HW1 - MATH411

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Problem 1

Find the maximum value of

$$\frac{x^2 + 2y + 3z}{\sqrt{x^2 + y^2 + z^2}}$$

as (x, y, z) varies among nonzero points in $\mathbb{R}^3$

Proof. Let $\vec{u} = (x, y, z)$ and $\vec{v} = (1, 2, 3)$. Then, by Cauchy-Schwarz:

$$\frac{x^2 + 2y + 3z}{\sqrt{x^2 + y^2 + z^2}} = \frac{\langle \vec{u}, \vec{v} \rangle}{||\vec{v}||} \leq \frac{||\vec{u}||||\vec{v}||}{||\vec{u}||} = ||\vec{v}|| = \sqrt{14}$$

So $\sqrt{14}$ is an upper bound for the expression, and it is attained when $\vec{u} = \lambda \vec{v}$; taking $\lambda = 1$ yields $\vec{u} = \vec{v}$, so $\vec{u} = (1, 2, 3)$. Then:

$$\frac{x^2 + 2y + 3z}{\sqrt{x^2 + y^2 + z^2}} = \frac{1(1) + 2(2) + 3(3)}{\sqrt{1^2 + 2^2 + 3^2}} = \frac{14}{\sqrt{14}} = \sqrt{14}$$

Thus $\sqrt{14}$ is the maximum value.

$\square$

Problem 2

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$. Prove that

$$\langle \mathbf{u}, \mathbf{v} \rangle = \frac{||\mathbf{u} + \mathbf{v}||^2 - ||\mathbf{u} - \mathbf{v}||^2}{4}$$

Proof. By the expansion formula covered in class

$$||\mathbf{u} + \mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2 + 2\langle \mathbf{u}, \mathbf{v} \rangle$$

We deduce that

$$\frac{||\mathbf{u} + \mathbf{v}||^2 - ||\mathbf{u} - \mathbf{v}||^2}{4} = \frac{(||\mathbf{u}||^2 + ||\mathbf{v}||^2 + 2\langle \mathbf{u}, \mathbf{v} \rangle) - |\mathbf{u} + \textbf{-}\mathbf{v}||^2}{4}$$

$$= \frac{(||\mathbf{u}||^2 + ||\mathbf{v}||^2 + 2\langle \mathbf{u}, \mathbf{v}\rangle) - (||\mathbf{u}||^2 + ||\mathbf{v}||^2 + 2\langle \mathbf{u}, \textbf{-v}\rangle)}{4}$$

$$= \frac{(||\mathbf{u}||^2 + ||\mathbf{v}||^2 + 2\langle \mathbf{u}, \mathbf{v}\rangle) - (||\mathbf{u}||^2 + ||\mathbf{v}||^2 - 2\langle \mathbf{u}, \mathbf{v}\rangle)}{4}$$

$$= \frac{4\langle \mathbf{u}, \mathbf{v}\rangle}{4} = \langle \mathbf{u}, \mathbf{v}\rangle$$

□

Problem 3

For points $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, define the function $p\colon \mathbb{R} \to \mathbb{R}$ by $p(t) = ||\mathbf{u} + t\mathbf{v}||^2$ for t in $\mathbb{R}$. Show that $p(t)$ is a quadratic polynomial that attains only nonnegative values. Use this to show that the discriminant is nonpositive and thus provide another proof of the Cauchy-Schwarz Inequality.

Proof. By definition of norm, we have that

$$||\mathbf{u} + t\mathbf{v}||^2 \geq 0$$

.

Using the formula from the previous problem mentioned in class, we have that

$$||\mathbf{u} + t\mathbf{v}||^2 = ||\mathbf{u}||^2 + ||t\mathbf{v}||^2 + 2\langle \mathbf{u}, t\mathbf{v}\rangle = t^2||\mathbf{v}||^2 + 2t\langle \mathbf{u}, t\mathbf{v}\rangle + ||\mathbf{u}||^2$$

which is a polynomial in t that only attains nonnegative values. Because the polynomial only attains nonnegative values, it either has a double real root or no real roots, meaning that the discriminant $b^2 - 4ac \leq 0$. Letting $a = ||\mathbf{v}||^2$, $b = 2\langle \mathbf{u}, \mathbf{v}\rangle$, and $c = ||\mathbf{u}||^2$, this is equivalent to:

