HW 8 - MATH403

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Problem 1 (Chapter 10, Exercise 24)

Suppose that $\phi\colon \mathbb{Z}_{50} \to \mathbb{Z}_{15}$ is a group homomorphism with $\phi(7) = 6$.

a. Determine $\phi(x)$.

b. Determine the image of ϕ.

c. Determine the kernel of ϕ.

d. Determine $\phi^{-1}(3)$. That is, determine the set of all elements that map to 3.

Proof.

a. $\phi(7) = 6 \implies 7k \equiv 6 \mod 15 \implies k = 3 \implies \phi(x) = 3x \mod 15$

b. $\mathrm{Im}(\phi) = \{3x \in \mathbb{Z}_{15} \mid x \in \mathbb{Z}_{50}\} = \{0, 3, 6, 9, 12\}$

c. $\mathrm{Ker}(\phi) = \{x \in \mathbb{Z}_{50} \mid 3x \equiv 0 \mod 15\} = \{0, 5, 10, 15, 20, 25, 30, 35, 40, 45\}$

d. $\phi^{-1}(3) = \{x \in \mathbb{Z}_{50} \mid 3x \equiv 3 \mod 15\} = \{1, 6, 11, 16, 21, 26, 31, 36, 41, 46\}$

$\square$

Problem 2 (Chapter 10, Exercise 30)

Suppose that ϕ is a homomorphism from a group G onto $\mathbb{Z}_6 \oplus \mathbb{Z}_2$ and that the kernel of ϕ has order 5. Explain why G must have normal subgroups of orders 5, 10, 15, 20, 30, and 60.

Proof. Since $\mathbb{Z}_6 \oplus \mathbb{Z}_2$ is Abelian, it has normal subgroups of orders 1, 2, 3, 4, 6 and 12 by Lagrange's theorem. If a subgroup K is normal in $\mathbb{Z}_6 \oplus \mathbb{Z}_2$, it follows that $\phi^{-1}(K)$ is normal in G. Because $|\mathrm{Ker}(\phi)| = 5$, it follows that $|\phi^{-1}(K)| = 5|K|$, which means that the possible orders of normal subgroups of G are 5, 10, 15, 20, 30, and 60. $\square$

Problem 3 (Chapter 10, Exercise 36)

Suppose that there is a homomorphism ϕ from $\mathbb{Z} \oplus \mathbb{Z}$ to a group G such that $\phi((3,2)) = a$ and $\phi((2,1)) = b$. Determine $\phi((4,4))$ in terms of a and b. Assume that the operation of G is addition.

Proof. $\phi(1,1) = \phi((3,2) - (2,1)) = \phi(3,2) - \phi(2,1) = a - b$, so that $\phi(4,4) = 4\phi(1,1) = 4(a-b)$. □

Problem 4 (Chapter 10, Exercise 38)

Let α be a homomorphism from G_1 to H_1 and β be a homomorphism from G_2 to H_2. Determine the kernel of the homomorphism γ from $G_1 \oplus G_2$ to $H_1 \oplus H_2$ defined by $\gamma(g_1, g_2) = (\alpha(g_1), \beta(g_2))$.

Proof. We want all 2-tuples (g_1, g_2) such that $\alpha(g_1) = e_{h_1}$ and $\beta(g_2) = e_{h_2}$. Let x be an arbitrary member of $\mathrm{Ker}(\alpha)$ and y be an arbitrary member of $\mathrm{Ker}(\beta)$; then $\mathrm{Ker}(\gamma)$ is the set of all possible 2-tuples (x, y). □

Problem 5 (Chapter 10, Exercise 40)

For each pair of positive integers m and n, we can define a homomorphism from $\mathbb{Z}$ to $\mathbb{Z}_m \oplus \mathbb{Z}_n$ by $x \to (x \mod m, x \mod n)$. What is the kernel when $(m,n) = (3,4)$? What is the kernel when $(m,n) = (6,4)$? Generalize.

