HW 6 - MATH411

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March 7, 2022

Problem 1 (Exercise 3, Page 370)

Suppose that the functions $f\colon \mathbb{R}^n \to \mathbb{R}$ and $g\colon \mathbb{R}^n \to \mathbb{R}$ are continuously differentiable. Find a formula for $\nabla(g \circ f)(x)$ in terms of $\nabla f(x)$ and $g'(f(x))$.

Proof. By definition of the partial derivative and the mean value theorem, we have:

$$\frac{\partial}{\partial x_i}(g \circ f)(x) = \lim_{t \to 0} \frac{g(f(x + te_i)) - g(f(x))}{t} = \lim_{t \to 0} g'(c_t)\frac{f(x + te_i) - f(x)}{t}$$

where c_t is a point strictly between $f(x)$ and $f(x+te_i)$. Because g is continuously differentiable, we have $\lim_{t\to 0} g'(c_t) = g'(f(x))$ and we also have $\lim_{t\to 0} c_t = f(x)$. Notice further that because f is also continuously differentiable, we have that:

$$\frac{\partial f}{\partial x_i} = \lim_{t \to 0} \frac{f(x + te_i) - f(x)}{t}$$

By rules for products of limits, we have that for each i:

$$\frac{\partial}{\partial x_i}(g \circ f)(x) = g'(f(x))\frac{\partial f}{\partial x_i}$$

which implies that

$$\nabla(g \circ f)(x) = g'(f(x)) \cdot \nabla f(x)$$

□

Problem 2 (Exercise 6, Page 370)

Define the function $f\colon \mathbb{R}^3 \to \mathbb{R}$ by

$$f(x, y, z) = xyz + x^2 + y^2 \qquad \text{for } (x, y, z) \text{ in } \mathbb{R}^3.$$

The Mean Value Theorem implies that there is a number θ with $0 < \theta < 1$ for which

$$f(1, 1, 1) - f(0, 0, 0) = \frac{\partial f}{\partial x}(\theta, \theta, \theta) + \frac{\partial f}{\partial y}(\theta, \theta, \theta) + \frac{\partial f}{\partial z}(\theta, \theta, \theta).$$

Find such a value of θ.

Proof. Notice that $f(1,1,1) = 3$ and $f(0,0,0) = 0$. Also notice that $\frac{\partial f}{\partial x} = yz + 2x$, $\frac{\partial f}{\partial y} = xz + 2y$, and $\frac{\partial f}{\partial z} = xy$. Thus our problem reduces to solving the following:

$$3 = (\theta^2 + 2\theta) + (\theta^2 + 2\theta) + (\theta^2) = 3\theta^2 + 4\theta \implies 3\theta^3 + 4\theta - 3 = 0$$

Solving this with the quadratic formula and taking the solution in the interval $(0,1)$ yields $\theta = -\frac{2}{3} + \frac{\sqrt{13}}{3}$. $\square$

Problem 3 (Exercise 7, Page 371)

Suppose that the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ has first-order partial derivatives and that $f(0,0) = 1$, while

$$\frac{\partial f}{\partial x}(x,y) = 2 \qquad \text{and} \qquad \frac{\partial f}{\partial y}(x,y) = 3 \qquad \text{for all } (x,y) \text{ in } \mathbb{R}^2$$

Prove that

$$f(x,y) = 1 + 2x + 3y \qquad \text{for all } (x,y) \text{ in } \mathbb{R}^2$$

Proof. We proceed by showing that if g is another function that satisfies the conditions, then g must necessarily be f. To show this, we prove that if a function's gradient is zero, then the function is constant, and we proceed by induction. The base case is the identity criterion from analysis of a single variable. Now assume all functions $g\colon \mathbb{R}^n \to \mathbb{R}$ that have first-order partial derivatives and a zero gradient are constant. We will show that all functions $h\colon \mathbb{R}^{n+1} \to \mathbb{R}$ that have first-order partial derivatives and a zero gradient are constant as well. Set $g_x\colon \mathbb{R}^n \to \mathbb{R}$ to be the function from $(v_1, \cdots, v_n)$ to $f(v_1, \cdots, v_n, x)$. Notice that

$$\nabla g_x = \left(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}\right) = 0$$

Thus by the inductive hypothesis, there exists $c_x \in \mathbb{R}$ such that $g(v) = c_x$ for all $v \in \mathbb{R}^n$. Define a mapping i such that $f(x_1, \cdots, x_{n+1} = i(x_{n+1})$. Then we have that $i'(x) = \frac{\partial f}{\partial x_{n+1}} = 0$, so that i and thus f are not constant.
Now suppose g is a function that satisfies the conditions as well. Notice that $\nabla g = \nabla f$, so that for a function $h = f - g$ it follows that $\nabla h = 0$, so that h is constant. Notice further that $h(0,0) = f(0,0) - g(0,0) = 0$, so that $f = g$, and thus $f(x,y) = 1 + 2x + 3y$ is the only possible function. $\square$

Problem 4 (Exercise 9, Page 371)

Define the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} (x/|y|)\sqrt{x^2 + y^2} & \text{if } y \neq 0 \\ 0 & \text{if } y = 0 \end{cases}$$

a. Prove that the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ is not continuous at the point $(0,0)$.

b. Prove that the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ has directional derivatives in all directions at the point $(0,0)$.

c. Prove that if c is any number, then there is a vector p of norm 1 such that

$$\frac{\partial f}{\partial p}(0,0) = c.$$

d. Does (c) contradict Corollary 13.18?

