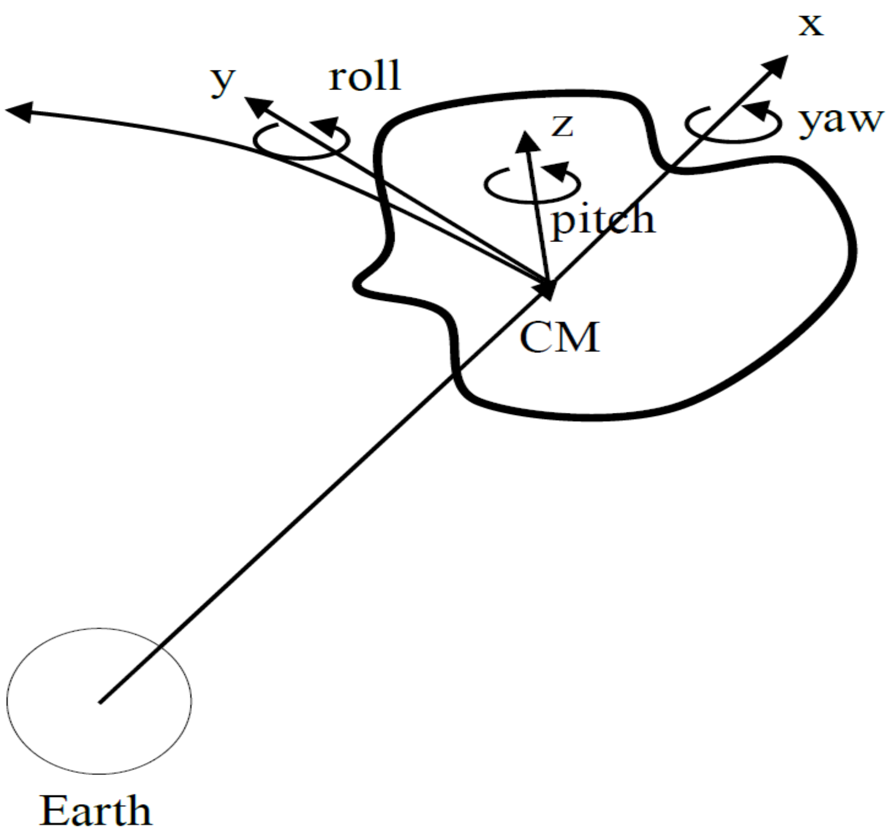
POLITECNICO
MILANO 1863

Spacecraft Attitude Dynamics

prof. Franco Bernelli

Moving reference frames and Gravity Gradient Torque

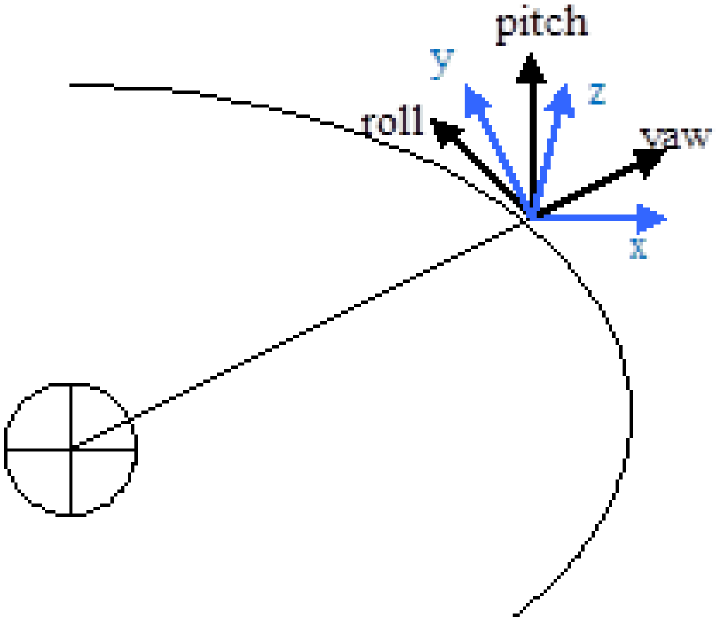
Rotating reference frames



$$A_{LVLH} = \begin{bmatrix} \cos\theta(t) & \sin\theta(t) & 0 \\ -\sin\theta\,(t) & \cos\theta(t) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Local Vertical Local Horizontal (LVLH) frame

