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Spacecraft Attitude Dynamics

prof. Franco Bernelli

Magnetic Field Modelling

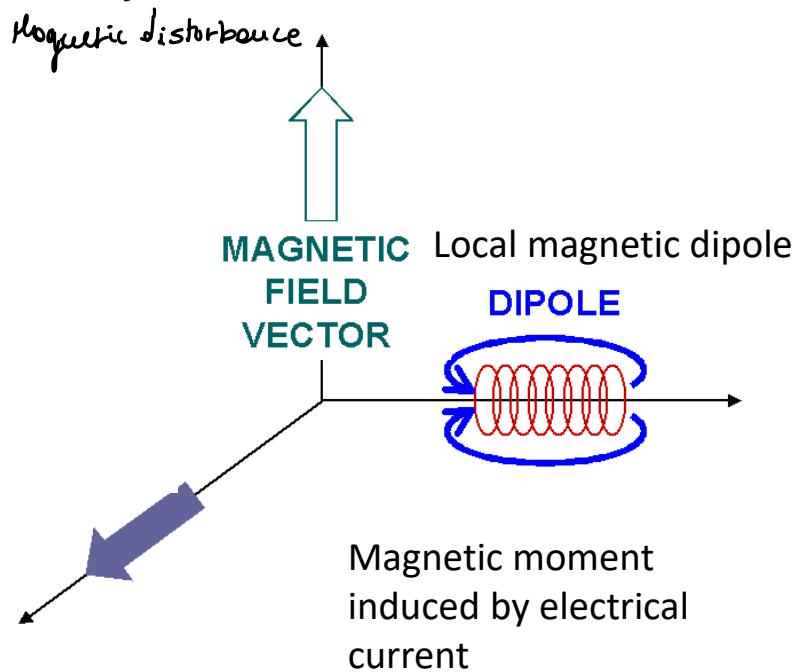
Important for s/c close to Earth

Important for sensor measure magnetic field + it can be used to produce torque for control → will more adopted than Solar radiation pressure or air drag.

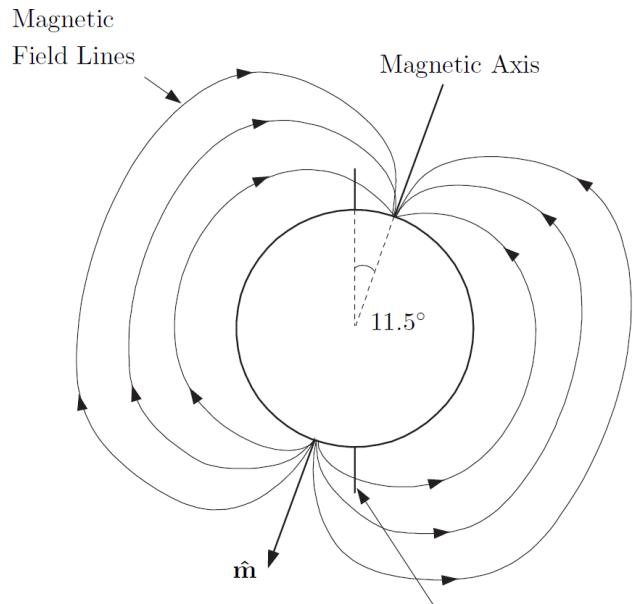
Magnetic field disturbance torque

$$\underline{M} = \underline{m} \wedge \underline{B}$$

→ Local magnetic field vector
→ magnetic dipole



It's a simple equation
but it is how to model the
vector \underline{B} and especially \underline{m}
Now let's see how to model \underline{B}



Looks pretty much like two fields produced by a dipole. The axis is tilted wrt spin axis.
the south of magnetic pole is at the north magnetic
Magnetic B-field pole.



Potential in terms of Schmidt quasi-normalized associated Legendre Polynomial

② Decay of the magnetic sphere

with distance

R = average equatorial radius r = distance from the center of Earth

$n+1$ n \rightarrow on the order of the shape factor because its importance decreases due to this expansion.

$$V(r, \theta, \varphi) = R \sum_{n=1}^k \left(\frac{R}{r} \right)^{n+1} \sum_{m=0}^n (g_n^m \cos(m\varphi) + h_n^m \sin(m\varphi)) P_n^m(\theta)$$

Polynomial function depends on the co-latitude

shape function by its self

r , θ and φ are the spherical coordinates of the position of the satellite from the center of the Earth, co-latitude and East longitude from Greenwich.

①

Usually the gravity field of the earth is modelled using harmonic functions \rightarrow These harmonic functions are basically shape functions that take into account the distributions of mass (Earth is more like an ellipsoid than a sphere). The combination of all these functions give us the gravity field \rightarrow The same think is done for the expansion of IAGA – International Association of Geomagnetism and Aeronomy the potential of the geomagnetic field. The shape function will respect the BCS \rightarrow e.g. have the same direction of the magnetic field at North or South magnetic pole.

IGRF – International Geomagnetic Reference Field

$$B_r = \frac{-\partial V}{\partial r}$$

$B_r \rightarrow$ radial component, positive outward

$$B_\theta = \frac{-1}{r} \frac{\partial V}{\partial \theta}$$

$B_\theta \rightarrow$ coelevation component, positive if directed towards south

$$B_\varphi = \frac{-1}{r \sin \theta} \frac{\partial V}{\partial \varphi}$$

$B_\varphi \rightarrow$ azimuth component, positive towards east



Potential in terms of Schmidt quasi-normalized associated Legendre Polynomial

expansion order of the model

(3)

How we can distinguish between magnetic field of different field?
The real shape of the model comes from the coefficients g and h .
They are computed from empirical observation of the magnetic field.
 $n+1$ we can distinguish the different planets by looking at
 n the coefficient g and h .

$$V(r, \theta, \varphi) = R \sum_{n=1}^k \left(\frac{R}{r} \right)^n \sum_{m=0}^{n+1} (g_n^m \cos(m\varphi) + h_n^m \sin(m\varphi)) P_n^m(\theta)$$

To compute g and h we need to get the same number of measurement of the defs of the system. We will have some necessary so we will get more points of measurement to do an approximation at the minimum squares to get the best results.

r , θ and φ are the spherical coordinates of the position of the satellite from the center of the Earth, co-latitude and East longitude from Greenwich.

IAGA – International Association of Geomagnetism and Aeronomy

IGRF – International Geomagnetic Reference Field

The vector of the magnetic field is computed from the potential using spherical coordinates.

gradient of potential function

$$B_r = \frac{-\partial V}{\partial r}$$

B_r -> radial component, positive outward

$$B_\theta = \frac{-1}{r} \frac{\partial V}{\partial \theta}$$

B_θ -> coelevation component, positive if directed towards south

$$B_\varphi = \frac{-1}{r \sin \theta} \frac{\partial V}{\partial \varphi}$$

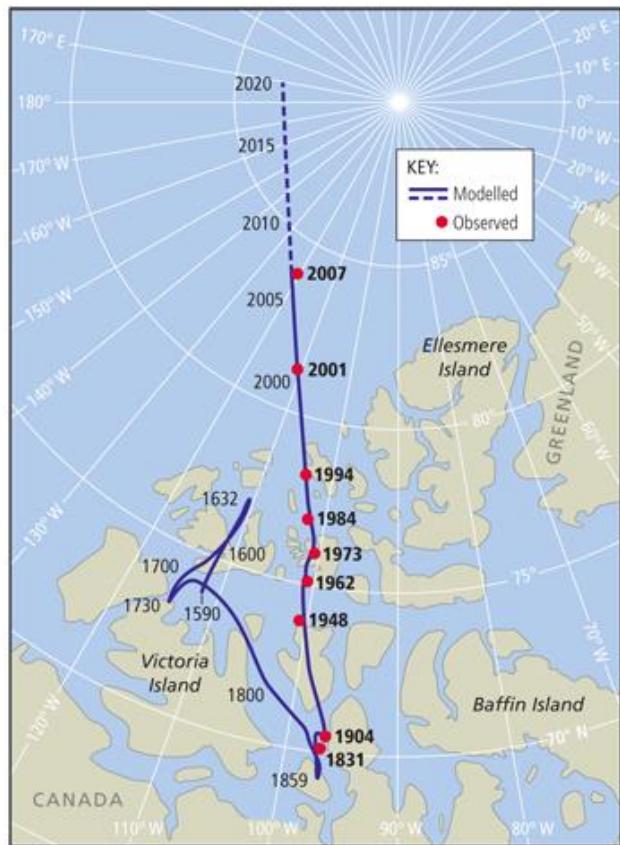
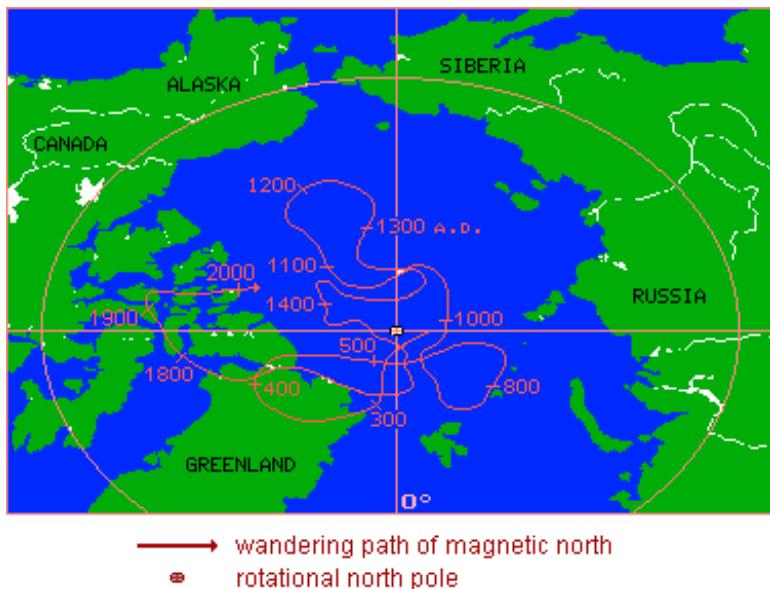
B_φ -> azimuth component, positive towards east



The wandering magnetic North

The magnetic pole is changing continually its relative position wrt the geographic north

TGPF → It is computed at a given epoch. The terms treat time with a time as the amplitude of the function $\sin(\omega t + \phi)$.



