HW 7 - MATH411

Danesh Sivakumar

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Problem 1 (Exercise 2, Page 412)

Define

$$F(x,y,z) = (xyz, x^2 + yz, 1 + 3x) \qquad \text{for } (x,y,z) \text{ in } \mathbb{R}^3.$$

Find the derivative matrix of the mapping $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ at the points $(1,2,3)$, $(0,1,0)$, and $(-1,4,0)$.

Proof. Note that the derivative matrix is

$$DF(x,y,z) = \begin{bmatrix} yz & xz & xy \\ 2x & z & y \\ 3 & 0 & 0 \end{bmatrix}$$

With this, we evaluate the following:

$$DF(1,2,3) = \begin{bmatrix} 6 & 3 & 2 \\ 2 & 3 & 2 \\ 3 & 0 & 0 \end{bmatrix}$$

$$DF(0,1,0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 3 & 0 & 0 \end{bmatrix}$$

$$DF(-1,4,0) = \begin{bmatrix} 0 & 0 & -4 \\ -2 & 0 & 4 \\ 3 & 0 & 0 \end{bmatrix}$$

□

Problem 2 (Exercise 3, Page 412)

Suppose that the mapping $F\colon \mathbb{R}^n \to \mathbb{R}^m$ is continuously differentiable and that the derivative matrix $DF(x)$ at each point x in $\mathbb{R}^n$ has all its entries equal to 0. Prove that the mapping $F\colon \mathbb{R}^n \to \mathbb{R}^m$ is constant; that is, there is some point c in $\mathbb{R}^m$ such that

$$F(x) = c \qquad \text{for every } x \text{ in } \mathbb{R}^n.$$

Proof. Fix x in $\mathbb{R}^n$ and suppose y is an arbitrary point in $\mathbb{R}^n$. By Theorem 15.29, we have that

$$F(y) - F(x) = A(y - x)$$

where A is the $m \times n$ matrix whose ith row is $\nabla F_i(x + \theta_i)$ (where θ_i is the number in the open interval (0, 1) that satisfies the Mean Value Theorem for each component function). But ∇F_i is equal to zero for any point in $\mathbb{R}^n$ because all of the derivative matrix's entries are zero for any point in $\mathbb{R}^n$, so that all of the entries in A are zero. This means that $A(y - x) = 0$; thus it follows that $F(y) = F(x)$, so that $F = c$ for some $c \in \mathbb{R}^m$.

$\square$

Problem 3 (Exercise 8, Page 413)

Suppose that the mapping $F \colon \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable. Suppose also that $F(0) = 0$ and that the derivative matrix $DF(0)$ has the property that there is some positive number c such that

$$||DF(0)h|| \geq c||h|| \qquad \text{for all } h \text{ in } \mathbb{R}^n.$$

Prove that there is some positive number r such that

$$||F(h)|| \geq c/2||h|| \qquad \text{if } ||h|| \leq r.$$

Proof. If $h = 0$, then the inequality follows trivially because $||F(0)|| = 0 \geq 0 = c/2||0||$ for any choice of r. Now suppose that $h \neq 0$. Because F is continuously differentiable at 0, by Theorem 15.31 we have

$$\lim_{h \to 0} \frac{||F(h) - DF(0)h||}{||h||} = 0$$

Denote this fraction as $g(h)$. Then by definition of functional limits, it follows that for any $\varepsilon > 0$, there exists $r > 0$ such that if $0 < ||h|| < r$ then $|g(h)| < \varepsilon$. In particular, set $\varepsilon = c/2$ and choose r accordingly; then we have by the reverse triangle inequality:

$$\frac{\big|||F(h)|| - ||DF(0)h||\big|}{||h||} \leq \frac{||F(h) - DF(0)h||}{||h||} < c/2$$

so that

$$-c/2 < \frac{||F(h)|| - ||DF(0)h||}{||h||} < c/2$$

In particular, notice that

$$-c/2 < \frac{||F(h)|| - ||DF(0)h||}{||h||} \leq \frac{||F(h)|| - c||h||}{||h||}$$

Multiplying both sides by $||h||$ and adding $c||h||$ yields

$$c/2||h|| < ||F(h)||$$

which was to be proven. $\square$

Problem 4 (Exercise 9, Page 413)

Suppose that the continuously differentiable mapping $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ is represented in components functions as

$$F(x,y) = (\psi(x,y), \phi(x,y)) \qquad \text{for } (x,y) \text{ in } \mathbb{R}^2.$$

