SCS 43XX

Quantum Mechanics in Computing

University of Colombo, School of Computing

Tutorial 7

Qubit state manipulation with Single Qubit Gates

1 Theory

A **qubit** is a two-level quantum system represented as:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \quad \text{where } |\alpha|^2 + |\beta|^2 = 1. \tag{1}$$

1.1 Basic Gates

- **Pauli-X (NOT Gate)**: $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
- **Pauli-Y**: $Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$
- **Pauli-Z (Phase Flip)**: $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
- **Hadamard (H)**: $H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

1.2 Rotation Gates

$$R_x(\theta) = \begin{bmatrix} \cos(\theta/2) & -i\sin(\theta/2) \\ -i\sin(\theta/2) & \cos(\theta/2) \end{bmatrix}, \tag{2}$$

$$R_y(\theta) = \begin{bmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{bmatrix}, \tag{3}$$

$$R_z(\theta) = \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix}. \tag{4}$$

1.3 Phase Gates

- **S-Gate**: $S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$
- **T-Gate**: $T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$

1.4 Measurement

The probability of measuring $|0\rangle$ is $|\alpha|^2$, and for $|1\rangle$ it is $|\beta|^2$.

1.5 Bloch Sphere Representation

A qubit is represented as:

$$|\psi\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle. \tag{5}$$

2 Activities

2.1 Basic Gate Applications

1. Compute $X|0\rangle$ and $X|1\rangle$.
2. Compute $Y|0\rangle$ and $Y|1\rangle$.
3. Compute $Z|+\rangle$ where $|+\rangle = H|0\rangle$.
4. Show that $H^2 = I$.
5. Compute $H|1\rangle$.
6. Show that $XZ = -ZX$.
7. Compute HXH.
8. Compute HYH.

2.2 Rotation Gates

1. Compute $R_x(\pi/2)|0\rangle$.
2. Compute $R_y(\pi/2)|1\rangle$.
3. Compute $R_z(\pi)|+\rangle$.
4. Show that $R_x(\pi) = X$.
5. Show that $R_y(\pi) = Y$.
6. Show that $R_z(\pi) = Z$.
7. Compute $R_x(\pi/2)R_y(\pi/2)|0\rangle$.
8. Compute $R_y(\pi/2)R_x(\pi/2)|0\rangle$.

2.3 Phase and Hadamard Gates

1. Compute $S|1\rangle$.
2. Compute $T|0\rangle$.
3. Show that $S^2 = Z$.
4. Compute $H|+\rangle$.
5. Compute $HS|0\rangle$.
6. Show that $SHS = H$.
7. Compute $T^2|1\rangle$.
8. Show that $T^4 = I$.

2.4 Measurement

1. If $|\psi\rangle = \frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle$, what is the probability of measuring $|1\rangle$?
2. If $|\psi\rangle = \cos(\theta)|0\rangle + \sin(\theta)|1\rangle$, what is the probability of measuring $|1\rangle$?
3. Given $H|0\rangle$, what is the probability of measuring $|0\rangle$?
4. Given $H|1\rangle$, what is the probability of measuring $|1\rangle$?
5. If a qubit is in state $|0\rangle$, what is the probability of measuring $|1\rangle$?
6. Compute the measurement probabilities for $R_x(\pi/3)|0\rangle$.
7. Compute the measurement probabilities for $R_y(\pi/3)|0\rangle$.

2.5 Bloch Sphere

1. Convert $|0\rangle$ and $|1\rangle$ into Bloch sphere form.
2. Express $|+\rangle$ and $|-\rangle$ on the Bloch sphere.
3. Compute the effect of $R_x(\pi/2)$ on $|0\rangle$ in Bloch sphere coordinates.
4. Compute the effect of $R_y(\pi/2)$ on $|1\rangle$ in Bloch sphere coordinates.
5. Compute the effect of $R_z(\pi/2)$ on $|+\rangle$ in Bloch sphere coordinates.
6. Show that X corresponds to a π rotation around the x-axis on the Bloch sphere.
7. Show that Y corresponds to a π rotation around the y-axis.
8. Show that Z corresponds to a π rotation around the z-axis.