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Exercise 11 (Solution)

Date issued: 7th January 2025

Homework Problem 11.1.

(a) (i) Let X, Y, Z be normed linear spaces and $F \colon X \to Y$ and $G \colon Y \to Z$ be Fréchet differentiable at $x \in X$ and $F(x) \in Y$, respectively. Show that $G \circ F \colon X \to Z$ is Fréchet differentiable at x.

(ii) Give an example of normed linear spaces X, Y, Z and functions $F \colon X \to Y$ and $G \colon Y \to Z$, that are Gâteaux differentiable at $x \in X$ and $F(x) \in Y$, respectively, where $G \circ F$ is not Gâteaux-differentiable at x.

(b) Let $F \colon X \to Y$ be a function between two linear spaces X and Y, and let $\|\cdot\|_X$ and $\|\cdot\|_{X'}$ as well as $\|\cdot\|_Y$ and $\|\cdot\|_{Y'}$ be norms on X and Y, respectively. Further, let $x \in X$.

(i) Show that if F is Fréchet differentiable with respect to $\|\cdot\|_X$ and $\|\cdot\|_Y$, then it is Fréchet differentiable with respect to $\|\cdot\|_{X'}$ and $\|\cdot\|_{Y'}$, if $\|\cdot\|_{X'}$ is stronger than $\|\cdot\|_X$ and $\|\cdot\|_{Y'}$ is weaker than $\|\cdot\|_Y$.

(ii) Show that the operator $u \mapsto \sin(u)$ is Fréchet-differentiable as an operator from $L^{p_1}(0,1)$ to $L^{p_2}(0,1)$ for $p_1, p_2 \in [1, \infty)$ if and only if $p_2 < p_1$.

Hint: Taylor's theorem will be helpful. In the proof of differentiability, consider $q \geqslant \frac{p}{2}$. In the counter example, consider step functions h and $u = 0$.

Solution.

(a) (i) We show that the derivative of $G \circ F$ at x is given by $G'(F(x))F'(x)$, as can be expected from a chain rule. To that end, note that for $h \in X$:

$$
\begin{aligned}
&G(F(x+h)) - G(F(x)) - G'(F(x))F'(x)h \\
=&G(F(x) + F'(x)h + r_F(x,h)) - G(F(x)) - G'(F(x))F'(x)h \\
=&G(F(x) + F'(x)h + r_F(x,h)) - G(F(x)) - G'(F(x))(F'(x)h + r_F(x,h)) + G'(F(x))r_F(x,h) \\
=&G'(F(x))r_F(x,h) + r_G(F(x), F'(x)h + r_F(x,h))
\end{aligned}
$$

and therefore

$$
\begin{aligned}
&\frac{\|G(F(x+h)) - G(F(x)) - G'(F(x))F'(x)h\|_Z}{\|h\|_X} \\
=&\frac{\|G'(F(x))r_F(x,h) + r_G(F(x), F'(x)h + r_F(x,h))\|_Z}{\|h\|_X} \\
\leqslant&\|G'(F(x))\| \frac{\|r_F(x,h)\|_Y}{\|h\|_X} + \frac{\|r_G(F(x), F'(x)h + r_F(x,h))\|_Z}{\|h\|_X} \\
=&\|G'(F(x))\| \frac{\|r_F(x,h)\|_Y}{\|h\|_X} + \frac{\|r_G(F(x), F'(x)h + r_F(x,h))\|_Z}{\|F'(x)h + r_F(x,h))\|_Y} \frac{\|F'(x)h + r_F(x,h))\|_Y}{\|h\|_X} \\
\leqslant&\|G'(F(x))\| \frac{\|r_F(x,h)\|_Y}{\|h\|_X} + \frac{\|r_G(F(x), F'(x)h + r_F(x,h))\|_Z}{\|F'(x)h + r_F(x,h))\|_Y} \frac{\|F'(x)\|\|h\|_X + \|r_F(x,h))\|_Y}{\|h\|_X}
\end{aligned}
$$

with the right hand side term converging to 0 as h to 0 because of the two Fréchet differentiabilities assumed on the data.

(ii) Consider the functions $F\colon \mathbb{R}^2 \to \mathbb{R}$ and $G\colon \mathbb{R} \to \mathbb{R}^2$ given by

$$
G(x_1, x_2) = \begin{cases} 1 & x_2 = x_1^2 \text{ and } x_1 > 0 \\ 0 & \text{else} \end{cases} \qquad F(x) = (x, x^2).
$$

At $x = 0$, $F(x) = (0,0)$ with Fréchet derivative $(1,0)^\intercal$ and $G(F(x)) = G(0) = 0$ with Gâteaux derivative $G'(0) = 0$. However, $G \circ F$ is the Heaviside function that is not directionally differentiable at 0.

