HW 6 - MATH403

Danesh Sivakumar

June 14, 2022

Problem 1 (Chapter 8, Exercise 20)

Find a subgroup of $\mathbb{Z}_{12} \oplus \mathbb{Z}_{18}$ that is isomorphic to $\mathbb{Z}_9 \oplus \mathbb{Z}_4$.

Proof. Note that $\mathbb{Z}_9 \oplus \mathbb{Z}_4$ is cyclic and of order 36 because 9 and 4 are coprime; also note that $3 \in \mathbb{Z}_{12}$ has order 4 and $2 \in \mathbb{Z}_{18}$ has order 9, so that $(3,2) \in \mathbb{Z}_{12} \oplus \mathbb{Z}_{18}$ has order $4 \cdot 9 = 36$. This immediately implies that the subgroup generated by this element $H = \langle (3,2) \rangle$ is cyclic and of order 36, so it is isomorphic to $\mathbb{Z}_9 \oplus \mathbb{Z}_4$. $\square$

Problem 2 (Chapter 8, Exercise 22)

Determine the number of elements of order 15 and the number of cyclic subgroups of order 15 in $\mathbb{Z}_{30} \oplus \mathbb{Z}_{20}$.

Proof. Note that elements of order 15 have the form (a,b) where $\text{lcm}(|a|, |b|) = 15$, where $|a|$ divides 30 and $|b|$ divides 20. There are three ways to make this happen: $(15,1)$, $(15,5)$ and $(3,5)$. This immediately implies that the number of elements of order 15 is $\varphi(15)\varphi(1)+\varphi(15)\varphi(5)+\varphi(3)\varphi(5) = 8\cdot1+8\cdot4+2\cdot4 = 48$. Note that each cyclic subgroup of order 15 has $\varphi(15) = 8$ generators; this implies that there are $48/8 = 6$ distinct cyclic subgroups of order 15, because picking any other generator from the same "group" of 8 generators would yield the same cyclic subgroup. $\square$

Problem 3 (Chapter 8, Exercise 60)

Give an example of an infinite non-Abelian group that has exactly six elements of finite order.

Proof. Note that $\mathbb{Z} \oplus S_3$ satisfies these properties. Because $\mathbb{Z}$ is infinite, the direct product is also infinite. S_3 is also non-Abelian; note that $(12)(23) = (21)(23) = (231)$ but $(23)(12) = (23)(21) = (213)$ and $(231) \neq (231)$. Thus the direct product is non-Abelian. But there are exactly six elements of finite order in this group; the only element of finite order in $\mathbb{Z}$ is 0 and all 6 of the elements in S_3 have finite order; this means that all 6 of the elements of the form $(0, \sigma)$ are the only elements of finite order in this infinite non-Abelian group. $\square$

Problem 4 (Chapter 9, Exercise 10)

Let $H = \{(1), (12)(34)\}$ in A_4.

a. Show that H is not normal in A_4.

b. Referring to the multiplication table for A_4 in Table 5.1 on page 105, show that, although $\alpha_6 H = \alpha_7 H$ and $\alpha_9 H = \alpha_{11} H$, it is not true that $\alpha_6 \alpha_9 H = \alpha_7 \alpha_{11} H$. Explain why this proves that the left cosets of H do not form a group under coset multiplication.

Proof.

a. Note that $(123) \in A_4$, but $(123)(12)(34)(123)^{-1} = (123)(12)(34)(321) = (13)(24) \notin H$.

b. From the multiplication table, we observe that $\alpha_6 H = \{(243), (142)\} = \alpha_7 H$ and $\alpha_9 H = \{(132), (234)\} = \alpha_{11} H$. However, we also note that $\alpha_6 \alpha_9 H = (243)(132)H = (12)(34)H = H$, but $\alpha_7 \alpha_{11} H = (142)(234)H = (14)(23)H \neq H$, so that $\alpha_6 \alpha_9 H \neq \alpha_7 \alpha_{11} H$. This shows that multiplication is not well-defined for the left cosets, meaning that the group operation fails to work; thus the left cosets of H do not form a group under coset multiplication.

□

Problem 5 (Chapter 9, Exercise 22)

Determine the order of $(\mathbb{Z} \oplus \mathbb{Z})/\langle(2, 2)\rangle$. Is the group cyclic?