$$(2\langle \mathbf{u}, \mathbf{v}\rangle)^2 - 4||\mathbf{u}||^2||\mathbf{v}||^2 \leq 0$$

$$\iff (2\langle \mathbf{u}, \mathbf{v}\rangle)^2 \leq 4||\mathbf{u}||^2||\mathbf{v}||^2$$

$$\iff 2\langle \mathbf{u}, \mathbf{v}\rangle \leq 2||\mathbf{u}||||\mathbf{v}||$$

$$\iff \langle \mathbf{u}, \mathbf{v}\rangle \leq ||\mathbf{u}||||\mathbf{v}||$$

which is the Cauchy-Schwarz inequality.

□

Problem 4

Suppose that the points $\mathbf{u_1}, \cdots, \mathbf{u_k}$ in $\mathbb{R}^n$ are an orthonormal set. For $\mathbf{u} = \alpha_1 \mathbf{u_1} + \cdots + \alpha_k \mathbf{u_k}$, show that

$$||\mathbf{u}|| = \sqrt{\sum_{i=1}^{k} \alpha_i^2}$$

Proof. Note that $||\mathbf{u}|| = ||\alpha_1 \mathbf{u_1} + \cdots + \alpha_k \mathbf{u_k}||$, so that $||\mathbf{u}||^2 = ||\alpha_1 \mathbf{u_1} + \cdots + \alpha_k \mathbf{u_k}||^2$. We first prove that $||\alpha_1 \mathbf{u_1} + \cdots + \alpha_k \mathbf{u_k}||^2 = ||\alpha_1 \mathbf{u_1}||^2 + \cdots + ||\alpha_k \mathbf{u_k}||^2$. by induction

To this end, we show that the base case holds. Because each of the $\mathbf{u_k}$ are orthonormal, each of the $\alpha_k \mathbf{u_k}$ are orthogonal, and from a theorem in class we deduce that $||\alpha_1 \mathbf{u_1} + \alpha_2 \mathbf{u_2}||^2 = ||\alpha_1 \mathbf{u_1}||^2 + ||\alpha_2 \mathbf{u_2}||^2$. Supposing that $||\alpha_1 \mathbf{u_1} + \cdots + \alpha_k \mathbf{u_k}||^2 = ||\alpha_1 \mathbf{u_1}||^2 + \cdots + ||\alpha_k \mathbf{u_k}||^2$, we aim to show $||\alpha_1 \mathbf{u_1} + \cdots + \alpha_k \mathbf{u_k} + \alpha_{k+1} \mathbf{u_{k+1}}||^2 = ||\alpha_1 \mathbf{u_1}||^2 + \cdots + ||\alpha_k \mathbf{u_k}||^2 + ||\alpha_{k+1} \mathbf{u_{k+1}}||^2$. Note that $\alpha_1 \mathbf{u_1} + \cdots + \alpha_k \mathbf{u_k}$ is orthogonal to $\alpha_{k+1} \mathbf{u_{k+1}}$, so that

$$\begin{aligned}
&||(\alpha_1 \mathbf{u_1} + \cdots + \alpha_k \mathbf{u_k}) + \alpha_{k+1} \mathbf{u_{k+1}}||^2 \\
&= ||\alpha_1 \mathbf{u_1} + \cdots + \alpha_k \mathbf{u_k}||^2 + ||\alpha_{k+1} \mathbf{u_{k+1}}||^2 \\
&= ||\alpha_1 \mathbf{u_1}||^2 + \cdots + ||\alpha_k \mathbf{u_k}||^2 + ||\alpha_{k+1} \mathbf{u_{k+1}}||^2
\end{aligned}$$

.

By properties of norms:

$$||\alpha_1 \mathbf{u_1} + \cdots + \alpha_k \mathbf{u_k}||^2 = ||\alpha_1 \mathbf{u_1}||^2 + \cdots + ||\alpha_k \mathbf{u_k}||^2$$

$$= \alpha_1^2 ||\mathbf{u_1}||^2 + \cdots + \alpha_k^2 ||\mathbf{u_k}||^2 = \sum_{i=1}^{k} \alpha_i^2 \mathbf{u_i}^2 = \sum_{i=1}^{k} \alpha_i^2 (1)^2 = \sum_{i=1}^{k} \alpha_i^2$$

.

Taking the square root of both sides gives the result.