Proof.

a. Notice that the sequence $\{(1/k, 1/k^2)\} \to (0,0)$ but $\{f(1/k, 1/k^2)\} = k\sqrt{1+k^2}$ which does not converge.

b. Let $p \in \mathbb{R}^2$ be arbitrary. Notice that f is homogeneous; that is that $f(tx, ty) = tf(x,y)$ for nonzero t. This means that

$$\frac{\partial f}{\partial p}(0,0) \lim_{t\to 0} \frac{f(tp)}{t} = \lim_{t\to 0} \frac{tf(p)}{t} = \lim_{t\to 0} f(p) = f(p)$$

so that the directional derivative exists at the origin for any vector in $\mathbb{R}^2$.

c. Let $u = (c, 1)$ with $c \in \mathbb{R}$. Let $p = u/||u||$, so that p is a unit vector. Then we have that

$$\frac{\partial f}{\partial p}(0,0) = f(p) = f(u/||u||) = \frac{f(u)}{||u||} = \frac{c||u||}{||u||} = c$$

d. Part (c) does not contradict Corollary 13.18 because f is not continuous.

$\square$

Problem 5 (Exercise 5, Page 378

Define

$$f(x,y) = e^{\sin(x-y)} \qquad \text{for } (x,y) \text{ in } \mathbb{R}^2$$

Find the affine function that is a first-order approximation to the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ at the point $(0,0)$.

Proof. Because f is continuously differentiable, we can take $f(0,0) = 1$ and find the gradient as follows:

$$\nabla f(x,y) = \left(e^{\sin(x-y)}\cos(x-y), -e^{\sin(x-y)}cos(x-y)\right)$$

so that $\nabla f(0,0) = (-1, 1)$; with Corollary 14.4 we deduce that the first order approximation is the function $g(x,y) = 1 + x - y$. $\square$

Problem 6 (Exercise 17, Page 379)

Define

$$f(x,y) = \begin{cases} \sin(y^2/x) \cdot \sqrt{x^2+y^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

a. Show that the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous at the point $(0,0)$ and has directional derivatives in every direction at $(0,0)$.

b. Show that there is no plane that is tangent to the graph of $f\colon \mathbb{R}^2 \to \mathbb{R}$ at the point $(0,0,f(0,0))$.

Proof.

a. Observe that

$$0 \leq |f(x,y)| \leq ||(x,y)||$$

for all $(x,y) \in \mathbb{R}^2$, so that by the squeeze theorem it follows that the limit $\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$ which means f is continuous at $(0,0)$. Now take $p = (x,y) \in \mathbb{R}^2$; note that for $y = 0$ it follows that $f(tp) = 0$, so that

$$\frac{\partial f}{\partial p}(0,0) = \lim_{t\to 0} \frac{f(tp)}{t} = 0$$

If $y \neq 0$, let $c = y^2/x$ so that $f(tp)/t = sin(tc)$. This is continuous and because $\lim_{t\to 0} tc = 0$, meaning:

$$\frac{\partial f}{\partial p}(0,) = \lim_{t\to 0} f(tp)/t = sin(0) = 0$$

which means that directional derivatives of f exist in every direction at (0, 0).

b. Suppose for the sake of contradiction that there is a tangent plane at $(0,0,f(0,0))$. Then there exists $\psi\colon \mathbb{R}^2 \to \mathbb{R}$ of the form $\psi(x,y) = a+bx+cy$ for $(x,y) \in \mathbb{R}^2$ such that

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)-\psi(x,y)}{||(x,y)||}$$

Note that $a = f(0,0) = 0$, $b = 0$, and $c = 0$, so that

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)-\psi(x,y)}{||(x,y)||} = \lim_{(x,y)\to(0,0)} \frac{f(x,y)}{||(x,y)||} = \lim_{(x,y)\to(0,0)} sin(y^2/x)$$

However, take $\{1/k^2, 1/k\} \to (0,0)$ but $\{f(1/k^2, 1/k)\} \to \sin(1) \neq 0$, which is a contradiction.

$\square$

Problem 7 (Exercise 18, Page 380)

Suppose that the continuous function $f\colon \mathbb{R}^2 \to \mathbb{R}$ has a tangent plane at the point $(x_0, y_0, f(x_0, y_0))$. Prove that the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ has directional derivatives in all directions at the point (x_0, y_0).

Proof. Because the tangent plane exists, we know there exists a function $\psi\colon \mathbb{R}^2 \to \mathbb{R}$ such that $\psi(x, y) = a + b(x - x_0) + c(y - y_0)$, and

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x, y) - \psi(x, y)}{||(x, y) - (x_0, y_0)||} = 0$$

so that

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) - \psi(x, y) = 0$$

The continuity of f and ψ implies that $f(x_0, y_0) - \psi(x_0, y_0) = f(x_0, y_0) - a = 0$ which means $a = f(x_0, y_0)$. For any $p \in \mathbb{R}^2$, any sequence $\{t_k\} \to 0$ will have $\{t_k p\} \to 0$ which means that $\{(x_0, y_0) + t_k p\} \to (x_0, y_0)$. This means that

$$||p|| \lim_{k\to\infty} \frac{f((x_0, y_0) + t_k p) - \psi((x_0, y_0) + t_k p)}{||(x_0, y_0) + t_k p - (x_0, y_0)||} = \lim_{k\to\infty} \frac{f((x_0, y_0) + t_k p) - \psi((x_0, y_0) + t_k p)}{t_k} = 0$$

Notice that

$$\lim_{k\to\infty} \frac{f((x_0, y_0) + t_k p) - f(x_0, y_0)}{t_k} = \lim_{k\to\infty} \frac{\langle (b, c), t_k p\rangle}{t_k} = \langle (b, c), p\rangle$$

This implies that

$$\frac{\partial f}{\partial p}(x_0, y_0) = \langle (b, c), p\rangle$$

so that f has directional derivatives in every direction at (x_0, y_0). $\square$