HW 8 - MATH411

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Problem 1 (Exercise 2, Page 427)

Define the function $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = x^3 + x + \cos x \qquad \text{for } x \text{ in } \mathbb{R}$$

At what points x in $\mathbb{R}$ does the Inverse Function Theorem apply? Prove that the function $f\colon \mathbb{R} \to \mathbb{R}$ is one-to-one and onto.

Proof. First, we will prove that $f'(x) = 3x^2 + 1 - \sin x$ is strictly positive and thus $f('x) > 0$ for all x in $\mathbb{R}$.
Case 1: $x = 0$
$f'(x) = 3(0)^2 + 1 - \sin 0 = 1 > 0$
Case 2: $x \neq 0$
$f'(x) = 3x^2 + 1 - \sin x \geq 3x^2 > 0$
Note that $f'(x)$ is continuous on $\mathbb{R}$ because it is the sum and product of continuous functions, and thus $f(x)$ is continuously differentiable. This means that the Inverse Function Theorem applies for all x in $\mathbb{R}$.
We claim $f\colon \mathbb{R} \to \mathbb{R}$ is one-to-one. To this end, let x and y in $\mathbb{R}$ be arbitrary; without loss of generality let $x < y$. By the Mean Value Theorem, for some $c \in (x, y)$:

$$\frac{f(y) - f(x)}{y - x} = f'(c) > 0$$

which implies that f is strictly increasing and thus one-to-one.
We claim $f\colon \mathbb{R} \to \mathbb{R}$ is onto. By the Intermediate Value Theorem, $f(\mathbb{R})$ is an interval. We claim that $f(\mathbb{R}) = \mathbb{R}$. It suffices to show that the image of $\mathbb{R}$ under f is unbounded in both directions. We claim

$$(1) \lim_{x \to \infty} f(x) = \infty \qquad \text{and} \qquad (2) \lim_{x \to -\infty} f(x) = -\infty$$

To prove (1), let $M > 0$ be arbitrary and take $N = \max\{\sqrt[3]{M}, 1\}$. Then for all $x > N$

$$f(x) = x^3 + x + \cos x > x^3 > M$$

To prove (2), let $M > 0$ be arbitrary and take $N = \max\{\sqrt[3]{M}, 1\}$. Then for all $x < -N$

$$f(x) = x^3 + x + \cos x < x^3 < -M$$

so that $f(\mathbb{R}) = \mathbb{R}$; thus f is onto. $\square$

Problem 2 (Exercise 5, Page 428)

Suppose that the continuously differentiable function $f\colon \mathbb{R} \to \mathbb{R}$ has the property that there is some positive number c such that

$$f'(x) \geq c \qquad \text{for every } x \text{ in } \mathbb{R}$$

Show that the function $f\colon \mathbb{R} \to \mathbb{R}$ is both one-to-one and onto.

Proof. We claim $f\colon \mathbb{R} \to \mathbb{R}$ is one-to-one. To this end, let x and y in $\mathbb{R}$ be arbitrary; without loss of generality let $x < y$. By the Mean Value Theorem, for some $d \in (x, y)$:

$$\frac{f(y) - f(x)}{y - x} = f'(d) \geq c > 0$$

which implies that f is strictly increasing and thus one-to-one.
We claim $f\colon \mathbb{R} \to \mathbb{R}$ is onto. By the Intermediate Value Theorem, $f(\mathbb{R})$ is an interval. We claim that $f(\mathbb{R}) = \mathbb{R}$. It suffices to show that the image of $\mathbb{R}$ under f is unbounded in both directions. We claim

$$(1) \lim_{x\to\infty} f(x) = \infty \qquad \text{and} \qquad (2) \lim_{x\to-\infty} f(x) = -\infty$$

Let $x \in \mathbb{R}$ be arbitrary.
Case 1: $x > 0$
By the Mean Value Theorem, for some $x_0 \in (0, x)$

$$\frac{f(x) - f(0)}{x - 0} = f'(x_0) \geq c \implies f(x) \geq cx + f(0)$$

This argument proves (1).
Case 2: $x < 0$
By the Mean Value Theorem, for some $x_0 \in (x, 0)$

$$\frac{f(x) - f(0)}{x - 0} = f'(x_0) \geq c \implies f(x) \leq cx + f(0)$$

This argument proves (2); thus f is onto. $\square$

Problem 3 (Exercise 8, Page 428)

For each of the following mappings $\mathbf{F}\colon \mathbb{R}^2 \to \mathbb{R}^2$, apply the Inverse Function Theorem at the point $(x_0, y_0) = (0, 0)$ and calculate the partial derivatives of the components of the inverse mapping at the point $(u_0, v_0) = \mathbf{F}(0, 0)$:

a. $\mathbf{F}(x, y) = (x + x^2 + e^{x^2y^2}, -x + y + \sin(xy))$ for (x, y) in $\mathbb{R}^2$

b. $\mathbf{F}(x, y) = (e^{x+y}, e^{x-y})$ for (x, y) in $\mathbb{R}^2$

Proof. Each of the mappings is continuously differentiable because each of the components is continuously differentiable:

a. $\mathbf{DF}(x,y) = \begin{bmatrix} 1+2x+2xy^2e^{x^2y^2} & 2yx^2e^{x^2y^2} \\ -1+y\cos(xy) & 1+x\cos(xy) \end{bmatrix}$
$J(0,0) = \det\mathbf{DF}(0,0) = 1 \neq 0$, so the Inverse Function Theorem applies and the partial derivatives of the components of the inverse mapping $\mathbf{F}^{-1} = (g(u,v), h(u,v))$ are the entries of the inverse matrix, namely:

$$\frac{\partial g}{\partial u}(u_0,v_0) = 1 \qquad \frac{\partial g}{\partial v}(u_0,v_0) = 0$$
$$\frac{\partial h}{\partial u}(u_0,v_0) = 1 \qquad \frac{\partial h}{\partial v}(u_0,v_0) = 1$$

b. $\mathbf{DF}(x,y) = \begin{bmatrix} e^{x+y} & e^{x+y} \\ e^{x-y} & -e^{x-y} \end{bmatrix}$
$J(0,0) = \det\mathbf{DF}(0,0) = -2 \neq 0$, so the Inverse Function Theorem applies and the partial derivatives of the components of the inverse mapping $\mathbf{F}^{-1} = (g(u,v), h(u,v))$ are the entries of the inverse matrix, namely:

$$\frac{\partial g}{\partial u}(u_0,v_0) = \frac{1}{2} \qquad \frac{\partial g}{\partial v}(u_0,v_0) = \frac{1}{2}$$
$$\frac{\partial h}{\partial u}(u_0,v_0) = \frac{1}{2} \qquad \frac{\partial h}{\partial v}(u_0,v_0) = -\frac{1}{2}$$

□

Problem 4 (Exercise 11, Page 428)

For a pair of real numbers a and b, consider the system of nonlinear equations

$$x + x^2\cos y + xye^{x^3y^2} = a$$
$$y + x^5 + y^3 - x^2\cos(xy) = b$$

Use the Inverse Function Theorem to show that there is some positive number r such that if $a^2 + b^2 < r^2$, then this system of equations has at least one solution.

Proof. Note that $(0,0)$ is a solution to the system of equations when $a, b = 0$. Define the mapping $\mathbf{F}\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $\mathbf{F}(x,y) = (x + x^2\cos y + xye^{x^3y^2}, y + x^5 + y^3 - x^2\cos(xy))$. The mapping is clearly continuously differentiable as its component functions are continuously differentiable; note that

$$\mathbf{DF}(x,y) = \begin{bmatrix} 1+2x\cos y + ye^{x^2y^2} + 3x^3y^3e^{x^3y^2} & -x^2\sin y + xe^{x^3y^2} + 2x^4y^2e^{x^3y^2} \\ 5x^4 - 2x\cos xy + yx^2\sin xy & 1+3y^2 - x^3\sin xy \end{bmatrix}$$

Because $\mathbf{DF}(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is invertible, the Inverse Function Theorem applies. This means that there is a neighborhood U of $(0,0)$ and a neighborhood V of $(0,0)$ such that $F\colon U \to V$ is bijective. Because V is open, there exists $r > 0$ such that $B_r(0,0) \subset V$. Thus, if $a^2 + b^2 < r^2$, then $(a,b) \in B_r(0,0)$. Thus $textbf F^{-1}(a,b)$ is a solution to the system of equations. □

Problem 5 (Exercise 13, Page 429)

Let the continuously differentiable mapping $\mathbf{F}\colon \mathbb{R}^2 \to \mathbb{R}^2$ be represented in component functions by $\mathbf{F}(x, y) = (\psi(x, y), \varphi(x, y))$ for (x, y) in $\mathbb{R}^2$. Suppose that the point (x_0, y_0) in $\mathbb{R}^2$ has the property that

$$\psi(x, y) \geq \psi(x_0, y_0) \qquad \text{for all } (x, y) \text{ in } \mathbb{R}^2$$

a. Explain analytically why the hypotheses of the Inverse Function Theorem cannot hold at (x_0, y_0).

b. Explain geometrically why the conclusion of the Inverse Function Theorem cannot hold at (x_0, y_0).

Proof.

a. The fact that $\psi(x, y) \geq \psi(x_0, y_0)$ for all (x, y) in $\mathbb{R}^2$ implies that (x_0, y_0) is a minimizer for $\psi\colon \mathbb{R}^2 \to \mathbb{R}$. This means that $\nabla\psi(x_0, y_0) = (0, 0)$, so that

$$\mathbf{DF}(x_0, y_0) = \begin{bmatrix} 0 & 0 \\ \frac{\partial \phi}{\partial x}(x_0, y_0) & \frac{\partial \phi}{\partial y}(x_0, y_0) \end{bmatrix}$$

and

$$J(x_0, y_0) = \det\mathbf{DF}(x_0, y_0) = 0(\frac{\partial \phi}{\partial y}(x_0, y_0)) - 0(\frac{\partial \phi}{\partial x}(x_0, y_0)) = 0$$

The determinant of the derivative matrix is equal to 0 at the point (x_0, y_0), meaning that $\mathbf{DF}(x_0, y_0)$ is not invertible, so the hypotheses of the Inverse Function Theorem cannot hold at (x_0, y_0).

b. The point $\mathbf{F}(x_0, y_0)$ is a boundary point of the image of F because every open ball will contain a point inside and outside the image. This implies that for any neighborhood V of $\mathbf{F}(x_0, y_0)$, there cannot exist a neighborhood U of (x_0, y_0) such that $\mathbf{F}\colon U \to V$ is a bijective mapping, because it cannot be onto. Thus the conclusion of the Inverse Function Theorem cannot hold at the point (x_0, y_0).

$\square$