Linearization about the LVLH frame (circular orbit)



$$A_{B/LVLH} = A_{123}$$

$$= \begin{bmatrix} \cos\alpha_z \cos\alpha_y & \cos\alpha_z \sin\alpha_y \sin\alpha_x + \sin\alpha_z \cos\alpha_x & -\cos\alpha_z \sin\alpha_y \cos\alpha_x + \sin\alpha_z \sin\alpha_x \\ -\sin\alpha_z \cos\alpha_y & -\sin\alpha_z \sin\alpha_y \sin\alpha_x + \cos\alpha_z \cos\alpha_x & \sin\alpha_z \sin\alpha_y \cos\alpha_x + \cos\alpha_z \sin\alpha_x \\ \sin\alpha_y & -\cos\alpha_y \sin\alpha_x & \cos\alpha_y \cos\alpha_x \end{bmatrix}$$

Linearization about the LVLH frame (circular orbit)

$$A = \begin{bmatrix} 1 & \alpha_z & -\alpha_y \\ -\alpha_z & 1 & \alpha_x \\ \alpha_y & -\alpha_x & 1 \end{bmatrix}$$

$$\begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} = \begin{bmatrix} 1 & \alpha_z & -\alpha_y \\ -\alpha_z & 1 & \alpha_x \\ \alpha_y & -\alpha_x & 1 \end{bmatrix} \begin{Bmatrix} \dot{\alpha}_x \\ \dot{\alpha}_y \\ \dot{\alpha}_z + n \end{Bmatrix}$$

$\alpha_x, \alpha_y, \alpha_z, \dot{\alpha}_x, \dot{\alpha}_y, \dot{\alpha}_z$ are assumed to be small in order to linearize

n is the nominal angular velocity of the satellite along its orbit (typically, one rotation per orbit)

Linearization about the LVLH frame

$$\begin{cases} \omega_x = \dot{\alpha}_x - \alpha_y n \\ \omega_y = \dot{\alpha}_y + \alpha_x n \\ \omega_z = \dot{\alpha}_z + n \end{cases}$$

$$\begin{cases} \dot{\omega}_x = \ddot{\alpha}_x - \dot{\alpha}_y n \\ \dot{\omega}_y = \ddot{\alpha}_y + \dot{\alpha}_x n \\ \dot{\omega}_z = \ddot{\alpha}_z \end{cases}$$

$$\longrightarrow$$

$$\dot{\omega}_x = \frac{I_y - I_z}{I_x}\omega_y\omega_z$$
$$\dot{\omega}_y = \frac{I_z - I_x}{I_y}\omega_x\omega_z$$
$$\dot{\omega}_z = \frac{I_x - I_y}{I_z}\omega_y\omega_x$$

Attitude dynamics relative to an LVLH frame (with circular orbit) Linear Equations

$$\begin{cases} I_x\ddot{\alpha}_x + n\left(I_z - I_y - I_x\right)\dot{\alpha}_y + n^2\left(I_z - I_y\right)\alpha_x = 0 \\ I_y\ddot{\alpha}_y + n\left(I_x + I_y - I_z\right)\dot{\alpha}_x + n^2(I_z - I_x)\alpha_y = 0 \\ I_z\ddot{\alpha}_z = 0 \end{cases}$$

Stability analysis and stability diagram

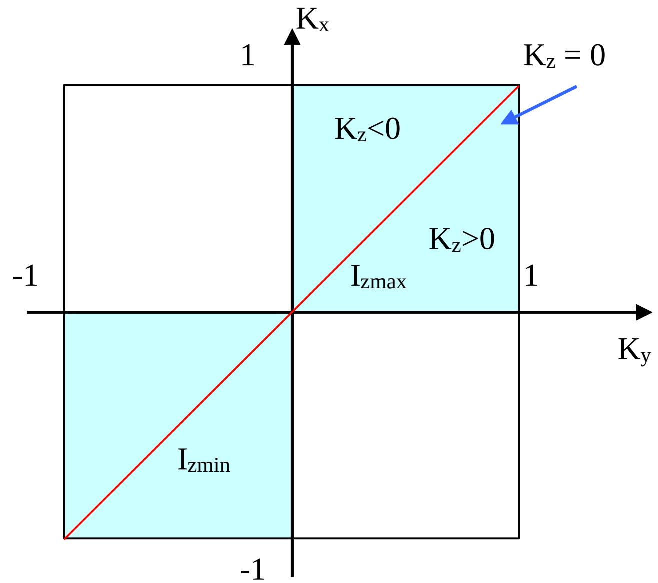
$$\begin{cases} \ddot{\alpha}_x + n(K_x - 1)\dot{\alpha}_y + n^2 K_x \alpha_x = 0 \\ \ddot{\alpha}_y + n\left(1 - K_y\right)\dot{\alpha}_x + n^2 K_y \alpha_y = 0 \end{cases}$$

