HW 11 - MATH411

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Problem 1 (Exercise 5, Page 481)

Let $\mathbf{I}$ be a generalized rectangle in $\mathbb{R}^n$ and suppose that the function $f\colon \mathbf{I} \to \mathbb{R}$ is integrable. Assume that $f(\mathbf{x}) \geq 0$ if $\mathbf{x}$ is a point in $\mathbf{I}$ with a rational component. Prove that $\int_{\mathbf{I}} f \geq 0$.

Proof. Because f is integrable, there exists an Archmedean sequence of partitions $\mathbf{P}_k$ of $\mathbf{I}$. Because the rationals are dense in $\mathbb{R}$, it follows that for each rectangle $\mathbf{J}$, $M(f, \mathbf{J}) \geq 0$. Thus, it follows that the upper Darboux sum $U(f, \mathbf{P}_k) \geq 0$. The fact that the function is integrable means that we can take the limit as k tends to ∞, resulting in

$$\int_{\mathbf{I}} f = \lim_{k\to\infty} U(f, \mathbf{P}_k) \geq 0$$

□

Problem 2 (Exercise 10, Page 482)

Let $\mathbf{I}$ be a generalized rectangle in $\mathbb{R}^2$ and suppose that the bounded function $f\colon \mathbf{I} \to \mathbb{R}$ has the value 0 on the interior of $\mathbf{I}$. Show that $f\colon \mathbf{I} \to \mathbb{R}$ is integrable and that $\int_{\mathbf{I}} f = 0$. Is the same result true for a generalized rectangle $\mathbf{I}$ in $\mathbb{R}^n$?

Proof. We prove the general case. Take the regular partition $\mathbf{P}_k$ of the generalized rectangle into k intervals in each component. Because f is bounded, take $M > 0$ such that $|f| < M$. Observe that all rectangles except two in each dimension will not share an edge with the boundary of $\mathbf{I}$, meaning that there are $(k-2)^n$ rectangles that do not share any edge with the boundary. Thus, the number of rectangles that have nonzero values is equal to $k^n - (k-2)^n$. Thus it follows that

$$0 \leq |U(f, \mathbf{P}_k) - L(f, \mathbf{P}_k)| \leq \sum_{\mathbf{J}\in\mathbf{P}_k} |M_{\mathbf{J}} - m_{\mathbf{J}}| \mathrm{vol}\mathbf{J} \leq \frac{2M(k^n - (k-2)^n)}{k^n}$$

$$\frac{2M(k^n - (k-2)^n)}{k^n} = 2M\left[1 - (1 - \frac{2}{k})^n\right] \to 0 \qquad \text{as } k \to \infty$$

Also note that

$$|U(f, \mathbf{P}_k)| \leq \sum_{\mathbf{J} \in \mathbf{P}_k} |M_\mathbf{J}| \text{vol} \mathbf{J} \leq M \frac{k^n - (k-2)^n}{k^n} \to 0 \qquad \text{as } k \to \infty$$

which means that by triangle inequality

$$|U(f, \mathbf{P}_k) - L(f, \mathbf{P}_k)| \leq |U(f, \mathbf{P}_k)| + |L(f, \mathbf{P}_k)| \to 0 \qquad \text{as } k \to \infty$$

meaning that

$$|L(f, \mathbf{P}_k)| \to 0 \qquad \text{as } k \to \infty$$

Because the limit of the upper and lower Darboux sums are equal, it follows from the definition of the Riemann integral that f is integrable, and

$$\int_\mathbf{I} f = \lim_{k \to \infty} U(f, \mathbf{P}_k) = 0$$

□

Problem 3 (Exercise 7, Page 482)

For the generalized rectangle $\mathbf{I} = [0,1] \times [0,1]$ in the plane $\mathbb{R}^2$, define

$$f(x,y) = \begin{cases} 5 & \text{if } (x,y) \text{ is in } \mathbf{I} \text{ and } x > 1/2 \\ 1 & \text{if } (x,y) \text{ is in } \mathbf{I} \text{ and } x \leq 1/2 \end{cases}$$

Use the Archimedes-Riemann Theorem to show that the function $f \colon \mathbf{I} \to \mathbb{R}$ is integrable.

Proof. Let $P_k = [(0, 1/2 - 1/3k, 1/2 + 1/3k, 1] \times [0,1]$. We have that $U(f, P_k) = (1/2 - 1/3k) + 5(2/3k) + 5(1/2 - 1/3k)$ and $L(f, P_k) = (1/2 - 1/3k) + (2/3k) + 5(1/2 - 1/3k)$. Thus $U(f, P_k) - L(f, P_k) = 1/2k \to 0$, thus by the Archimedes Riemann theorem f is integrable. □

Problem 4 (Exercise 11, Page 482)

For the rectangle $\mathbf{I} = [0,1] \times [0,1]$ in the plane $\mathbb{R}^2$, define the function $f \colon \mathbf{I} \to \mathbb{R}$ by

$$f(x,y) = xy \qquad \text{for } (x,y) \text{ in } \mathbf{I}$$

Use the Archimedes-Riemann Theorem to evaluate $\int_\mathbf{I} f$.

