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Exercise 9

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Homework Problem 9.1. (Traces in L^p)

Let $\Omega \coloneqq B_1^{\|\cdot\|_2}(0) \subseteq \mathbb{R}^2$. Show that there can not be an extension of the trace map $\tau \colon C(\overline{\Omega}) \to C(\partial\Omega)$ to a continous map on $L^2(\Omega)$.

Homework Problem 9.2. (The Lax-Milgram lemma)

(a) Let $n \in \mathbb{N}$, $b \in \mathbb{R}^n$ and a symmetric $A \in R^{n \times n}$ such that $x^\top Ax > c\|x\|_2^2$ for a $c \in \mathbb{R}_>$. Use the Lax-Milgram lemma to show that the linear system $Ax = b$ has a unique solution $x \in \mathbb{R}^n$.

(b) Let H be a Hilbert space and let $A \colon H \mapsto H$ be a bounded, linear operator such that $(Ax, x) \geq 0$ for every $x \in H$. Use the Lax-Milgram lemma to show that the operator $\mathrm{id} + \alpha A \colon H \to H$ is bijective for every $\alpha \geqslant 0$. Show boundedness of $(\mathrm{id} + \alpha A)^{-1}$.

You are not expected to turn in your solutions.