$$K_x = \frac{I_z - I_y}{I_x}$$

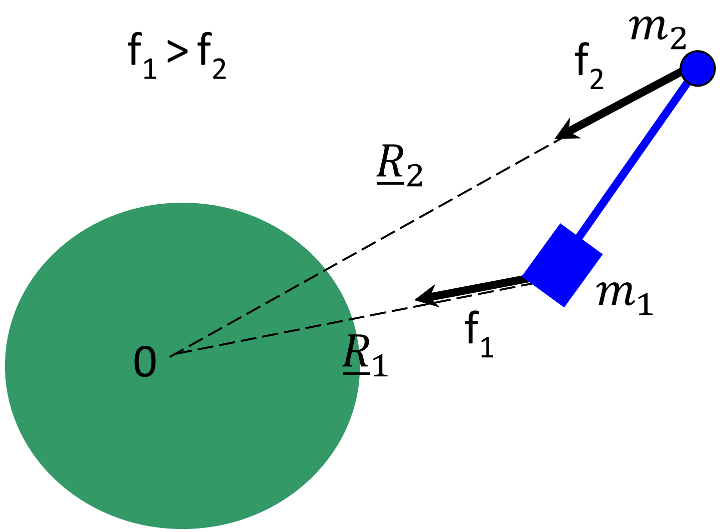
$$K_y = \frac{I_z - I_x}{I_y}$$

$$K_z = \frac{I_y - I_x}{I_z}$$

$$\lambda^2 = \frac{-n^2\left(1 + K_x K_y\right) \pm n^2\sqrt{\left(1 - K_x K_y\right)^2}}{2}$$



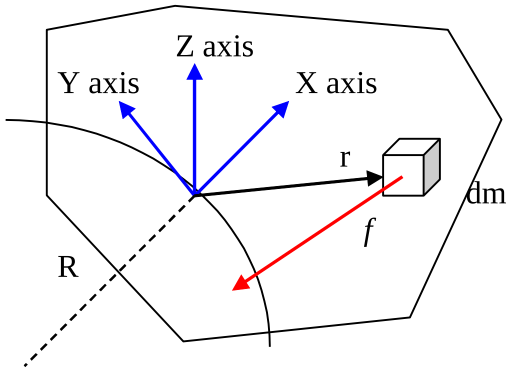
Gravity gradient disturbance torque



$$M_1 = -\underline{r}_{1-c} \times \frac{G m_t m_1}{|R_1|^3} (\underline{R}_1)$$

$$M_2 = -\underline{r}_{2-c} \times \frac{G m_t m_2}{|R_2|^3} (\underline{R}_2)$$

Gravity Gradient - *M*



$$dM = -\underline{r} \wedge \frac{Gm_t dm}{|R + r|^3}(\underline{R} + \underline{r})$$

$$M = -\int_B \underline{r} \wedge \frac{Gm_t}{|R + r|^3}(\underline{R} + \underline{r})dm$$

Approximate

$$M = -\frac{Gm_t}{R^3}\int_B \underline{r} \wedge \left(1 - 3\frac{\underline{R} \cdot \underline{r}}{R^2}\right)(\underline{R} + \underline{r})dm$$

$$\boxed{M = \frac{3Gm_t}{R^5}\int_B (\underline{r} \cdot \underline{R})(\underline{r} \wedge \underline{R})dm}$$