Define the function $g\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$g(x,y) = \frac{1}{2}[(\psi(x,y))^2 + (\phi(x,y))^2] \qquad \text{for } (x,y) \text{ in } \mathbb{R}^2.$$

a. Show that

$$Dg(x_0,y_0) = [DF(x_0,y_0)]^T F(x_0,y_0).$$

b. Use (a) to prove that if (x_0,y_0) is a minimizer of the function $g\colon \mathbb{R}^2 \to \mathbb{R}$ and the matrix $DF(x_0,y_0)$ is invertible, then

$$F(x_0,y_0) = 0.$$

Proof.

a. Notice that

$$Dg(x_0,y_0) = \nabla g(x_0,y_0) = \begin{bmatrix} \psi(x_0,y_0)\frac{\partial\psi}{\partial x_0} + \phi(x_0,y_0)\frac{\partial\phi}{\partial x_0} \\ \psi(x_0,y_0)\frac{\partial\psi}{\partial y_0} + \phi(x_0,y_0)\frac{\partial\phi}{\partial y_0} \end{bmatrix}$$

$$\psi(x_0,y_0)\begin{bmatrix} \frac{\partial\psi}{\partial x_0} \\ \frac{\partial\psi}{\partial y_0} \end{bmatrix} + \phi(x_0,y_0)\begin{bmatrix} \frac{\partial\phi}{\partial x_0} \\ \frac{\partial\phi}{\partial y_0} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial\psi}{\partial x_0} & \frac{\partial\phi}{\partial x_0} \\ \frac{\partial\psi}{\partial y_0} & \frac{\partial\phi}{\partial y_0} \end{bmatrix}\begin{bmatrix} \psi(x_0,y_0) \\ \phi(x_0,y_0) \end{bmatrix} = \begin{bmatrix} \frac{\partial\psi}{\partial x_0} & \frac{\partial\psi}{\partial y_0} \\ \frac{\partial\phi}{\partial x_0} & \frac{\partial\phi}{\partial y_0} \end{bmatrix}^T \begin{bmatrix} \psi(x_0,y_0) \\ \phi(x_0,y_0) \end{bmatrix}$$

$$= [DF(x_0,y_0)]^T F(x_0,y_0)$$

b. Note that because (x_0,y_0) is a minimizer of g, it follows that $Dg(x_0,y_0) = 0$; because $DF(x_0,y_0)$ is invertible, it follows that its transpose is also invertible, so that

$$F(x_0,y_0) = ([DF(x_0,y_0)]^T)^{-1} Dg(x_0,y_0) = 0$$

□

Problem 5 (Exercise 2, Page 419)

Suppose that the function $h\colon \mathbb{R}^3 \to \mathbb{R}$ is continuously differentiable. Define the function $\eta\colon \mathbb{R}^3 \to \mathbb{R}$ by

$$\eta(u,v,w) = (3u+2v)h(u^2,v^2,uvw) \qquad \text{for } (u,v,w) \text{ in } \mathbb{R}^3.$$

Find $D_1\eta(u,v,w)$, $D_2\eta(u,v,w)$, and $D_3\eta(u,v,w)$.

Proof.

$$D_1\eta(u,v,w) = 3h(u^2,v^2,uvw)+(3u+2v)\left[\frac{\partial h}{\partial u}(u^2,v^2,uvw)2u + \frac{\partial h}{\partial w}(u^2,v^2,uvw)vw\right]$$

$$D_2\eta(u,v,w) = 2h(u^2,v^2,uvw)+(3u+2v)\left[\frac{\partial h}{\partial v}(u^2,v^2,uvw)2v + \frac{\partial h}{\partial w}(u^2,v^2,uvw)uw\right]$$

$$D_3\eta(u,v,w) = (3u+2v)\left[\frac{\partial h}{\partial w}(u^2,v^2,uvw)uv\right]$$

□

Problem 6 (Exercise 5, Page 419)

Let $\mathcal{O}$ be an open subset of the plane $\mathbb{R}^2$ and let the mapping $F\colon \mathcal{O} \to \mathbb{R}^2$ be represented by $F(x,y) = (u(x,y), v(x,y))$ for (x,y) in $\mathcal{O}$. Then the mapping $F\colon \mathcal{O} \to \mathbb{R}^2$ is called a *Cauchy-Riemann mapping* provided that each of the functions $u\colon \mathcal{O} \to \mathbb{R}$ and $v\colon \mathcal{O} \to \mathbb{R}$ has continuous second-order partial derivatives and

$$\frac{\partial u}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y) \qquad \text{and} \qquad \frac{\partial u}{\partial y}(x,y) = -\frac{\partial v}{\partial x}(x,y) \qquad \text{for all } (x,y) \text{ in } \mathcal{O}.$$

