HW 3 - MATH411

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Problem 1

Show that the set $\{u \text{ in } \mathbb{R}^n \mid u_n > 0\}$ is open in $\mathbb{R}^n$.

Proof. Define $f(u) = p_n(u) = u_n$, where p_n is the nth component projection. By Proposition 11.1, the projection function is continuous, so that the set $\{u \text{ in } \mathbb{R}^n \mid f(u) > 0\}$ is open in $\mathbb{R}^n$ by Corollary 11.13.

□

Problem 2

Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and suppose that the function $f\colon \mathcal{O} \to \mathbb{R}$ is continuous. Suppose that u is a point in $\mathcal{O}$ at which $f(u) > 0$. Prove that there is an open ball B about u such that $f(v) > f(u)/2$ for all v in B.

Proof. Because f is continuous on $\mathcal{O}$, letting $\varepsilon = f(u)/2$, it follows that $||f(v) - f(u)|| < f(u)/2$ if $||v - u|| < \delta_1$. Because $\mathcal{O}$ is open in $\mathbb{R}^n$, it follows that for some $r > 0$, there exists $B_r(u) \subseteq \mathcal{O}$. Take $\delta = \min\{\delta_1, r\}$, then it follows that if $||v - u|| < \delta$ (or equivalently, $v \in B_\delta(u)$) then $||f(v) - f(u)|| < \epsilon = f(u)/2$. Rewriting this, we have $-f(u)/2 < f(v) - f(u) < f(u)/2$; adding $f(u)$ to both sides of the left inequality shows that for all $v \in B_\delta(u), f(v) > f(u)/2$, which was to be shown. □

Problem 3

Let A be a subset of $\mathbb{R}^n$. The *characteristic function* of the set A is the function $f\colon \mathbb{R}^n \to \mathbb{R}$ defined by

$$f(u) = \begin{cases} 1 & \text{if } u \text{ is in A} \\ 0 & \text{if } u \text{ is not in A} \end{cases}$$

Prove that this characteristic function is continuous at each interior point of A and at each exterior point of A but fails to be continuous at each boundary point of A.

Proof. Recall that a point u is an interior point of A if there exists an $r > 0$ such that $B_r(u) \subseteq A$. Also recall that u is an exterior point of A if there exists an $r > 0$ such that $B_r(u) \subseteq \mathbb{R}^n \setminus A$. Finally, recall that u is a boundary point of A if there exists an $r > 0$ such that $B_r(u)$ contains points in A and $\mathbb{R}^n \setminus A$. To this end:

- Let $u \in \text{int } A \subseteq A$. Then by definition, there exists $r > 0$ such that $B_r(u) \subseteq A$. Take any $\{u_k\}$ converging to u; by definition of convergence, there exists $K \in \mathbb{N}$ such that for all $k \geq K$, $u_k \in B_r(u)$, so that $f(u_k) = 1$ for all $k \geq K$. Note also that $f(u) = 1$, so that $\{u_k\} \to u \implies \{f(u_k)\} \to f(u)$, so that f is continuous at any $u \in \text{int } A$.
- Let $u \in \text{ext } A \subseteq \mathbb{R}^n \setminus A$. Then by definition, there exists $r > 0$ such that $B_r(u) \subseteq \mathbb{R}^n \setminus A$. Take any $\{u_k\}$ converging to u; by definition of convergence, there exists $K \in \mathbb{N}$ such that for all $k \geq K$, $u_k \in B_r(u)$, so that $f(u_k) = 0$ for all $k \geq K$. Note also that $f(u) = 0$, so that $\{u_k\} \to u \implies \{f(u_k)\} \to f(u)$, so that f is continuous at any $u \in \text{ext } A$.
- Let $u \in \text{bd } A$. Construct the sequences $\{u_k\}$ and $\{v_k\}$ by taking $u_k \in A \cap B_{1/k}(u)$ and $v_k \in (\mathbb{R}^n \setminus A) \cap B_{1/k}(u)$ for each $k \in \mathbb{N}$. Then it follows that $f(u_k) = 1$ and $f(v_k) = 0$ for each natural number k, but both $\{u_k\}$ and $\{u_k\}$ converge to u, which disproves the continuity of f at u.

$\square$

Problem 4

Let $A \subseteq \mathbb{R}^n$ and $u \in A$. Suppose that $F, G \colon A \to \mathbb{R}^m$ are continuous mappings at u.

a) Define the function $f \colon A \to \mathbb{R}$ by

$$f(v) = \langle F(v), G(v) \rangle, \quad v \in A$$

Prove that f is continuous at u.

b) Define the function $g \colon A \to \mathbb{R}$ by

$$g(v) = ||F(v)||, \quad v \in A$$

Prove that g is continuous at u.