(b) (i) Simple use of the norm estimates gives us

$$
\begin{aligned}
\frac{\|F(x+h) - F(x) - F'(x)h\|_{Y'}}{\|h\|_{X'}} &\leqslant C_Y \frac{\|F(x+h) - F(x) - F'(x)h\|_Y}{\|h\|_{X'}} \\
&\leqslant \frac{C_Y}{C_X} \frac{\|F(x+h) - F(x) - F'(x)h\|_Y}{\|h\|_X} \xrightarrow{\|h\|_{X'} \to 0} 0.
\end{aligned}
$$

Note that $\|h\|_X \leqslant C_X \|h\|_{X'} \xrightarrow{\|h\|_{X'} \to 0} 0$.

(ii) The sin function is smooth as a function on the reals, any directional derivative of the superposition operator therefore needs to coincide with this derivative in a pointwise sense. From Taylor's theorem, we know that

$$\sin(u+h) = \sin(u) + \cos(u)\,h + r(u,h).$$

for all $u, h \in \mathbb{R}$. Applying this expansion at each point $x \in [0,1]$ for functions $u(x), h(x) \in L^{p_1}$, we therefore obtain

$$\sin(u(x)+h(x)) = \sin(u(x)) + \cos(u(x))\,h(x) + r(u(x),h(x)).$$

At the constant zero function $u = 0$ and for $h_\varepsilon \coloneqq \chi_{[0,\varepsilon]}$ and obtain that

$$r(0, h_\varepsilon(x)) = \sin(h_\varepsilon(x)) - h_\varepsilon(x) = C h_\varepsilon(x) = C\chi_{[0,\varepsilon]}$$

for a constant $C \in \mathbb{R}$. Accordingly

$$\frac{\|\sin(0+h_\varepsilon) - \sin(0) - \cos(0)h_\varepsilon\|_{L^{p_2}}}{\|h_\varepsilon\|_{L^{p_1}}} = \frac{\|r\|_{L^{p_2}}}{\|h_\varepsilon\|_{L^{p_1}}} = \frac{\|Ch_\varepsilon\|_{L^{p_2}}}{\|h_\varepsilon\|_{L^{p_1}}} = \frac{\|C\chi_{[0,\varepsilon]}\|_{L^{p_2}}}{\|\chi_{[0,\varepsilon]}\|_{L^{p_1}}} = c\varepsilon^{\frac{1}{p_2}-\frac{1}{p_1}}.$$

This shows the nondifferentiability at 0 for $p_2 \geqslant p_1$ because of non-zero-convergence with $\varepsilon \longrightarrow 0$.

When $p_2 < p_1$, the above is not a contradiction to Fréchet differentiability. Instead, we note the following:

$$\begin{aligned}
|r(u(x),h(x))| &= |\sin(u(x)+h(x)) - \sin(u(x)) - \cos(u(x))h(x)| \\
&\leqslant |\sin(u(x)+h(x)) - \sin(u(x))| + |\cos(u(x))h(x)| \\
&\leqslant 2|h(x)| \\
|r(u(x),h(x))| &= |\sin(u(x)+h(x)) - \sin(u(x)) - \cos(u(x))h(x)| \\
&= |-\sin(u(x) + \underbrace{\xi}_{\in[0,1]} h(x))h(x)^2| \\
&\leqslant |h(x)|^2
\end{aligned}$$

where the first line holds due to boundedness of $|\cos|$ by 1 and the corresponding Lipschitz-continuity of sin with the same constant and the second estimate is a direct application of taylors theorem. Accordingly, there is a $C > 0$ such that $|r(u(x),h(x))| \le C|h(x)|$ and $|r(u(x),h(x))| \le C|h(x)|^2$ for arbitrary $u, h \in L^{p_1}$. So if we additionally assume that $p_2 \geqslant \frac{p_1}{2}$, and therefore $1 < \frac{p_1}{p_2} \leqslant 2$, we therefore obtain that $|r(u(x),h(x))| \le C|h(x)|^{\frac{p_1}{p_2}}$. Accordingly, we have that

$$\|r(u,h)\|_{L^{p_2}}^{p^2} = \int_0^1 |r(u(x),h(x))|_2^p \,\mathrm{d}x \leqslant C^{p_2} \int_0^1 |h|^{p_1}\,\mathrm{d}x = C^{p_2}\|h\|_{L^{p_1}}^{p^1},$$

and therefore

$$\begin{aligned}
\frac{\|\sin(u+h) - \sin(u) - \cos(u)h\|_{L^{p_2}}}{\|h\|_{L^{p_1}}} &= \frac{\|r(u,h)\|_{L^{p_2}}}{\|h\|_{L^{p_1}}} \\
&\leqslant c\frac{\|h\|_{L^{p_1}}^{\frac{p_1}{p_2}}}{\|h\|_{L^{p_1}}} \\
&= c\|h\|_{L^{p_1}}^{\frac{p_1}{p_2}-1} \xrightarrow{\|h\|_{p_1}\to 0} 0.
\end{aligned}$$

When $q < \frac{p}{2}$, the result follows because of part (i) of the exercise.

You are not expected to turn in your solutions.