HW2 - MATH411

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Problem 1

Determine which of the following subsets of $\mathbb{R}$ are open in $\mathbb{R}$, closed in $\mathbb{R}$, or neither open nor closed in $\mathbb{R}$. Justify your conclusions:

(a) $A = \mathbb{Q}$, the set of rational numbers

(b) $A = \{u \in \mathbb{R} \mid u^2 > 4\}$

Determine which of the following subsets of $\mathbb{R}^2$ are open in $\mathbb{R}^2$, closed in $\mathbb{R}^2$, or neither open nor closed in $\mathbb{R}^2$. Justify your conclusions:

(a) $A = \{u = (x, y) \mid x^2 + y^2 = 1\}$

(b) $A = \{u = (x, y) \mid x \text{ is rational}\}$

Proof. For the sets in $\mathbb{R}$:

(a) A is neither open nor closed in $\mathbb{R}$. To prove that it is not open, note that the interior of A is empty, as any open ball centered at $u \in A$ contains irrationals by the density of $\mathbb{Q}$ in $\mathbb{R}$, so int $A = \varnothing \neq A$, so A is not open. To prove that it is not closed, consider the sequence $\{u_k\} = (1 + \frac{1}{k})^k$. Each $u_k \in A$ but $\{u_k\} \to e \notin A$.

(b) A is open. Trivially, int $A \subseteq A$. For any point $u \in A$, we have that $|u| > 2$, so that $|u| - 2 > 0$. Set $r = \frac{|u|-2}{2}$, then it follows that $B_r(u) \subseteq A$, so that $A \subseteq$ int A; thus int $A = A$, so A is open in $\mathbb{R}$. A is not closed in R; consider $\{u_k\} = (2 + \frac{1}{k})^2$. All of the $u_k \in A$, but $\{u_k\} \to 4 \notin A$.

For the sets in $\mathbb{R}^2$: For the sets in $\mathbb{R}$:

(a) A is closed. To prove this, suppose that $\{u_k\} \in A$ converges to u. By a theorem in class, it follows that $||u_k|| \to ||u||$, but $||u_k|| = 1$ for all k, which means that $||u|| = 1$. This implies that u is on the unit circle, so that $u \in A$, showing that A is closed. A is not open, because any open ball will contain points outside of A.

(b) A is neither open nor closed by the same reasoning of (a) in $\mathbb{R}$; since the x- component of any $u \in A$ is neither closed nor open in $\mathbb{R}$, component-wise convergence of sequences implies that there exists $\{u_k\} \in A$ such that $\{u_k\} \to u \notin A$, so that A is not closed. Furthermore, any open ball centered at $u \in A$ contains an irrational x-component, so A is not open.

$\square$

Problem 2

Let A be a subset of $\mathbb{R}^n$ and let w be a point in $\mathbb{R}^n$. The *translate* of A by w is denoted by $w + A$ and is defined by

$$w + A = \{w + u \mid u \text{ in } A\}$$

(a) Show that A is open if and only if $w + A$ is open.

(b) Show that A is closed if and only if $w + A$ is closed.

Proof. (a) If A is open, then for any $u \in A$, it follows that $B_r(u) \in A$, so that $w + B_r(u) \in w + A$. We must show that $w + B_r(u) = B_r(w + u)$. To this end, note that:

$$x \in w + B_r(u) \iff x - w \in B_r(u) \iff ||(x - w) - u|| < r$$

$$\iff ||x - (w + u)|| < r \iff x \in B_r(w + u)$$

so that A open $\iff$ $w + A$ open.

(b) Note that A closed $\iff$ A^c open $\iff$ $w + A^c$ open $\iff$ $(w + A)^c$ open $\iff$ $w + A$ closed. Additionally, if $\{u_k\} \in A$ converges to $u \in A$, it follows that $\{w + u_k\} = w + \{u_k\} = w + u \in w + A$ (because w is fixed); a similar argument holds in the reverse direction, because of the fact that w is fixed.

$\square$

Problem 3

Let A and B be subsets of $\mathbb{R}^n$ with $A \subseteq B$.

(a) Prove that int $A \subseteq$ int B.

