

LECTURE NOTES

INFINITE DIMENSIONAL OPTIMIZATION

Roland Herzog*

2024-11-10

*Interdisciplinary Center for Scientific Computing, Heidelberg University, 69120 Heidelberg, Germany
(roland.herzog@iwr.uni-heidelberg.de, <https://scoop.iwr.uni-heidelberg.de/team/roland-herzog>).

Material for approximately 25 lectures, 14 weeks.

In these lecture notes we use colored markup for **definitions** and **alerts**.

Expert Knowledge: topic

A block like this contains further information that are not subject to examination.

Contents

o	Introduction	5
§ 1	Motivating Examples	6
§ 2	Normed Linear Spaces	11
§ 2.1	Open and Closed Sets	12
§ 2.2	Banach Spaces	14
§ 2.3	Comparison of Norms	15
§ 2.4	Compactness	17
§ 2.5	Lebesgue Spaces	22
§ 2.6	Sobolev Spaces	24
§ 3	Inner Product Spaces	28
§ 4	Continuous Functions	30
§ 4.1	Linear Operators	31
§ 4.2	Continuous Embeddings	34
§ 4.3	The Dual Space	35
§ 4.4	The Dual Space of a Hilbert Space	36
1	Convex Infinite-Dimensional Optimization with Applications in Imaging	37
2	Optimal Control of Partial Differential Equations	38

Chapter 0 Introduction

We will consider in this class optimization problems of the following kind:

$$\begin{array}{ll} \text{Minimize} & f(x), \quad \text{where } x \in X \\ \text{subject to} & h(x) = 0. \end{array}$$

In this problem, $f: X \rightarrow \mathbb{R}$ is called the **objective function** and $h: X \rightarrow Y$ is the **equality constraint**. The **optimization variable** x is sought in some **optimization space** X .

Inequality constraints may be added to the above problem, either

- explicitly in the form $g(x) \leq 0$ or, more generally, in the form $g(x) \in K \subseteq Z$,
- or implicitly, by imposing $x \in C \subseteq X$ or allowing f to take values in $\mathbb{R} \cup \{\infty\}$.

Often, K is a cone and C is a convex set.

What are reasonable choices for the “spaces” X, Y, Z ?

- (1) To define the notion of global minimizers, no structure at all is required, so X, Y, Z can be general **sets**.
- (2) To define the notion of local minimizers, the space X of optimization variables must carry a **topology** since we require the concept of neighborhoods.
- (3) Statements about the existence of global minimizers build on notions of continuity and compactness.¹ Therefore, **topological spaces** are required for this purpose as well.
- (4) To formulate first-order optimality conditions, we need to be able to differentiate. A convenient setting for this are **normed linear spaces**.
- (5) For algorithmic purposes, derivatives need to be converted into directions, e. g., directions of largest/smallest directional derivatives over the unit sphere. For this purpose, **normed linear spaces** or even **Hilbert spaces**, are convenient.

Based on these considerations, we will consider only **normed linear spaces** over the field of real numbers \mathbb{R} (§ 2).²

We may anticipate a couple of differences compared to optimization over finite-dimensional linear spaces, as well as a number of questions that we will want to answer throughout the course:

- (1) Different norms on an infinite-dimensional linear space are, in general, not equivalent to each other.
- (2) How do we differentiate functions defined on infinite-dimensional normed linear space?

¹Compare, for instance, the **Weierstrass extreme value theorem**: a continuous function $f: X \rightarrow \mathbb{R}$ attains its minimum (and its maximum) on a compact set $C \subseteq X$.

²We use the term **linear space** instead of the synonymous **vector space**.

- (3) Can we formulate optimization algorithms on infinite-dimensional spaces?
- (4) If so, then when and how do we discretize in order to realize them numerically?

§ 1 MOTIVATING EXAMPLES

Example 1.1 (Brachistochrone problem).

In a 1696 article, Johann Bernoulli posted the following problem:

Given two points A and B in a vertical plane, what is the curve traced out by a point acted on only by gravity, which starts at A and reaches B in the shortest time?

This problem is known as the **Brachistochrone problem** (ancient Greek: *βράχιστος χρόνος*). In modern terms, it can be formulated as follows. Suppose that the points have coordinates $A = (0, 0)$ and $B = (b_1, b_2)$ with $b_2 \geq 0$. Let $g > 0$ denote the gravitational constant.

We are seeking a function $\gamma: [0, b_1] \rightarrow \mathbb{R}$ whose graph defines the curve from A to B . Using the principle of conservation of (potential plus kinetic) energy, we may express the speed of the particle at horizontal position x in terms of its height $\gamma(x)$. Skipping the details, this eventually leads to the following optimization problem:

$$\begin{aligned} \text{Minimize} \quad & f(\gamma) := \int_0^{b_1} \frac{\sqrt{1 + \gamma'(x)^2}}{\sqrt{2g\gamma(x)}} dx, \quad \text{where } \gamma \in X \\ \text{s. t.} \quad & \gamma(0) = 0 \\ & \text{and } \gamma(b_1) = b_2 \\ \text{as well as} \quad & \gamma \geq 0 \text{ on } [0, b_1]. \end{aligned} \tag{1.1}$$

Here X is a suitable vector space of functions $\gamma: [0, b_1] \rightarrow \mathbb{R}$, e. g., $X = C^1(0, b_1) \cap C([0, b_1])$, the space of continuous functions on $[0, b_1]$ whose restriction to the open interval $(0, b_1)$ is continuously differentiable. An alternative is the **Sobolev space** $X = H^1(0, b_1)$ of square integrable functions with square integrable weak derivative on $(0, b_1)$.³

(Quiz 1.1: Does the gravitational constant impact optimal curves?) One can show that the (unique) minimizer of (1.1) satisfies a first-order necessary optimality condition, which comes in the form of a differential equation:

$$\frac{1}{2} \sqrt{\frac{1 + \gamma'(x)^2}{\gamma(x)^3}} + \frac{d}{dx} \frac{\gamma'(x)}{\sqrt{\gamma(x) (1 + \gamma'(x)^2)}} = 0.$$

The solutions of this equation satisfy

$$\gamma(x) (1 + \gamma'(x)^2) = C \quad \text{in } (0, b_1) \tag{1.2}$$

for some $C > 0$, and it has infinite slope initially:

$$\lim_{x \searrow 0} \gamma'(x) = \infty.$$

³We will introduce Sobolev spaces later; see § 2.6.

The unique solution is given by the curve

$$t \mapsto \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C \begin{pmatrix} t - \sin(t) \\ 1 - \cos(t) \end{pmatrix} \quad \text{for } t \in [0, T], \quad (1.3)$$

where $C > 0$ and $T \in (0, 2\pi]$ are determined by the conditions $x(T) = b_1$ and $y(T) = b_2$.

This curve is a segment of a **cycloid** with radius C . △

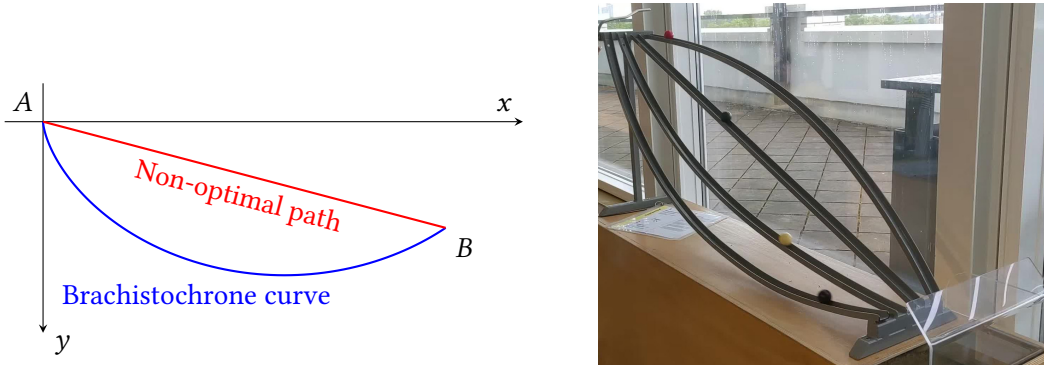


Figure 1.1: Some non-optimal curve $\gamma: [0, b_1] \rightarrow \mathbb{R}$ from A to B (left) as well as the unique global minimizer of the Brachistochrone problem (1.1), given by the segment of a cycloid (left). Image of an experimental device on display at **Technoseum Mannheim** (right), shot by Roland Herzog.

Remark 1.2 (on the Brachistochrone problem).

The first-order optimality condition of the Brachistochrone problem come in the form of a differential equation (1.2). This is typical for optimization problems whose variables are functions and whose objectives involve derivatives of those functions. As a result, minimizers may be more regular than suggested by the optimization space X . This is indeed the case in the Brachistochrone problem (1.1), where the unique minimizer turns out to be a $C^\infty(0, b_1)$ -function. △

Expert Knowledge: The origins of the calculus of variations

The Brachistochrone problem belongs to a class of problems referred to as **calculus of variations**, where optimization variables are functions and objectives are typically integrals involving values of the function and its derivative(s). This term was coined in 1766 by Leonhard Euler. The first-order optimality conditions for calculus of variations problems are referred to as **Euler-Lagrange equations**.

Newton's problem of minimal resistance from 1687 is considered the first problem of this type, and the Brachistochrone problem (1696) is second. That problem attracted the attention of Johann Bernoulli's brother Jakob, as well as of Isaac Newton, Gottfried Leibniz, Ehrenfried Walther von Tschirnhaus and Guillaume de l'Hôpital, who all turned in solutions.

Example 1.3 (Fermat's principle in optics).

Suppose that $n: \mathbb{R}^2 \rightarrow \mathbb{R}_{>0}$ is the material dependent refractive index of an optical material. Let $\gamma: [0, b_1] \rightarrow \mathbb{R}$ denote a function whose graph defines a curve through this material. Then the optical length of this curve is defined by

$$\int_0^{b_1} n(x, \gamma(x)) \sqrt{1 + \gamma'(x)^2} dx.$$

Fermat's principle stipulates that the path a ray of light will take minimizes the optical length. Suppose that the end points of that path are $A = (0, 0)$ and $B = (b_1, b_2)$. Then we obtain the following optimization problem:

$$\begin{aligned} \text{Minimize} \quad & f(\gamma) := \int_0^{b_1} n(x, \gamma(x)) \sqrt{1 + \gamma'(x)^2} dx, \quad \text{where } \gamma \in X \\ \text{s. t.} \quad & \gamma(0) = 0 \\ \text{and} \quad & \gamma(b_1) = b_2. \end{aligned} \tag{1.4}$$

In the particular case where the refractive index is piecewise constant on slabs, the unique global minimizer of (1.4) satisfies **Snell's law**, which states that the incident angles θ_+ , θ_- (measured against the normal) of two neighboring slabs satisfy the relation $n_+ \sin(\theta_+) = n_- \sin(\theta_-)$, see Figure 1.2.

Similar as in Example 1.1, every minimizer satisfies a first-order optimality condition that amounts to a differential equation:

$$-\frac{n(x, \gamma(x)) \gamma'(x)}{\sqrt{1 + \gamma'(x)^2}} + n_y(x, \gamma(x)) \sqrt{1 + \gamma'(x)^2} = 0.$$

In this case, however, the discontinuous coefficient n may limit the regularity of an optimal path. Again, for piecewise constant refractive index, an optimal curve will be piecewise linear with discontinuous derivative at optical interfaces. \triangle

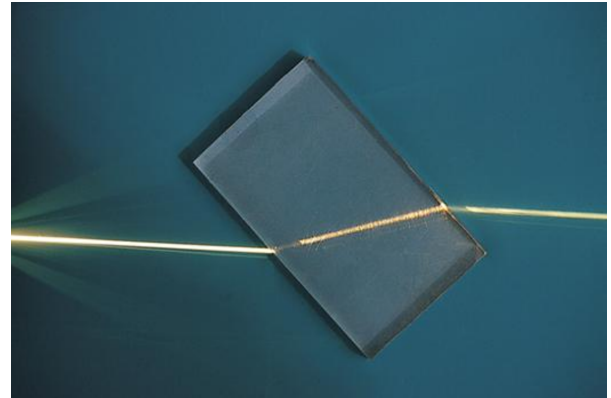
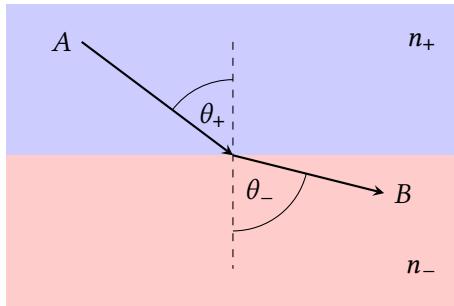


Figure 1.2: Illustration of Snell's law of refraction (left) as a special case of Example 1.3. Image (right) obtained from <https://en.wikipedia.org/wiki/Refraction>, released into the public domain by creator ajizai.

