HW 3 - MATH403

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Problem 1 (Chapter 5, Exercise 1)

Let

$$\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 5 & 4 & 6 \end{bmatrix} \quad \text{and} \quad \beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 4 & 3 & 5 \end{bmatrix}$$

Compute each of the following

a. α^{-1}

b. $\beta\alpha$

c. $\alpha\beta$

Proof. **a.**

$$\alpha^{-1} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 5 & 4 & 6 \end{bmatrix}$$

b.

$$\beta\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 2 & 3 & 4 & 5 \end{bmatrix}$$

c.

$$\alpha\beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 2 & 1 & 5 & 3 & 4 \end{bmatrix}$$

□

Problem 2 (Chapter 5, Exercise 2)

Let

$$\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 5 & 1 & 7 & 8 & 6 \end{bmatrix} \quad \text{and} \quad \beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 3 & 8 & 7 & 6 & 5 & 2 & 4 \end{bmatrix}$$

Write α, β and $\alpha\beta$ as

a. products of disjoint cycles;

b. products of 2-cycles.

Proof. Note that

$$\alpha\beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 4 & 6 & 8 & 7 & 1 & 3 & 5 \end{bmatrix}$$

Then for α we have:

a. $\alpha = (12345)(678)$

b. $\alpha = (15)(14)(13)(12)(68)(67)$

For β we have:

a. $\beta = (1)(23847)(56)$

b. $\beta = (27)(24)(28)(23)(56)$

For $\alpha\beta$ we have:

a. $\alpha\beta = (12485736)$

b. $\alpha\beta = (16)(13)(17)(15)(18)(14)(12))$

□

Problem 3 (Chapter 5, Exercise 10)

Show that A_8 contains an element of order 15.

Proof. Consider the permutation $\sigma = (12345)(678)$. Writing σ as a product of transpositions shows that $\sigma = (15)(14)(13)(12)(68)(67)$, which is a product of an even number of transpositions; thus $\sigma \in A_8$. Note that because (12345) and (678) are disjoint cycles whose product is σ, it follows that the order of σ is the least common multiple of the orders of its disjoint cycles; the order of each n-cycle is n, so $|(12345)| = 5$ and $|(678)| = 3$, so that $|\sigma| = \operatorname{lcm}(5, 3) = 15$. Thus, σ is an element of A_8 whose order is 15.

□

Problem 4 (Chapter 5, Exercise 14)

Suppose that α is a 6-cycle and β is a 5-cycle. Determine whether $\alpha^5\beta^4\alpha^{-1}\beta^{-3}\alpha^5$ is even or odd. Show your reasoning.

Proof. Let $\alpha = (a_1a_2a_3a_4a_5a_6)$ and $\beta = (b_1b_2b_3b_4b_5)$, with α and β not necessarily disjoint. Writing α as a product of transpositions yields
$\alpha = (a_1a_6)(a_1a_5)(a_1a_4)(a_1a_3)(a_1a_2)$ and $\beta = (b_1b_5)(b_1b_4)(b_1b_3)(b_1b_2)$, so that α is the product of 5 transpositions and β is the product of 4 transpositions. Using the exponents of the expression, we deduce that it can be written as the product of $5(5) + 4(4) + 1(5) + 3(4) + 5(5) = 83$ transpositions, which is odd. Note that if r transpositions are "redundant" (i.e. can be replaced with ε), it follows that r is even, so the parity does not change.

□

Problem 5 (Chapter 5, Exercise 26)

Let α and β belong to S_n. Prove that $\alpha^{-1}\beta^{-1}\alpha\beta$ is an even permutation.

Proof. It is helpful to establish that an element of S_n and its inverse have the same parity. This is because for any $\sigma \in S_n$, it follows that $\sigma\sigma^{-1} = \varepsilon$, and because ε is always the product of an even number of transpositions, it follows that σ and σ^{-1} both have the same parity. Also, note that the product of two transpositions with the same parity is even, and the product of two transpositions with different parity is odd; this follows from the parity of sums of pairs of integers. We will use the notation $\sigma = \text{(parity)}$ to denote the parity of a permutation. We break this down to four cases:

1. Suppose α is even and β is even. Then it follows that α^{-1} is even and β^{-1} is even, so that $\alpha^{-1}\beta^{-1}\alpha\beta = \text{(even)(even)(even)(even)} = \text{(even)(even)(even)} = \text{(even)(even)} = \text{(even)}$.

2. Suppose α is even and β is odd. Then it follows that α^{-1} is even and β^{-1} is odd, so that $\alpha^{-1}\beta^{-1}\alpha\beta = \text{(even)(odd)(even)(odd)} = \text{(even)(odd)(odd)} = \text{(even)(even)} = \text{(even)}$.

3. Suppose α is odd and β is odd. Then it follows that α^{-1} is odd and β^{-1} is odd, so that $\alpha^{-1}\beta^{-1}\alpha\beta = \text{(odd)(odd)(odd)(odd)} = \text{(odd)(odd)(even)} = \text{(odd)(odd)} = \text{(even)}$.

4. Suppose α is odd and β is even. Then it follows that α^{-1} is odd and β^{-1} is even, so that $\alpha^{-1}\beta^{-1}\alpha\beta = \text{(odd)(even)(odd)(even)} = \text{(odd)(even)(odd)} = \text{(odd)(odd)} = \text{(even)}$.

