HW 10 - MATH411

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Problem 1 (Exercise 4, Page 447)

Consider the equation

$$e^{2x-y} + \cos(x^2 + xy) - 2 - 2y = 0 \qquad (x, y) \text{ in } \mathbb{R}^2$$

Does the set of solutions of this equation in a neighborhood of the solution $(0, 0)$ implicitly define one of the components of the point (x, y) as a function of the other component? If so, compute the derivative of this function (these functions?) at the point 0.

Proof. Yes. Let $f(x, y) = e^{2x-y} + \cos(x^2 + xy) - 2 - 2y$; then because f is a composition, sum and product of continuously differentiable functions, it follows that f is continuously differentiable. Notice that $f(0, 0) = 0$, and

$$\nabla f(x, y) = (2e^{2x-y} - (2x + y)\sin(x^2 + xy), -e^{2x-y} - x\sin(x^2 + xy) - 2)$$

and

$$\nabla f(0, 0) = (2, -3) \neq (0, 0)$$

so that by Dini's Theorem, there exists a neighborhood of $(0, 0)$ such that one of the components is implicitly defined by the other component (and vice versa). To calculate the derivative, use the following formulas:

$$\frac{\partial f}{\partial x}(x, g(x)) + \frac{\partial f}{\partial y}(x, g(x))g'(x) = 0$$

and

$$\frac{\partial f}{\partial x}(f(y), y)f'(y) + \frac{\partial f}{\partial y}(f(y), y)) = 0$$

which yields that

$$g'(0) = \frac{2}{3} \qquad \text{and} \qquad f'(0) = \frac{3}{2}$$

□

Problem 2 (Exercise 7, Page 447)

Let $\mathcal{O}$ be an open subset of the plane and suppose that the function $f\colon \mathcal{O} \to \mathbb{R}$ is continuously differentiable. At the point (x_0, y_0) in $\mathcal{O}$, suppose that $f(x_0, y_0) = 0$ and that $\nabla f(x_0, y_0) \neq (0, 0)$. Show that the vector $\nabla f(x_0, y_0)$ is orthogonal to the tangent line at (x_0, y_0) of the implicitly defined function.

Proof. Dini's theorem implies that

$$\nabla f(x_0, y_0) = \left(\frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0) \right)$$

and

$$\frac{\partial f}{\partial x}(x_0, g(x_0)) + \frac{\partial f}{\partial y}(x_0, g(x_0)) g'(x_0) = 0$$

so that in a neighborhood of (x_0, y_0), there is an implicitly defined function defined by $y = g(x)$. Rewriting the above expression yields that

$$\langle \nabla f(x_0, y_0), (1, g'(x_0)) \rangle = 0$$

But $(1, g'(x_0)$ is the tangent line at (x_0, y_0) by definition of the implicitly defined function. $\square$

Problem 3 (Exercise 8, Page 447)

Let $\mathcal{O}$ be an open subset of the plane and suppose that the function $f\colon \mathcal{O} \to \mathbb{R}$ is continuously differentiable. At the point (x_0, y_0) in $\mathcal{O}$, suppose that $f(x_0, y_0) = 0$ and that

$$\frac{\partial f}{\partial x}(x_0, y_0) \neq 0, \qquad \frac{\partial f}{\partial y}(x_0, y_0) \neq 0.$$

Show that the two functions implicitly defined by Dini's theorem, when their domains are properly chosen, are inverses of each other.

Proof. Suppose the two functions are $f(y)$ and $g(x)$. Dini's theorem implies that when the domains are chosen such that $|x - x_0| \leq r$ and $|x - x_0| \leq r$ for some $r > 0$ when $f(x_0, y_0) = 0$, it follows that $y = g(x)$. Then, for all $x \in B_r(x_0)$ and $y \in B_r(y_0)$ it follows that

$$f(g(x)) = f(y) = x$$

meaning that f and g are inverses of each other. $\square$

Problem 4 (Exercise 14, Page 448)

In addition to the assumptions of Dini's Theorem, assume also that the function $f\colon \mathcal{O} \to \mathbb{R}$ has continuous second-order partial derivatives.

a. Verify formula (17.13).

b. Moreover, suppose that

$$\frac{\partial f}{\partial x}(x_0, y_0) = 0 \qquad \text{and} \qquad \frac{\partial f}{\partial y}(x_0, y_0)\frac{\partial^2 f}{\partial x^2}(x_0, y_0) > 0.$$

Prove that the graph of $g\colon I \to \mathbb{R}$ lies below the line $y = y_0$ if I is chosen sufficiently small.

Proof. a. Note that Dini's theorem states that

$$\frac{\partial f}{\partial x}(x, g(x)) + \frac{\partial f}{\partial y}(x, g(x))g'(x) = 0$$

Because f has continuous second order partial derivatives, we can differentiate the expression again; differentiating the first term yields

$$\frac{\partial^2 f}{\partial x^2}(x, g(x)) + \frac{\partial^2 f}{\partial x \partial y}(x, g(x))g'(x)$$

and differentiating the second term yields

$$\frac{\partial^2 f}{\partial x \partial y}(x, g(x))g'(x) + \frac{\partial^2 f}{\partial y^2}(x, g(x))g'(x) \cdot g'(x) + \frac{\partial^2 f}{\partial y^2}(x, g(x))g''(x)$$