End of Class 1

Example 1.4 (signal denoising).

Suppose a signal $s: [0, T] \rightarrow \mathbb{R}$ is given.⁴ In case the signal is noisy, we may formulate an optimization problem to try and find a denoised signal $y: [0, T] \rightarrow \mathbb{R}$:

$$\text{Minimize } f(y) := \int_0^T |y(t) - s(t)|^2 dt + \beta \int_0^T |\dot{y}(t)|^2 dt, \quad \text{where } y \in X. \quad (1.5)$$

The dot denotes the time derivative. A suitable function space for this problem is the Sobolev space $X = H^1(0, T)$.

The second term in the objective penalizes “fast variations” in the signal. The parameter $\beta > 0$ balances the two summands in the objective and thus determines the degree of denoising.

We will be able to show later that the first-order optimality conditions for (1.5) involve the second-order differential equation

$$-\beta \ddot{y}(t) + y(t) = s(t), \quad (1.6)$$

which shows that the minimizer will indeed be a smoothed version of the noisy signal s . More precisely, we can expect the solution to gain two orders of differentiation compared to the data s . In particular, the solution will not admit any discontinuities. Therefore, one often prefers a “less powerful” regularization term, such as the **total variation** of the function y . We will come back to this type of problem in the context of image denoising problems in Chapter 1. \triangle

Example 1.5 (crane trolley optimal control problem).

Consider a load on rope of length ℓ hanging from a crane trolley system (Figure 1.3). We denote the position of the trolley relative to the origin by s . The position of the load relative to the trolley is denoted by z . The trolley has mass M and the load has mass m . A controllable force u acts on the trolley.

This system is described by a second-order differential equation for the positions (s, z) . It can be derived by working out Newton’s law, force equals mass times acceleration. We convert it here to a first-order system of differential equations in terms of $x = (s, \dot{s}, z, \dot{z})$, where the dot denotes the time derivative. Assuming small angles θ , the differential equations can be taken as linear and the system reads

$$\begin{pmatrix} \dot{s} \\ \ddot{s} \\ \dot{z} \\ \ddot{z} \end{pmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{m}{M} \frac{g}{\ell} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{m+M}{M} \frac{g}{\ell} & 0 \end{bmatrix}}_{=:A} \begin{pmatrix} s \\ \dot{s} \\ z \\ \dot{z} \end{pmatrix} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{M} \end{bmatrix}}_{=:B} u \quad (1.7)$$

or, in short, $\dot{x} = Ax + Bu$. Notice that we have omitted the (t) argument everywhere for brevity.

We wish to steer the system from an initial state $x(0) = (0, 0, 0, 0)^T$ to a terminal state $x(T) = (E, 0, 0, 0)^T$ in as short a time T as possible. This leads us to the preliminary optimization problem

$$\begin{aligned} &\text{Minimize } \int_0^T 1 dt, \quad \text{where } (u, x, T) \in U \times X \times \mathbb{R} \\ &\text{s. t. } \dot{x} = Ax + Bu \quad \text{in } [0, T] \\ &\text{and } x(0) = (0, 0, 0, 0)^T \\ &\text{and } x(T) = (E, 0, 0, 0)^T \\ &\text{as well as } T > 0. \end{aligned} \quad (1.8)$$

⁴Think, for instance, of an audio signal sampled with a certain frequency, say, 48 kHz into a piecewise constant function.

This preliminary problem formulation has some issues. Due to the terminal time T being an optimization variable, we cannot fix function spaces for the **control** u and the **state** x since they depend on T .

There is, however, an easy remedy to this. We can renormalize the unknown time interval $[0, T]$ to the fixed interval $[0, 1]$. Replacing the unknowns x and u by their counterparts on the fixed interval, the dynamics need to be rescaled and the problem becomes

$$\begin{aligned}
 &\text{Minimize} && \int_0^1 T \, dt, \quad \text{where } (u, x, T) \in U \times X \times \mathbb{R} \\
 &\text{s. t.} && \dot{x} = \frac{1}{T}(Ax + Bu) \quad \text{in } [0, 1] \\
 &&& \text{and } x(0) = (0, 0, 0, 0)^\top \\
 &&& \text{and } x(1) = (E, 0, 0, 0)^\top \\
 &\text{as well as} && T > 0.
 \end{aligned} \tag{1.9}$$

We can now fix suitable function spaces⁵, e. g., $U = L^2(0, 1)$ and $X = H^1(0, 1)^4$. A problem such as (1.9), in which a **state** function x depends on the choice of the **control** function u through a differential equation, is termed an **optimal control problem**. We will see more of these in Chapter 2.

Unfortunately, problem (1.9) as stated will not have a solution. (**Quiz 1.2:** Can you see why?) We may fix this by imposing bounds on the control function, e. g., by adding the pointwise inequality constraints

$$u(t) \in [-u_{\max}, u_{\max}],$$

with some $u_{\max} > 0$ to problem (1.9), or by adding a cost term such as

$$\beta \int_0^1 |u(t)| \, dt$$

to the objective. △

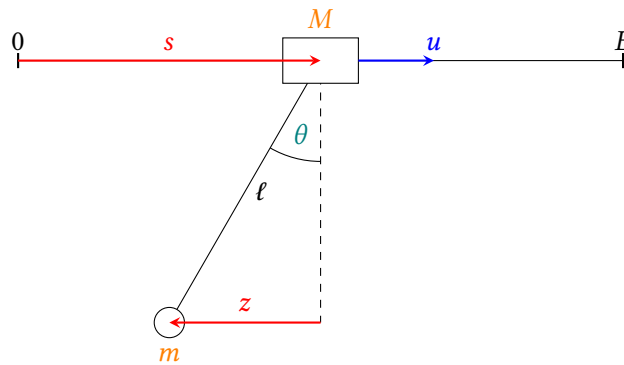


Figure 1.3: Illustration of the crane trolley problem (Example 1.5).

⁵Again, we will introduce these Lebesgue and Sobolev spaces later; see §§ 2.5 and 2.6.

§ 2 NORMED LINEAR SPACES

In this section we recap the notion of a normed linear space. We will also introduce Lebesgue and Sobolev spaces as our prime examples of normed linear spaces.

Definition 2.1 (linear space).

An algebraic structure $(V, +, \cdot)$ with two operations⁶

$$\begin{aligned} +: V \times V &\rightarrow V && \text{(addition)} \\ \cdot: \mathbb{R} \times V &\rightarrow V && \text{(S-multiplication)} \end{aligned}$$

is said to be a **linear space** over the field of real numbers \mathbb{R} if

- (i) $(V, +)$ is an Abelian group.
- (ii) The S-multiplication satisfies the mixed distributive laws

$$\begin{aligned} \alpha(u + v) &= (\alpha u) + (\alpha v) \\ (\alpha + \beta)v &= (\alpha v) + (\beta v) \end{aligned}$$

as well as the mixed associative law

$$(\alpha \beta)v = \alpha(\beta v)$$

for all $\alpha, \beta \in \mathbb{R}$ and $u, v \in V$. Moreover, the neutral element $1 \in \mathbb{R}$ w.r.t. multiplication in \mathbb{R} is also neutral w.r.t. S-multiplication:

$$1v = v. \quad \triangle$$

All linear spaces will be over the field of real numbers \mathbb{R} and we will not explicitly mention that. We already anticipated that in order to be able to differentiate functions $f: V \rightarrow \mathbb{R}$ or, more generally, $f: V \rightarrow W$, we will require linear spaces to be **normed**.

Definition 2.2 (normed linear space).

Suppose that V is a linear space.

- (i) A map $\|\cdot\|: V \rightarrow \mathbb{R}$ is said to be a **norm on V** if the following conditions hold:

$$\|u\| \geq 0, \quad \text{and } \|u\| = 0 \Rightarrow u = 0 \quad \text{positive definiteness} \quad (2.1a)$$

$$\|\alpha u\| = |\alpha| \|u\| \quad \text{absolute homogeneity} \quad (2.1b)$$

$$\|u + v\| \leq \|u\| + \|v\| \quad \text{triangle inequality or subadditivity} \quad (2.1c)$$

for all $u, v \in V$ and all $\alpha \in \mathbb{R}$.

- (ii) The pair $(V, \|\cdot\|)$ is said to be a **(real) normed linear space** or **normed vector space**. \triangle

⁶The dot \cdot for S-multiplication is usually not written, just as the multiplication symbol in \mathbb{R} is usually not written.

Expert Knowledge: from topological to normed linear spaces

We have the inclusions

- Every normed linear space is a metric space.
- Every metric space is a topological space.

A topological space is defined by a collection of its subsets that are called the open sets. Topological spaces admit notions of convergence and limits, closure and compactness of sets, as well as notions of continuity of functions.

Metric spaces are spaces with a notion of distance. The metric induces a topology.

Normed spaces are spaces with a notion of length. The norm induces a metric.

We will not discuss general topological spaces in full generality but restrict ourselves to normed linear spaces.

§ 2.1 OPEN AND CLOSED SETS

Definition 2.3 (balls, spheres, open sets, closed sets).

Suppose that $(V, \|\cdot\|)$ is a normed linear space.

- (i) For $\varepsilon > 0$, the set

$$B_\varepsilon(x) := \{y \in V \mid \|y - x\| < \varepsilon\}$$

is said to be the **open ε -ball** about x of radius ε . In particular, $B_1(0)$ is termed the **open unit ball**.

- (ii) A point $x \in E$ of a subset $E \subseteq V$ is said to be an **interior point** of E if there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq E$. The subset of interior points of E is called the **interior** of E and it is denoted by $\text{int } E$.

- (iii) A set $U \subseteq V$ is said to be **open** if every $x \in U$ is an interior point of U , i. e., if $\text{int } U = U$.

- (iv) A set $A \subseteq V$ is said to be **closed** if its complement $V \setminus A$ is open.

- (v) For $\varepsilon > 0$, the set

$$\overline{B_\varepsilon(x)} := \{y \in V \mid \|y - x\| \leq \varepsilon\}$$

is said to be the **closed ε -ball** about x of radius ε . In particular, $\overline{B_1(0)}$ is termed the **closed unit ball**.

- (vi) The **closure** of a subset $E \subseteq V$ is

$$\text{cl } E := \bigcap \{A \subseteq V \mid A \text{ is closed and } E \subseteq A\}. \quad (2.2)$$

- (vii) The **boundary** of a subset $E \subseteq V$ is $\partial E := \text{cl } E \setminus \text{int } E$, i. e., the closure minus the interior of E .

- (viii) The set

$$\partial B_\varepsilon(x) := \{y \in V \mid \|y - x\| = \varepsilon\}$$

is said to be the **ε -sphere** about x of radius ε . In particular, $\partial B_1(0)$ is termed the **unit sphere** of V . △

It is not difficult to show that the interior of a set is open and the closure of a set is closed. **In fact, a set E is open if and only if $E = \text{int } E$, and a set A is closed if and only if $A = \text{cl } A$. Also, a set A is closed if and only if $A = \partial A$.** The boundary of a set is also closed. (**Quiz 2.1:** Can you show this?)

The following result was inserted after the class.

Lemma 2.4 (characterization of the closure⁷).

Suppose that $(V, \|\cdot\|)$ is a normed linear space and $E \subseteq V$. Then

$$\begin{aligned} \text{cl } E &= \{y \in V \mid \text{for any } \varepsilon > 0 \text{ there exists } x \in E \text{ such that } \|x - y\| < \varepsilon\} \\ &= \{y \in V \mid \text{for any } \varepsilon > 0, B_\varepsilon(y) \cap E \neq \emptyset\} \\ &= \{y \in V \mid \text{there exists a sequence } (x^{(k)}) \text{ in } E \text{ converging to } y\}. \end{aligned} \quad (2.3)$$

Proof.

□

The following lemma (**inserted after the class**) confirms that the nomenclature and symbols related to balls and spheres is meaningful:

Lemma 2.5 (openness, closedness, boundary of balls and spheres).