Gaussian coefficients (nT)

		IGRF 1995		IGRF 2000	
n	m	g_n^m	h_n^m	g_n^m	h_n^m
1	0	-29682	-	-29615	-
1	1	-1789	5318	-1728	5186
2	0	-2197	-	-2267	-
2	1	3074	-2356	3072	-2478
2	2	1685	-425	1672	-458
3	0	1329	-	1341	-
3	1	-2268	-263	-2290	-227
3	2	1249	302	1253	296
3	3	769	-406	715	-492
4	0	941	-	935	-
4	1	782	262	787	272
4	2	291	-232	251	-232
4	3	-421	98	-405	119
4	4	116	-301	110	-304

		IGRF 1995		IGRF 2000	
n	m	g_n^m	h_n^m	g_n^m	h_n^m
5	0	-210	-	-217	-
5	1	352	44	351	44
5	2	237	157	222	172
5	3	-122	-152	-131	-134
5	4	-167	-64	-169	-40
5	5	-26	99	-12	107
6	0	66	-	72	-
6	1	64	-16	68	-17
6	2	65	77	74	64
6	3	-172	67	-161	65
6	4	2	-57	-5	-61
6	5	17	4	17	1
6	6	-94	28	-91	44

		IGRF 1995		IGRF 2000	
n	m	g_n^m	h_n^m	g_n^m	h_n^m
7	0	78	-	79	-
7	1	-67	-77	-74	-65
7	2	1	-25	0	-24
7	3	29	3	33	6
7	4	4	22	9	24
7	5	8	16	7	15
7	6	10	-23	8	-25
7	7	-2	-3	-2	-6
8	0	24	-	25	-
8	1	4	12	6	12
8	2	-1	-20	-9	-22
8	3	-9	7	-8	8
8	4	-14	-21	-17	-21
8	5	4	12	9	15

		IGRF 1995		IGRF 2000	
n	m	g_n^m	h_n^m	g_n^m	h_n^m
8	6	5	10	7	9
8	7	0	-17	-8	-16
8	8	-7	-10	-7	-3
9	0	4	-	5	-
9	1	9	-19	9	-20
9	2	1	15	3	13
9	3	-12	11	-8	12
9	4	9	-7	6	-6
9	5	-4	-7	-9	-8
9	6	-2	9	-2	9
9	7	7	7	9	4
9	8	0	-8	-4	-8
9	9	-6	1	-8	5



Magnetic field

$$\mathbf{B} = -\nabla V$$

Earth rotating-fixed frame Frame

$$b_r = \sum_{n=1}^k \left(\frac{R}{r}\right)^{n+2} (n+1) \sum_{m=0}^n (g_n^m \cos m\phi + h_n^m \sin m\phi) P_n^m(\theta)$$

$$b_\theta = - \sum_{n=1}^k \left(\frac{R}{r}\right)^{n+2} \sum_{m=0}^{n+2} (g_n^m \cos m\phi + h_n^m \sin m\phi) \frac{\partial P_n^m(\theta)}{\partial \theta}$$

$$b_\phi = - \frac{1}{\sin \theta} \sum_{n=1}^k \left(\frac{R}{r}\right)^{n+2} \sum_{m=0}^n (-g_n^m \sin m\phi + h_n^m \cos m\phi) P_n^m(\theta)$$

It is a designation of which order should be used to model the magnetic field → it depends on the application. Notice that the coefficient tends to decrease as we go up with the order of approximations.

Schmidt quasi-normalized associated Legendre Polynomial

Normalized form to compute the polynomials.

$$P_n^m = \left[\frac{2(n-m)!}{(n+m)!} \right]^{1/2} P_{n,m}$$

$$P_n(v) = \frac{1}{2^n n!} \left(\frac{d}{dv} \right)^n (v^2 - 1)^n$$

$$P_{n,m}(v) = (1 - v^2)^{1/2m} \frac{d^m}{dv^m} (P_n(v))$$

$$P_0(v) = 1$$

$$P_1(v) = v$$

$$P_2(v) = (3v^2 - 1)/2$$

$$P_3(v) = (5v^3 - 3v)/2$$

Davis, J., "Mathematical Modeling of Earth's Magnetic Field" Virginia Tech, Technical Note, 2004.



Magnetic field

$$\mathbf{B} = -\nabla V$$

Alternative normalization due to Gauss

Based on the previous model but with a reorganization of the coefficients. Different normalization strategy.
different way to express them to avoid confusion.

$$B_r = \frac{-\partial V}{\partial r} = \sum_{n=1}^k \left(\frac{R}{r}\right)^{n+2} (n+1) \sum_{m=0}^n \left(g^{n,m} \cos(m\varphi) + h^{n,m} \sin(m\varphi) \right) P^{n,m}(\theta)$$

$$B_\theta = \frac{-1}{r} \frac{\partial V}{\partial \theta} = - \sum_{n=1}^k \left(\frac{R}{r}\right)^{n+2} \sum_{m=0}^n \left(g^{n,m} \cos(m\varphi) + h^{n,m} \sin(m\varphi) \right) \frac{\partial P^{n,m}(\theta)}{\partial \theta}$$

$$B_\varphi = \frac{-1}{r \sin \theta} \frac{\partial V}{\partial \varphi} = \frac{-1}{\sin \theta} \sum_{n=1}^k \left(\frac{R}{r}\right)^{n+2} \sum_{m=0}^n m \left(-g^{n,m} \sin(m\varphi) + h^{n,m} \cos(m\varphi) \right) P^{n,m}(\theta)$$

$$g^{n,m} = S_{n,m} g_n^m$$
$$h^{n,m} = S_{n,m} h_n^m$$

mathematical convention from one method
to the other.

for the previous model the normalization
is due to shrink and the φ variable
comes out from lagrange.



Magnetic field

$$\mathbf{B} = -\nabla V$$

Alternative normalization due to Gauss

Using the Gauss normalization permits us to calculate the coefficients in an recursive way
The code is sequential \rightarrow it can ease the computation part.
In short all the terms are different from one another and they must be computed one by one.

$$S_{0,0} = 1$$

$$S_{n,0} = S_{n-1,0} \frac{(2n-1)}{n} \quad \text{for } n \geq 1$$

$$S_{n,m} = S_{n,m-1} \left[\frac{(\delta_m^1 + 1)(n-m+1)}{(n+m)} \right]^{\frac{1}{2}} \quad \text{for } m \geq 1$$

NOTE

We can compute all the terms in one single loop because the terms are recursive.

$$P^{0,0} = 1$$

\hookrightarrow This for the Gauss model.

$$P^{n,n} = \sin \theta P^{n-1,n-1}$$

while the coding of the order change

$$P^{n,m} = \cos \theta P^{n-1,m} - K^{n,m} P^{n-2,m}$$

every time with the order \rightarrow

\hookrightarrow it is more difficult to code and optimized.

$$K^{n,m} = 0 \quad \text{for } n = 1$$

$$K^{n,m} = \frac{(n-1)^2 - m^2}{(2n-1)(2n-3)} \quad \text{for } n > 1$$



Magnetic field

$$\mathbf{B} = -\nabla V$$

Earth rotating fixed frame Frame

$$b_r = \sum_{n=1}^k \left(\frac{R}{r}\right)^{n+2} (n+1) \sum_{m=0}^n (g_n^m \cos m\phi + h_n^m \sin m\phi) P_n^m(\theta)$$

$$b_\theta = - \sum_{n=1}^k \left(\frac{R}{r}\right)^{n+2} \sum_{m=0}^n (g_n^m \cos m\phi + h_n^m \sin m\phi) \frac{\partial P_n^m(\theta)}{\partial \theta}$$

$$b_\phi = - \frac{1}{\sin \theta} \sum_{n=1}^k \left(\frac{R}{r}\right)^{n+2} \sum_{m=0}^n (-g_n^m \sin m\phi + h_n^m \cos m\phi) P_n^m(\theta)$$

The magnetic field is given in the inertial frame of earth rotating fixed reference frame \rightarrow but it is vector so it is necessary to change the magnetic field in the inertial frame and there can be transformed especially in the body frame of reference.

