SCS 43XX

Quantum Mechanics in Computing

University of Colombo, School of Computing

Tutorial 8

Qubit State Manipulation with Multiple Qubit Gates

1 Multi-Qubit Quantum Gates

1.1 Two-Qubit Gates

1.1.1 Controlled-NOT (CNOT) Gate

The **CNOT gate** acts on two qubits: a *control* qubit and a *target* qubit. If the control qubit is $|1\rangle$, the target qubit flips (i.e., $|0\rangle \leftrightarrow |1\rangle$). If the control is $|0\rangle$, nothing changes.

$$CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Example:

$$CNOT\,|01\rangle = |11\rangle\,, \quad CNOT\,|10\rangle = |10\rangle$$

1.1.2 Controlled-Z (CZ) Gate

The **Controlled-Z (CZ) gate** applies a phase flip (Z gate) to the target qubit only if the control qubit is $|1\rangle$.

$$CZ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Example:

$$CZ\,|10\rangle = |10\rangle\,, \quad CZ\,|11\rangle = -\,|11\rangle$$

1.1.3 SWAP Gate

The **SWAP gate** exchanges the states of two qubits.

$$SWAP = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example:

$$SWAP\,|10\rangle = |01\rangle\,, \quad SWAP\,|11\rangle = |11\rangle$$

1.2 Creating Entangled States

Applying a **Hadamard gate (H)** followed by a **CNOT** can create **Bell states**, which are maximally entangled states.
Example:

$$H\,|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$CNOT\left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle\right) = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

1.3 Multi-Qubit Gates

1.3.1 Toffoli (CCNOT) Gate

The **Toffoli gate** (controlled-controlled-NOT) is a three-qubit gate where the third qubit flips if both control qubits are $|1\rangle$.

$$Toffoli = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Example:

$$Toffoli\,|110\rangle = |111\rangle\,, \quad Toffoli\,|100\rangle = |100\rangle$$

1.3.2 Fredkin (Controlled-SWAP) Gate

The **Fredkin gate** swaps the last two qubits if the control qubit is $|1\rangle$.

$$Fredkin = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Example:

$$Fredkin\,|010\rangle = |010\rangle\,, \quad Fredkin\,|110\rangle = |101\rangle$$

1.4 Quantum Fourier Transform (QFT)

The **Quantum Fourier Transform (QFT)** is the quantum analog of the discrete Fourier transform. It transforms computational basis states into superpositions.
For a 3-qubit system:

$$QFT\,|011\rangle = \frac{1}{2}\left(|000\rangle + e^{2\pi i/4}\,|001\rangle + e^{2\pi i/2}\,|010\rangle + e^{3\pi i/2}\,|011\rangle\right)$$

QFT is essential for quantum algorithms like Shor's factoring algorithm.

2 Activities

Two-Qubit Gates

1. Apply the **CNOT** gate to the state $|01\rangle$. What is the resulting state?
2. Apply the **Controlled-Z** (CZ) gate to the state $|10\rangle$. What is the result?
3. Apply the **SWAP** gate to the state $|10\rangle$. What is the resulting state?
4. Consider the state $|00\rangle + |11\rangle$. Apply a **CNOT** gate with the first qubit as the control and the second qubit as the target. What is the resulting state?
5. Apply a **Hadamard** gate on the first qubit of the state $|10\rangle$, and then apply a **CNOT** gate. What is the resulting state?

Multi-Qubit Gates

1. Apply the **Toffoli** gate to the state $|110\rangle$. What is the resulting state?
2. Consider the state $|000\rangle$. Apply the **Fredkin** gate (controlled-SWAP). What is the resulting state?
3. Given the state $|0001\rangle$, apply a **multi-controlled-NOT** gate, with the first three qubits as control and the last qubit as the target. What is the resulting state if the first three qubits are all $|1\rangle$?
4. Apply the **Quantum Fourier Transform (QFT)** on the 3-qubit state $|011\rangle$. Provide the resulting state.
5. Given the state $|0001\rangle$, apply the **multi-controlled-Z** gate with the first two qubits as control and the last qubit as the target. What is the result if both control qubits are $|1\rangle$?