The parity of $\alpha^{-1}\beta^{-1}\alpha\beta$ is even in every case, so it is always an even permutation.

□

Problem 6 (Chapter 5, Exercise 34)

If α and β are distinct 2-cycles, what are the possibilities for $|\alpha\beta|$?

Proof. If α and β are distinct 2-cycles, there are two cases:

1. α and β both move the same element (i.e. they are not disjoint), so that WLOG $\alpha = (ac)$ and $\beta = (ab)$. This implies that $\alpha\beta = (ac)(ab) = (abc)$, so that $|\alpha\beta| = 3$. Note that $(ab) = (ba)$ and b, c are arbitrary, so that generality holds.

2. α and β both move distinct elements (i.e. they are disjoint), so that $\alpha = (ab)$ and $\beta = (cd)$. This implies that $\alpha\beta = (ab)(cd)$; because $\alpha\beta$ is the product of two disjoint 2-cycles, it follows that $|\alpha\beta| = \operatorname{lcm}(2, 2) = 2$

Thus $|\alpha\beta| = 3$ or $|\alpha\beta| = 2$.

□

Problem 7 (Chapter 5, Exercise 46)

Show that in S_7, the equation $x^2 = (1234)$ has no solutions but the equation $x^3 = (1234)$ has at least two.

Proof. Note that $(1234) = (14)(13)(12)$, so that it is an odd permutation. But x^2 is an even permutation regardless of the parity of x, and a permutation can never be both even and odd; thus (even) $= x^2 = (1234) =$ (odd) is impossible and thus has no solutions.
However, $x = (4321)$ satisfies $x^3 = (1234)$; this is because $x^{-1} = x^3$, since the order of x is 4, meaning $x^4 = \varepsilon$. Note that because disjoint cycles commute and the order of any 3-cycle is 3, it follows that $x = (4321)(567)$ also satisfies the equation; we have that $x^3 = (4321)(4321)(4321)(567)(567)(567) = (1234)\varepsilon = (1234)$. Thus, the equation $x^3 = (1234)$ has at least two solutions in S_7. $\square$

Problem 8 (Chapter 5, Exercise 53)

Show that A_5 has 24 elements of order 5, 20 elements of order 3, and 15 elements of order 2.

Proof. For the elements of order 5, note that the only form is $\sigma = (a_1a_2a_3a_4a_5)$; this has $\frac{5\cdot4\cdot3\cdot3\cdot2\cdot1}{5} = 24$ possibilities.
For the elements of order 3, note that the only form is $\sigma = (a_1a_2a_3)$, which has $\frac{5\cdot4\cdot3}{3} = 20$ possibilities.
For the elements of order 2, note that the only form is $\sigma = (a_1a_2)(a_3a_4)$; $\sigma = (a_1a_2)$ does not work because it is not in A_5. We have $\frac{5\cdot4\cdot3\cdot2}{2\cdot2\cdot2} = 15$ possibilities. $\square$

Problem 9 (Chapter 5, Exercise 58)

Show that for $n \geq 3$, $Z(S_n) = \{\varepsilon\}$.

Proof. We want to show that the only element of S_n that commutes with every element in S_n is ε; that is, for any non-identity element $\sigma \in S_n$, it follows that $\tau\sigma \neq \sigma\tau$ for some $\tau \in S_n$. Since the center always contains the identity, it suffices to show that any non-identity element cannot be in the center. We proceed by contradiction; suppose that there exists $\sigma \in Z(S_n), \sigma \neq \varepsilon$. Take distinct $i, j \in \{1, \cdots, n\}$ such that $\sigma(i) = j$. Now construct τ with the following property: take $k \in \{1, \cdots, n\}$ with $k \neq i, j$ such that $\tau(j) = k$ and τ fixes i; the existence of such a k follows from the stipulation that $n \geq 3$. It follows that $(\sigma\tau)(i) = \sigma(\tau(i)) = \sigma(i) = j$ and $(\tau\sigma)(i) = \tau(\sigma(i)) = \tau(j) = k$. Since j and k were chosen to be distinct, we have that $\sigma\tau \neq \tau\sigma$ for this choice of τ, contradicting the supposition that a non-identity σ existed in the center of S_n; thus $Z(S_n) = \{\varepsilon\}$. $\square$

Problem 10 (Chapter 5, Exercise 64)

Find five subgroups of S_5 of order 24.

Proof. We construct each subgroup by fixing an element, and permuting each of the other elements; this works because the order of S_4 is 24. To this end, let H_n be the subgroup obtained by fixing n and permuting all other elements with the actions of S_4:

1. $H_1 =$ actions of S_4 on the set $\{2, 3, 4, 5\}$
2. $H_2 =$ actions of S_4 on the set $\{1, 3, 4, 5\}$
3. $H_3 =$ actions of S_4 on the set $\{1, 2, 4, 5\}$
4. $H_4 =$ actions of S_4 on the set $\{1, 2, 3, 5\}$
5. $H_5 =$ actions of S_4 on the set $\{1, 2, 3, 4\}$

These are each indeed subgroups, because they are closed and contain inverses by virtue of S_4 being a group, and each $H_n \subseteq S_5$.

$\square$