Proof. a) Because $F(v)$ and $G(v)$ are continuous at u, it follows that the components of $F(v)$ and $G(v)$ are also continuous at u, so that $F_1(v) \cdots F_m(v)$ and $G_1(v) \cdots G_m(v)$ are continuous at u, where $F_i(v)$ and $G_i(v)$ denote the ith component functions of F and G at v respectively. Then, by definition of scalar product, we have:

$$\langle F(v), G(v) \rangle = \langle (F_1(v), \cdots, F_m(v)), (G_1(v), \cdots, G_m(v)) \rangle$$

$$= F_1(v)G_1(v) + \cdots + F_m(v)G_m(v)$$

By continuity of sums and products of continuous functions, it follows that $\langle F(v), G(v)\rangle$ is continuous at u.

b) The norm is a continuous function and $F(v)$ is continuous at u, so that by the continuity of composition of continuous functions $||F(v)||$ is continuous at u.

□

Problem 5

Suppose that the functions $f\colon \mathbb{R}^n \to \mathbb{R}$ and $g\colon \mathbb{R}^n \to \mathbb{R}$ are both continuous. Prove that the set $\{u \text{ in } \mathbb{R}^n \mid f(u) = g(u) = 0\}$ is closed in $\mathbb{R}^n$.

Proof. Consider the set $B = \{u \text{ in } \mathbb{R}^n \mid f(u) = 0\}$. Because the singleton 0 is closed in $\mathbb{R}$, it follows that $f^{-1}((0)) = B$ is closed in $\mathbb{R}^n$.

Consider the set $C = \{u \text{ in } \mathbb{R}^n \mid g(u) = 0\}$. Because the singleton 0 is closed in $\mathbb{R}$, it follows that $g^{-1}((0)) = C$ is closed in $\mathbb{R}^n$.

Since the intersection of closed sets is closed, it follows that $B \cap C$ is closed, and $B \cap C = \{u \text{ in } \mathbb{R}^n \mid f(u) = g(u) = 0\}$

□

Problem 6

Give a counter example to the following incorrect statement. "If $f\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous in each variable separately, then f is continuous". Here, f is continuous in x if for each fixed $y \in \mathbb{R}$, the function $x \mapsto f(x, y)$ is continuous; the continuity of f in y is defined similarly.

Proof. Consider $f\colon \mathbb{R}^2 \to \mathbb{R}$:

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if x, y } \neq 0 \\ 0 & \text{if x and y } = 0 \end{cases}$$

If $y = 0$, then $f(x, 0) = 0$ because the numerator is 0, so the function is always 0 and thus continuous. Fix $y = c \neq 0$; then $f(x, c) = \frac{xc}{x^2+c}$. This is a quotient of continuous functions with a nonzero denominator, so it is continuous.
We show that f is not continuous. If we approach (0, 0) along the line $y = x$, we have that $\lim_{(x,y)\to(0,0)} f(x, y) = \frac{x^2}{x^2+x^2} = \frac{1}{2}$, but $f(0, 0) = 0 \neq \frac{1}{2}$; thus f is not continuous at $(0, 0)$. □

Problem 7

Let A be a bounded set in $\mathbb{R}^n$. Prove that for any $a \in \mathbb{R}^n$, there exists $s > 0$ such that $A \subseteq B_s(a)$

Proof. By definition, for any $u \in A$, we have that $||u|| \leq M$ for some $M > 0$. Set $s = 2M + ||a|| + 1$. Then, by the triangle inequality:

$$||u - a|| \leq ||u|| + ||a|| \leq M + ||a|| < 2M + ||a|| < 2M + ||a|| + 1$$

So that $u \in B_s(a)$ if $u \in A$.

$\square$

Problem 8

Let $A_1, \cdots, A_n$ be subsets of $\mathbb{R}$. Consider the Cartesian product

$$A = A_1 \times \cdots \times A_n \coloneqq \{x = (x_1, \cdots, x_n) \colon x_j \in A_j \text{ for all } 1 \leq j \leq n\} \subseteq \mathbb{R}^n$$

1. Prove that A is closed in $\mathbb{R}^n$ if each A_j is closed in $\mathbb{R}$.
2. Prove that A is open in $\mathbb{R}^n$ if each A_j is open in $\mathbb{R}$.

Proof.

1. Take a sequence $\{u_k\} \in A$ converging to u. We have that $p_i(u) = \lim_{k\to\infty} p_i(u_k)$; because each A_i is closed, it follows that $\lim_{k\to\infty} p_i(u_k) = p_i(u) \in A_i$. Since each $p_i(u) \in A_i$, it follows that $u \in A$, so A is closed.
2. Let $u \in A$ be arbitrary. Because each A_j is open, for any $u_j \in A_j$ there exists an open ball $B_{r_j}(u_j) \subseteq A_j$. Take $r = \min(r_1, \cdots, r_n)$; then it follows that each $u_j \in B_r(u_j) \subseteq A_j$; this implies that $u \in A$ if $u \in B_r(u)$, so A is open.

$\square$