratrion of Snell's law of refraction (left) as a special case of Example 1.3. Image (right) obtained from `https://en.wikipedia.org/wiki/Refraction`, released into the public domain by creator ajizai.

End of Class 1

Example 1.4 (signal denoising).

Suppose a signal $s\colon [0,T] \to \mathbb{R}$ is given.[4] In case the signal is noisy, we may formulate an optimization problem to try and find a denoised signal $y\colon [0,T] \to \mathbb{R}$:

$$\text{Minimize} \quad f(y) \coloneqq \int_0^T |y(t) - s(t)|^2 \, \mathrm{d}t + \beta \int_0^T |\dot{y}(t)|^2 \, \mathrm{d}t, \quad \text{where } y \in X. \tag{1.5}$$

The dot denotes the time derivative. A suitable function space for this problem is the Sobolev space $X = H^1(0,T)$.

The second term in the objective penalizes "fast variations" in the signal. The parameter $\beta > 0$ balances the two summands in the objective and thus determines the degree of denoising.

We will be able to show later that the first-order optimality conditions for (1.5) involve the second-order differential equation

$$-\beta\, \ddot{y}(t) + y(t) = s(t), \tag{1.6}$$

which shows that the minimizer will indeed be a smoothed version of the noisy signal s. More precisely, we can expect the solution to gain two orders of differentiation compared to the data s. In particular, the solution will not admit any discontinuities. Therefore, one often prefers a "less powerful" regularization term, such as the **total variation** of the function y. We will come back to this type of problem in the context of image denoising problems in **??**. △

Example 1.5 (crane trolley optimal control problem)**.**
Consider a load on rope of length ℓ hanging from a crane trolley system (Figure 1.3). We denote the position of the trolley relative to the origin by s. The position of the load relative to the trolley is denoted by z. The trolley has mass M and the load has mass m. A controllable force u acts on the trolley.

This system is described by a second-order differential equation for the positions (s, z). It can be derived by working out Newton's law, force equals mass times acceleration. We convert it here to a first-order system of differential equations in terms of $x = (s, \dot{s}, z, \dot{z})$, where the dot denotes the time derivative. Assuming small angles θ, the differential equations can be taken as linear and the system reads

$$\begin{pmatrix} \dot{s} \\ \ddot{s} \\ \dot{z} \\ \ddot{z} \end{pmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{m}{M}\frac{g}{\ell} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{m+M}{M}\frac{g}{\ell} & 0 \end{bmatrix}}_{=:A} \begin{pmatrix} s \\ \dot{s} \\ z \\ \dot{z} \end{pmatrix} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{M} \end{bmatrix}}_{=:B} u \tag{1.7}$$

or, in short, $\dot{x} = A\,x + B\,u$. Notice that we have omitted the (t) argument everywhere for brevity.

We wish to steer the system from an initial state $x(0) = (0,0,0,0)^\intercal$ to a terminal state $x(T) = (E,0,0,0)^\intercal$ in as short a time T as possible. This leads us to the preliminary optimization problem

$$\begin{aligned} \text{Minimize} \quad & \int_0^T 1 \, \mathrm{d}t, \quad \text{where } (u, x, T) \in U \times X \times \mathbb{R} \\ \text{s.t.} \quad & \dot{x} = A\,x + B\,u \quad \text{in } [0,T] \\ \text{and} \quad & x(0) = (0,0,0,0)^\intercal \\ \text{and} \quad & x(T) = (E,0,0,0)^\intercal \\ \text{as well as} \quad & T > 0. \end{aligned} \tag{1.8}$$

[4]Think, for instance, of an audio signal sampled with a certain frequency, say, 48 kHz into a piecewise constant function.

This preliminary problem formulation has some issues. Due to the terminal time T being an optimization variable, we cannot fix function spaces for the **control** u and the **state** x since they depend on T.

There is, however, an easy remedy to this. We can renormalize the unknown time interval $[0, T]$ to the fixed interval $[0, 1]$. Replacing the unknowns x and u by their counterparts on the fixed interval, the dynamics need to be rescaled and the problem becomes

$$\begin{aligned} \text{Minimize} \quad & \int_0^1 T \, \mathrm{d}t, \quad \text{where } (u, x, T) \in U \times X \times \mathbb{R} \\ \text{s. t.} \quad & \dot{x} = \frac{1}{T}(A\,x + B\,u) \quad \text{in } [0, 1] \\ \text{and} \quad & x(0) = (0, 0, 0, 0)^\intercal \\ \text{and} \quad & x(1) = (E, 0, 0, 0)^\intercal \\ \text{as well as} \quad & T > 0. \end{aligned} \tag{1.9}$$

We can now fix suitable function spaces[5], e. g., $U = L^2(0, 1)$ and $X = H^1(0, 1)^4$. A problem such as (1.9), in which a **state** function x depends on the choice of the **control** function u through a differential equation, is termed an **optimal control problem**. We will see more of these in **??**.