Inertial Frame \rightarrow Transformation depends only on the point of observation.

$$b_1 = (b_r \cos(\delta) + b_\theta \sin(\delta)) \cos(\alpha) - b_\phi \sin(\alpha)$$

$$b_2 = (b_r \cos(\delta) + b_\theta \sin(\delta)) \sin(\alpha) + b_\phi \cos(\alpha)$$

$$b_3 = (b_r \sin(\delta) - b_\theta \cos(\delta))$$

$$\underbrace{b_N}_{\text{Body Frame}} =$$

r = radii direction
 φ = longitude
 δ = colatitude

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} \cos \delta \cos \alpha & -\sin \alpha & \cos \alpha \sin \delta \\ \sin \delta \cos \alpha & \cos \alpha & \sin \alpha \sin \delta \\ -\sin \alpha & 0 & \cos \alpha \end{bmatrix} \begin{bmatrix} b_r \\ b_\theta \\ b_\phi \end{bmatrix}$$

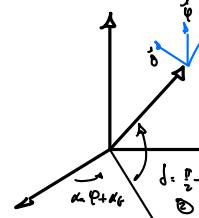
ANR

Body Frame

$$\underline{b}_B = A_B/N \underline{b}_N$$

$$\underline{b}_B = A_{BN}^{-1} A_{NR} \begin{bmatrix} b_r \\ b_\theta \\ b_\phi \end{bmatrix}$$

$$A_{BN}^{-1} = A_{NR} = \begin{bmatrix} \cos \delta \cos \alpha & -\sin \alpha & \cos \alpha \sin \delta \\ \sin \delta \cos \alpha & \cos \alpha & \sin \alpha \sin \delta \\ -\sin \alpha & 0 & \cos \alpha \end{bmatrix} \xrightarrow{\text{det} = 0} A_{NR}$$



$$\delta = \frac{\pi}{2} - \theta$$

$$\alpha = \phi + \alpha_G$$

By definition

$$\begin{bmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{det} = 0} \begin{bmatrix} \cos \delta & 0 & -\sin \delta \\ 0 & 1 & 0 \\ \sin \delta & 0 & \cos \delta \end{bmatrix} \xrightarrow{\text{det} = 0} \begin{bmatrix} \cos \delta \cos \alpha & -\sin \alpha & \cos \alpha \sin \delta \\ \sin \delta \cos \alpha & \cos \alpha & \sin \alpha \sin \delta \\ -\sin \alpha & 0 & \cos \alpha \end{bmatrix}$$



Simple dipole model of the Earth's Magnetic Field ($n=1$)

Use this model only if we have confidence in the precision of the dipole model

alternative form if we know the position of the spacecraft wrt the Earth fixed frame of reference and also the epoch.

$$\underline{b}_N = \frac{R^3 H_0}{r^3} [3(\hat{m} \cdot \hat{r})\hat{r} - \hat{m}]$$

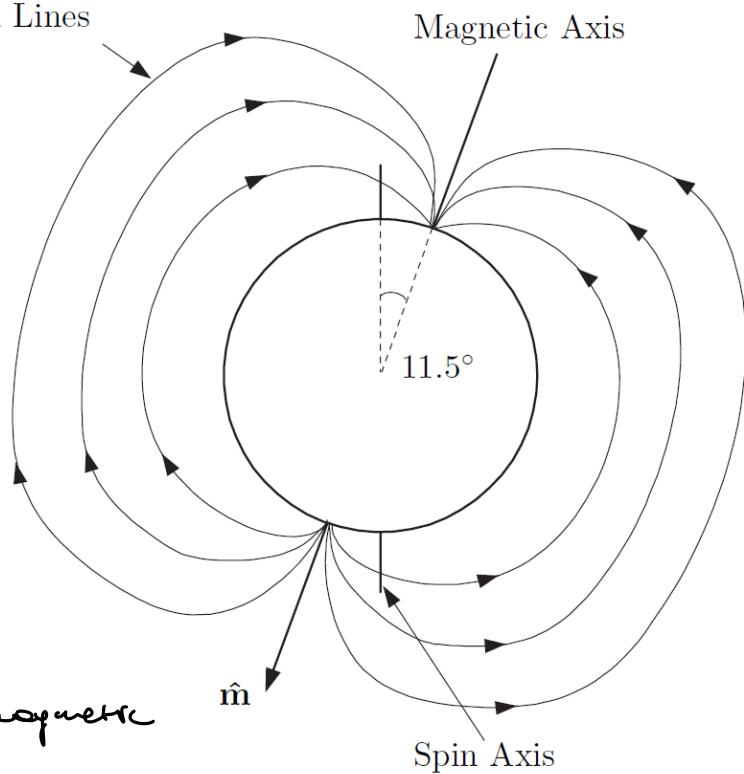
if we take the coefficient up to $n=0$ we get this terms from the Schmidt - Legendre approx.

$$H_0 = ((g_1^0)^2 + (g_1^1)^2 + (h_1^1)^2)^{1/2}$$

$$\hat{m} = \begin{bmatrix} \sin 11.5^\circ \cos \omega_\oplus t \\ \sin 11.5^\circ \sin \omega_\oplus t \\ \cos 11.5^\circ \end{bmatrix}$$

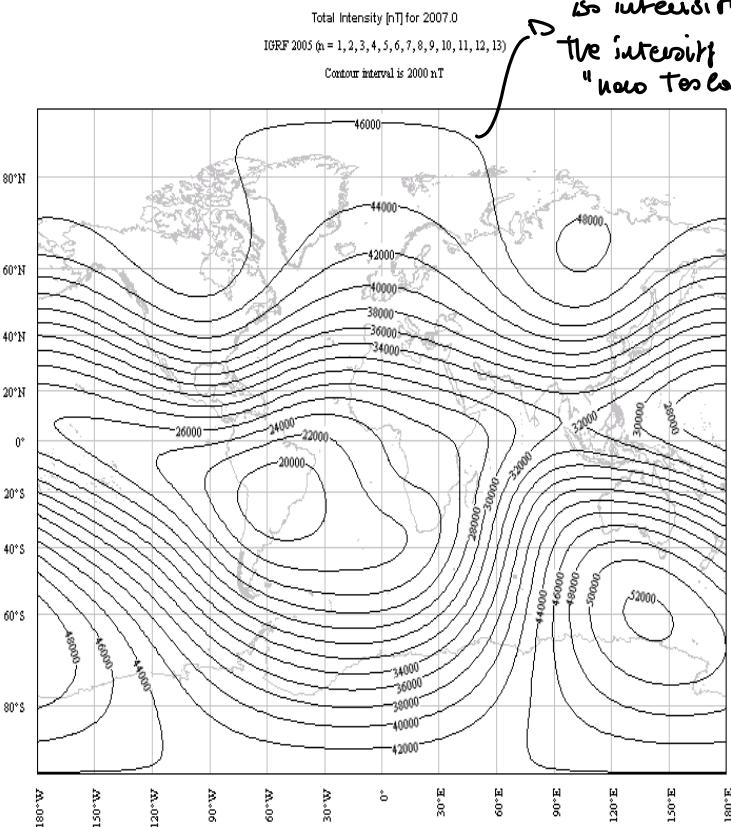
to indicate of magnetic dipole axis.

Magnetic
Field Lines

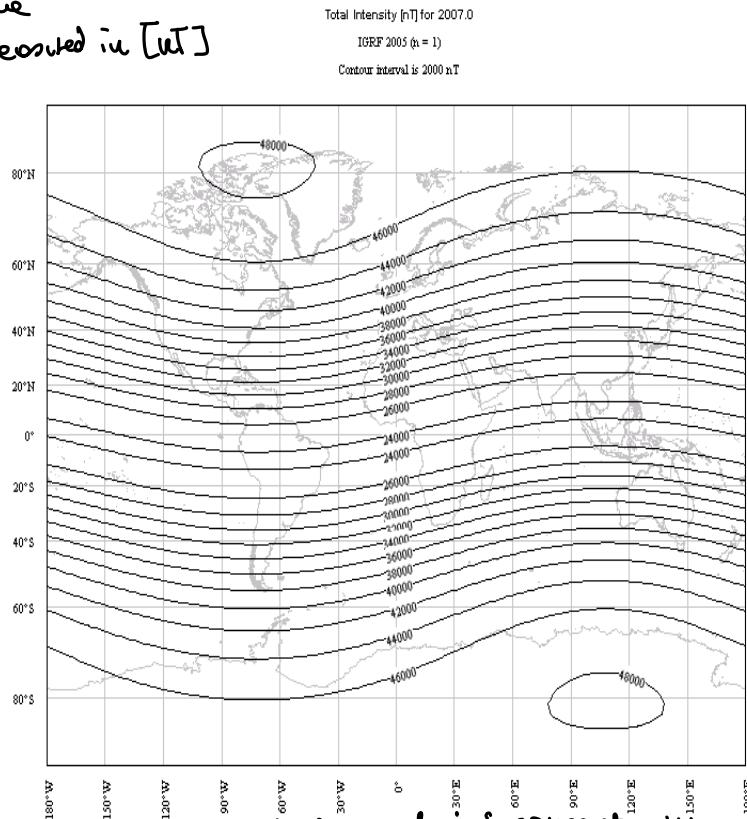


Magnetic field Model at 500km altitude

Order 13 → considered as "exact model"



Dipole model

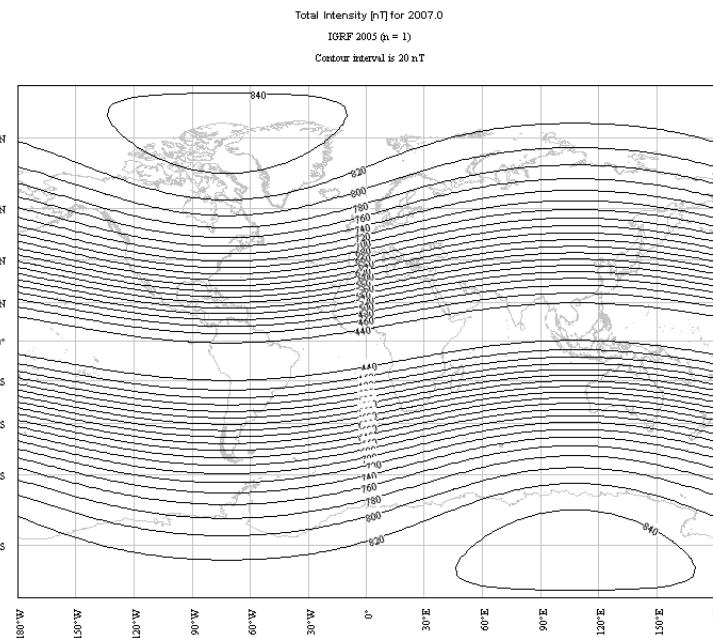
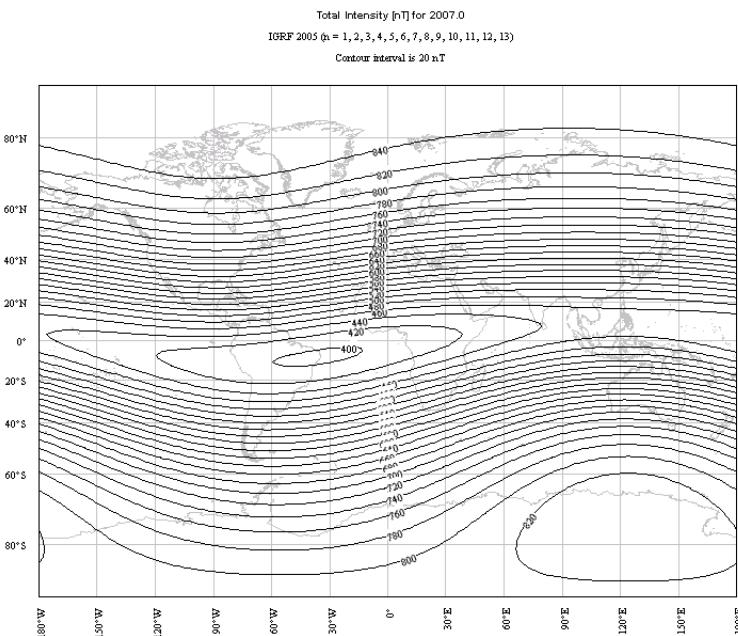


more or less the dipole model is incorrect with exception of few zone → polar regions.



