COSC2627 Discrete Structures
School of Computer Science and IT
RMIT University
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Week 1 Tutorial
Sets

1 Overview of Tutorials, Group Formation

Much of the material in this course is conceptually challenging. In tutorials, you will form teams of 3 or 4 people to work on each week's problems.

In each tutorial, your tutor will be present to answer questions and provide advice, but you are expected to work on the problems for each week in your groups. You should discuss each problem with your group, and work actively together to solve each problem. This may involve asking questions of each other, looking up material in notes or online, asking your tutor, writing out solutions and making sure that each member of the group understands what is being done. Working on problems this way is the main "learning engine" for this material.

Aim

The aim of this week's tutorial is to build tute groups and to ensure that you understand some of the basic concepts of sets. Sets are used in all sorts of contexts, so you need to be "literate" in sets.

Group Reinforcement

- Form yourselves into your groups. You should re-arrange the furniture in the room as necessary in order to be able to sit around a table or number of tables facing each other.
- Make sure that you know everyone else's name. One way to test this is to point to each member of the group and state their name without hesitation. When everyone in the group can do this, you can move on.

Notation

A reminder about some set notation:

- $x \in A$ means that x *is an element of the set* A
- $\emptyset$ is the *empty set*, i.e., the set with no elements in it. Note that $|\emptyset| = 0$.
- $A \subseteq B$ means that A is a *subset* of B, i.e. every element of A is also an element of B. This includes the case that $A = B$.
- $A \supseteq B$ means that B is a *subset* of A, i.e. every element of B is also an element of A. This includes the case that $A = B$.
- $A \subset B$ means that A is a *proper subset* of B, i.e. $A \subseteq B$ and $A \neq B$.
- $A \supset B$ means that B is a *proper subset* of A, i.e. $B \subseteq A$ and $A \neq B$.
- $A \cap B$ is the *intersection* of A and B, i.e. $A \cap B = \{x | x \in A \text{ and } x \in B\}$.
- $A \cup B$ is the *union* of A and B, i.e. $A \cup B = \{x | x \in A \text{ or } x \in B\}$.
- $\overline{A}$ is the *complement* of A, i.e. $\{x \in U | x \notin A\}$ where U is the universe (so all elements are elements of U).
- $A \backslash B$ is the *relative complement* of A and B, i.e. $A \backslash B = \{x | x \in A \text{ and } x \notin B\}$.
- $A \Delta B$ is the *symmetric difference* of A and B which is the elements that are in either set **but not both**, i.e. $A \Delta B = (A \cup B) \backslash (A \cap B)$.

- $A \times B$ is the *Cartesian product* of A and B, i.e. $A \times B = \{(x, y) | x \in A, y \in B\}$.
- $\mathcal{P}(A)$ is the *powerset* of A, i.e. $\mathcal{P}(A)$ is the set of all subsets of A.

Some special sets.

- $\mathbb{N} = \mathbb{N}_0 = \{0, 1, 2, 3, 4, \ldots\}$ *non-negative integers*
- $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ *integers*
- $\mathbb{Q} = \{\frac{p}{q} \mid p \in \mathbb{Z}, q \in \mathbb{N}^{+}\}$ *rationals*
- $\mathbb{R} = \{real \ \ numbers\}$ (includes all rationals and irrationals)
- $\emptyset \subseteq \mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$

The * modifier (or sometimes $^+$) indicates *positive* elements of the set, so $\mathbb{N}^* = \mathbb{Z}^+ = \{1, 2, 3, 4, \ldots\}$, $\mathbb{Q}^* = \{x \in \mathbb{Q} \mid x > 0\}$, etc.

The $^-$ modifier indicates *negative* elements of the set, so $\mathbb{Z}^-\{\ldots, -3, -2, -1\}$, $\mathbb{Q}^- = \{x \in \mathbb{Q} \mid x < 0\}$, etc.

2 Set elements

For each statement below, indicate whether it is true or false.

1. $\sqrt{5}/2 \in \mathbb{R}\backslash\mathbb{Q}$
2. $\dfrac{13}{6-3} \in \mathbb{N}$
3. $\sqrt{2} + 2 \in \mathbb{R}\backslash\mathbb{Q}$
4. $\dfrac{10}{4-1} \in \mathbb{N}$
5. The sum of two real numbers is always a real number.
6. The product of a rational number and a natural number is always a natural number.
7. The difference of two natural numbers is always a rational number.
8. The difference of two natural numbers is always a natural number.
9. The product of a real number and a rational number is always a real number.
10. The product of two rational numbers is always a rational number.
11. $\mathbb{R}^- \subseteq \mathbb{Q}$
12. $\mathbb{N}^* \subseteq \mathbb{Z}^+$
13. $\mathbb{N}^* \subseteq \mathbb{Z}$
14. $\mathbb{Q}^+ \subseteq \mathbb{N}$
15. $\dfrac{7}{14} \in \mathbb{Q}\backslash\mathbb{N}$
16. $\dfrac{3}{4} \in \mathbb{Q}\backslash\mathbb{N}$
17. $\{1, 2\} \in \{\{\emptyset\}, \{1\}, \{2\}, \{1, 2\}\}$
18. $\{2\} \in \{\{\emptyset\}, \{1\}, \{2\}, \{1, 2\}\}$

3 Set operations

1. Given a word W we define $L(W)$ as the set of letters of W. We let $D = L(discrete)$ and $M = L(mathematics)$.

   Are the following statements true or false? Justify your answer.