Combining these two yields the desired formula

$$\frac{\partial^2 f}{\partial x^2}(x, g(x)) + 2\frac{\partial^2 f}{\partial x \partial y}(x, g(x))g'(x) + \frac{\partial^2 f}{\partial y^2}(x, g(x))[g'(x)]^2 + \frac{\partial f}{\partial y}(x, g(x))g''(x) = 0$$

b. Formula (17.13) implies that

$$g''(x_0) = \frac{-\frac{\partial^2 f}{\partial x^2}(x_0, g(x_0)) + 2\frac{\partial^2 f}{\partial x \partial y}(x_0, g(x_0))g'(x_0) + \frac{\partial^2 f}{\partial y^2}(x_0, g(x_0))[g'(x_0)]^2}{\frac{\partial f}{\partial y}(x_0, g(x_0))}$$

Notice that when $|y - y_0| < r$ for the $r > 0$ as defined by Dini's theorem, we have that for the sufficiently small interval

$$g''(x_0) = \frac{-\frac{\partial^2 f}{\partial x^2}(x_0, y_0) + 2\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0)g'(x_0) + \frac{\partial^2 f}{\partial y^2}(x_0, y_0)[g'(x_0)]^2}{\frac{\partial f}{\partial y}(x_0, y_0)}$$

thus demonstrating that the graph lies below the line $y = y_0$.

$\square$

For the following exercises, use the Implicit Function Theorem to analyze the solutions of the given systems of equations near the solution $\mathbf{0}$.

Problem 5 (Exercise 2, Page 453)

$$\begin{cases} a^3 + a^2 b + \sin(a+b+c) = 0 \\ \ln(1+a^2) + 2a + (bc)^4 = 0 \qquad (a,b,c) \text{ in } \mathbb{R}^3 \end{cases}$$

Proof. Observe that $(0,0,0)$ is a solution, and $\mathbb{R}^3$ is open. Also, note that $F(a,b,c) = (a^3 + a^2 b + \sin(a+b+c), \ln(1+a^2) + 2a + (bc)^4)$ is a continuously differentiable mapping by the continuous differentiability of the component functions. Notice that the derivative matrix is

$$DF = \begin{bmatrix} 3a^2 + 2a\cos(a+b+c) & a^2 + \cos(a+b+c) & \cos(a+b+c) \\ \frac{2a}{1+a^2} + 2 & 4(bc)^3 \cdot c & 4(bc)^3 \cdot b \end{bmatrix}$$

and $DF(0) = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 0 \end{bmatrix}$, meaning that the derivative matrix whose components are for a and b is equal to $\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$ and is invertible. Thus, the implicit function theorem allows us to select $r > 0$ such that $(f(c), g(c), c)$ is a solution of the system of equations if $c < r$ and $a^2 + b^2 < r^2$, and f and g are continuously differentiable functions. If $(a,b,c) \in \mathbb{R}^3$ is a solution of the system and $a^2 + b^2 < r^2$ and $c < r$, then $a = f(c)$ and $b = g(c)$. $\square$

Problem 6 (Exercise 3, Page 454)

$$\begin{cases} (uv)^4 + (u+s)^3 + t = 0 \\ \sin(uv) + e^{v+t^2} - 1 = 0 \qquad (u,v,s,t) \text{ in } \mathbb{R}^4 \end{cases}$$

Proof. Observe that $(0,0,0,0)$ is a solution, and $\mathbb{R}^4$ is open. Also, note that $F(u,v,s,t) = ((uv)^4 + (u+s)^3 + t, \sin(uv) + e^{v+t^2} - 1)$ is a continuously differentiable mapping by the continuous differentiability of the component functions. Notice that the derivative matrix is

$$DF = \begin{bmatrix} 2(uv) \cdot v + 3(u+s)^2 & 2(uv) \cdot u & 3(u+s)^2 & 1 \\ v\cos(uv) & u\cos(uv) + e^{v+t^2} & 0 & 2t \cdot e^{v+t^2} \end{bmatrix}$$

and $DF(0) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, meaning that the derivative matrix whose components are for v and t is equal to $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and is invertible. Thus, the implicit function theorem allows us to select $r > 0$ such that $(u, f(u,s), s, g(u,s))$ is a solution of the system of equations if $u^2 + s^2 < r^2$, and f and g are continuously differentiable functions. If $(u,v,s,t) \in \mathbb{R}^4$ is a solution of the system and $u^2 + s^2 < r^2$ and $v^2 + t^2 < r^2$, then $v = f(u,s)$ and $t = g(u,s)$. $\square$

Problem 7 (Exercise 5, Page 454)

$e^{x^2} + y^2 + z - 4xy^3 - 1 = 0 \qquad (x, y, z)$ in $\mathbb{R}^3$

Proof. Note that $(0, 0, 0)$ is a solution to the given system. Also note that $\mathbb{R}^3$ is open. Also, $f(x, y, z) = e^{x^2} + y^2 + z - 4xy^3 - 1$ is continuously differentiable by the composition, sum and products of continuously differentiable functions. The derivative matrix is

$$DF = \begin{bmatrix} 2x \cdot e^{x^2} - 4y^3 & 2y - 12xy^2 & 1 \end{bmatrix}$$

and $DF(0) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$. Because the last component is nonzero, the implicit function theorem allows us to select $r > 0$ such that if $x^2 + y^2 < r^2$, then $(x, y, g(x, y))$ is a solution to the system of equations. If $(x, y, z) \in \mathbb{R}^3$ is a solution and $z < r$, then $z = g(x, y)$. $\square$