Suppose that $(V, \|\cdot\|)$ is a normed linear space.

- (i) Open balls $B_\varepsilon(x)$ are open sets.
- (ii) Closed balls $\overline{B_\varepsilon(x)}$ are closed sets.
- (iii) Open balls and closed balls are related via

$$\overline{B_\varepsilon(x)} = \text{cl } B_\varepsilon(x) \quad \text{and} \quad B_\varepsilon(x) = \text{int } \overline{B_\varepsilon(x)}. \quad (2.4)$$

- (iv) Spheres and balls are related via

$$\partial B_\varepsilon(x) = \partial(B_\varepsilon(x)) = \partial(\overline{B_\varepsilon(x)}). \quad (2.5)$$

Proof.

□

End of Class 2

End of Week 1

⁷We can read this result as “The closure of a set E consists of the **accumulation points** of E .”

§ 2.2 BANACH SPACES

Since norms furnish a linear space with a topology, they also bring about a notion of convergence.

Definition 2.6 (convergent sequence, Cauchy sequence).

Suppose that $(V, \|\cdot\|)$ is a normed linear space.

- (i) A sequence⁸ $(x^{(k)})$ in V is said to **converge to** $x \in V$ in case $\|x^{(k)} - x\| \rightarrow 0$ in \mathbb{R} . We then write $x^{(k)} \rightarrow x$ or $\lim_{k \rightarrow \infty} x^{(k)} = x$ and call x a **limit point** or **limit** of the sequence $(x^{(k)})$.
In other words, $x^{(k)} \rightarrow x$ means: for every $\varepsilon > 0$ there exists an index k_ε such that $\|x^{(k)} - x\| < \varepsilon$ holds for all $k \geq k_\varepsilon$.
- (ii) A sequence $(x^{(k)})$ in V is said to **converge** if there exists some $x \in V$ such that $x^{(k)} \rightarrow x$.
- (iii) A sequence $(x^{(k)})$ in V is said to be a **Cauchy sequence** in V if, for every $\varepsilon > 0$, there exists an index k_ε such that $\|x^{(k)} - x^{(\ell)}\| < \varepsilon$ holds for all $k, \ell \geq k_\varepsilon$. △

Lemma 2.7 (properties of convergent sequences).

Suppose that $(V, \|\cdot\|)$ is a normed linear space and that $(x^{(k)})$ is a sequence in V .

- (i) Suppose that $(x^{(k)})$ converges. Then its limit is unique.
- (ii) Suppose that $(x^{(k)})$ converges. Then it is a Cauchy sequence.

Proof. This proof is addressed in [homework problem 2.3](#). □

The converse of [statement \(ii\)](#) is not true in general. Therefore, spaces in which it is true deserve special mention:

Definition 2.8 (complete normed linear space, Banach space, complete subset).

Suppose that $(V, \|\cdot\|)$ is a normed linear space.

- (i) The space $(V, \|\cdot\|)$ is said to be **complete** or a **Banach space** if every Cauchy sequence in V converges.
- (ii) A subset $A \subseteq V$ is said to be **complete** if every Cauchy sequence in A converges to a limit in A . △

The following result was inserted after the class.

Lemma 2.9 (in Banach spaces, completeness is closedness).

Suppose that $(V, \|\cdot\|)$ is a Banach space. The $A \subseteq V$ is complete if and only if A is closed.

Proof. This proof is addressed in [homework problem 2.2](#). □

The following result was inserted after the class.

⁸The exact index set of a sequence does not matter. We will allow any interval of the integers \mathbb{Z} which is bounded below but not bounded above. In other words, any subset of \mathbb{Z} of the form $\{k_0, k_0 + 1, k_0 + 2, \dots\}$.

Lemma 2.10 (complete sets are closed).

Suppose that $(V, \|\cdot\|)$ is a normed linear space and $E \subseteq V$. If E is complete, then E is closed.

Proof. Suppose that $(x^{(k)})$ is a sequence in E converging to some $x \in V$. Then this sequence is a Cauchy sequence in E . Since E is complete, $(x^{(k)})$ converges to a limit $y \in E$. By uniqueness of the limit, we have $x = y \in E$. By the characterization (2.3) of the closure, we have $E = \text{cl } E$. \square

§ 2.3 COMPARISON OF NORMS

We wish to be able to compare two different norms on the same linear space. The following definition allows us to do that.

Definition 2.11 (partial ordering of norms).

Suppose that V is a linear space and that $\|\cdot\|_a$ and $\|\cdot\|_b$ are two norms on V .

- (i) The norm $\|\cdot\|_a$ is said to be **weaker** than the norm $\|\cdot\|_b$ if there exists a constant $c > 0$ such that

$$\|x\|_a \leq c \|x\|_b \quad \text{holds for all } x \in V. \quad (2.6)$$

In this case, we also say that $\|\cdot\|_b$ is **stronger** than $\|\cdot\|_a$. We write $\|\cdot\|_a \leq \|\cdot\|_b$ or $\|\cdot\|_b \geq \|\cdot\|_a$.

- (ii) The norms $\|\cdot\|_a$ and $\|\cdot\|_b$ are said to be **equivalent** if both $\|\cdot\|_a \leq \|\cdot\|_b$ and $\|\cdot\|_b \leq \|\cdot\|_a$ hold, i. e., if there exist constants $c_1, c_2 > 0$ such that

$$c_1 \|x\|_a \leq \|x\|_b \leq c_2 \|x\|_a \quad \text{holds for all } x \in V. \quad (2.7)$$

\triangle

The following result was corrected.^{RH}

Lemma 2.12 (openness, closedness, completeness and the Cauchy property are preserved under weaker norms).

Suppose that V is a linear space and that $\|\cdot\|_a$ and $\|\cdot\|_b$ are two norms on V such that $\|\cdot\|_a \leq \|\cdot\|_b$. Then the following hold:

- (i) For any open ball $B_\varepsilon^{\|\cdot\|_a}(x)$ in the weaker norm $\|\cdot\|_a$, there exists an open ball $B_\delta^{\|\cdot\|_b}(x)$ in the **stronger** norm $\|\cdot\|_b$ such that $B_\delta^{\|\cdot\|_b}(x) \subseteq B_\varepsilon^{\|\cdot\|_a}(x)$.
(The **stronger** norm has the smaller balls and more open sets.)
- (ii) If $U \subseteq V$ is open in the weaker norm $\|\cdot\|_a$, then U is open in the **stronger** norm $\|\cdot\|_b$.
(The **stronger** norm defines the finer topology.)
- (iii) If $A \subseteq V$ is closed in the weaker norm $\|\cdot\|_a$, then A is closed in the **stronger** norm $\|\cdot\|_b$.
- (iv) If $E \subseteq V$ is bounded in the **stronger** norm $\|\cdot\|_b$, then E is bounded in the weaker norm $\|\cdot\|_a$.
- (v) If $K \subseteq V$ is totally bounded in the **stronger** norm $\|\cdot\|_b$, then K is totally bounded in the weaker norm $\|\cdot\|_a$.
- (vi) If $K \subseteq V$ is compact in the **stronger** norm $\|\cdot\|_b$, then K is compact in the weaker norm $\|\cdot\|_a$.
- (vii) If $(x^{(k)})$ converges in the **stronger** norm $\|\cdot\|_b$, then $(x^{(k)})$ converges in the weaker norm $\|\cdot\|_a$ (to the same limit point).

(viii) If $(x^{(k)})$ is a Cauchy sequence in the **stronger** norm $\|\cdot\|_b$, then $(x^{(k)})$ is a Cauchy sequence in the weaker norm $\|\cdot\|_a$.

Proof. This proof is addressed in [homework problem 3.1](#). □

Theorem 2.13 (in finite-dimensional normed linear spaces, all norms are equivalent).

Suppose that V is a finite-dimensional linear space. If $\|\cdot\|_a$ and $\|\cdot\|_b$ are two norms on V , then $\|\cdot\|_a$ and $\|\cdot\|_b$ are equivalent.

Proof. Suppose that $\{v^{(1)}, \dots, v^{(n)}\}$ is a basis of V . Then every $x \in V$ can be uniquely written as $x = \sum_{j=1}^n x_j v^{(j)}$. The map $x \mapsto \|x\|_\infty := \left\| \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right\|_\infty = \max\{|x_1|, \dots, |x_n|\}$ is a norm on V .

It is enough to prove that the norms $\|\cdot\|_a$ and $\|\cdot\|_\infty$ are equivalent norms on V since equivalence of norms is an equivalence relation.

Step 1: To show $\|\cdot\|_a \lesssim \|\cdot\|_\infty$, we estimate:

$$\begin{aligned} \|x\|_a &= \left\| \sum_{j=1}^n x_j v^{(j)} \right\|_a \\ &\leq \sum_{j=1}^n |x_j| \|v^{(j)}\|_a \\ &\leq \|x\|_\infty \sum_{j=1}^n \|v^{(j)}\|_a \\ &=: c \|x\|_\infty. \end{aligned}$$

Step 2: We show that $\|\cdot\|_\infty \lesssim \|\cdot\|_a$.

Suppose that this is not the case. Then there exists a sequence $(x^{(k)})$ in V such that $\|x^{(k)}\|_\infty > k \|x^{(k)}\|_a$. We can assume that $\|x^{(k)}\|_\infty = 1$ holds. (**Quiz 2.2:** Why?)

On the other hand, for all $j = 1, \dots, n$, the j -th coefficients $\{x_j^{(k)} \mid k \in \mathbb{N}\}$ belong to the compact interval $[-1, 1]$. Therefore, we can find a subsequence $x^{(k^{(\ell)})}$ such that $x_j^{(k^{(\ell)})}$ converges to some x_j^* for all $j = 1, \dots, n$. Moreover, for at least one index $j_0 \in \{1, \dots, n\}$, we have $|x_{j_0}^{(k^{(\ell)})}| = 1$ for infinitely many indices $\ell \in \mathbb{N}$. We pass to this subsequence without re-labeling it. This shows $|x_{j_0}^*| = 1$ by continuity of the absolute value function.

We define $x^* := \sum_{j=1}^n x_j^* v^{(j)}$. The estimate

$$\begin{aligned} \|x^*\|_a &\leq \|x^* - k^{(\ell)}\|_a + \|k^{(\ell)}\|_a \\ &\leq c \|x^* - k^{(\ell)}\|_\infty + \frac{1}{k^{(\ell)}} \quad \text{by step 1} \\ &\rightarrow 0 + 0 \quad \text{as } \ell \rightarrow \infty \end{aligned}$$

shows $x^* = 0$, i. e., all coefficients x_j^* are zero. This contradicts $|x_{j_0}^*| = 1$. □

Note: As a consequence of this theorem, we do not necessarily need to specify the norm when we talk about a finite-dimensional linear space. In particular, all norms on \mathbb{R} are equivalent, with the absolute value $|\cdot|$ as the standard norm.

As a consequence of [Theorem 2.13](#), we can show:

Lemma 2.14 (finite-dimensional subspaces are complete and thus closed).

Suppose that $(V, \|\cdot\|)$ is a normed linear space. Every finite-dimensional subspace $Y \subseteq V$ is complete and thus closed.

Proof. Suppose that $\{y^{(1)}, \dots, y^{(n)}\}$ is a basis of Y . By [Theorem 2.13](#), the norms $\|\cdot\|$ and $\|\cdot\|_\infty$ are equivalent on Y , where $\|x\|_\infty := \max\{|x_1|, \dots, |x_n|\}$ when $x = \sum_{j=1}^n x_j y^{(j)}$.

Suppose now that $(x^{(k)})$ is a Cauchy sequence in Y . The elements of $(x^{(k)})$ have a representation

$$x^{(k)} = \sum_{j=1}^n x_j^{(k)} y^{(j)}.$$

Then for any $j = 1, \dots, n$, the sequence $\{x_j^{(k)}\}$ is a Cauchy sequence in $(\mathbb{R}, |\cdot|)$. Therefore, $x_j^{(k)} \rightarrow x_j^*$ for some $x_j^* \in \mathbb{R}$. We thus obtain

$$x^{(k)} = \sum_{j=1}^n x_j^{(k)} y^{(j)} \rightarrow \sum_{j=1}^n x_j^* y^{(j)} \in Y.$$

This shows that $(x^{(k)})$ converges in Y . Therefore, Y is a complete subset of V and thus closed by [Lemma 2.10](#). \square

Note: In particular, if V itself is finite-dimensional, then it is complete and thus closed.