Exercise: Evaluate the Gravity Gradient in the body Frame

$$M = \frac{3Gm_t}{R^5}\int_B \left(\underline{r}\cdot\underline{R}\right)\left(\underline{r}\wedge\underline{R}\right)dm \qquad \underline{R}_B = R[c_1 \quad c_2 \quad c_3]^T \qquad \underline{r} = x\underline{b}_1 + y\underline{b}_2 + z\underline{b}_3$$

$$\underline{M} = \frac{3Gm_t}{R^3}\int_B \begin{pmatrix} (y^2-z^2)c_2c_3 \\ (z^2-x^2)c_1c_3 \\ (x^2-y^2)c_1c_2 \end{pmatrix} dm = \frac{3Gm_t}{R^3}\begin{Bmatrix} \left(I_z-I_y\right)c_2c_3 \\ (I_x-I_z)c_1c_3 \\ \left(I_y-I_x\right)c_1c_2 \end{Bmatrix}$$

therefore

$$\begin{cases} I_x\dot{\omega}_x + \left(I_z-I_y\right)\omega_z\omega_y = \dfrac{3Gm_t}{R^3}\left(I_z-I_y\right)c_3c_2 \\ I_y\dot{\omega}_y + (I_x-I_z)\omega_x\omega_z = \dfrac{3Gm_t}{R^3}(I_x-I_z)c_1c_3 \\ I_z\dot{\omega}_z + \left(I_y-I_x\right)\omega_y\omega_x = \dfrac{3Gm_t}{R^3}\left(I_y-I_x\right)c_2c_1 \end{cases}$$

Linearized equations in trivial LVLH frame (circular orbit)

$$I_x\dot{\omega}_x + \left(I_z - I_y\right)\omega_z\omega_y = \frac{3Gm_t}{R^3}\left(I_z - I_y\right)c_3c_2$$
$$I_y\dot{\omega}_y + (I_x - I_z)\omega_x\omega_z = \frac{3Gm_t}{R^3}(I_x - I_z)c_1c_3$$
$$I_z\dot{\omega}_z + \left(I_y - I_x\right)\omega_y\omega_x = \frac{3Gm_t}{R^3}\left(I_y - I_x\right)c_2c_1$$

$$\begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = \begin{bmatrix} 1 & \alpha_z & -\alpha_y \\ -\alpha_z & 1 & \alpha_x \\ \alpha_y & -\alpha_x & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

$$n^2 = \frac{Gm_t}{R^3}$$

$$\begin{cases} I_x\ddot{\alpha}_x + \left(I_z - I_y - I_x\right)n\dot{\alpha}_y + \left(I_z - I_y\right)n^2\alpha_x = 0 \\ I_y\ddot{\alpha}_y + \left(I_x + I_y - I_z\right)n\dot{\alpha}_x + (I_z - I_x)n^2\alpha_y = 3n^2(I_x - I_z)\alpha_y \\ I_z\ddot{\alpha}_z = -3n^2\left(I_y - I_x\right)\alpha_z \end{cases}$$

Stability conditions

$$\ddot{\alpha}_x + (K_x - 1)n\dot{\alpha}_y + K_x n^2 \alpha_x = 0$$
$$\ddot{\alpha}_y + \left(1 - K_y\right)n\dot{\alpha}_x + 4K_y n^2 \alpha_y = 0$$