Magnetic field Model at 10000km altitude

If we go at or are higher altitude from the Earth \rightarrow the dipole model is much more accurate except for the equatorial region. @ higher altitude is seems reasonable to use lower Order 13 order approximation and if we are enough for every we could use the dipole model approximation.

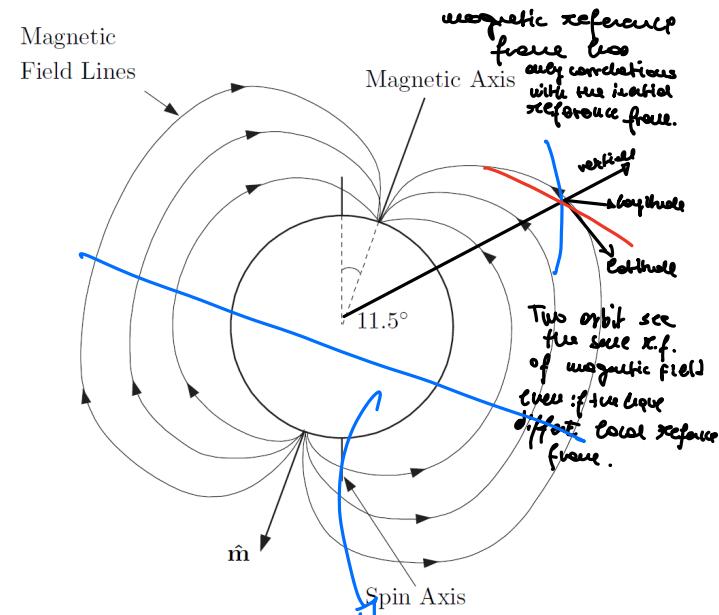
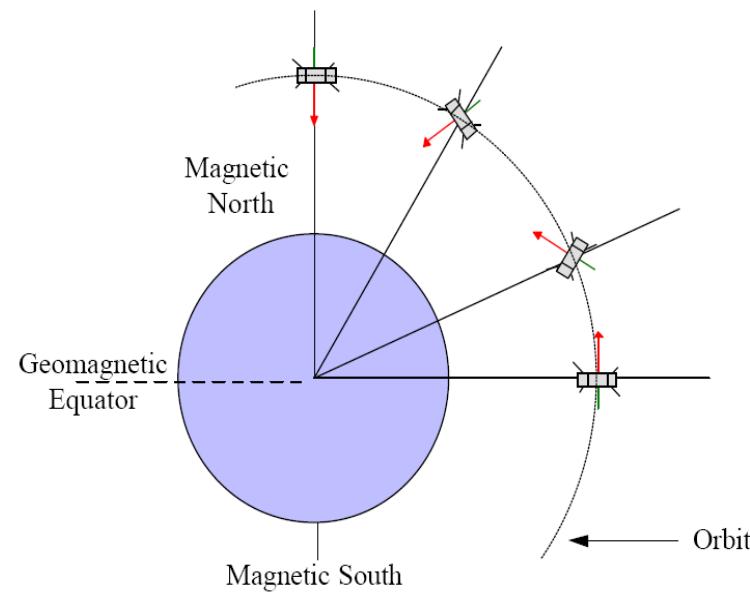


It is also important to model correctly the magnetic properties of SIC.



Permanent magnet for passive pointing

The magnet will tend to align with the local magnetic field



The only orbit that does have a constant magnetic field is the equatorial w.r.t the magnetic reference frame.



Passive control methods

Type	Advantages	Disadvantages
Spin-stabilised (~1° accuracy)	Simple, passive, long-life, provides scan motion, gyroscopic stability for large burns, inertial pointing	Poor manoeuvrability, low solar cell efficiency (cover entire drum),
Aerodynamic (~5° accuracy)	Simple, passive, low cost, useful for Earth pointing <i>Not fast</i> Aerodynamic forces → drag → orbit decay <i>accuracy for short period of time.</i>	Poor yaw stability, effective only in low altitude, poor accuracy, degrades orbit
Dual-spin stabilised (~0.1° accuracy)	Provides both fixed pointing (on de-spun platform) and scanning motion, gyroscopic stability for large burns	Require de-spin mechanism, low solar cell efficiency (cover entire drum), cost can approach 3-axis if high accuracy
Gravity-gradient (~5° accuracy)	Simple, low cost totally passive, long-life, provides simple passive Earth pointing mode <i>require less disturbance payload etc.</i>	Low accuracy, almost no manoeuvrability, poor yaw stability, require deployment mechanism
Magnetic (~1° accuracy)	Simple, low cost, can be passive with use of permanent magnet or active with use of electromagnets	Poor accuracy (uncertainty in Earth's magnetic field), magnetic interference with science payload

↳ Different type of orientation capability → save direction of local magnetic field



Mathematical Modeling of Earth's Magnetic Field

TECHNICAL NOTE

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May 12, 2004

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1 Introduction

In controlling satellites, one viable method of stabilization and control is the use of Earth's magnetic field. There are obvious advantages and disadvantages to this method. The controllers can be small and do not deteriorate over time, but they also can only produce torques perpendicular to the local magnetic field. An in depth analysis of this problem can be found in the thesis "A Nonlinear Magnetic Controller for Three-Axis Stability of Nanosatellites" by Kristin Makovec.¹ The purpose of this paper is to present a model of Earth's magnetic field such that further analysis of this method of control could be completed.

The Earth's magnetic field generally resembles the field around a magnetized sphere, or a tilted dipole seen in Figure 1 below. As of 1999, the dipole axis was tilted approximately 11.5° from the spin axis, and drifting approximately $0.2^\circ/\text{yr}$. Its strength at the Earth's surface varies from approximately 30000nT near the equator to 60000nT near the poles. Further, there exists a low magnetic intensity field at approximately 25°S and 45°W known as the Brazilian Anomaly. A high exists at 10°N and 100°E , and the two of these together suggest that not only is the dipole axis tilted, but it does not quite pass through the center of the Earth.¹

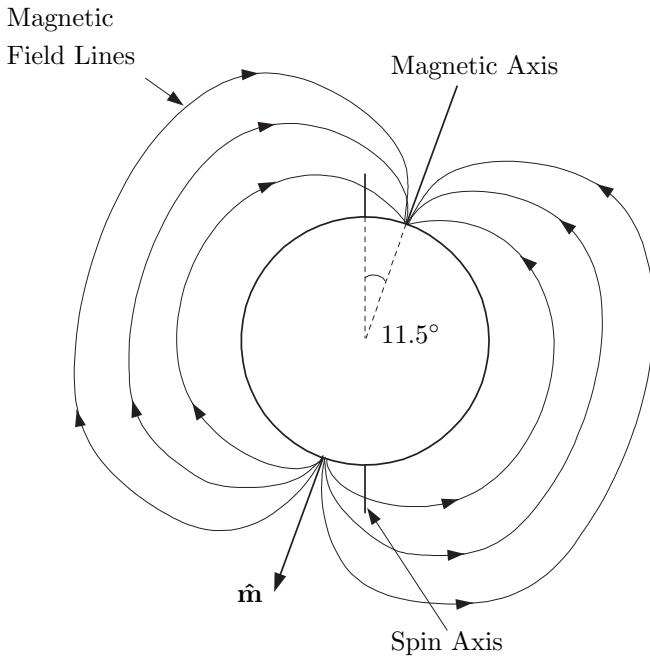


Figure 1: Magnetic Field Model¹

1.1 Modeling the magnetic field

The accepted model for Earth's magnetic field is the International Geomagnetic Reference Field, put forth by the International Association of Geomagnetism and Aeronomy (IAGA), and is the central topic of this paper. An overview of this model can be found on the website of the IAGA's Working Group V-MOD.²

The IGRF is essentially a set of Gaussian coefficients, g_n^m and h_n^m , that are put forth every 5 years by IAGA for use in a spherical harmonic model. At each of these epoch years, the group considers several proposals and typically adopts a compromise that best fits the data available. The coefficients for a given epoch year are referred to by IGRF and then the year, as in IGRF2000. The model includes both the coefficients for the epoch year and secular variation variables, which track the change of these coefficients in nanoTessla per year. These secular variation coefficients are used to extrapolate the Gaussian coefficients to the date in question. Once data becomes available about the actual magnetic field for a given epoch year, the model is adjusted and becomes the Definitive Geomagnetic Reference Field, or DGRF.²

Typically the IGRF consists of 120 coefficients for each epoch year, with 80 secular variation coefficients. However, due to unprecedented geomagnetic data available, the IAGA released a new set of values for IGRF2000 in July 2003 in the 9th-generation IGRF. This new model expanded to increase the precision of the coefficients to one-tenth of a nanoTessla (up from one nanoTessla), and increased the number of coefficients to degree 13 (rather than 10).²

1.2 Random variation in the magnetic field

The many variations in the magnetic field around the Earth that can not be accurately modeled due to their random nature are thoroughly described in Reference 1, while a brief overview is provided here. It is important to be aware of these variations when considering spacecraft design. First, there are temporal variations that occur about every 27 days when the active solar area of the Sun faces the Earth. These variations last between seconds and days, and are particularly bad when the Earth is near equinox in March-April and September-October.