§ 2.4 COMPACTNESS

Compactness of sets plays a major role in topology, analysis, and also in optimization.

Definition 2.15 (compact, sequentially compact and totally bounded sets).

Suppose that $(V, \|\cdot\|)$ is a normed linear space and $E \subseteq V$ is some subset.

- (i) A collection $(U_i)_{i \in I}$ of open subsets $U_i \subseteq V$ is said to be an **open cover** of E if $E \subseteq \bigcup_{i \in I} U_i$ holds.
- (ii) A subset $K \subseteq V$ is said to be **compact** if every open cover $(U_i)_{i \in I}$ of K contains a finite subcover, i. e., there exist a finite number of indices $i_1, \dots, i_N \in I$ such that $K \subseteq \bigcup_{j=1}^N U_{i_j}$.
- (iii) A subset $K \subseteq V$ is said to be **sequentially compact** if every sequence $(x^{(k)})$ in K contains a convergent subsequence whose limit belongs to K .⁹

⁹Stated equivalently, $(x^{(k)})$ has an accumulation point in K .

- (iv) A subset $K \subseteq V$ is said to be **totally bounded** if for any $\varepsilon > 0$, there exist finitely many $x^{(1)}, \dots, x^{(N)} \in K$ such that $\{B_\varepsilon(x^{(1)}), \dots, B_\varepsilon(x^{(N)})\}$ covers K . \triangle

The verification of compactness via [Definition 2.15 \(ii\)](#) can be cumbersome. The following results can help.

Lemma 2.16 (compact sets are closed and bounded).

Suppose that $(V, \|\cdot\|)$ is a normed linear space and $K \subseteq V$ is a compact subset. Then K is closed and bounded.

Proof. We prove both properties independently.

Step 1: We show that K is closed.

The statement is true when $K = V$ (**Quiz 2.3:** Is it clear to you?), so suppose $K \subsetneq V$ from now on. Suppose that $z \in V \setminus K$ is a point of the complement of K . We need to show that there exists an open ball $B_\varepsilon(z) \subseteq V \setminus K$.

For any $x \in K$, define $\varepsilon_x := \frac{1}{2}\|x - z\|$. In view of $z \notin K$ and the positive definiteness of the norm, we have $\varepsilon_x > 0$. The open balls $B_{\varepsilon_x}(x)$ and $B_{\varepsilon_x}(z)$ are disjoint since for any point y in their intersection, the triangle inequality would imply the contradiction

$$\|x - z\| \leq \|x - y\| + \|y - z\| < \varepsilon_x + \varepsilon_x = \|x - z\|.$$

The sets $\{B_{\varepsilon_x}(x) \mid x \in K\}$ form an open cover of K . Since K is compact, finitely many of these suffice, say, those with center points $x^{(1)}, \dots, x^{(N)} \in K$. As we noticed above, $B_{\varepsilon_{x^{(j)}}}(x^{(j)})$ and $B_{\varepsilon_{x^{(j)}}}(z)$ are disjoint for all $j = 1, \dots, N$. Let $\varepsilon := \min\{\varepsilon_{x^{(1)}}, \dots, \varepsilon_{x^{(N)}}\}$. Then $B_\varepsilon(z)$ is disjoint from all $B_{\varepsilon_{x^{(j)}}}(x^{(j)})$ and hence from K .

Step 2: We show that K is bounded.

Fix $x \in V$ arbitrarily and consider the open balls $\{B_i(x) \mid i \in \mathbb{N}\}$. Since every element of K has a finite distance from the point x , this collection of open balls covers K . Since K is compact, a finite number of these suffice, say,

$$\{B_{i(1)}(x), \dots, B_{i(N)}(x)\}.$$

These being balls with the same center, one of them is largest, say, $B_{i(*)}(x)$, which alone covers K . \square

Theorem 2.17 (in normed linear spaces, the notions of compact and sequentially compact sets coincide).

Suppose that $(V, \|\cdot\|)$ is a normed linear space and $K \subseteq V$ is some subset. Then the following are equivalent:

- (i) K is compact.
- (ii) K is sequentially compact.
- (iii) K is complete and totally bounded.

Proof. **Statement (i) \Rightarrow statement (ii):** Suppose that $(x^{(n)})$ is a sequence in K that **does not possess a convergent subsequence with limit in K . In other words, $(x^{(n)})$** does not have an accumulation point in K . Therefore, for any $x \in K$, there exists $\varepsilon_x > 0$ such that $x^{(k)} \in B_x(\varepsilon_x)$ holds only for finitely many indices k . The sets $\{B_{\varepsilon_x}(x) \mid x \in K\}$ form an open cover of K . By the compactness of K , there exists a finite subcover

$$\{B_{\varepsilon_{x^{(1)}}}(x^{(1)}), \dots, B_{\varepsilon_{x^{(N)}}}(x^{(N)})\}$$

of K . By construction, $x^{(k)} \in B_{\varepsilon_{x^{(i)}}}(x^{(i)})$ holds only for finitely many indices k . That is, $x^{(k)} \in \bigcup_{i=1}^N B_{\varepsilon_{x^{(i)}}}(x^{(i)})$ also holds only for finitely many indices k . Therefore, finally, $x^{(k)} \in K$ also holds only for finitely many indices k . This contradicts $(x^{(n)})$ being a sequence in K .

Statement (ii) \Rightarrow statement (iii): Suppose now that K is sequentially compact. Then, by definition, every sequence in K contains a convergent subsequence whose limit belongs to K . In particular, this is true for any Cauchy sequence in K , hence K is complete.

To show that K is totally bounded, suppose that $\varepsilon > 0$. If $K = \emptyset$, nothing is to be done, so suppose $K \neq \emptyset$. Pick a point $x^{(1)} \in K$. In case $K \subseteq B_\varepsilon(x^{(1)})$, we are done. Otherwise, pick a point $x^{(2)} \in K \setminus B_\varepsilon(x^{(1)})$. In case $K \subseteq B_\varepsilon(x^{(1)}) \cup B_\varepsilon(x^{(2)})$, we are done. Otherwise, continue in the same way. If this process produced an infinite sequence $(x^{(k)})$, its members would satisfy $\|x^{(k)} - x^{(\ell)}\| \geq \varepsilon$ for all $k \neq \ell$. Therefore, this sequence in K cannot have a convergent subsequence, contradicting the assumption that K is sequentially compact. Consequently, the process above terminates after finitely many steps, showing $K \subseteq \bigcup_{i=1}^N B_{\varepsilon_{x^{(i)}}}(x^{(i)})$. That is, K is totally bounded.

Statement (iii) \Rightarrow statement (i): We proceed by contradiction. Suppose that $(U_i)_{i \in I}$ is an open cover of K that does not possess a finite subcover.

Since K is totally bounded, K can be covered by a finite number of open balls of radius 1 with centers in K . For at least one of these, say, $B_1(x^{(0)})$, the intersection $B_1(x^{(0)}) \cap K$ cannot be covered by a finite subfamily of $(U_i)_{i \in I}$. (Otherwise, K itself could be covered by a finite subfamily of $(U_i)_{i \in I}$, which we assumed is not the case.)

Now consider $B_1(x^{(0)}) \cap K$. As a subset of K , this set is again totally bounded and thus can be covered by a finite number of open balls of radius $1/2$ with centers in $B_1(x^{(0)}) \cap K$. Again, for at least one of these, say, $B_{1/2}(x^{(2)})$, the intersection $B_{1/2}(x^{(2)}) \cap K$ cannot be covered by a finite subfamily of $(U_i)_{i \in I}$. (Otherwise, $B_1(x^{(0)}) \cap K$ itself could be covered by a finite subfamily of $(U_i)_{i \in I}$, which we know is not the case.)

Repeating this process, we obtain a sequence of balls $B_{2^{-k}}(x^{(k)})$, for none of which $B_{2^{-k}}(x^{(k)}) \cap K$ is covered by a finite subfamily of $(U_i)_{i \in I}$. The centers satisfy $x^{(k+1)} \in B_{2^{-k}}(x^{(k)}) \cap K$. Therefore, the sequence $(x^{(k)})$ is a Cauchy sequence in K since $\|x^{(k)} - x^{(\ell)}\| < 2^{1-k}$ holds for all $\ell \geq k$. (**Quiz 2.4:** Can you fill in the details?) Since K was assumed to be a complete subset of V , this Cauchy sequence converges and its limit x^* belongs to K .

This implies that x^* belongs to some member of the family $(U_i)_{i \in I}$, say, $x \in U_{i^*}$. Since U_{i^*} is open, there exists $\varepsilon > 0$ such that

$$x^* \in B_\varepsilon(x^*) \subseteq U_{i^*}$$

holds. We can find an index $N \in \mathbb{N}$ such that $2^{-N} < \varepsilon/2$ and

$$\|x^{(N)} - x^*\| < \frac{\varepsilon}{2}$$

holds. Consequently, for any $y \in B_{2^{-N}}(x^{(N)})$, we have

$$\|y - x^*\| \leq \|y - x^{(N)}\| + \|x^{(N)} - x^*\| < 2^{-N} + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which means that we have

$$B_{2^{-N}}(x^{(N)}) \subseteq B_\varepsilon(x^*) \subseteq U_{i^*}.$$

This, however, contradicts the fact that for none of the balls $B_{2^{-k}}(x^{(k)})$, the intersection $B_{2^{-k}}(x^{(k)}) \cap K$ can be covered by a finite subfamily of $(U_i)_{i \in I}$.

Consequently, the assumption that there exists an open cover $(U_i)_{i \in I}$ of K that does not possess a finite subcover, cannot be true. This shows that K is compact. \square

The notion of compactness is very strong in infinite-dimensional normed linear spaces. As a consequence, only “few” sets are compact.

Theorem 2.18 (compactness of the unit ball).

Suppose that $(V, \|\cdot\|)$ is a normed linear space. Then the following are equivalent:

- (i) The closed unit ball $\overline{B_1(0)}$ is compact.
- (ii) The unit sphere $\partial B_1(0)$ is compact.
- (iii) $\dim(V)$ is finite.

Notice that this theorem holds independently of which particular norm is chosen on the linear space V !

The proof of [Theorem 2.18](#) uses the following result:

Lemma 2.19 (**Riesz lemma**).

Suppose that $(V, \|\cdot\|)$ is a normed linear space. Moreover, let $Y \subsetneq V$ be a closed proper subspace of V . Then for any $\theta \in (0, 1)$, there exists $x_\theta \in V$ of unit norm $\|x_\theta\| = 1$ such that

$$\theta \leq \|x_\theta - y\| \quad \text{for all } y \in Y. \quad (2.8)$$

Note: Read this as: “You can find a vector x_θ on the unit sphere that is at least the distance θ away from any point in the subspace Y .” This result is sometimes written equivalently as¹⁰

$$\theta \leq \text{dist}_Y(x_\theta) \leq 1.$$

Proof. Pick any $v \in V \setminus Y$ and define $R := \inf\{\|v - y\| \mid y \in Y\}$. By [Lemma 2.4](#), $\text{dist}_Y(x) = 0$ if and only if $x \in \text{cl } Y$. Therefore, we have $R = \text{dist}_Y(v) > 0$. Due to $\theta < 1$, we can find $y_\theta \in Y$ such that

$$0 < \|v - y_\theta\| \leq \frac{R}{\theta} \quad (2.9)$$

holds. We define

$$x_\theta := \frac{v - y_\theta}{\|v - y_\theta\|}.$$

¹⁰The **distance** of a point x to a set Y in a normed linear space is defined as $\text{dist}_Y(x) := \inf\{\|x - y\| \mid y \in Y\}$.

Then we have $\|x_\theta\| = 1$ and, for any $y \in Y$,

$$\begin{aligned}\|x_\theta - y\| &= \left\| \frac{v - y_\theta}{\|v - y_\theta\|} - y \right\| \\ &= \frac{1}{\|v - y_\theta\|} \|v - \underbrace{(y_\theta + \|v - y_\theta\| y)}_{\in Y}\| \\ &\geq \frac{R}{\|v - y_\theta\|}.\end{aligned}$$

Together with (2.9), this proves (2.8). \square

End of Class 4

Proof of Theorem 2.18:

Item (i) \Rightarrow item (iii): When the closed unit ball $\overline{B_1(0)}$ is compact, then it is also totally bounded by Theorem 2.17. Thus, it can be covered by finitely many balls of radius $1/2$:

$$\overline{B_1(0)} \subseteq \bigcup_{i=1}^N B_{1/2}(y^{(i)}).$$

Define $Y := \text{span}\{y^{(1)}, \dots, y^{(N)}\}$. Then by Lemma 2.14, Y is a closed subspace of V .