Unfortunately, problem (1.9) as stated will not have a solution. (**Quiz 1.2:** Can you see why?) We may fix this by imposing bounds on the control function, e. g., by adding the pointwise inequality constraints

$$u(t) \in [-u_{\max}, u_{\max}],$$

with some $u_{\max} > 0$ to problem (1.9), or by adding a cost term such as

$$\beta \int_0^1 |u(t)| \, \mathrm{d}t$$

to the objective. △

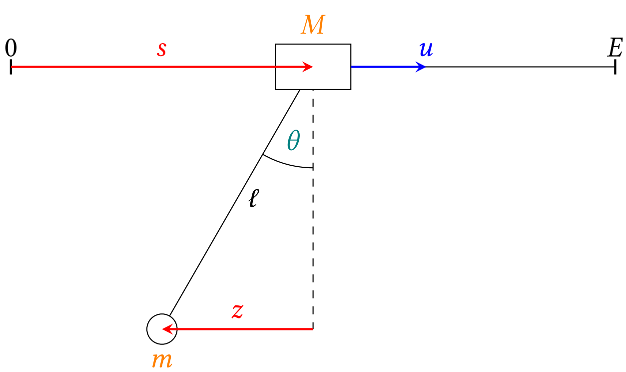


Figure 1.3: Illustration of the crane trolley problem (Example 1.5).

[5]Again, we will introduce these Lebesgue and Sobolev spaces later.

§ 2 Normed Linear Spaces

In this section we recap the notion of a normed linear space. We will also introduce Lebesgue and Sobolev spaces as our prime examples of normed linear spaces.

Definition 2.1 (linear space)**.**
An algebraic structure $(V, +, \cdot)$ with two operations[6]

$$\begin{aligned} +\colon V \times V &\to V && \text{(addition)} \\ \cdot\colon \mathbb{R} \times V &\to V && \text{(S-multiplication)} \end{aligned}$$

is said to be a **linear space** over the field of real numbers $\mathbb{R}$ if

(i) $(V, +)$ is an Abelian group.

(ii) The S-multiplication satisfies the mixed distributive laws

$$\begin{aligned} \alpha\,(u + v) &= (\alpha\,u) + (\alpha\,v) \\ (\alpha + \beta)\,v &= (\alpha\,v) + (\beta\,v) \end{aligned}$$

as well as the mixed associative law

$$(\alpha\,\beta)\,v = \alpha\,(\beta\,v)$$

for all $\alpha, \beta \in \mathbb{R}$ and $u, v \in V$. Moreover, the neutral element $1 \in \mathbb{R}$ w.r.t. multiplication in $\mathbb{R}$ is also neutral w.r.t. S-multiplication:

$$1\,v = v. \qquad \triangle$$

All linear spaces will be over the field of real numbers $\mathbb{R}$ and we will not explicitly mention that. We already anticipated that in order to be able to differentiate functions $f\colon V \to \mathbb{R}$ or, more generally, $f\colon V \to W$, we will require linear spaces to be normed.

Definition 2.2 (normed linear space)**.**
Suppose that V is a linear space.

(i) A map $\|\cdot\|\colon V \to \mathbb{R}$ is said to be a **norm on V** if the following conditions hold:

$$\|u\| \geqslant 0, \quad \text{and } \|u\| = 0 \;\Rightarrow\; u = 0 \qquad \textbf{positive definiteness} \tag{2.1a}$$

$$\|\alpha\,u\| = |\alpha|\,\|u\| \qquad \textbf{absolute homogeneity} \tag{2.1b}$$

$$\|u + v\| \leqslant \|u\| + \|v\| \qquad \textbf{triangle inequality} \text{ or } \textbf{subadditivity} \tag{2.1c}$$

for all $u, v \in V$ and all $\alpha \in \mathbb{R}$.

(ii) The pair $(V, \|\cdot\|)$ is said to be a **(real) normed vector space**. $\triangle$

[6] The dot $\cdot$ for S-multiplication is usually not written, just as the multiplication symbol in $\mathbb{R}$ is usually not written.

Expert Knowledge: from topological to normed linear spaces

We have the inclusions

- Every normed linear space is a metric space.
- Every metric space is a topological space.

A topological space is defined by a collection of its subsets that are called the open sets. Topological spaces admit notions of convergence and limits, closure and compactness of sets, as well as notions of continuity of functions.

Metric spaces are spaces with a notion of distance. The metric induces a topology.

Normed spaces are spaces with a notion of length. The norm induces a metric.

We will not discuss general topological spaces in full generality but restrict ourselves to normed linear spaces.

End of Class 2

End of Week 1