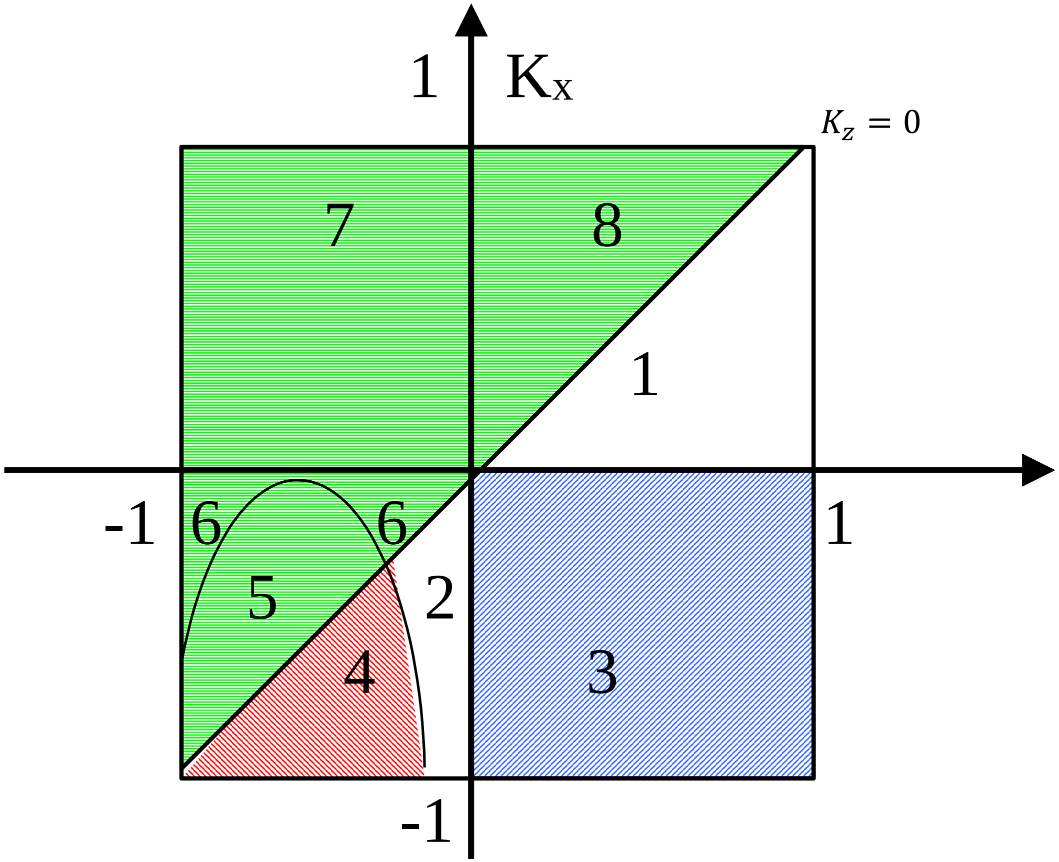
$$K_x = \frac{I_z - I_y}{I_x}$$
$$K_y = \frac{I_z - I_x}{I_y}$$

Eigenvalues of the matrix

$$\lambda^4 + n^2\lambda^2\left(1 + 3K_x + K_xK_y\right) + 4n^4K_xK_y = 0$$

$$x^2 + bx + c = 0$$
$$\begin{cases} c > 0 \\ \dfrac{-b \pm \sqrt{b^2 - 4c}}{2} < 0 \\ b^2 - 4c > 0 \end{cases}$$
$$\left(1 + 3K_y + K_yK_x\right)^2 > 16K_yK_x$$
$$K_xK_y > 0$$

Stability analysis of the linear equations in x-y plane



$$K_x = \frac{I_z - I_y}{I_x}$$
$$K_y = \frac{I_z - I_x}{I_y}$$
$$K_z = \frac{I_y - I_x}{I_z}$$

1 and 2 stability
3 and 4 y axis instability
5 and 7 x, y, z axis instability
6 and 8 z axis instability