Proof. When $(m,n) = (3,4)$, $\mathrm{Ker}(\phi) = \langle 12 \rangle$ and when $(m,n) = (6,4)$, $\mathrm{Ker}(\phi) = \langle 12 \rangle$. We show that the kernel is $\langle \mathrm{lcm}\,(m,n) \rangle$. Indeed, if $x \in \mathrm{Ker}(\phi)$, then $x \equiv 0 \mod n$ and $x \equiv 0 \mod m$ so that x is a common multiple of both m and n. Conversely, suppose that $x \in \langle \mathrm{lcm}\,(m,n) \rangle$. Because $\mathrm{lcm}\,(m,n)|x$, it follows that $m|x$ and $n|x$ so that $\phi(x) = (0,0)$ and thus $x \in \mathrm{Ker}(\phi)$. □

Problem 6 (Chapter 10, Exercise 42)

(Third Isomorphism Theorem) If M and N are normal subgroups of G and $N \leq M$, prove that $(G/N)/(M/N) \cong G/M$. Think of this as a form of "cancelling out" the N in the numerator and denominator.

Proof. Define a mapping ϕ from G/N to G/M by $\phi(gN) = \phi(gM)$. This is well defined because $xN = yN \implies y^{-1}x \in N \leq M$ so that $y^{-1}x \in M$ and thus $xM = yM$. This is a homomorphism because $\phi(xN)\phi(yN) = xMyM = xyM = \phi(xyN) = \phi(xNyN)$. Because $|N| \leq |M|$, it follows that $|G/N| \geq |G/M|$, meaning that the map is surjective. Thus it follows that $\mathrm{Ker}(\phi) = \{gN \in G/N \mid gM = M\} = M/N$, and thus by the First Isomorphism Theorem $(G/N)/(M/N) \cong G/M$. □

Problem 7 (Chapter 10, Exercise 52)

Show that a homomorphism defined on a cyclic group is completely determined by its action on a generator of the group.

Proof. If g is a generator of G, then every element $x \in G$ has the form g^n, so that $\phi(x) = \phi(g^n) = \phi(g)^n$ by the homomorphism property; this implies that the homomorphism is completely determined by where it takes the generator. $\square$

Problem 8 (Chapter 9, Exercise 56)

Prove that the mapping from $\mathbb{R}$ under addition to $SL(2, \mathbb{R})$ that takes x to

$$\begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$$

is a group homomorphism. What is the kernel of the homomorphism?

Proof. Note that

$$\phi(x)\phi(y) = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \begin{bmatrix} \cos y & \sin y \\ -\sin y & \cos y \end{bmatrix}$$

$$= \begin{bmatrix} \cos x \cos y - \sin x \sin y & \cos x \sin y + \sin x \cos y \\ -\sin x \cos y - \cos x \sin y & -\sin x \sin y + \cos x \cos y \end{bmatrix}$$

$$= \begin{bmatrix} \cos x + y & \sin x + y \\ -\sin x + y & \cos x + y \end{bmatrix} = \phi(x + y)$$

so that the operation preserving property holds and thus the mapping is a homomorphism. The kernel is all angles that are a multiple of 2π because the identity is the identity matrix I_2 and the mapping is equivalent to rotating counterclockwise about the origin. $\square$

Problem 9 (Chapter 10, Exercise 62)

Determine all homomorphisms from $\mathbb{Z}$ onto S_3. Determine all homomorphisms from $\mathbb{Z}$ to S_3.

Proof. There is no homomorphism ϕ from $\mathbb{Z}$ onto S_3 because $\phi(\mathbb{Z})$ is Abelian and S_3 is not Abelian. There are six elements in S_3 and the homomorphisms are completely determined by $\phi(1)$, so that there are six homomorphisms $\square$

Problem 10 (Chapter 10, Exercise 66)

Let p be a prime. Determine the number of homomorphisms from $\mathbb{Z}_p \oplus \mathbb{Z}_p$ into $\mathbb{Z}_p$.

Proof. Note that the homomorphism is completely determined by $\phi(1,0)$ and $\phi(0,1)$ because those are the generators. Any element in $\mathbb{Z}_p$ has order p or 1, so that $\phi(1,0)$ and $\phi(0,1)$ can be any element in $\mathbb{Z}_p$; thus we deduce that there are p^2 homomorphisms.

$\square$