The second type of variation is diurnal variations which occur due to partial movement within the ionosphere. These polar (auroral) and equatorial electrojets can have significant impact on the magnetic field. The auroral electrojet can cause changes on the order of 1000nT to 1500nT at the Earth's surface, while the equatorial electrojets cause disturbances on the order of 220nT between the altitudes of 96 and 130 kilometers.

The last variation is a result of magnetic storms, which occur during solar flares. Solar flares are very closely related to Sun activity and so magnetic storms generally follow the same 27 day pattern seen in the general temporal variations, although they occur less frequently. During the first phase of the magnetic storm the effect is around 50nT, however, during the main phase, which lasts several hours, the variation is as much as 400nT.

1.3 Breakdown of this paper

This paper is broken down as follows. Section 2 gives the equations used to formulate the spherical harmonic model used in the IGRF. Section 3 provides recursive formulations of the formulas in Section 2 that can be used in computer modeling. Section 4 concludes with specific information about this computer model, including its limitations and possible improvements.

2 Mathematics of the IGRF

According to physics, the magnetic field, \mathbf{B} , is defined as the negative gradient of the scalar potential function V , such that

$$\mathbf{B} = -\nabla V \quad (1)$$

Although a simple dipole model gives a good approximation of the geomagnetic field, it can be modeled more closely using a spherical harmonic model of the scalar potential as given in Equation (2). This is the equation about which the IGRF is based.

$$V(r, \theta, \phi) = a \sum_{n=1}^k \left(\frac{a}{r}\right)^{n+1} \sum_{m=0}^n (g_n^m \cos m\phi + h_n^m \sin m\phi) P_n^m(\theta) \quad (2)$$

Here, a is the reference radius of the Earth ($a = 6371.2\text{km}$), r , θ , and ϕ are the geocentric coordinates (r is the radius in kilometers, θ is the co-latitude ($\theta = 90^\circ - \text{latitude}$), and ϕ is the longitude. The coefficients g_n^m and h_n^m are Gaussian coefficients put forth by the IAGA for the IGRF, and $P_n^m(\theta)$ represents the Schmidt quasi-normalized associated Legendre functions of degree n and order m . The input to this function is actually $\cos\theta$, rather than θ , but this has been dropped for brevity.

2.1 Derivatives of the scalar potential function

Since the magnetic field strength is calculated by taking the partial derivatives of V , the following equations are necessary from Wertz.³

$$B_r = \frac{-\partial V}{\partial r} = \sum_{n=1}^k \left(\frac{a}{r}\right)^{n+2} (n+1) \sum_{m=0}^n (g_n^m \cos m\phi + h_n^m \sin m\phi) P_n^m(\theta) \quad (3a)$$

$$B_\theta = \frac{-1}{r} \frac{\partial V}{\partial \theta} = - \sum_{n=1}^k \left(\frac{a}{r}\right)^{n+2} \sum_{m=0}^n (g_n^m \cos m\phi + h_n^m \sin m\phi) \frac{\partial P_n^m(\theta)}{\partial \theta} \quad (3b)$$

$$B_\phi = \frac{-1}{r \sin \theta} \frac{\partial V}{\partial \phi} = \frac{-1}{\sin \theta} \sum_{n=1}^k \left(\frac{a}{r}\right)^{n+2} \sum_{m=0}^n m(-g_n^m \sin m\phi + h_n^m \cos m\phi) P_n^m(\theta) \quad (3c)$$

where B_r , B_θ , and B_ϕ represent the field strength in local tangential coordinates, and the other variables defined as before.

2.2 Legendre polynomials

In order to calculate the magnetic field, one must first calculate the associated Legendre polynomials. Legendre polynomials are a set of orthogonal polynomials that also satisfy the zero mean condition. The following equations for the Legendre polynomials and associated Legendre polynomials come from Schaub.⁴

Regular Legendre polynomials $P_n(v)$ are calculated to satisfy the following equation:

$$(1 - 2vx + x^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(v)x^n \quad (4)$$

When solved, this becomes Rodrigues' formula:

$$P_n(v) = \frac{1}{2^n n!} \left(\frac{d}{dv} \right)^n (v^2 - 1)^n \quad (5)$$

For reference, this equation yields the following first four Legendre polynomials:

$$P_0(v) = 1 \quad (6a)$$

$$P_1(v) = v \quad (6b)$$

$$P_2(v) = (3v^2 - 1)/2 \quad (6c)$$

$$P_3(v) = (5v^3 - 3v)/2 \quad (6d)$$

2.3 Associated Legendre polynomials

2.3.1 Calculating associated Legendre polynomials

The above Legendre polynomials are related to the associated Legendre polynomials through the following equation:

$$P_{n,m}(v) = (1 - v^2)^{1/2m} \frac{d^m}{dv^m} (P_n(v)) \quad (7)$$

Note that for all m greater than n , the associated Legendre polynomial is equal to zero. Following are the associated Legendre polynomials through the third degree, which can be verified using equations 6 and 7.

$$P_{1,1}(v) = \sqrt{1 - v^2} \quad P_{2,1}(v) = 3v\sqrt{1 - v^2} \quad P_{3,1}(v) = \frac{3}{2}\sqrt{1 - v^2}(5v^2 - 1) \quad (8a)$$

$$P_{2,2}(v) = 3(1 - v^2) \quad P_{3,2}(v) = 15v(1 - v^2) \quad (8b)$$

$$P_{3,3}(v) = 15(1 - v^2)^{3/2} \quad (8c)$$

In the case of the magnetic field model, v is replaced by $\cos \theta$, such that the above equations become

$$P_{1,1}(v) = \sin \theta \quad P_{2,1}(v) = 3 \cos \theta \sin \theta \quad P_{3,1}(v) = \frac{3}{2} \sin \theta (5 \cos^2 \theta - 1) \quad (9a)$$

$$P_{2,2}(v) = 3 \sin^2 \theta \quad P_{3,2}(v) = 15 \cos \theta \sin^2 \theta \quad (9b)$$

$$P_{3,3}(v) = 15 \sin^3 \theta \quad (9c)$$

2.3.2 Note about proper notation

The formulas in Equation (9) represent traditional associated Legendre polynomials that have not been normalized in any way, and they are represented by $P_{n,m}$. There are two commonly used normalizations that must also be accounted for as described by Jacobs.⁵ The first is the Gaussian normalized associated Legendre polynomials, $P^{n,m}$. These are related to the non-normalized set by the equation

$$P^{n,m} = \frac{2^n!(n-m)!}{(2n)!} P_{n,m} \quad (10)$$

In magnetic field modeling, the Schmidt quasi (semi) - normalized form, P_n^m is used, related by the equation

$$P_n^m = \left[\frac{2(n-m)!}{(n+m)!} \right]^{1/2} P_{n,m} \quad (11)$$

The recursive formulas for the Gaussian normalized associated Legendre polynomials are given by Wertz³ and used in this model, so the relationship between the Gaussian normalized and Schmidt quasi-normalized polynomials is explained in detail in the next section.

2.4 Schmidt quasi-normalization

The relationship between the Gaussian normalized associated Legendre polynomials and the Schmidt quasi-normalized is given by Wertz³ as

$$P_n^m = S_{n,m} P^{n,m} \quad (12)$$

where $P^{n,m}$ is the Gaussian associated Legendre polynomial, and $S_{n,m}$ is defined by

$$S_{n,m} = \left[\frac{(2-\delta_m^0)(n-m)!}{(n+m)!} \right]^{1/2} \frac{(2n-1)!!}{(n-m)!} \quad (13)$$

The Kronecker delta is defined as $\delta_i^j = 1$ if $i = j$ and $\delta_i^j = 0$ otherwise. Further, $(2n-1)!! \equiv 1 \cdot 3 \cdot 5 \cdots (2n-1)$.

Due to the fact that these normalization values can be calculated irrespective of the value of θ at which the associated Legendre polynomials are calculated, it is much simpler to instead normalize the model coefficients, g_n^m and h_n^m , recognizing that

$$g^{n,m} = S_{n,m} g_n^m \quad (14a)$$

$$h^{n,m} = S_{n,m} h_n^m \quad (14b)$$

This allows the normalization to only occur once, and then use those new coefficients can be used to compute the field strength at whatever location is desired.

2.5 Final result

In order for the results of Equation (3) to be effective in satellite work, they must be converted to geocentric inertial components, using the following equation:

$$B_x = (B_r \cos \delta + B_\theta \sin \delta) \cos \alpha - B_\phi \sin \alpha \quad (15a)$$

$$B_y = (B_r \cos \delta + B_\theta \sin \delta) \sin \alpha + B_\phi \cos \alpha \quad (15b)$$

$$B_z = (B_r \sin \delta + B_\theta \cos \delta) \quad (15c)$$

Where δ is the latitude measured positive North from the equator, and α is the local sidereal time of the location in question.

3 Recursive formulas for computation

In order to produce efficient computer code, the preceding formulas must be decomposed into recursive formulas as seen in Wertz.³ This section provides those recursive relationships for use in coding.