□

Problem 5

Let $\{\mathbf{u_k}\}$ be a sequence in $\mathbb{R}^n$ that converges to the point $\mathbf{u}$. Prove that

$$\lim_{k \to \infty} \langle \mathbf{u_k}, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle$$

Proof. Because $\{\mathbf{u_k}\}$ converges to $\mathbf{u}$, it follows that $\{\mathbf{u_k}\}$ converges component-wise to $\mathbf{u}$, so that:

$$\lim_{k \to \infty} \langle \mathbf{u_k}, \mathbf{v} \rangle = \lim_{k \to \infty} [p_1(\mathbf{u_k}) \mathbf{v_1} + \cdots + p_n(\mathbf{u_k}) \mathbf{v_n}] = p_1(\mathbf{u}) \mathbf{v_1} + \cdots + p_n(\mathbf{u}) \mathbf{v_n}$$

$$= \mathbf{u_1 v_1} + \cdots \mathbf{u_n v_n} = \langle \mathbf{u}, \mathbf{v} \rangle$$

□

Problem 6

Let $\{\mathbf{u_k}\}$ be a sequence in $\mathbb{R}^n$ and let $\mathbf{u}$ be a point in $\mathbb{R}^n$. Suppose that for every $\mathbf{v}$ in $\mathbb{R}^n$,

$$\lim_{k \to \infty} \langle \mathbf{u_k}, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle.$$

Prove that $\{\mathbf{u_k}\}$ converges to $\mathbf{u}$.

Proof. Let $\{e_1, \cdots e_i\} \subset \mathbb{R}^n$ be the set of vectors where each e_i is the vector whose i-th component is 1 and the others are all 0. Because

$$\lim_{k \to \infty} \langle \mathbf{u_k}, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle.$$

holds for every $\mathbf{v}$, we have that

$$\lim_{k \to \infty} \langle \mathbf{u_k}, e_i \rangle = \langle \mathbf{u}, e_i \rangle$$

so that for each $i \in \{1, \cdots, n\}$

$$\lim_{k \to \infty} p_i(\mathbf{u_k}) = p_i(\mathbf{u})$$

which implies that each component of $\{\mathbf{u_k}\}$ converges to the respective component in $\mathbf{u}$, so that by the component-wise convergence theorem, $\{\mathbf{u_k}\}$ converges to $\mathbf{u}$.

□

Problem 7

Suppose that $\{\mathbf{u_k}\}$ is a sequence of points in $\mathbb{R}^n$ that converges to the point $\mathbf{u}$ and that $||\mathbf{u}|| = r > 0$. Prove that there is an index K such that

$$||\mathbf{u_k}|| > r/2 \qquad \text{if } k \geq K$$

Proof. Because $\mathbf{u_k}$ converges to $\mathbf{u}$, by a theorem in class it follows that $\lim_{k \to \infty} ||u_k|| = ||u||$. By convergence of real sequences, for all $\epsilon > 0$, there exists $K \in \mathbb{N}$ such that

$$k \geq K \implies |||u_k|| - ||u||| < \epsilon$$

Letting $\epsilon = r/2$, we observe that because $u = r$

$$-r/2 < ||u_k|| - r < r/2 \implies r/2 < ||u_k|| < 3r/2$$

The left side of the inequality shows that for all $k \geq K$, we have that $r/2 < ||u_k||$.

$\square$

Problem 8

Let $\{u_k\}_{k\geq 1}$ be a sequence in $\mathbb{R}^n$ and $\{a_k\}_{k\geq 1}$ be a sequence in $\mathbb{R}$. Prove that if

$$\lim_{k\to\infty} u_k = u \in \mathbb{R}^n \quad \text{and} \quad \lim_{k\to\infty} a_k = a \in \mathbb{R}$$

then

$$\lim_{k\to\infty} (a_k u_k) = au$$

Proof. Because $\lim_{k\to\infty} u_k = u$, u_k converges component-wise, so that $\lim_{k\to\infty} p_i(u_k) = p_i(u)$. This means for each $i \in \{1, \cdots, n\}$

$$\lim_{k\to\infty} (a_k u_k) = \lim_{k\to\infty} [a_k p_1(u_k) + \cdots a_k p_n(u_k)]$$

$$= ap_1(u) + \cdots + ap_n(u) = a \sum_{i=1}^{n} p_i(u) = au$$

$\square$