(b) Is it necessarily true that bd $A \subseteq$ bd B?

Proof. (a) Let $u \in$ int A be arbitrary. Then for some $r > 0$ there exists an open ball such that $u \in B_r(u) \subseteq A$. But $A \subseteq B$, so that in particular $B_r(u) \subseteq B$, meaning that $u \in$ int B, which implies that int $A \subseteq$ int B

(b) No. Consider two concentric closed circles in $\mathbb{R}^2$; the intersection of the boundaries is empty, so neither can be contained in the other.

$\square$

Problem 4

For a subset A is $\mathbb{R}^n$, the *closure* of A, denoted by cl A, is defined by

$$\text{cl } A = \text{int } A \cup \text{bd } A$$

Prove that $A \subseteq \text{cl } A$ and that $A = \text{cl } A$ if and only if A is closed in $\mathbb{R}^n$.

Proof. Let $u \in A$ be arbitrary. Consider an open ball of radius $r > 0$ centered at u. There are two cases: (1) $B_r(u) \subseteq A$ or (2) $B_r(u)$ is not contained in A. For (1), it follows that $u \in \text{int } A$, so that in particular $u \in \text{int } A \cup \text{bd } A$. For (2), it follows that $B_r(u)$ contains a point in A (namely u) and a point not in A, so that $u \in \text{bd } A$ and in particular $u \in \text{int } A \cup \text{bd } A$. Thus, in either case, $u \in \text{cl } A$, so that $A \subseteq \text{cl } A$.
Suppose that $A = \text{cl } A$. In particular, $\text{bd} A \subseteq A$, so that A is closed in $\mathbb{R}^n$.
Suppose that A is closed in $\mathbb{R}^n$. We must show $\text{bd} A \subseteq A$. Let $u \in \text{bd } A$. Then there exists a sequence in A such that $\{u_k\} \in B_{1/k}(u)$, which clearly converges to $u \in A$ because A is closed.

□

Problem 5

Fix a point V in $\mathbb{R}^n$ and define the function $f\colon \mathbb{R}^n \to \mathbb{R}$ by

$$f(u) = \langle u, v \rangle \qquad \text{for } u \in \mathbb{R}^n$$

Prove that the function $f\colon \mathbb{R}^n \to \mathbb{R}$ is continuous.

Proof. The projection functions p_i are continuous, so that in particular $p_i(u)$ and $p_i(v)$ are continuous for $i \in \{1, \cdots, n\}$. Use this fact to rewrite f as:

$$f(u) = \langle u, v \rangle = \langle (p_1(u), \cdots, p_n(u)), (p_1(v), \cdots, p_n(v)) \rangle$$

$$= p_1(u)p_1(v) + \cdots + p_n(u)p_n(v)$$

By the continuity of sums and products of continuous functions, the final result is continuous; thus f is continuous.

□

Problem 6

Suppose that the function $f\colon \mathbb{R}^n \to \mathbb{R}$ is continuous and that $f(u) > 0$ if the point u in $\mathbb{R}^n$ has at least one rational component. Prove that $f(u) \geq 0$ for all points u in $\mathbb{R}^n$.

Proof. For every $u \in \mathbb{R}^n$ with a rational component, we have that $f(u) > 0$. In particular, let $u = (u_1, \cdots, u_n) \in \mathbb{R}^n$. Let $\varepsilon > 0$ be arbitrary. Then, because f is continuous, there exists a $\delta > 0$ such that $||u - v|| < \delta \implies ||f(u) - f(v)|| < \varepsilon$. By the density of $\mathbb{Q}$ in $\mathbb{R}$, it follows that there exists v_1 such that $u_1 < v_1 < u_1 + \delta$, where $0 < v_1 - u_1 < \delta$. Then, constructing v to be the same as u except for the first component, we have that:

$$||u - v|| = \sqrt{(u - v_1)^2 + \cdots + (u - v_n)^2} = \sqrt{(u - v_1)^2 + \cdots + 0} = ||u - v_1|| < \delta$$

Putting this together yields:

$$f(u) - f(v) < \varepsilon \implies f(u) > f(v) - epsilon \implies f(u) \geq 0$$

$\square$