3.1 Associated Legendre Polynomials in Recursive Form

The Gaussian normalized associated Legendre polynomials are given in recursive form as

$$P^{0,0} = 1 \quad (16a)$$

$$P^{n,n} = \sin \theta P^{n-1,m-1} \quad (16b)$$

$$P^{n,m} = \cos \theta P^{n-1,m} - K^{n,m} P^{n-2,m} \quad (16c)$$

where $K^{n,m}$ is defined as follows:

$$K^{n,m} = 0, n=1 \quad (17a)$$

$$K^{n,m} = \frac{(n-1)^2 - m^2}{(2n-1)(2n-3)}, n > 1 \quad (17b)$$

3.2 Schmidt quasi-normalization factors in recursive form

The Schmidt quasi-normalization factors can be found recursively using the following formulas:

$$S_{0,0} = 1 \quad (18a)$$

$$S_{n,0} = S_{n-1,0} \left[\frac{2n-1}{n} \right] \quad (18b)$$

$$S_{n,m} = S_{n,m-1} \sqrt{\frac{(n-m+1)(\delta_m^1 + 1)}{n+m}} \quad (18c)$$

Again, it is important to realize that the efficiency of the code can be increased by computing these normalization factors only once and applying them to the g and h coefficients to be stored as the values input into the code actually computing the magnetic field.

3.3 Recursive derivatives of associated Legendre polynomials

In order to calculate B_θ , one must calculate the derivatives of the associated Legendre polynomials, which can be done recursively using the following three equations:

$$\frac{\partial P^{0,0}}{\partial \theta} = 0 \quad (19a)$$

$$\frac{\partial P^{n,n}}{\partial \theta} = \sin \theta \frac{\partial P^{n-1,n-1}}{\partial \theta} + \cos \theta P^{n-1,n-1}, n \geq 1 \quad (19b)$$

$$\frac{\partial P^{n,m}}{\partial \theta} = \cos \theta \frac{\partial P^{n-1,m}}{\partial \theta} - \sin \theta P^{n-1,m} - K^{n,m} \frac{\partial P^{n-2,m}}{\partial \theta} \quad (19c)$$

4 Model implementation in MATLAB code

4.1 General explanation

The MATLAB script, ‘magnet.m’, written to implement this model is shown in Appendix A.1. The implementation was fairly straight-forward, using the recursive formulas shown above and looping through the values of n and m to create the appropriate sums. Currently the inputs to this model are radius, in kilometers from the center of the Earth, latitude, measured in degrees from the equator, longitude, also measured in degrees positive East from Greenwich, and time in decimal days since January 1, 2000. Inputting the time allows the secular variation coefficients to be accounted for in the code. The output is the local tangential components of the magnetic field. The script also reads in the values of the IGRF coefficients that have already been Schmidt quasi-normalized and placed in two files called ‘igrfSg.txt’ and ‘igrfSh.txt’, which are the output of the Schmidt quasi-normalization code, ‘ghnorm.m’, shown in Appendix A.3. The input and output text files from ‘ghnorm.m’ are shown in Appendix B. Two other subroutines are shown in Appendix A.2. The first, ‘msph2cart.m’ relates the local spherical coordinates output by ‘magnet.m’ to local Cartesian coordinates through Equation (20) shown below. The second subroutine, ‘msph2inert.m’ relates the local spherical coordinates to geocentric inertial coordinates through Equation (15).

4.2 Model validation

This model was validated using the magnetic field calculator available through the British Geological Survey website listed in Reference 6. This calculator uses the 9th generation coefficients, and takes the same inputs as the MATLAB script. Several different locations and dates were tested, and all were found to be in exact agreement. The output of this website and other programs available on the internet are in local tangential Cartesian coordinates, X (North), Y (East), and Z (Down). Here X , Y , and Z relate to B_r , B_θ , and B_ϕ through the equations

$$X = -B_\theta \cos \epsilon - B_r \sin \epsilon \quad (20a)$$

$$Y = B_\phi \quad (20b)$$

$$Z = B_\theta \sin \epsilon - B_r \cos \epsilon \quad (20c)$$

where ϵ is a term used to correct for the oblateness of the Earth.³ In the programs online, this term is set to zero and it is also not used in determining the geocentric inertial components, so its precise definition was irrelevant for the purpose of this paper.

To further validate the model, a visualization was done using MATLAB’s isosurface function to create a representation of the magnetic field for a field strength magnitude of 25000nT, seen in Figure 2. In that figure, the blue sphere represents the surface of the Earth, the red surface represents an isosurface where the magnitude of the magnetic field strength is 25000nT, and the green spot shows the location of the Brazilian Anomaly. As one can see, the surface is continuous, and a dimple is clearly visible near the Brazilian Anomaly, further verifying the accuracy of the code.

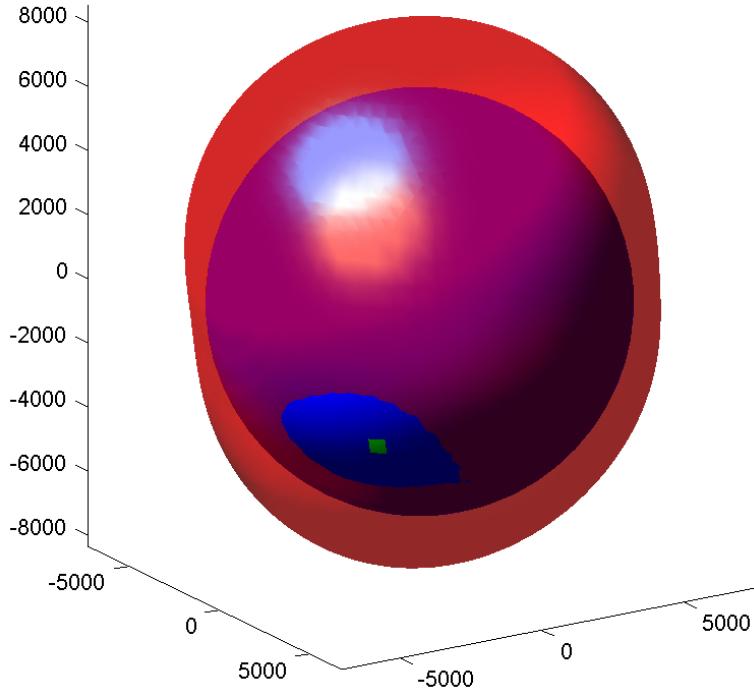


Figure 2: Magnetic Field Visualization

4.3 Model limitations

The accuracy of IGRF2000 8th generation was 50nT root mean square, while the secular variation could introduce error of 20nT/year.⁷ With the increase in the number of coefficients and the increase in precision of those coefficients with the 9th generation model, the error should be significantly reduced, however, no estimate of that is available. In addition, the error decreases as one moves away from the surface of the Earth, as the magnetic field becomes smoother and approaches the simple tilted dipole distribution. By a distance of four Earth radii, the error has decreased by a factor of 3 or 4.¹

As with all approximations, the IGRF model has a few significant limitations. First, it fails to take into account the temporal variations that occur due to factors such as solar activity. Disturbances in the ionosphere or magnetosphere can cause variations up to 1000nT during a magnetic storm.⁷ Second, it only applies up to a radius of approximately 6.6 Earth radii, beyond which point the magnetic effects of the solar wind can no longer be neglected.¹ Lastly, the secular variation coefficients are used to linearly extrapolate the magnetic field model. While this provides a good approximation and does not significantly increase the error, it is important to understand that the instantaneous time rate of change of the magnetic field is not linear.⁷

4.4 Improvements

Due to the fact that most new code is being written in the C language in the aerospace industry, the first improvement would be converting the MATLAB script into C. Towards that end, Appendix C contains tables of the values of $P^{n,m}$ and $\frac{\partial P^{n,m}}{\partial \theta}$ for all n and m at the location $\theta = 30^\circ$, which can be used for debugging. To further improve this model for satellite work, it may be helpful to implement code that would allow for random variation of the magnetic field in accordance to some of the factors listed in Section 1. In addition, since the tilted dipole model is simply the first degree spherical harmonic model ($n = 1$), it would be helpful to be able to specify the order of the harmonic that one wishes to use if calculation time is of importance.

References

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- [6] British Geological Survey. The igrf model. <http://www.geomag.bgs.ac.uk/gifs/igrf.html>.
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A MATLAB code

A.1 magnet.m - calculates magnetic field strength in local spherical coordinates

```
function [Br,Bt,Bp] = magnet(r,theta,phi,days)
% Inputs
%   r      Geocentric radius
%   theta  Latitude measured in degrees positive from equator
%   phi    Longitude measured in degrees positive east from Greenwich
%   days   Decimal days since January 1, 2000
%
% Outputs - magnetic field strength in local tangential coordinates
%   Br     B in radial direction
%   Bt     B in theta direction
%   Bp     B in phi direction

% Checks to see if located at either pole to avoid singularities
if (theta>-0.00000001 & theta<0.00000001)
    theta=0.00000001;
elseif(theta<180.00000001 & theta>179.99999999)
    theta=179.99999999;
end

% The angles must be converted from degrees into radians
theta=(90-theta)*pi/180;
phi = phi*pi/180;

a=6371.2; % Reference radius used in IGRF

% This section of the code simply reads in the g and h Schmidt
% quasi-normalized coefficients
[gn, gm, gvali, gsvi] = textread('igrfSg.txt','%f %f %f %f');
[hn, hm, hvali, hsvi] = textread('igrfSh.txt','%f %f %f %f');
N=max(gn);
g=zeros(N,N+1);
h=zeros(N,N+1);
for x=1:length(gn)
    g(gn(x),gm(x)+1) = gvali(x) + gsvi(x)*days/365;
    h(hn(x),hm(x)+1) = hvali(x) + hsvi(x)*days/365;
end

% Initialize each of the variables
% Br       B in the radial direction
% Bt       B in the theta direction
```

```

% Bp      B in the phi direction
% P      The associated Legendre polynomial evaluated at cos(theta)
%
% The nomenclature for the recursive values generally follows
% the form P10 = P(n-1,m-0)
% dP      The partial derivative of P with respect to theta

Br=0; Bt=0; Bp=0;
P11=1; P10=P11;
dP11=0; dP10=dP11;
dP20=0; P20=1;
for m=0:N
    for n=1:N
        if m<=n
            % Calculate Legendre polynomials and derivatives recursively
            if n==m
                P2 = sin(theta)*P11;
                dP2 = sin(theta)*dP11 + cos(theta)*P11;
                P11=P2; P10=P11; P20=0;
                dP11=dP2; dP10=dP11; dP20=0;
            elseif n==1
                P2 = cos(theta)*P10;
                dP2 = cos(theta)*dP10 - sin(theta)*P10;
                P20=P10; P10=P2;
                dP20=dP10; dP10=dP2;
            else
                K = ((n-1)^2-m^2)/((2*n-1)*(2*n-3));
                P2 = cos(theta)*P10 - K*P20;
                dP2 = cos(theta)*dP10 - sin(theta)*P10 - K*dP20;
                P20=P10; P10=P2;
                dP20=dP10; dP10=dP2;
            end
            % Calculate Br, Bt, and Bp
            Br = Br + (a/r)^(n+2)*(n+1)*...
                  ((g(n,m+1)*cos(m*phi) + h(n,m+1)*sin(m*phi))*P2);
            Bt = Bt + (a/r)^(n+2)*...
                  ((g(n,m+1)*cos(m*phi) + h(n,m+1)*sin(m*phi))*dP2);
            Bp = Bp + (a/r)^(n+2)*...
                  (m*(-g(n,m+1)*sin(m*phi) + h(n,m+1)*cos(m*phi))*P2);
        end
    end
end
[Br,Bt,Bp] = [Br;-Bt;-Bp/sin(theta)];