Suppose that $Y \subseteq V$ is a *proper* subspace. The Riesz lemma 2.19 then implies that there exists $x_\theta \in V$ of unit norm such that $\text{dist}_Y(x_\theta) \geq \theta := \frac{3}{4}$. Moreover, x_θ belongs to one of the covering balls, say, $B_{1/2}(y^{(j)})$. Therefore, we have

$$\text{dist}_Y(x_\theta) \leq \|x_\theta - y^{(j)}\| < \frac{1}{2},$$

which contradicts $\text{dist}_Y(x_\theta) \geq \frac{3}{4}$. Therefore, $Y = V$ and $\dim(V)$ is finite.

Item (ii) \Rightarrow item (iii): The proof is the same as above.

Item (iii) \Rightarrow item (i): The closed unit ball $\overline{B_1(0)}$ is a clearly a closed subset of V . Suppose that $\dim(V) = n \in \mathbb{N}_0$ and that $\{v^{(1)}, \dots, v^{(n)}\}$ is a basis of V . Then V is complete by Lemma 2.14. When we show that $\overline{B_1(0)}$ is totally bounded w.r.t. $\|\cdot\|$, then it is compact by Theorem 2.17. By the equivalence of norms (Theorem 2.13), we may equivalently show that $\overline{B_1(0)}$ is totally bounded w.r.t. $\|\cdot\|_\infty$.

Suppose that $\|\cdot\|_\infty \leq c \|\cdot\|$ holds for $c > 0$. Consider $\varepsilon > 0$. We claim that

$$\overline{B_1(0)} \subseteq \overline{B_c^{\|\cdot\|_\infty}(0)} \subseteq \bigcup_{\substack{q \in \varepsilon \mathbb{Z}^n \\ \|q\|_\infty \leq c + \varepsilon/2}} B_\varepsilon^{\|\cdot\|_\infty} \left(\sum_{j=1}^n q_j v^{(j)} \right),$$

holds. Notice that the right-hand side is a finite union of open balls of radius ε . The first inequality is clear. For the second inequality, consider a point $x \in \overline{B_c^{\|\cdot\|_\infty}(0)}$, whose coordinates x_j then satisfy $|x_j| \leq c$. For $j = 1, \dots, n$, find $q_j \in \varepsilon \mathbb{Z}$ closest to x_j . This implies $|x_j - q_j| \leq \varepsilon/2$ and thus

$$\left\| x - \sum_{j=1}^n q_j v^{(j)} \right\|_\infty \leq \frac{\varepsilon}{2} < \varepsilon.$$

In other words, x belongs to the open ball $B_\varepsilon^{\|\cdot\|_\infty} \left(\sum_{j=1}^n q_j v^{(j)} \right)$. Due to $|x_j| \leq c$, we will have $|q_j| \leq c + \varepsilon/2$.

This proves the claim.

Item (iii) \Rightarrow item (ii): The proof is the same as above. \square

Note: The proof **item (iii) \Rightarrow item (i)** can be easily extended to show that every bounded set in a finite-dimensional normed linear space is totally bounded.

Remark 2.20 (there is nothing special about *unit* balls).

For any $r > 0$, the closed ball $\overline{B_r(0)}$ is compact if and only if $\dim(V)$ is finite. The same holds for spheres. \triangle

§ 2.5 LEBESGUE SPACES

Literature: Rudin, 1987, Chapter 3

Lebesgue spaces are prominent examples of Banach spaces. All references to a measure will mean the Lebesgue measure on \mathbb{R}^d . We will state results in this subsection without proof.

Definition 2.21 (Lebesgue spaces).

Suppose that $\Omega \subset \mathbb{R}^n$ is an open set and $p \in [1, \infty)$.

- (i) A measurable function $f: \Omega \rightarrow \mathbb{R}$ is said to be **Lebesgue integrable of index p** or simply **p -integrable** if $|f|^p$ is integrable on Ω .
- (ii) A measurable function $f: \Omega \rightarrow \mathbb{R}$ is said to be **essentially bounded** if it is bounded except on a set of measure zero.
- (iii) Two measurable functions $f, g: \Omega \rightarrow \mathbb{R}$ are said to be **equivalent** if they coincide except on a set of measure zero.
- (iv) The **Lebesgue space** $L^p(\Omega)$ is defined as the set of equivalence classes¹¹ of measurable functions $f: \Omega \rightarrow \mathbb{R}$ that are Lebesgue integrable of index p :

$$L^p(\Omega) := \{[f] \mid f: \Omega \rightarrow \mathbb{R} \text{ is Lebesgue integrable of index } p\}. \quad (2.10)$$

- (v) The **Lebesgue space** $L^\infty(\Omega)$ is defined as the set of equivalence classes of measurable functions $f: \Omega \rightarrow \mathbb{R}$ that are essentially bounded:

$$L^\infty(\Omega) := \{[f] \mid f: \Omega \rightarrow \mathbb{R} \text{ is essentially bounded}\}. \quad (2.11)$$

\triangle

It is customary to denote the equivalence class of a function f by f itself. We will do so from now on.

¹¹The construction is that of a quotient space: we begin with the vector space of p -integrable functions and factor out the subspace of functions which are almost everywhere zero. Recall that “**almost everywhere**” means “except on a set of measure zero”.

Theorem 2.22 (Lebesgue spaces as Banach spaces).

Suppose that $\Omega \subset \mathbb{R}^n$ is an open set.

- (i) For $p \in [1, \infty)$, the Lebesgue space $L^p(\Omega)$ is a Banach space when equipped with the norm

$$\|f\|_{L^p(\Omega)} := \left(\int_{\Omega} |f|^p \right)^{1/p}. \quad (2.12)$$

- (ii) The Lebesgue space $L^\infty(\Omega)$ is a Banach space when equipped with the norm

$$\|f\|_{L^\infty(\Omega)} := \operatorname{ess\,sup}_{x \in \Omega} |f(x)| := \inf \{M \geq 0 \mid |f(x)| \leq M \text{ for almost all } x \in \Omega\}. \quad (2.13)$$

- (iii) For any $p \in [1, \infty]$, the triangle inequality $\|f+g\|_{L^p(\Omega)} \leq \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)}$ for all $f, g \in L^p(\Omega)$ is called the **Minkowski inequality**.

Example 2.23 (functions in L^p).

- (i) On $\Omega = \mathbb{R}$, non-zero constant functions belong to $L^\infty(\mathbb{R})$ but not to any $L^p(\mathbb{R})$ with $p < \infty$.
- (ii) On $\Omega = (-1, 1)$, the absolute power function $x \mapsto |x|^\alpha$ belongs to $L^p((-1, 1))$ if and only if $\alpha p > -1$.¹² For instance, the inverse square root function $x \mapsto 1/\sqrt{|x|} = x^{-1/2}$ belongs to $L^p((-1, 1))$ if and only if $p < 2$.
- (iii) More generally, on the open unit ball $\Omega = B_1(0) \subseteq \mathbb{R}^d$, the function $x \mapsto |x|^\alpha$ belongs to $L^p(\Omega)$ if and only if $\alpha p > -d$ holds.¹³ \triangle

Lemma 2.24 (Hölder's inequality).

Suppose that $\Omega \subset \mathbb{R}^d$ is an open set. Moreover, let $p, q \in [1, \infty]$ be such that $\frac{1}{p} + \frac{1}{q} = 1$.¹⁴ For all $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, the product fg belongs to $L^1(\Omega)$, and the estimate

$$\|fg\|_{L^1(\Omega)} \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)} \quad (2.14)$$

holds. Inequality (2.14) is known as **Hölder inequality**.

Lemma 2.25 (comparison of norms on Lebesgue spaces).

Suppose that $\Omega \subset \mathbb{R}^d$ is an open and **bounded** set. For $1 \leq p \leq q \leq \infty$, the space $L^q(\Omega)$ is a subspace of $L^p(\Omega)$ (**and a proper subspace if $1 \leq p < q \leq \infty$**). Moreover, the L^q -norm is stronger than the L^p -norm:

$$\|f\|_{L^p(\Omega)} \leq |\Omega|^{\frac{q-p}{pq}} \|f\|_{L^q(\Omega)} \quad \text{for all } f \in L^q(\Omega), \quad (2.15)$$

where $|\Omega|$ denotes the Lebesgue measure (d -dimensional volume) of Ω . When $q = \infty$, the expression $\frac{q-p}{pq}$ is to be understood as $1/p$ (for $p < \infty$) or as 0 (for $p = \infty$).

Note: Lemma 2.25 states that the higher the index of a Lebesgue space on a bounded domain, the smaller the space and the stronger the norm.

¹²With the convention that $\alpha \infty = \infty$ for $\alpha > 0$ and $\alpha \infty = -\infty$ for $\alpha < 0$ as well as $0 \infty = 0$.

¹³Here $|\cdot|$ denotes the Euclidean norm on \mathbb{R}^d .

¹⁴Such numbers p, q are called **conjugate exponents**. The convention here is that $1/\infty = 0$ so that 1 and ∞ are conjugate.

Example 2.26 (comparison of norms on Lebesgue spaces).

Suppose that $\Omega \subset \mathbb{R}^d$ is an open and **bounded** set.

- (i) $\|f\|_{L^1(\Omega)} \leq |\Omega|^{1/2} \|f\|_{L^2(\Omega)}$ for all $f \in L^2(\Omega)$.
- (ii) $\|f\|_{L^2(\Omega)} \leq |\Omega|^{1/2} \|f\|_{L^\infty(\Omega)}$ for all $f \in L^\infty(\Omega)$.
- (iii) $\|f\|_{L^1(\Omega)} \leq |\Omega| \|f\|_{L^\infty(\Omega)}$ for all $f \in L^\infty(\Omega)$.

△

End of Class 5

End of Week 3

§ 2.6 SOBOLEV SPACES

Lebesgue spaces are not sufficient to deal with optimization problems whose objective functions involve derivatives of the unknown, as is the case in the Brachistochrone problem (Example 1.1), Fermat's principle in optics (Example 1.3), the signal denoising problem (Example 1.4), and the optimal control example (Example 1.5). Sobolev spaces are the natural setting for such problems. In brief, they consist of functions whose derivatives up to a certain order are in a Lebesgue space. The notion of derivative is meant in a *weak sense*.

Derivatives of multivariate functions are conveniently described using multi-indices.

Definition 2.27 (multi-index).

Suppose $d \in \mathbb{N}$.

- (i) A **multi-index** of length d is a tuple $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$.
- (ii) The **order** of a multi-index $\alpha = (\alpha_1, \dots, \alpha_d)$ is defined as $|\alpha| := \alpha_1 + \dots + \alpha_d$.
- (iii) We associate with a multi-index $\alpha = (\alpha_1, \dots, \alpha_d)$ the derivative operator $D^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}}$. The **order** of D^α is defined as $|\alpha|$.
- (iv) In particular, we have $D^{(0, \dots, 0)} = \text{id}$ and

$$D_i := \frac{\partial}{\partial x_i} = D^{(0, \dots, 0, 1, 0, \dots, 0)}$$

for $i = 1, \dots, d$.

△

Definition 2.28 (function spaces $C^k(\Omega)$, $C_c^k(\Omega)$ and $C^k(\text{cl } \Omega)$).

Suppose that $\Omega \subset \mathbb{R}^d$ is an open set.

- (i) For $f: \Omega \rightarrow \mathbb{R}$, the set

$$\text{supp } f := \text{cl}\{x \in \Omega \mid f(x) \neq 0\} \quad (2.16)$$

is called the **support** of f .

- (ii) For $k \in \mathbb{N}_0$, the set of all **k -times continuously partially differentiable functions** on Ω is denoted by $C^k(\Omega)$. This means that all partial derivatives of order $\leq k$ exist and are continuous functions on Ω .

Moreover, $C^\infty(\Omega) := \bigcap_{k \in \mathbb{N}_0} C^k(\Omega)$ is the set of all **infinitely often continuously partially differentiable functions** on Ω .

(iii) For $k \in \mathbb{N}_0$, the set $C_c^k(\Omega)$ consists of all **functions** $f \in C^k(\Omega)$ **with compact support**, i. e., $\text{supp } f$ is a compact subset of Ω .