Proof. Note that $(1, 0) + \langle(2, 2)\rangle$ is never in the form $(2k, 2k)$ for some $k \in \mathbb{Z}$, so it has infinite order; thus the group has infinite order. In order for it to be cyclic, it must be isomorphic to $\mathbb{Z}$, but this is not the case because $(1, 1)+\langle(2, 2)\rangle$ generates elements of order 2 (adding any of these elements to itself twice yields an element in the form $(2k, 2k)$), but $\mathbb{Z}$ has no elements of order 2; thus the group is not cyclic. □

Problem 6 (Chapter 9, Exercise 26)

Let $H = \{1, 17, 41, 49, 73, 89, 97, 113\}$ under multiplication modulo 120. Write H as an external direct product of groups of the form $\mathbb{Z}_{2^k}$. Write H as an internal direct product of nontrivial subgroups.

Proof. Notice that H has 8 elements, meaning that it is isomorphic to $\mathbb{Z}_8$, $\mathbb{Z}_4 \oplus \mathbb{Z}_2$, or $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Note further that 1 is the only element of order 1, $\{17, 73, 97, 113\}$ are the elements of order 4 and $\{41, 49, 89\}$ are the elements of order 2. Because no elements have order 8 and there are elements of order 4, it follows that H is isomorphic to $\mathbb{Z}_4 \oplus \mathbb{Z}_2$ as an external direct product. To express H as an internal direct product, take any elements of order 4 and 2, say

$\langle 17\rangle \times \langle 41\rangle$. Clearly the intersection of both of these subgroups is trivial and these subgroups are normal in H because multiplication modulo n is commutative, so this internal direct product works; it is also isomorphic to our choice of external direct product. $\square$

Problem 7 (Chapter 9, Exercise 38)

Prove that for every positive integer n, $\mathbb{Q}/\mathbb{Z}$ has an element of order n.

Proof. We want to find a coset $\frac{p}{q}\mathbb{Z}$ such that $n\frac{p}{q}\mathbb{Z} = \mathbb{Z}$ and it is the smallest positive solution to this relation; clearly taking $\frac{1}{n}\mathbb{Z}$ works, and this holds for any positive integer. $\square$

Problem 8 (Chapter 9, Exercise 48)

If G is a group and $|G\colon Z(G)| = 4$, prove that $G/Z(G) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Proof. Note that G could possibly be isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ or $\mathbb{Z}_4$. Assume it is isomorphic to $\mathbb{Z}_4$; then by the G/Z theorem, G is Abelian because $\mathbb{Z}_4$ is cyclic. But this implies that $|G\colon Z(G)| = |G|/|Z(G)| = |G|/|G| = 1$, which is a contradiction; thus $G/Z(G)$ must be isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. $\square$

Problem 9 (Chapter 9, Exercise 52)

Let G be an Abelian group and let H be the subgroup consisting of all elements of G that have finite order. Prove that every nonidentity element in G/H has infinite order.

Proof. Pick a nonidentity element $gH \in G/H$ such that $gH \neq H$; that is g has infinite order. Suppose that gH has finite order n; then $H = (gH)^n = (g^n)H$, which implies that g^n has finite order; but this implies that g also has finite order, which contradicts our assumption that $gH \neq H$; thus every nonidentity element has infinite order. $\square$

Problem 10 (Chapter 9, Exercise 58)

If N and M are normal subgroups of G, prove that NM is also a normal subgroup of G.

Proof. First, we prove that NM is a subgroup of G with the one-step subgroup test. Clearly NM is nonempty, because $ee = e \in NM$. Take $n_1m_1, n_2m_2, \in NM$; then $(n_1m_1)(n_2m_2)^{-1} = n_1m_1m_2^{-1}n_2^{-1}$. The normality of M implies that there exists $m_3 \in M$ such that $m_3 = n_2m_1m_2^{-1}n_2^{-1}$; this means that $n_1m_1m_2^{-1}n_2^{-1} = n_1n_2^{-1}m_3 \in NM$, so that $NM \leq G$. Because N and M are normal in G, it follows that for any $g \in G$ that $gNg^{-1} \subset N$ and $gMg^{-1} \subset M$; thus $gNMg^{-1} = gNg^{-1}gMg^{-1} \subset NM$; thus NM is a normal subgroup of G. $\square$

Problem 11 (Chapter 9, Exercise 72)

Let G be a group and H an odd-order subgroup of G of index 2. Show that H contains every element of G of odd order.

Proof. Suppose $g \in G$ has odd order; then g^2 is a generator of $\langle g \rangle$ because 2 is coprime to any odd integer. But the fact that H is of index 2 implies that the order of G/H is 2, meaning $g^2H = H$. This means $g^2 \in H$, so that $g \in \langle g^2 \rangle \leq H$ must also be contained in H. $\square$