   (a) $L(sister) \subseteq D \cap M$
   (b) $D \backslash M \subseteq L(dream)$
   (c) $L(drama) = D \triangle M$

2. Let $A = \{a, c, e, f, l, m\}$ $B = \{b, d, e, f, m, n\}$ and $C = \{a, h, l, q, z\}$. Find

   (a) $(A \cup B) \cap C$
   (b) $(B \cap C) \cup (A \cap B)$
   (c) $(A \Delta B) \cap (B \Delta C)$

3. Let $A = \{a, c, d, e, g, h, j, z\}$, $B = \{b, d, h, k, q, r, t, z\}$ and $C = \{g, h, l, m, y\}$. Find

   (a) $(A \cap B) \backslash C$
   (b) $A \cap (B \backslash C)$
   (c) $(A \Delta B) \cap (B \Delta C)$

4. Let $A = \{1, 5, 6, 10\}$, $B = \{1, 3, 6, 12\}$ and $C = \{2, 3, 12, 15\}$. Find

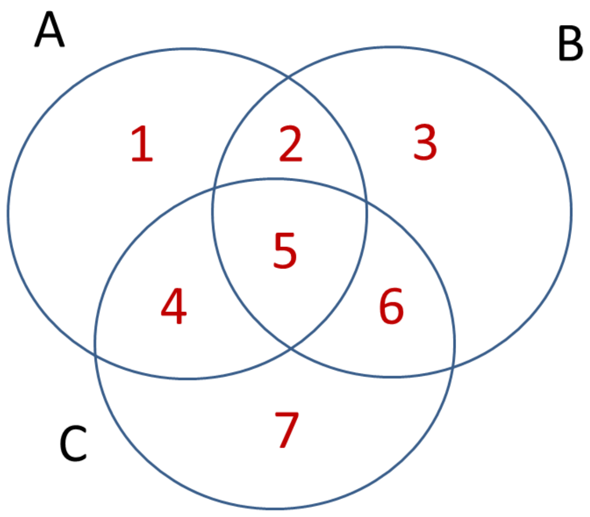
   (a) $(A \cap B) \cup (C \cap A)$
   (b) $(C \Delta A) \cap (A \Delta B)$

5. Let $A = \{a, e, l, m\}$, $B = \{b, e, m, n\}$ and $C = \{a, h, l, z\}$. Find

   (a) $(B \cap C) \cup (A \cap B)$
   (b) $(A \Delta B) \cap (B \Delta C)$

4 Set properties

1. Let A, B and C denote sets. In the following sentences complete the space with =, $\subseteq$ or $\supseteq$.



(a) $A \cup (B \backslash C)...(A \cup B) \backslash C$
(b) $A \Delta B...(A \backslash B) \cup (B \backslash A)$
(c) $(A \Delta C) \cap (B \backslash C)$ ... $A \cap B$
(d) $(A \cup B) \backslash C$ ... $(A \Delta B) \backslash C$
(e) $(A \Delta B) \cap (B \Delta C)$ ... $(A \cap C) \backslash B$
.
(f) $(A \Delta C) \cap (B \cap A)$ ... $A \cap (B \Delta C)$
(g) $(A \Delta B) \cap (B \Delta C)$ ... $B \backslash (A \cup C)$
(h) $(A \cup C) \Delta (B \cup C)...A \Delta B$

2. How can you justify each of your conclusions above? What general method can you use to show this? (**Hint:** Venn diagrams)

5 Set Definitions

Sets can be defined either explicitly, such as $S = \{1, 4, 6, 9\}$ or implicitly, such as $\{x \mid x$ is a positive integer, x is even and x is prime $\}$.