$$\left(1+3K_y+K_yK_x\right)^2 > 16K_yK_x$$

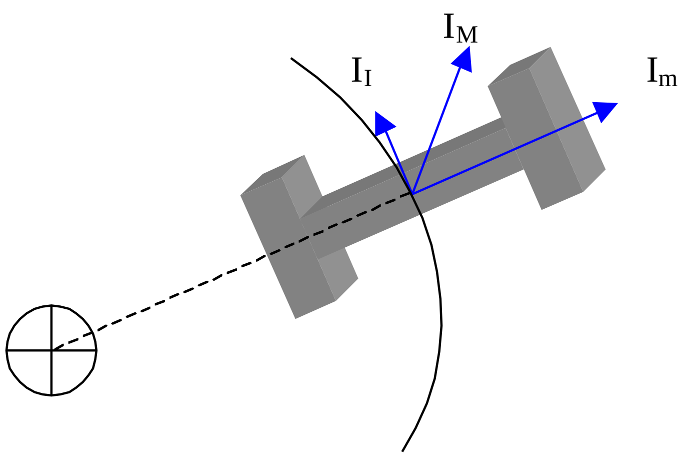
$$K_xK_y > 0$$

$$K_z > 0$$

Stability conditions

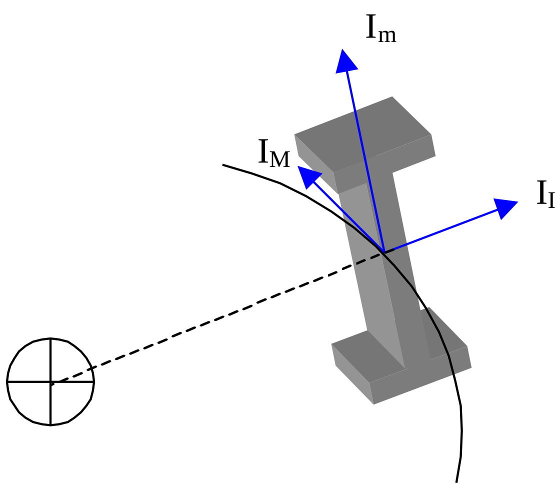
Region 1

$I_z > I_y > I_x$



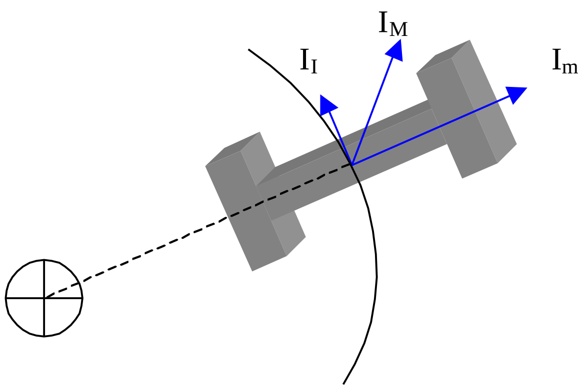
Region 2

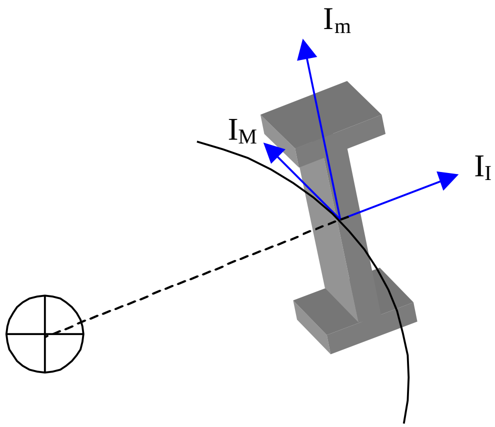
$I_y > I_x > I_z$



$$1 + 3K_x + K_yK_x > 4\sqrt{K_yK_x}$$

Stable conditions





Robustness to stability

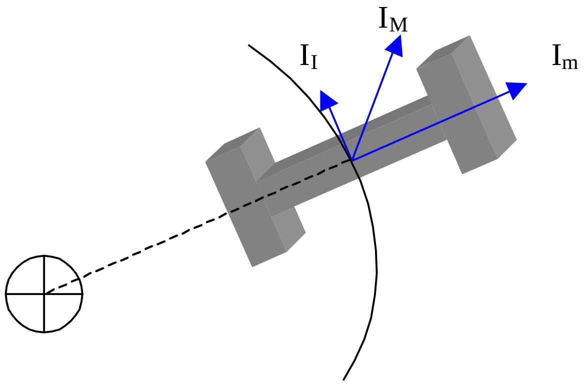
Nonlinear form

$$\underset{-}{M} = \frac{3Gm_t}{R^3}\begin{bmatrix}(I_z - I_y)c_3c_2 \\ (I_x - I_z)c_1c_3 \\ (I_y - I_x)c_2c_1\end{bmatrix}$$

Linear form

$$\underset{-}{M} = 3n^2\begin{Bmatrix}0 \\ (I_x - I_z)\alpha_y \\ (I_x - I_y)\alpha_z\end{Bmatrix}$$

Stable conditions



special case for which the satellite mass tends to be aligned with the yaw axis -> $I_z = I_y$

$$I_x \to 0 \Rightarrow K_y \to 1$$
$$K_z \to 1$$
$$K_x = 0$$

$$\omega_{ry} = 2n$$
$$\omega_p = n\sqrt{3}$$