```

A.2 MATLAB code for coordinate transformations (uses output from magnet.m as input)

A.2.1 msph2cart.m - converting from spherical to Cartesian coordinates

```
function [X,Y,Z] = msph2cart(Br,Bt,Bp)
% Inputs - magnetic field strength (B) in local tangential coordinates
%   Br      B in radial direction
%   Bt      B in theta direction
%   Bp      B in phi direction
% Outputs - magnetic field strength (B) in local tangential coordinates
%   X      B in "north" direction
%   Y      B in "east" direction
%   Z      B in down direction
% Variables
%   e      epsilon, the term used to correct for the oblateness of the
%          Earth. In all programs online, e=0
e = 0.0*pi/180;
X = -Bt*cos(e) - Br*sin(e);
Y = Bp;
Z = Bt*sin(e) - Br*cos(e);
```

A.2.2 msph2inert.m - converting from tangential spherical to geocentric inertial Cartesian coordinates

```
function [Bx,By,Bz] = msph2inert(Br,Bt,Bp,LST,lat)
% Inputs
%   Br      B in radial direction | Magnetic field strength (B)
%   Bt      B in theta direction | in local tangential coordinates
%   Bp      B in phi direction |
%   LST     Local sidereal time of location (in degrees)
%   lat     Latitude measured positive north from equator (in degrees)
% Outputs
%   Bx      B in x-direction | Magnetic field strength (B)
%   By      B in y-direction | in geocentric inertial coordinates
%   Bz      B in z-direction |

% Angle conversion to radians
lat=lat*pi/180; LST=LST*pi/180;

% Coordinate transformation
Bx = (Br*cos(lat)+Bt*sin(lat))*cos(LST) - Bp*sin(LST);
By = (Br*cos(lat)+Bt*sin(lat))*sin(LST) + Bp*cos(LST);
Bz = (Br*sin(lat)+Bt*cos(lat));
```

*we ke
another function*

A.3 ghnorm.m - Schmidt quasi-normalization code

```

function ghnorm()
[gh, n, m, val, sv] = textread('igrf.txt', '%s %f %f %f %f');
N=max(n);
g=zeros(N,N+1);
h=zeros(N,N+1);
hsv=zeros(N,N+1);
gsv=zeros(N,N+1);
for x=1:length(gh)
    if strcmp(gh(x), 'g')
        g(n(x),m(x)+1) = val(x);
        gsv(n(x),m(x)+1) = sv(x);
    else
        h(n(x),m(x)+1) = val(x);
        hsv(n(x),m(x)+1) = sv(x);
    end
end
count=1;
S = zeros(N,N+1);
for n=1:N
    for m=0:n
        if m>1
            S(n,m+1) = S(n,m)*((n-m+1)/(n+m))^0.5;
        elseif m>0
            S(n,m+1) = S(n,m)*(2*(n-m+1)/(n+m))^0.5;
        elseif n==1
            S(n,1) = 1;
        else
            S(n,1) = S(n-1,1)*(2*n-1)/(n);
        end
        gS(count,1) = n; gS(count,2)=m;
        gS(count,3)=g(n,m+1)*S(n,m+1); gS(count,4)=gsv(n,m+1)*S(n,m+1);
        hS(count,1) = n; hS(count,2)=m;
        hS(count,3)=h(n,m+1)*S(n,m+1); hS(count,4)=hsv(n,m+1)*S(n,m+1);
        count=count+1;
    end
end
dlmwrite('igrfSg.txt',gS,'t')
dlmwrite('igrfSh.txt',hS,'t')

```

B Coefficient text files

B.1 IGRF coefficients before normalization

g/h	n	m	2000.0	SV	g/h	n	m	2000.0	SV	g/h	n	m	2000.0	SV
g	1	0	-29619.4	13.3	g	6	0	72.3	0.4	g	8	4	-16.6	-0.4
g	1	1	-1728.2	11.6	g	6	1	68.2	0.3	h	8	4	-21.5	0.4
h	1	1	5186.1	-21.2	h	6	1	-17.4	-0.7	g	8	5	9.1	0.2
g	2	0	-2267.7	-14.4	g	6	2	74.2	0.7	h	8	5	15.5	0.1
g	2	1	3068.4	-3.7	h	6	2	63.7	-1.8	g	8	6	7.0	0.5
h	2	1	-2481.6	-22.7	g	6	3	-160.9	1.9	h	8	6	8.9	-0.3
g	2	2	1670.9	-3.6	h	6	3	65.1	-0.2	g	8	7	-7.9	-0.7
h	2	2	-458.0	-11.1	g	6	4	-5.9	-1.7	h	8	7	-14.9	0.4
g	3	0	1339.6	-1.1	h	6	4	-61.2	-0.4	g	8	8	-7.0	0.4
g	3	1	-2288.0	-3.5	g	6	5	16.9	-0.5	h	8	8	-2.1	0.4
h	3	1	-227.6	5.6	h	6	5	0.7	-0.2	g	9	0	5.0	0.0
g	3	2	1252.1	-1.2	g	6	6	-90.4	0.7	g	9	1	9.4	0.0
h	3	2	293.4	-4.5	h	6	6	43.8	1.5	h	9	1	-19.7	0.0
g	3	3	714.5	-8.5	g	7	0	79.0	0.2	g	9	2	3.0	0.0
h	3	3	-491.1	-8.0	g	7	1	-74.0	-0.1	h	9	2	13.4	0.0
g	4	0	932.3	-2.7	h	7	1	-64.6	0.7	g	9	3	-8.4	0.0
g	4	1	786.8	2.2	g	7	2	0.0	-0.3	h	9	3	12.5	0.0
h	4	1	272.6	1.7	h	7	2	-24.2	0.3	g	9	4	6.3	0.0
g	4	2	250.0	-8.0	g	7	3	33.3	1.1	h	9	4	-6.2	0.0
h	4	2	-231.9	1.0	h	7	3	6.2	0.1	g	9	5	-8.9	0.0
g	4	3	-403.0	4.5	g	7	4	9.1	0.7	h	9	5	-8.4	0.0
h	4	3	119.8	5.1	h	7	4	24.0	0.3	g	9	6	-1.5	0.0
g	4	4	111.3	-1.9	g	7	5	6.9	0.5	h	9	6	8.4	0.0
h	4	4	-303.8	-0.3	h	7	5	14.8	-0.8	g	9	7	9.3	0.0
g	5	0	-218.8	-1.4	g	7	6	7.3	-0.3	h	9	7	3.8	0.0
g	5	1	351.4	0.7	h	7	6	-25.4	-0.1	g	9	8	-4.3	0.0
h	5	1	43.8	-0.3	g	7	7	-1.2	0.5	h	9	8	-8.2	0.0
g	5	2	222.3	-2.6	h	7	7	-5.8	0.2	g	9	9	-8.2	0.0
h	5	2	171.9	1.5	h	8	0	24.4	0.1	h	9	9	4.8	0.0
g	5	3	-130.4	-1.2	g	8	1	6.6	0.2	g	10	0	-2.6	0.0
h	5	3	-133.1	2.0	h	8	1	11.9	-0.2	g	10	1	-6.0	0.0
g	5	4	-168.6	0.0	g	8	2	-9.2	-0.5	h	10	1	1.7	0.0
h	5	4	-39.3	3.8	h	8	2	-21.5	0.1	g	10	2	1.7	0.0
g	5	5	-12.9	-0.2	g	8	3	-7.9	0.2	h	10	2	0.0	0.0
h	5	5	106.3	-0.5	h	8	3	8.5	0.3	g	10	3	-3.1	0.0