Prove that if the function $\omega\colon \mathbb{R}^2 \to \mathbb{R}$ is harmonic and the mapping $F\colon \mathcal{O} \to \mathbb{R}^2$ is a Cauchy-Riemann mapping, then the function $\omega \circ F\colon \mathcal{O} \to \mathbb{R}$ is also harmonic.

Proof. Because ω is harmonic, it follows that when ω is a function of u and v

$$\frac{\partial^2 \omega}{\partial u^2} + \frac{\partial^2 \omega}{\partial v^2} = 0$$

Because the second partials of u and v are continuous, it follows that the mixed derivatives are equal, so

$$\frac{\partial v}{\partial y \partial x} = \frac{\partial v}{\partial x \partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y \partial x} = \frac{\partial u}{\partial x \partial y}$$

Thus, using the chain rule and equality from the Cauchy-Riemann relations, we have that:

$$\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} = \left[\frac{\partial^2 \omega}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial \omega}{\partial u}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 \omega}{\partial v^2}\left(\frac{\partial v}{\partial x}\right)^2 + \frac{\partial \omega}{\partial v}\frac{\partial^2 v}{\partial x^2}\right]$$

$$+\left[\frac{\partial^2\omega}{\partial u^2}\left(\frac{\partial u}{\partial y}\right)^2+\frac{\partial\omega}{\partial u}\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2\omega}{\partial v^2}\left(\frac{\partial v}{\partial y}\right)^2+\frac{\partial\omega}{\partial v}\frac{\partial^2 v}{\partial y^2}\right]$$

$$=\left(\frac{\partial^2\omega}{\partial u^2}+\frac{\partial^2\omega}{\partial v^2}\right)\left(\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right)$$

$$+\frac{\partial\omega}{\partial u}\left(\frac{\partial^2 v}{\partial x\partial y}\right)+\frac{\partial\omega}{\partial u}\left(-\frac{\partial^2 v}{\partial y\partial x}\right)+\frac{\partial\omega}{\partial w}\left(\frac{\partial^2 u}{\partial y\partial x}\right)+\frac{\partial\omega}{\partial w}\left(-\frac{\partial^2 u}{\partial x\partial y}\right)=0$$

so that ω is harmonic in x and y, meaning $\omega\circ F$ is harmonic.

□

Problem 7 (Exercise 9, Page 420)

Let $\mathcal{O}=\{(x,y,z) \text{ in } \mathbb{R}^3 \mid x^2+y^2+z^2>0\}$ and define the function $u\colon\mathcal{O}\to\mathbb{R}$ by

$$u(p)=\frac{1}{||p||}\qquad\text{for } p \text{ in } \mathcal{O}.$$

Prove that

$$\frac{\partial^2 u}{\partial x^2}(x,y,z)+\frac{\partial^2 u}{\partial y^2}(x,y,z)+\frac{\partial^2 u}{\partial z^2}(x,y,z)=0\qquad\text{for every } (x,y,z) \text{ in } \mathcal{O}.$$

Proof. Note that

$$u(x,y,z)=\frac{1}{(x^2+y^2+z^2)^{1/2}}\qquad\text{for every } (x,y,z) \text{ in } \mathcal{O}$$

so that with the chain rule:

$$\frac{\partial u}{\partial x}=-\frac{x}{(x^2+y^2+z^2)^{3/2}}\quad\frac{\partial u}{\partial y}=-\frac{y}{(x^2+y^2+z^2)^{3/2}}\quad\frac{\partial u}{\partial z}=-\frac{z}{(x^2+y^2+z^2)^{3/2}}$$

Direct computations with the chain rule show that

$$\frac{\partial^2 u}{\partial x^2}=\frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}\quad\frac{\partial^2 u}{\partial y^2}=\frac{2y^2-x^2-z^2}{(x^2+y^2+z^2)^{5/2}}\quad\frac{\partial^2 u}{\partial z^2}=\frac{2z^2-y^2-x^2}{(x^2+y^2+z^2)^{5/2}}$$

thus

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$$

□