1. Give at least three examples of elements in each of the following sets.

   (a) $\{x \in \mathbb{N} \mid x > 3$ and $x < 10\}$
   (b) $\{y \in \mathbb{Q} \mid 0 < y < 1\}$
   (c) $\{x \in \{\text{strings}\} \mid x$ contains c or x ends with a $\}$
   (d) $\{x \in \mathbb{N} \mid x - 2$ is prime$\}$

2. Given an explicit definition of each of the following sets.

   (a) $\{x \in \mathbb{N} \mid x$ is prime and $x < 20\}$
   (b) $\{x \in \mathbb{R} \mid x^2 - 3x + 2 = 0\}$
   (c) $\{x \in \mathbb{N} \mid x^2 + 4x + 4 = 0\}$
   (d) $\{x$ is a string containing $a,\ b,$ or $c \mid x$ has length 2 or less$\}$

6 Sets??

A barbarous medieval monarch, who hated beards, once decreed that all men in the kingdom must shave. This was back in the days when shaving meant an appalling ritual involving putting sharp blades very close to one's own throat, and so naturally many were afraid to do this. The king, who was not a total idiot and did not want to see needless bloodletting, therefore made the following two decrees:

1. Any man who could (safely) shave himself **must shave himself.**

2. Any man who could not (safely) shave himself (and only such men) **must be shaved by the royal barber**.

(Clearly the king could have just omitted the first decree, so that the royal barber would shave everyone, but he thought that would overwork both the barber and the royal treasury).

All went well, and one day the king went to visit the barber. The king was very impressed with the many clean-shaven men he saw and the professional skill of the barber. However, whilst in the barber shop, he noted the barber's clean-shaven face and asked, "Barber, who shaves you?"

The barber hesitated, as he knew that he was in a pickle. On the one hand, as a professional barber, he is quite capable of shaving himself, and so according to the first decree, he must shave himself. On the other hand, the second decree says that the barber only shaves those who cannot shave themselves, so if the barber shaves himself, then he cannot shave himself (by definition).

1. What does the barber say to the king?

2. Can you resolve this paradox? It seems that the barber has to both shave himself and not shave himself. What can you do to help him?

3. What other paradoxes do you know? Can you find any similar ones? (**Hint:** liar)

This tale (or variants of it) is known as the *paradox of the barber*, and is a popular version of what is known to mathematicians as *Russell's paradox*. Russell's paradox was originally stated along the lines of *the set of all sets which are not members of themselves*. If we call this set R, then we have that both $R \in R$ and $R \notin R$. This showed that sets had to be defined very carefully to avoid such paradoxes. If you want to know the state of the art in defining sets, look up *Zermelo-Fraenkel set theory* (but hang onto your hat, and be very clear that such concepts go well beyond what will be studied in this course).

7 Potpourri

1. For the sets $A = \{a, b, c, d\}$ and $B = \{x, y, c\}$, compute $A * B$ where $*$ is a set operation from the set $\{\times, \setminus, \cap, \cup\}$. Check if $|A * B| = |B * A|$.

2. Provide an *extensional* definition for each of the following languages:
   - $L_1 = \{a^n \mid n \text{ is even and positive}\}$
   - $L_2 = \{a^{3^n} \mid n \geq 0\}$
   - $L_3 = \{a^{2n} \mid 0 < n < 3\}$
   - $L_4 = \{w \mid w \in \{a, b\}^*, \text{ where the length of } w \text{ is a power of } 2\}$

3. Provide an *intensional* definition for each of the following sets/languages:
   (a) $L_a = \{aa, aaa, aaaaa, aaaaaaaaa, aaaaaaaaaaaaaaaaa, \ldots\}$
   (b) $L_b = \{242, 262, 22422, 22622, \ldots\}$
   (c) $L_c = \{a, b, aaa, aab, aba, abb, baa, bab, bba, bbb, aaaaaa, aaaaab, aaaaba, \ldots, bbbbbb, \ldots\}$

4. Consider set $S = \{a, b\}$. How many strings of length 2 can be built from S? How many of length 3? In general, how many of length n?

5. Give 5 positive numbers that belong to the set $\{x \mid x = 3 \times y \text{ or } x = 5 \times y, y \in \mathbb{Z}\}$, where $\mathbb{Z}$ is the set of integers.

6. Give 5 positive numbers that belong to the set $\{x \mid x = 3 \times y \text{ and } x = 5 \times y, y \in \mathbb{Z}\}$, where $\mathbb{Z}$ is the set of integers.

7. Give all the elements of the power set of $\{2, 3, 4\}$.

8. Let f be a function on a set of numbers such that f returns the sum of elements in set $\mathcal{X}$. For example, $f(\{10, 15\}) = 25$ and $f(\{\emptyset\}) = 0$. Give the elements of set $\mathcal{Y}$ such that $\mathcal{Y} = \{x \mid x \in 2^{\{1,3,5,10\}}, f(x) < 10\}$. The notation $2^{\mathcal{X}}$ denotes the power set of set $\mathcal{X}$.

9. Intuitively explain why the size of the power set of a set $\mathcal{X}$ is $2^{|\mathcal{X}|}$.

8 You say more?

If you did all the above exercises, there are lots of books and online resources where you can find additional exercises for you to practice and get better and better! For example, you can check the exercises in the Chapter 1 (Sets) of the Book of Proofs.