Moreover, $C_c^\infty(\Omega) := \bigcap_{k \in \mathbb{N}_0} C_c^k(\Omega)$ is the set of all **infinitely often continuously partially differentiable functions** on Ω **with compact support**.

(iv) For $k \in \mathbb{N}_0$, $C^k(\text{cl } \Omega)$ denotes the set of all k -times continuously partially differentiable functions $f: \Omega \rightarrow \mathbb{R}$ such that all partial derivatives of order $\leq k$ extend continuously to $\text{cl } \Omega$. \triangle

Note: Given $k \in \mathbb{N}_0$ and $f \in C^k(\Omega)$, the support of all partial derivatives $D^\alpha f$ of order $|\alpha| \leq k$ is contained in the support of f .

Lemma 2.29 (properties of derivatives).

Suppose that $\Omega \subset \mathbb{R}^d$ is an open set. Moreover, suppose that $\alpha \in \mathbb{N}_0^d$ is a multi-index of order $m \in \mathbb{N}_0$ and $k \geq m$. Then the following holds:

(i) The derivative operator D^α is well-defined as a map

$$D^\alpha: C^k(\Omega) \rightarrow C^{k-m}(\Omega).$$

(ii) The order of differentiation does not matter, i. e., for any decomposition of the multi-index $\alpha = \beta + \gamma$, we have

$$D^\alpha f = D^\beta (D^\gamma f) = D^\gamma (D^\beta f)$$

for all $f \in C^k(\Omega)$.

Proof. **Statement (i)** follows immediately from the fact that higher-order partial derivatives are derivatives of lower-order partial derivatives. **Statement (ii)** is a consequence of the commutativity of partial derivatives by Schwarz' theorem. \square

Example 2.30 (function spaces $C^k(\Omega)$ and $C^k(\text{cl } \Omega)$).

(i) For $\Omega = (0, 1)$, the function $x \mapsto 1/x$ belongs to $C^\infty(\Omega)$ but not to $C(\text{cl } \Omega)$ since it does not extend continuously to 0.

(ii) For $\Omega = (0, 1)$, the function $x \mapsto \sqrt{x}$ belongs to $C^\infty(\Omega)$ and to $C(\text{cl } \Omega)$ but not to $C^1(\text{cl } \Omega)$ since the derivative $1/(2\sqrt{x})$ does not extend continuously to 0. \triangle

Lemma 2.31 (integration by parts).

Suppose that $\Omega \subset \mathbb{R}^d$ is an open set and $f \in C^1(\Omega)$. Then for any $i = 1, \dots, d$, we have

$$\int_{\Omega} (D_i f) g \, dx = - \int_{\Omega} f (D_i g) \, dx \quad \text{for all } g \in C_c^1(\Omega). \quad (2.17)$$

Note: The supports of both integrands are compact subsets of Ω and the integrands are continuous, so that the integrals are well-defined.

Proof. Suppose that $C = (a_1, b_1) \times \cdots \times (a_d, b_d) \subseteq \mathbb{R}^d$ is an open and bounded **box** containing the compact set $\text{supp } g$. Define

$$\Phi(x) := \begin{cases} f(x) g(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^d \setminus \Omega. \end{cases}$$

Then by the product rule, $\Phi \in C^1(\mathbb{R}^d)$ and $\text{supp } \Phi \subseteq \text{supp } g \subseteq C$ and thus also $\text{supp } D_1 \Phi \subseteq C$.

For notational convenience, we consider only the case $i = 1$. By the fundamental theorem of calculus, we have

$$\int_{a_1}^{b_1} D_1 \Phi(x_1, x_2, \dots, x_d) \, dx_1 = \Phi(b_1, x_2, \dots, x_d) - \Phi(a_1, x_2, \dots, x_d)$$

for any $x_2, \dots, x_d \in \mathbb{R}$. Plugging in the definition of Φ , this amounts to

$$\begin{aligned} & \int_{a_1}^{b_1} (D_1 f)(x_1, x_2, \dots, x_d) g(x_1, x_2, \dots, x_d) \, dx_1 + \int_{a_1}^{b_1} f(x_1, x_2, \dots, x_d) (D_1 g)(x_1, x_2, \dots, x_d) \, dx_1 \\ &= f(b_1, x_2, \dots, x_d) \underbrace{g(b_1, x_2, \dots, x_d)}_{=0} - f(a_1, x_2, \dots, x_d) \underbrace{g(a_1, x_2, \dots, x_d)}_{=0}. \end{aligned}$$

Notice that the right-hand side is zero since the points where g is being evaluated are outside of $\text{supp } g$. We now see that

$$\begin{aligned} & \int_{\Omega} (D_1 f) g \, dx + \int_{\Omega} f (D_1 g) \, dx \\ &= \int_C (D_1 f) g \, dx + \int_C f (D_1 g) \, dx \\ &= \int_{a_d}^{b_d} \cdots \int_{a_1}^{b_1} (D_1 f)(x_1, x_2, \dots, x_d) g(x_1, x_2, \dots, x_d) \, dx_1 \cdots dx_d \\ &\quad + \int_{a_d}^{b_d} \cdots \int_{a_1}^{b_1} f(x_1, x_2, \dots, x_d) (D_1 g)(x_1, x_2, \dots, x_d) \, dx_1 \cdots dx_d \quad \text{by Fubini's theorem} \\ &= 0, \end{aligned}$$

which concludes the proof. \square

By induction, we can generalize [Lemma 2.31](#) to higher-order derivatives:

Corollary 2.32 (integration by parts for higher-order derivatives).

Suppose that $\Omega \subset \mathbb{R}^d$ is an open set and $f \in C^k(\Omega)$. Then for any multi-index $\alpha \in \mathbb{N}_0^d$ of order $k \in \mathbb{N}_0$, we have

$$\int_{\Omega} (D^\alpha f) g \, dx = (-1)^{|\alpha|} \int_{\Omega} f (D^\alpha g) \, dx \quad \text{for all } g \in C_c^k(\Omega). \quad (2.18)$$

Formula (2.18) describes properties of classical derivatives for sufficiently smooth functions. These properties serve as a motivation for the definition of a more general notion of derivative, applicable to a much larger class of functions.

Definition 2.33 (weak derivative).

Suppose that $\Omega \subset \mathbb{R}^d$ is an open set.

- (i) For any $A \subseteq \Omega$, the **characteristic function** $\chi_A: \Omega \rightarrow \mathbb{R}$ is defined as $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ if $x \notin A$.
- (ii) The set $L^1_{\text{loc}}(\Omega)$ denotes the set of all (equivalence classes of) functions $f: \Omega \rightarrow \mathbb{R}$ such that $f \chi_K \in L^1(\Omega)$ for all compact subsets $K \subseteq \Omega$.
- (iii) Suppose that $f \in L^1_{\text{loc}}(\Omega)$ and $\alpha \in \mathbb{N}_0^d$ is a multi-index. A function $w \in L^1_{\text{loc}}(\Omega)$ is called the α -th **weak derivative** of f if

$$\int_{\Omega} f D^{\alpha} \varphi \, dx = (-1)^{|\alpha|} \int_{\Omega} w \varphi \, dx \quad \text{for all } \varphi \in C_c^{\infty}(\Omega) \quad (2.19)$$

holds. In this case, we write $w = D^{\alpha} f$. Δ

Note: For any $\varphi \in C_c^{\infty}(\Omega)$, both v and $D^{\alpha} v$ have compact support in Ω . The function class $L^1_{\text{loc}}(\Omega)$ is therefore a natural setting so that the integrals in (2.19) are well-defined.

Remark 2.34 (weak derivative).

- (i) The α -th weak derivative $D^{\alpha} f$ of a function $f \in L^1_{\text{loc}}(\Omega)$ is unique (if it exists).
- (ii) The existence of a weak derivative $D^{\alpha} f$ does not imply the existence of weak derivatives $D^{\alpha'} f$ for multi-indices $\alpha' \leq \alpha$.
- (iii) If both $D^{\alpha} f$ and $D^{\alpha+\beta} f$ exist, then $D^{\alpha+\beta} f = D^{\beta}(D^{\alpha} f)$.
- (iv) If both $D^{\alpha} f$ and $D^{\beta}(D^{\alpha} f)$ exist, then $D^{\alpha+\beta} f = D^{\beta}(D^{\alpha} f)$. Δ

Example 2.35 (weak derivative).

The function $f: \Omega := (-1, 1) \rightarrow \mathbb{R}$ defined by $f(x) = |x|$ has the weak first-order derivative

$$w(x) = \begin{cases} -1 & \text{if } x < 0, \\ 1 & \text{if } x > 0. \end{cases}$$

But f does not have a weak second-order derivative in $L^1_{\text{loc}}(\Omega)$. Δ

We can now define the Sobolev spaces.

Definition 2.36 (Sobolev spaces).

Suppose that $\Omega \subset \mathbb{R}^d$ is an open set. The **Sobolev space** of differentiability index $k \in \mathbb{N}_0$ and index $p \in [1, \infty]$ is defined as

$$W^{k,p}(\Omega) := \{f \in L^p(\Omega) \mid D^{\alpha} f \in L^p(\Omega) \text{ for all } |\alpha| \leq k\}. \quad (2.20)$$

Δ

We re-iterate that the elements of a Sobolev space are actually equivalence classes of functions but we continue to use the simplified notation. For $k = 0$, the Sobolev spaces agree with the Lebesgue spaces: $W^{0,p}(\Omega) = L^p(\Omega)$

Remark 2.37 (alternative definition of Sobolev spaces).

For $k \in \mathbb{N}_0$ and $p \in [1, \infty)$, Sobolev spaces can be defined alternatively via a process of completion:

$$H^{k,p}(\Omega) := \text{cl}(C^\infty(\Omega) \cap W^{k,p}(\Omega)) \text{ w.r.t. the norm } \|\cdot\|_{W^{k,p}(\Omega)}.$$

The paper [Meyers, Serrin, 1964](#) with the title “ $H = W$ ” shows that $H^{k,p}(\Omega) = W^{k,p}(\Omega)$; see also [Adams, Fournier, 2003](#), Theorem 3.17. \triangle

Theorem 2.38 (Sobolev spaces as Banach spaces).

Suppose that $\Omega \subset \mathbb{R}^n$ is an open set.

- (i) For $k \in \mathbb{N}_0$ and $p \in [1, \infty)$, the Sobolev space $W^{k,p}(\Omega)$ is a Banach space when equipped with the norm

$$\|f\|_{W^{p,k}(\Omega)} := \left(\sum_{|\alpha| \leq k} \int_{\Omega} |D^\alpha f|^p \right)^{1/p} = \left(\sum_{|\alpha| \leq k} \|D^\alpha f\|_{L^p(\Omega)}^p \right)^{1/p}. \quad (2.21)$$

- (ii) The Sobolev space $W^{k,\infty}(\Omega)$ is a Banach space when equipped with the norm

$$\|f\|_{W^{k,\infty}(\Omega)} := \max_{|\alpha| \leq k} \text{ess sup}_{x \in \Omega} |D^\alpha f(x)| = \max_{|\alpha| \leq k} \|D^\alpha f\|_{L^\infty(\Omega)}. \quad (2.22)$$

Example 2.39 (Sobolev spaces).

By [Example 2.35](#), the function defined by $f(x) = |x|$ on $\Omega = (-1, 1)$ belongs to $W^{1,\infty}(\Omega)$. However, it does not belong to any $W^{2,p}(\Omega)$ for any $p \in [1, \infty]$. \triangle

End of Class 6

§ 3 INNER PRODUCT SPACES

In this section we introduce the notion of an inner product space, which is a concept more specific than a normed linear space.

Definition 3.1 (inner product space).

Suppose that V is a linear space.

- (i) A map $(\cdot, \cdot): V \times V \rightarrow \mathbb{R}$ is said to be an **inner product on V** if the following conditions hold:

$$(\alpha_1 u_1 + \alpha_2 u_2, v) = \alpha_1 (u_1, v) + \alpha_2 (u_2, v) \quad \text{linearity in the first argument} \quad (3.1a)$$

$$(u, \alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 (u, v_1) + \alpha_2 (u, v_2) \quad \text{linearity in the second argument} \quad (3.1b)$$

$$(u, v) = (v, u) \quad \text{symmetry} \quad (3.1c)$$

$$(u, u) \geq 0, \quad \text{and } (u, u) = 0 \Rightarrow u = 0 \quad \text{positive definiteness} \quad (3.1d)$$

for all $u, u_1, u_2, v, v_1, v_2 \in V$ and all $\alpha_1, \alpha_2 \in \mathbb{R}$.

