

## Tutorial 3

### Unitary and Hermitian Matrices. Eigenvectors

Unitary matrices, denoted as  $U$ , are used in quantum computing to represent quantum gates, ensuring that

$$UU^\dagger = I$$

where  $U^\dagger$  is the conjugate transpose of  $U$ .

Hermitian matrices, denoted as  $H$ , represent observable quantities in quantum mechanics, satisfying:

$$H = H^\dagger$$

**Conjugate:** For a matrix  $A$ , the conjugate  $A^*$  is obtained by taking the complex conjugate of each element in  $A$ .

**Conjugate Transpose:** The conjugate transpose (or Hermitian adjoint) of a matrix  $A$ , denoted as  $A^\dagger$ , is obtained by taking the transpose of  $A$  and then taking the complex conjugate of each element.

1. Let  $A = \begin{bmatrix} 1 & 0 & 4 \\ 17 & 10 & 18 \end{bmatrix}$ , find  $A^T$ .
2. Let  $A = \begin{bmatrix} 3 + 2i & 2e^{\pi i/3} & 4 \\ 1 - 7i & 10 & -3i \end{bmatrix}$ , find  $A^*$  (Hint:  $z = re^{i\theta}$ ,  $z^* = re^{-i\theta}$ ).
3. Let  $A = \begin{bmatrix} 3 + 2i & 2e^{\pi i/3} & 4 \\ 1 - 7i & 10 & -3i \end{bmatrix}$ , find  $A^\dagger$ .
4. Is the matrix  $A = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$ , Unitary? Prove.
5. Is the matrix  $A = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$ , Hermitian? Prove.

Eigenvectors of a matrix  $A$  are non-zero vectors  $v$  that satisfy:

$$Av = \lambda v,$$

where  $\lambda$  is the eigenvalue.

Eigenvalues are scalars that indicate how the eigenvector  $v$  is scaled during the transformation by  $A$ .

1. When we apply a matrix to one of its eigenvectors, what happens to the eigenvector geometrically?
2. Find the eigenvalue when the matrix  $\begin{bmatrix} 1 & 9 \\ 4 & 1 \end{bmatrix}$  applied to its eigenvector  $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$
3. Geometrically what happens to the vector when the eigenvalue is greater than 1 and less than 1 respectively
4. Find the eigenvalues and eigenvectors of the matrices  $\begin{bmatrix} -2 & 1 & 1 \\ 6 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$  and  $\begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 2 \\ -1 & -1 & -1 \end{bmatrix}$
5. Show that the Hermitian matrix  $\begin{bmatrix} 0 & 0 & i \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$  has only two real eigenvalues and the three eigenvectors are orthonormal.