Gate Applications and State Transformations

1. Apply the **Hadamard** gate to the state $|0\rangle$, followed by a **CNOT** gate. What is the resulting state?
2. Apply the **CNOT** gate to the state $|01\rangle$ and then apply a **Hadamard** gate on the first qubit. What is the resulting state?
3. Consider the state $|0\rangle$. Apply a **Hadamard** gate, then a **Controlled-Z** gate with the second qubit as the target, and finally another **Hadamard** gate on the first qubit. What is the resulting state?
4. Starting with the state $|10\rangle$, apply the **CNOT** gate followed by the **SWAP** gate. What is the resulting state?
5. Given the state $|00\rangle$, apply the **Controlled-X** gate with the first qubit as control and the second qubit as the target. Then apply the **Hadamard** gate on the first qubit. What is the resulting state?

Advanced Operations and Entanglement

1. Consider the state $|01\rangle$. Apply a **CNOT** gate, followed by a **Hadamard** gate on the first qubit. What is the resulting state?
2. Create the **Bell state** $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ starting from $|00\rangle$ using a **Hadamard** gate and a **CNOT** gate. What are the intermediate steps and the resulting state?
3. Given the state $|010\rangle$, apply a **Fredkin** gate. What is the resulting state?
4. Apply a **Toffoli** gate to the state $|101\rangle$. What is the resulting state?
5. Apply the **Quantum Fourier Transform** (QFT) to the state $|110\rangle$. What is the resulting state?

Two-Qubit Superposition States

1. Apply a Hadamard gate to the state $|01\rangle$. What is the resulting state?
2. Consider the state $|0\rangle + |1\rangle$. Apply a CNOT gate with the first qubit as the control and the second qubit as the target. What is the resulting state?
3. Apply a Hadamard gate to the state $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, and then apply a CNOT gate. What is the resulting state?
4. Starting with the state $|+\rangle$, apply a CNOT gate with the first qubit as the control and the second qubit as the target. What is the result?
5. Consider the state $|0\rangle + i|1\rangle$. Apply a Hadamard gate on the first qubit. What is the resulting state?

Three-Qubit Superposition States

1. Consider the state $\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$. Apply a Hadamard gate to the first qubit. What is the resulting state?
2. Apply a Hadamard gate to each qubit of the state $|000\rangle$. What is the resulting state?
3. Given the state $\frac{1}{\sqrt{2}}(|001\rangle + |110\rangle)$, apply a CNOT gate with the first qubit as the control and the second qubit as the target. What is the resulting state?
4. Consider the state $|+\rangle \otimes |0\rangle$, where $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. Apply a CNOT gate with the first qubit as the control and the second qubit as the target. What is the resulting state?
5. Given the state $\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$, apply a Toffoli gate. What is the resulting state?

Four-Qubit Superposition States

1. Consider the state $\frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle)$. Apply a Hadamard gate to each qubit. What is the resulting state?
2. Apply a CNOT gate with the first qubit as the control and the second qubit as the target, followed by a Hadamard gate on the first qubit. What is the resulting state when applied to the state $|+\rangle \otimes |000\rangle$?
3. Starting with the state $\frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle)$, apply a Fredkin gate. What is the resulting state?

4. Given the state $|0\rangle \otimes \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$, apply a CNOT gate to the second and third qubits. What is the resulting state?

5. Consider the state $\frac{1}{\sqrt{2}}(|0000\rangle + |0110\rangle + |1001\rangle + |1111\rangle)$. Apply a Quantum Fourier Transform (QFT). What is the resulting state?

Superposition and Entanglement

1. Apply a Hadamard gate to the state $|0\rangle|0\rangle$, followed by a CNOT gate, and then apply a Hadamard gate again to the first qubit. What is the resulting state if you start with $|0\rangle|0\rangle$?

2. Consider the state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Apply a Hadamard gate to the first qubit, and then a CNOT gate with the first qubit as the control. What is the resulting state?

3. Starting with the state $|0\rangle + |1\rangle|0\rangle + |1\rangle$, apply a CNOT gate and then apply a Hadamard gate to both qubits. What is the resulting state?

4. Consider the state $\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$. Apply a Hadamard gate to the first qubit, then apply a CNOT gate. What is the resulting state?

5. Given the state $|0\rangle \otimes |+\rangle \otimes |1\rangle$, apply a CNOT gate with the second qubit as control and the third qubit as target. What is the resulting state?