- (ii) The pair $(V, (\cdot, \cdot))$ is said to be a **(real) inner product space**.

- (iii) Two vectors $u, v \in V$ are said to be **orthogonal** if $(u, v) = 0$. \triangle

In brief, an inner product is a bilinear form on V that is symmetric and positive definite.

An inner product induces a norm on the linear space under consideration:

Lemma 3.2 (inner product induces norm).

Suppose that $(V, (\cdot, \cdot))$ is an inner product space. Then

$$\|u\| := \sqrt{(u, u)} \quad (3.2)$$

defines a norm on V .

Proof. The proof is part of ??.

□

Definition 3.3 (Hilbert space).

An inner product space $(V, (\cdot, \cdot))$ is said to be a **Hilbert space** if the norm induced by the inner product is complete (see Definition 2.8). \triangle

Note: In other words, a Hilbert space is a Banach space whose norm is induced by an inner product.

Example 3.4 (Hilbert space).

- (i) In \mathbb{R}^n , inner products are in a bijective correspondence with symmetric positive definite matrices. Every inner product on \mathbb{R}^n has the form

$$(u, v)_M = u^\top M v$$

for some symmetric positive definite matrix $M \in \mathbb{R}^{n \times n}$. The induced norm is then given by

$$\|u\|_M = \sqrt{u^\top M u}.$$

- (ii) Every finite-dimensional inner product space is a Hilbert space.

- (iii) Suppose that $\Omega \subset \mathbb{R}^d$ is an open set. The Lebesgue space $L^2(\Omega)$ carries the inner product

$$(f, g)_{L^2(\Omega)} := \int_{\Omega} f g \, dx, \quad (3.3)$$

which induces the norm (2.12) for $p = 2$. Since $L^2(\Omega)$ is complete, it is a Hilbert space.

- (iv) More generally, suppose that $\Omega \subset \mathbb{R}^d$ is an open set and $k \in \mathbb{N}_0$. The Sobolev space $W^{k,2}(\Omega)$ carries the inner product

$$(f, g)_{W^{k,2}(\Omega)} := \sum_{|\alpha| \leq k} \int_{\Omega} (D^\alpha f) (D^\alpha g) \, dx, \quad (3.4)$$

which induces the norm (2.21) for $p = 2$. Since $W^{k,2}(\Omega)$ is complete, it is a Hilbert space. It is customary to denote the inner product space $W^{k,2}(\Omega)$ by $H^k(\Omega)$. In particular, $H^0(\Omega) = L^2(\Omega)$. \triangle

Lemma 3.5 (Cauchy-Schwarz inequality).

Suppose that $(V, (\cdot, \cdot))$ is an inner product space. Then for all $u, v \in V$, we have

$$|(u, v)| \leq \|u\| \|v\|. \quad (3.5)$$

Equality holds if and only if u and v are linearly dependent, i. e., $\alpha u + \beta v = 0$ and not both α and β are zero.

Proof. When $v = 0$, then (3.5) holds with equality, and $\{u, v\}$ is linearly dependent.

For the rest of the proof, assume $v \neq 0$. For $\beta \in \mathbb{R}$ we have

$$\begin{aligned} 0 &\leq (u - \beta v, u - \beta v) && \text{due to positive definiteness} \\ &= (u, u) - 2\beta (u, v) + \beta^2 (v, v) && \text{due to bilinearity and symmetry.} \end{aligned}$$

Here $(v, v) > 0$ due to positive definiteness, and we set $\beta := \frac{(u, v)}{(v, v)}$. This implies

$$\begin{aligned} 0 &\leq (u, u) - 2 \frac{(u, v)}{(v, v)} (u, v) + \frac{(u, v)^2}{(v, v)} \\ &= (u, u) - \frac{(u, v)^2}{(v, v)}. \end{aligned}$$

Multiplication by $(v, v) > 0$ yields (3.5).

We have to investigate when equality holds in (3.5). We can continue to assume $v \neq 0$. When $\{u, v\}$ is linearly dependent, then we have $u = \delta v$ for some $\delta \in \mathbb{R}$. Bilinearity then implies $(u, u) = \delta^2 (v, v)$ and

$$(u, v)^2 = (\delta (v, v))^2 = \delta^2 (v, v)^2 = (u, u) (v, v),$$

hence equality (3.5).

Conversely, suppose that equality holds in (3.5), i. e., $(u, v)^2 = (u, u) (v, v)$. Setting $\beta := \frac{(u, v)}{(v, v)}$ and applying the same manipulations as above, we find that

$$\begin{aligned} 0 &= (u, u) - \frac{(u, v)^2}{(v, v)} \\ &= (u, u) - 2\beta (u, v) + \beta^2 (v, v) \\ &= (u - \beta v, u - \beta v). \end{aligned}$$

The positive definiteness implies $u - \beta v = 0$, and thus $\{u, v\}$ is linearly dependent. \square

§ 4 CONTINUOUS FUNCTIONS

The continuity of functions between normed linear spaces can be defined via sequences.

Definition 4.1 (continuity).

Suppose that X and Y are normed linear spaces. A map $F: X \rightarrow Y$ is said to be **continuous at $x \in X$** if for all sequences $(x^{(k)})$ in X with $x^{(k)} \rightarrow x$ in X , we have $F(x^{(k)}) \rightarrow F(x)$ in Y . It is said to be **continuous (on X)** if it is continuous at every $x \in X$. \triangle

Lemma 4.2 (equivalent definition of continuity).

Suppose that X and Y are normed linear spaces. A map $F: X \rightarrow Y$ is continuous if and only if for all $x \in X$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $\|x - y\|_X < \delta$ implies $\|F(x) - F(y)\|_Y < \varepsilon$.

Proof.

□

§ 4.1 LINEAR OPERATORS

Linear maps between normed linear spaces are of particular importance.

Definition 4.3 (linear operator, bounded linear operator).

Suppose that X and Y are normed linear spaces.

- (i) A function $A: X \rightarrow Y$ is said to be a **linear map** or **linear operator** if

$$A(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 A(x_1) + \alpha_2 A(x_2) \quad (4.1)$$

holds for all $x_1, x_2 \in X$ and all $\alpha_1, \alpha_2 \in \mathbb{R}$.

- (ii) A linear operator $A: X \rightarrow Y$ is said to be **bounded** if there exists $C \geq 0$ such that

$$\|A(x)\|_Y \leq C \|x\|_X \quad \text{for all } x \in X. \quad (4.2)$$

The number

$$\|A\|_{\mathcal{L}(X,Y)} := \inf \{C \geq 0 \mid (4.2) \text{ holds}\} \quad (4.3)$$

is called the **operator norm** of A .

△

Note: It is easy to see that the interval $\{C \geq 0 \mid (4.2) \text{ holds}\}$ is closed, and thus the infimum in (4.3) is actually a minimum.

Lemma 4.4 (alternative definitions of the operator norm).

Suppose that X and Y are normed linear spaces and $A: X \rightarrow Y$ is a bounded linear operator. The operator norm satisfies

$$\|A\|_{\mathcal{L}(X,Y)} = \sup_{\|x\|_X=1} \|A(x)\|_Y = \sup_{\|x\|_X \leq 1} \|A(x)\|_Y = \sup_{x \neq 0} \frac{\|A(x)\|_Y}{\|x\|_X}. \quad (4.4)$$

Lemma 4.5 (boundedness is continuity).

Suppose that X and Y are normed linear spaces and $A: X \rightarrow Y$ is a linear operator. Then the following are equivalent:

- (i) A is continuous at 0.
- (ii) A is continuous on X .
- (iii) A is Lipschitz continuous.
- (iv) A is bounded.

Proof.

□

End of Class 7

End of Week 4

Convergence in the operator norm implies pointwise convergence:

Lemma 4.6 (convergence in the operator norm implies pointwise convergence).

Suppose that X and Y are normed linear spaces and $(A^{(k)})$ is a sequence of bounded linear operators $X \rightarrow Y$. If $A^{(k)}$ converges to $A \in \mathcal{L}(X, Y)$ in the operator norm, then $A^{(k)}(x)$ converges to $A(x)$ for all $x \in X$.

Proof.

□

Lemma 4.7 (existence of unbounded operators).

Suppose that X and Y are normed linear spaces with $\dim(Y) \geq 1$. Then the following are equivalent:

- (i) X is finite-dimensional.
- (ii) Every linear operator $A: X \rightarrow Y$ is continuous.

Proof. **Statement (i) \Rightarrow statement (ii):** Suppose that $\dim(X) = n \in \mathbb{N}_0$ and that $\{v^{(1)}, \dots, v^{(n)}\}$ is a basis of V . For any $x \in X$, we can write $x = \sum_{j=1}^n x_j v^{(j)}$ and thus $A(x) = \sum_{j=1}^n x_j A(v^{(j)})$. We estimate

$$\|A(x)\|_Y = \left\| \sum_{j=1}^n x_j A(v^{(j)}) \right\|_Y \leq \|x\|_\infty \sum_{j=1}^n \|A(v^{(j)})\|_Y =: C \|x\|_\infty,$$

where $C \geq 0$ is a constant. By [Theorem 2.13](#), the norms $\|\cdot\|_\infty$ and $\|\cdot\|_X$ are equivalent, and thus A is continuous.

\neg Statement (i) $\Rightarrow \neg$ statement (ii): Suppose that X is infinite-dimensional, i. e., at least of countable dimension. Suppose that $(v^{(i)})_{i \in I}$ is a basis for X . Without loss of generality, $\mathbb{N} \subseteq I$. Pick a non-zero element $y \in Y$ and define the linear operator $A: X \rightarrow Y$ by $A(x^{(k)}) = k \|x^{(k)}\|_X y$ for $k \in \mathbb{N}$, and $A(v^{(i)}) = 0$ for $i \in I \setminus \mathbb{N}$. Then A is not bounded since $\|A(x^{(k)})\|_Y = k \|y\|_Y$ for all $k \in \mathbb{N}$. □

The set of all linear operators $X \rightarrow Y$ forms itself a linear space, which we denote by $L(X, Y)$. Addition and scalar multiplication are defined pointwise. The subset of *bounded* linear operators forms a subspace:

Theorem 4.8 (subspace of bounded linear operators).

Suppose that X and Y are normed linear spaces.

- (i) The set of all bounded linear operators $X \rightarrow Y$ is a linear subspace of the space of all linear operators $X \rightarrow Y$. We denote it by $\mathcal{L}(X, Y)$.
- (ii) The operator norm (4.3) is a norm on $\mathcal{L}(X, Y)$.
- (iii) If Y is a Banach space, then $\mathcal{L}(X, Y)$ is a Banach space.
- (iv) If $\mathcal{L}(X, Y)$ is a Banach space and $\dim(X) \geq 1$, then Y is a Banach space.

Proof. **Statement (i)** and **statement (ii)**: We use the subspace criterion to show that $\mathcal{L}(X, Y)$ is a linear subspace of $L(X, Y)$. The zero operator is bounded, so $\mathcal{L}(X, Y)$ is nonempty. With $A \in \mathcal{L}(X, Y)$, we have $\alpha A \in \mathcal{L}(X, Y)$ since

$$\|\alpha A\|_{\mathcal{L}(X, Y)} = \sup_{\|x\|_X=1} \|\alpha A(x)\|_Y = \sup_{\|x\|_X=1} |\alpha| \|A(x)\|_Y = |\alpha| \sup_{\|x\|_X=1} \|A(x)\|_Y.$$

This proves the absolute homogeneity of the operator norm. Also, for $A, B \in \mathcal{L}(X, Y)$, we have

$$\begin{aligned} \|A + B\|_{\mathcal{L}(X, Y)} &= \sup_{\|x\|_X=1} \|A(x) + B(x)\|_Y \\ &\leq \sup_{\|x\|_X=1} \|A(x)\|_Y + \|B(x)\|_Y \\ &\leq \sup_{\|x\|_X=1} \|A(x)\|_Y + \sup_{\|x\|_X=1} \|B(x)\|_Y \\ &= \|A\|_{\mathcal{L}(X, Y)} + \|B\|_{\mathcal{L}(X, Y)} \end{aligned}$$

and thus $A + B \in \mathcal{L}(X, Y)$ and the triangle inequality holds. Finally, $\|A\|_{\mathcal{L}(X, Y)} \geq 0$ is clear, and $\|A\|_{\mathcal{L}(X, Y)} = 0$ implies $\|A(x)\|_Y = 0$ for all $x \in X$, and thus $A = 0$, the zero element of $\mathcal{L}(X, Y)$.