Proof. Let $\mathbf{P}_k$ be the regular partition of $[0,1] \times [0,1]$. Each rectangle in the partition has area $1/k^2$ and for indices i and j between 1 and k, we set

$$\mathbf{J} = \left[\frac{i-1}{k}, \frac{i}{k}\right] \times \left[\frac{j-1}{k}, \frac{j}{k}\right]$$

then

$$m(f, \mathbf{J}) = \frac{(i-1)(j-1)}{k^2} \quad \text{and} \quad M(f, \mathbf{J}) = \frac{ij}{k^2}.$$

It follows that

$$U(f, \mathbf{P}_k) = \sum_{\mathbf{J} \in \mathbf{P}_k} M(f, \mathbf{J}) \operatorname{vol} \mathbf{J} = \sum_{1 \le i,j \le k} \frac{ij}{k^4} = \frac{1}{k^4} \sum_{i=1}^{k} i \left[\sum_{j=1}^{k} j \right] = \frac{1}{k^4} \left[\frac{k(k+1)}{2} \right]^2$$

Similarly, we note that

$$L(f, \mathbf{P}_k) = \frac{1}{k^4} \left[\frac{k(k-1)}{2} \right]^2.$$

We observe that

$$\lim_{k \to \infty} U(f, \mathbf{P}_k) = 1/4 \qquad \text{and} \qquad \lim_{k \to \infty} L(f, \mathbf{P}_k) = 1/4.$$

Thus we deduce that $\{\mathbf{P}_k\}$ is an Archimedean sequence and it follows from the Archimedes-Riemann Theoerem that f is integrable and

$$\int_{\mathbf{I}} f = 1/4.$$

$\square$

Problem 5 (Exercise 8, Page 489)

Let $\{\mathbf{u}_k\}$ be a convergent sequence in $\mathbb{R}^n$. Show that the set $\{\mathbf{u}_k \mid k \in \mathbb{N}\}$ has Jordan content 0.

Proof. Suppose that $\{\mathbf{u}_k\}$ converges to $\mathbf{u}$. Let $\varepsilon > 0$ be arbitrary and take $r > 0$ such that $r < \left(\frac{\varepsilon}{2}\right)^n$. By the convergence of $\{\mathbf{u}_k\}$, there exists K such that for all $k \ge K$ we have that $\mathbf{u}_k \in B_{r/2}(\mathbf{u})$. We can cover this open ball with the rectangle

$$R_{K+1} = \left[u_1 - \frac{r}{2}, u_1 + \frac{r}{2}\right] \times \cdots \times \left[u_n - \frac{r}{2}, u_n + \frac{r}{2}\right]$$

where $u = (u_1, \cdots, u_n)$ in components. It follows that $B_{r/2}(\mathbf{u}) \subset R_{K+1}$, so that this rectangle covers all of the elements after index K. For the finitely many elements with index $1 \le i \le K$, define R_i to be the rectangle of side length $\frac{r}{K^{\frac{1}{n}}}$ with each u_i in the center of R_i. Now let $\mathcal{F}$ be the collection of rectangles $\{R_1, \cdots, R_{K+1}\}$. This covers the set $\{\mathbf{u}_k \mid k \in \mathbb{N}\}$ and the sum of volumes is

$$\sum_{i=1}^{K+1} \operatorname{vol} \mathbf{R}_i = K \cdot \left(\frac{r}{K^{\frac{1}{n}}}\right)^n + r^n = 2r^n < \varepsilon$$

$\square$

Problem 6 (Exercise 10, Page 489)

Let $\mathbf{I}$ be a generalized rectangle in $\mathbb{R}^n$ and suppose that the function $f\colon \mathbf{I} \to \mathbb{R}$ is continuous. Assume that $f(\mathbf{x}) \geq 0$ for all points $\mathbf{x}$ in $\mathbf{I}$. Prove that if $\int_{\mathbf{I}} f = 0$, then $f(\mathbf{x}) = 0$ for all points $\mathbf{x}$ in $\mathbf{I}$. Is continuity necessary for this to hold?

Proof. Suppose not; that is, there exists a point $\mathbf{x}_0$ such that $f(\mathbf{x}_0) > 0$. By the continuity of f, there exists an open ball that contains a rectangle $\mathbf{J}$ on which $f(\mathbf{x}) > 0$ for all $x \in \mathbf{J}$. Because J is a rectangle, it is compact and thus attains its minimum value $m > 0$. The continuity of f implies that it is integrable, so that there exists a sequence of Archimedean partitions such that

$$\int_{\mathbf{I}} f = \lim_{k\to\infty} L(f, \mathbf{P}_k) = \lim_{k\to\infty} \sum_{\mathbf{R}\in\mathbf{P}_k} m_{\mathbf{R}} \mathrm{vol}\mathbf{R} \geq \lim_{k\to\infty} \sum_{\mathbf{R}\in\mathbf{J}} m_{\mathbf{R}} \mathrm{vol}\mathbf{R}$$

$$\geq \lim_{k\to\infty} \sum_{\mathbf{R}\in\mathbf{J}} m \mathrm{vol}\mathbf{R} = \lim_{k\to\infty} m \mathrm{vol}\mathbf{J} = m \mathrm{vol}\mathbf{J} > 0$$

But this contradicts the assumption that the integral of f is 0.
The condition that f be continuous is necessary; consider a function $f\colon \mathbf{I} \to \mathbb{R}$ such that $f(\mathbf{x}) = 0$ for every $\mathbf{x} \in \mathbf{I}$ except for a single $\mathbf{x}_0$ at which $f(\mathbf{x}_0) = 1$. This function is integrable because the single point at which the function is discontinuous has Jordan content 0, and the integral is equal to 0. $\square$