Statement (iii): Suppose first that Y is a Banach space and that $(A^{(n)})$ is a Cauchy sequence in $\mathcal{L}(X, Y)$. That is, for every $\varepsilon > 0$, there exists an index k_ε such that $\|A^{(k)} - A^{(\ell)}\| < \varepsilon$ holds for all $k, \ell \geq k_\varepsilon$.

Step 1: We construct the candidate $A: X \rightarrow Y$ for the limit of $A^{(n)}$.

For any fixed $x \in X$, we have

$$\|A^{(k)}(x) - A^{(\ell)}(x)\|_Y = \|[A^{(k)} - A^{(\ell)}](x)\|_Y \leq \|A^{(k)} - A^{(\ell)}\|_{\mathcal{L}(X, Y)} \|x\|_X.$$

Therefore, the sequence $(A(x)^{(k)})$ is Cauchy in Y . Since Y is complete, we can define the pointwise limit $A(x) := \lim_{k \rightarrow \infty} A_k(x)$.

Step 2: We show that A is linear.

For any $x, y \in X$ and $\alpha, \beta \in \mathbb{R}$, we have

$$\begin{aligned} A(\alpha x + \beta y) &= \lim_{k \rightarrow \infty} A^{(k)}(\alpha x + \beta y) && \text{by definition of } A \\ &= \lim_{k \rightarrow \infty} [\alpha A^{(k)}(x) + \beta A^{(k)}(y)] && \text{by linearity of } A^{(k)} \\ &= \alpha \lim_{k \rightarrow \infty} A^{(k)}(x) + \beta \lim_{k \rightarrow \infty} A^{(k)}(y) && \text{by linearity of the limit, all limits exist} \\ &= \alpha A(x) + \beta A(y) && \text{by definition of } A. \end{aligned}$$

Step 3: We show that A is bounded.

For any $x \in X$, we have

$$\begin{aligned} \|A(x)\|_Y &\leq \|A(x) - A^{(k)}(x)\|_Y + \|A^{(k)}(x)\|_Y && \text{by the triangle inequality} \\ &\leq \|A(x) - A^{(k)}(x)\|_Y + \|A^{(k)}\|_{\mathcal{L}(X, Y)} \|x\|_X. \end{aligned}$$

Since every Cauchy sequence is bounded (**Quiz 4.1**: Can you prove it?), we have

$$\leq \|A(x) - A^{(k)}(x)\|_Y + C \|x\|_X.$$

By letting $k \rightarrow \infty$, we find that $\|A(x)\|_Y \leq C \|x\|_X$, with C independent of x . That is, A is bounded.

Step 4: We show that $A^{(k)} \rightarrow A$ in $\mathcal{L}(X, Y)$.

Let $\varepsilon > 0$. Since $(A^{(k)})$ is Cauchy, there exists k_ε such that $\|A^{(k)} - A^{(\ell)}\|_{\mathcal{L}(X, Y)} < \varepsilon$ for all $k, \ell \geq k_\varepsilon$. Now let $x \in X$ be arbitrary. We estimate

$$\begin{aligned} \|A^{(k)}(x) - A^{(\ell)}(x)\|_Y &\leq \|A^{(k)} - A^{(\ell)}\|_{\mathcal{L}(X, Y)} \|x\|_X \\ &\leq \varepsilon \|x\|_X \quad \text{for all } k, \ell \geq k_\varepsilon. \end{aligned}$$

Passing to the limit $\ell \rightarrow \infty$, we obtain

$$\|A^{(k)}(x) - A(x)\|_Y \leq \varepsilon \|x\|_X \quad \text{for all } k \geq k_\varepsilon.$$

This shows $\|A^{(k)} - A\|_{\mathcal{L}(X, Y)} \leq \varepsilon$ for all $k \geq k_\varepsilon$, i. e., $A^{(k)} \rightarrow A$ in $\mathcal{L}(X, Y)$.

Statement (iv): Suppose that $\mathcal{L}(X, Y)$ is a Banach space and $\dim(X) \geq 1$. Then there exists a non-zero bounded linear map $f: X \rightarrow \mathbb{R}$. (**Quiz 4.2:** How do we see this?) In particular, we have $f(x_0) = 1$ for some $x_0 \in X$.

Now define a family $(A_y)_{y \in Y}$ of bounded linear operators $X \rightarrow Y$ by

$$A_y(x) := f(x) y \quad \text{for all } x \in X.$$

Every A_y is indeed bounded since

$$\|A_y(x)\|_Y = |f(x)| \|y\|_Y \leq \|f\|_{\mathcal{L}(X, \mathbb{R})} \|y\|_Y \|x\|_X$$

and thus $\|A_y\|_{\mathcal{L}(X, Y)} \leq \|f\|_{\mathcal{L}(X, \mathbb{R})} \|y\|_Y$. Suppose now that $(y^{(k)})$ is a Cauchy sequence in Y . Then

$$\|A_{y^{(k)}} - A_{y^{(\ell)}}\|_{\mathcal{L}(X, Y)} \leq \|f\|_{\mathcal{L}(X, \mathbb{R})} \|y^{(k)} - y^{(\ell)}\|_Y$$

and therefore, $(A_{y^{(k)}})$ is a Cauchy sequence in $\mathcal{L}(X, Y)$. Since $\mathcal{L}(X, Y)$ is complete, there exists a limit $A \in \mathcal{L}(X, Y)$. But this and **Lemma 4.6** imply

$$y^{(k)} = A_{y^{(k)}}(x_0) \rightarrow A(x_0) \in Y,$$

and thus $(y^{(k)})$ converges, i. e., Y is complete. □

§ 4.2 CONTINUOUS EMBEDDINGS

Definition 4.9 (continuous embedding, isomorphism).

Suppose that X and Y are normed linear spaces.

- (i) An injective linear map $A: X \rightarrow Y$ that is also bounded is said to be a **continuous embedding** of X into Y . In this case, the space X is said to be **continuously embedded** into Y .
- (ii) A bijective linear map $A: X \rightarrow Y$ that is also bounded and whose inverse is bounded is said to be an **isomorphism** of X onto Y . In this case, the spaces X and Y are said to be **isomorphic**.
- (iii) An isomorphism $A: X \rightarrow Y$ such that $\|A(x)\|_Y = \|x\|_X$ for all $x \in X$ is said to be an **isometric isomorphism** or an **isometry** of X onto Y . In this case, the spaces X and Y are said to be **isometric**. △

Remark 4.10 (continuous embedding, isomorphism).

(i) In many cases, $X \subseteq Y$ algebraically as a subspace, and we consider the linear inclusion map $i: X \rightarrow Y$ with $i(x) = x$, which is clearly injective. Notice that the inclusion map is continuous if and only if $\|x\|_Y = \|i(x)\|_Y \leq C \|x\|_X$, i. e., if and only if $\|\cdot\|_Y$ is weaker on X than $\|\cdot\|_X$. We denote the continuous embedding of X into Y by $X \hookrightarrow Y$.

(ii) A surjective linear map $A: X \rightarrow Y$ is an isomorphism if and only if there exist constants $c, C > 0$ such that

$$c \|x\| \leq \|A(x)\|_Y \leq C \|x\|_X \quad \text{for all } x \in X$$

holds.

(iii) A surjective linear map $A: X \rightarrow Y$ is an isometry if and only if

$$\|x\| = \|A(x)\|_Y \quad \text{for all } x \in X$$

holds.

(iv) Two isomorphic normed linear spaces X and Y do not need to be distinguished in terms of their structure, up to the equivalence of norms. Two isometric normed linear spaces X and Y do not need to be distinguished at all. \triangle

Example 4.11 (continuous embeddings).

Suppose that $\Omega \subset \mathbb{R}^d$ is an open and **bounded** set. Then we have the following continuous embeddings of Sobolev spaces:

$$\begin{array}{ccccccc} L^\infty(\Omega) & \hookrightarrow & \dots & \hookrightarrow & L^2(\Omega) & \hookrightarrow & \dots & \hookrightarrow & L^1(\Omega) \\ \uparrow & & & & \uparrow & & & & \uparrow \\ W^{1,\infty}(\Omega) & \hookrightarrow & \dots & \hookrightarrow & W^{1,2}(\Omega) & \hookrightarrow & \dots & \hookrightarrow & W^{1,1}(\Omega) \\ \uparrow & & & & \uparrow & & & & \uparrow \\ W^{2,\infty}(\Omega) & \hookrightarrow & \dots & \hookrightarrow & W^{2,2}(\Omega) & \hookrightarrow & \dots & \hookrightarrow & W^{2,1}(\Omega) \\ \vdots & & & & \vdots & & & & \vdots \end{array}$$

The inclusions in horizontal direction rely on the boundedness of Ω , while the inclusions in vertical direction hold for any open set Ω . Moreover, there are further embeddings in “north-westerly” direction due to the Sobolev embedding theorem, which allow differentiability to be traded for higher integrability indices. \triangle

§ 4.3 THE DUAL SPACE

Definition 4.12 (algebraic and topological dual space).

Suppose that X is a normed linear space.

(i) The **algebraic dual space** of X is the space

$$X' := L(X, \mathbb{R}) \tag{4.5}$$

of linear maps $X \rightarrow \mathbb{R}$, also known as **linear functionals** on X .

(ii) The **topological dual space** of X is the space

$$X^* := \mathcal{L}(X, \mathbb{R}) \quad (4.6)$$

of **continuous** linear functionals on X . \triangle

Clearly, X^* is a linear subspace of X' , and in fact a proper subspace if and only if X is infinite-dimensional (Lemma 4.7). Since \mathbb{R} is complete, X^* is always a Banach space by Theorem 4.8. Since we use the absolute value as the norm on \mathbb{R} , the dual space X^* is equipped with the operator norm

$$\|f\|_{X^*} = \sup_{\|x\|_X=1} |f(x)|.$$

Given $f \in X^*$ and $x \in X$, we often use the notation

$$\langle f, x \rangle_{X^*, X} := f(x).$$

The bracket $\langle \cdot, \cdot \rangle_{X^*, X}$ is a bilinear form on $X^* \times X$ and it is called the **dual pairing** of X and X^* . In the future, we will often simply say **dual space** instead of **topological dual space**.

Example 4.13 (dual spaces).

- (i) Suppose that $\Omega \subset \mathbb{R}^d$ is an open and bounded set. Moreover, let $p, q \in [1, \infty)$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then the dual space of $L^p(\Omega)$ is isometrically isomorphic to $L^q(\Omega)$.

In this representation of $L^p(\Omega)^*$, the dual pairing is given by

$$\langle f, g \rangle := \int_{\Omega} f g \, dx \quad \text{for } f \in L^p(\Omega) \text{ and } g \in L^q(\Omega). \quad (4.7)$$

- (ii) The dual space of $L^\infty(\Omega)$ does not have a similarly simple representation. It is isometrically isomorphic to the space of finitely additive signed measures on Ω that are absolutely continuous w.r.t. the Lebesgue measure. \triangle

§ 4.4 THE DUAL SPACE OF A HILBERT SPACE

The ability to represent the dual of a normed linear space as concretely as for L^p spaces is a rather special property of a normed linear space. However, it is always possible for Hilbert spaces.

Theorem 4.14 (Riesz representation theorem).

Suppose that H is a Hilbert space. Then the dual space of H is isometrically isomorphic to H itself, via the isomorphism

$$\Phi: H \ni u \mapsto (u, \cdot)_H \in H^*. \quad (4.8)$$

Proof. \square

End of Class 8

End of Week 5

Chapter 1 Convex Infinite-Dimensional Optimization with Applications in Imaging

Chapter 2 Optimal Control of Partial Differential Equations

Bibliography

Adams, R.; J. Fournier (2003). *Sobolev Spaces*. 2nd ed. New York: Academic Press.

Meyers, N.; J. Serrin (1964). “ $H=W$ ”. *Proceedings of the National Academy of Sciences* 51, pp. 1055–1056.

DOI: [10.1073/pnas.51.6.1055](https://doi.org/10.1073/pnas.51.6.1055).

Rudin, W. (1987). *Real and Complex Analysis*. McGraw-Hill.