g/h	n	m	2000.0	SV	g/h	n	m	2000.0	SV	g/h	n	m	2000.0	SV
h	10	3	4.0	0.0	h	11	10	-1.9	0.0	g	13	4	-0.4	0.0
g	10	4	-0.5	0.0	g	11	11	4.0	0.0	h	13	4	-0.4	0.0
h	10	4	4.9	0.0	h	11	11	-0.9	0.0	g	13	5	1.3	0.0
g	10	5	3.7	0.0	g	12	0	-2.2	0.0	h	13	5	-1.0	0.0
h	10	5	-5.9	0.0	g	12	1	-0.3	0.0	g	13	6	-0.4	0.0
g	10	6	1.0	0.0	h	12	1	-0.4	0.0	h	13	6	-0.1	0.0
h	10	6	-1.2	0.0	g	12	2	0.2	0.0	g	13	7	0.7	0.0
g	10	7	2.0	0.0	h	12	2	0.3	0.0	h	13	7	0.7	0.0
h	10	7	-2.9	0.0	g	12	3	0.9	0.0	g	13	8	-0.4	0.0
g	10	8	4.2	0.0	h	12	3	2.5	0.0	h	13	8	0.3	0.0
h	10	8	0.2	0.0	g	12	4	-0.2	0.0	g	13	9	0.3	0.0
g	10	9	0.3	0.0	h	12	4	-2.6	0.0	h	13	9	0.6	0.0
h	10	9	-2.2	0.0	g	12	5	0.9	0.0	g	13	10	-0.1	0.0
g	10	10	-1.1	0.0	h	12	5	0.7	0.0	h	13	10	0.3	0.0
h	10	10	-7.4	0.0	g	12	6	-0.5	0.0	g	13	11	0.4	0.0
g	11	0	2.7	0.0	h	12	6	0.3	0.0	h	13	11	-0.2	0.0
g	11	1	-1.7	0.0	g	12	7	0.3	0.0	g	13	12	0.0	0.0
h	11	1	0.1	0.0	h	12	7	0.0	0.0	h	13	12	-0.5	0.0
g	11	2	-1.9	0.0	g	12	8	-0.3	0.0	g	13	13	0.1	0.0
h	11	2	1.3	0.0	h	12	8	0.0	0.0	h	13	13	-0.9	0.0
g	11	3	1.5	0.0	g	12	9	-0.4	0.0					
h	11	3	-0.9	0.0	h	12	9	0.3	0.0					
g	11	4	-0.1	0.0	g	12	10	-0.1	0.0					
h	11	4	-2.6	0.0	h	12	10	-0.9	0.0					
g	11	5	0.1	0.0	g	12	11	-0.2	0.0					
h	11	5	0.9	0.0	h	12	11	-0.4	0.0					
g	11	6	-0.7	0.0	g	12	12	-0.4	0.0					
h	11	6	-0.7	0.0	h	12	12	0.8	0.0					
g	11	7	0.7	0.0	g	13	0	-0.2	0.0					
h	11	7	-2.8	0.0	g	13	1	-0.9	0.0					
g	11	8	1.7	0.0	h	13	1	-0.9	0.0					
h	11	8	-0.9	0.0	g	13	2	0.3	0.0					
g	11	9	0.1	0.0	h	13	2	0.2	0.0					
h	11	9	-1.2	0.0	g	13	3	0.1	0.0					
g	11	10	1.2	0.0	h	13	3	1.8	0.0					

B.2 IGRF coefficients after Schmidt quasi-normalization

B.2.1 IGRF g -coefficients after Schmidt quasi-normalization

n	m	2000.0	SV	n	m	2000.0	SV	n	m	2000.0	SV
1	0	-29619.40	13.30	8	0	1226.67	5.03	11	5	15.84	0.00
1	1	-1728.20	11.60	8	1	442.41	13.41	11	6	-65.88	0.00
2	0	-3401.55	-21.60	8	2	-515.96	-28.04	11	7	34.72	0.00
2	1	5314.62	-6.41	8	3	-327.21	8.28	11	8	38.69	0.00
2	2	1447.04	-3.12	8	4	-443.82	-10.69	11	9	0.88	0.00
3	0	3349.00	-2.75	8	5	134.96	2.97	11	10	3.26	0.00
3	1	-7005.54	-10.72	8	6	48.06	3.43	11	11	2.32	0.00
3	2	2424.68	-2.32	8	7	-19.80	-1.75	12	0	-1452.43	0.00
3	3	564.86	-6.72	8	8	-4.39	0.25	12	1	-269.11	0.00
4	0	4078.81	-11.81	9	0	474.80	0.00	12	2	159.03	0.00
4	1	4354.14	12.17	9	1	1197.59	0.00	12	3	584.30	0.00
4	2	978.28	-31.31	9	2	325.95	0.00	12	4	-97.38	0.00
4	3	-842.94	9.41	9	3	-697.06	0.00	12	5	300.62	0.00
4	4	82.31	-1.41	9	4	355.17	0.00	12	6	-104.15	0.00
5	0	-1723.05	-11.03	9	5	-299.85	0.00	12	7	35.12	0.00
5	1	3572.54	7.12	9	6	-26.10	0.00	12	8	-17.56	0.00
5	2	1708.42	-19.98	9	7	70.06	0.00	12	9	-10.22	0.00
5	3	-613.69	-5.65	9	8	-11.11	0.00	12	10	-0.94	0.00
5	4	-374.04	0.00	9	9	-4.99	0.00	12	11	-0.56	0.00
5	5	-9.05	-0.14	10	0	-469.11	0.00	12	12	-0.23	0.00
6	0	1043.83	5.78	10	1	-1459.72	0.00	13	0	-253.92	0.00
6	1	1289.19	5.67	10	2	358.18	0.00	13	1	-1557.16	0.00
6	2	1108.86	10.46	10	3	-512.37	0.00	13	2	464.26	0.00
6	3	-1603.02	18.93	10	4	-58.44	0.00	13	3	128.31	0.00
6	4	-32.20	-9.28	10	5	273.49	0.00	13	4	-393.65	0.00
6	5	39.32	-1.16	10	6	41.32	0.00	13	5	904.64	0.00
6	6	-60.72	0.47	10	7	40.09	0.00	13	6	-180.62	0.00
7	0	2118.19	5.36	10	8	34.37	0.00	13	7	187.00	0.00
7	1	-2624.75	-3.55	10	9	0.80	0.00	13	8	-57.12	0.00
7	2	0.00	-8.69	10	10	-0.65	0.00	13	9	20.42	0.00
7	3	681.93	22.53	11	0	930.01	0.00	13	10	-2.84	0.00
7	4	112.38	8.64	11	1	-792.86	0.00	13	11	4.01	0.00
7	5	42.60	3.09	11	2	-777.19	0.00	13	12	0.00	0.00
7	6	17.68	-0.73	11	3	491.95	0.00	13	13	0.06	0.00
7	7	-0.78	0.32	11	4	-23.95	0.00				

B.2.2 IGRF h -coefficients after Schmidt quasi-normalization

n	m	2000.0	SV	n	m	2000.0	SV	n	m	2000.0	SV
1	0	0.00	0.00	8	0	0.00	0.00	11	5	142.58	0.00
1	1	5186.10	-21.20	8	1	797.67	-13.41	11	6	-65.88	0.00
2	0	0.00	0.00	8	2	-1205.77	5.61	11	7	-138.89	0.00
2	1	-4298.26	-39.32	8	3	352.07	12.43	11	8	-20.48	0.00
2	2	-396.64	-9.61	8	4	-574.83	10.69	11	9	-10.58	0.00
3	0	0.00	0.00	8	5	229.87	1.48	11	10	-5.17	0.00
3	1	-696.88	17.15	8	6	61.10	-2.06	11	11	-0.52	0.00
3	2	568.17	-8.71	8	7	-37.35	1.00	12	0	0.00	0.00
3	3	-388.25	-6.32	8	8	-1.32	0.25	12	1	-358.81	0.00
4	0	0.00	0.00	9	0	0.00	0.00	12	2	238.54	0.00
4	1	1508.56	9.41	9	1	-2509.85	0.00	12	3	1623.05	0.00
4	2	-907.45	3.91	9	2	1455.91	0.00	12	4	-1265.98	0.00
4	3	250.58	10.67	9	3	1037.29	0.00	12	5	233.81	0.00
4	4	-224.66	-0.22	9	4	-349.53	0.00	12	6	62.49	0.00
5	0	0.00	0.00	9	5	-283.00	0.00	12	7	0.00	0.00
5	1	445.30	-3.05	9	6	146.14	0.00	12	8	0.00	0.00
5	2	1321.09	11.53	9	7	28.63	0.00	12	9	7.66	0.00
5	3	-626.40	9.41	9	8	-21.19	0.00	12	10	-8.49	0.00
5	4	-87.19	8.43	9	9	2.92	0.00	12	11	-1.11	0.00
5	5	74.58	-0.35	10	0	0.00	0.00	12	12	0.45	0.00
6	0	0.00	0.00	10	1	413.59	0.00	13	0	0.00	0.00
6	1	-328.91	-13.23	10	2	0.00	0.00	13	1	-1557.16	0.00
6	2	951.95	-26.90	10	3	661.12	0.00	13	2	309.50	0.00
6	3	648.58	-1.99	10	4	572.67	0.00	13	3	2309.64	0.00
6	4	-333.96	-2.18	10	5	-436.10	0.00	13	4	-393.65	0.00
6	5	1.63	-0.47	10	6	-49.58	0.00	13	5	-695.88	0.00
6	6	29.42	1.01	10	7	-58.13	0.00	13	6	-45.15	0.00
7	0	0.00	0.00	10	8	1.64	0.00	13	7	187.00	0.00
7	1	-2291.34	24.83	10	9	-5.84	0.00	13	8	42.84	0.00
7	2	-700.85	8.69	10	10	-4.39	0.00	13	9	40.84	0.00
7	3	126.97	2.05	11	0	0.00	0.00	13	10	8.52	0.00
7	4	296.37	3.70	11	1	46.64	0.00	13	11	-2.01	0.00
7	5	91.38	-4.94	11	2	531.76	0.00	13	12	-1.42	0.00
7	6	-61.51	-0.24	11	3	-295.17	0.00	13	13	-0.50	0.00
7	7	-3.75	0.13	11	4	-622.74	0.00				

C Intermediate values for debugging purposes

C.1 $P^{n,m}$ at $\theta = 30^\circ$

n	m						
	0	1	2	3	4	5	6
0	1.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00
1	5.000E-01	8.660E-01	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00
2	-8.333E-02	4.330E-01	7.500E-01	0.000E+00	0.000E+00	0.000E+00	0.000E+00
3	-1.750E-01	4.330E-02	3.750E-01	6.495E-01	0.000E+00	0.000E+00	0.000E+00
4	-6.607E-02	-7.732E-02	8.036E-02	3.248E-01	5.625E-01	0.000E+00	0.000E+00
5	1.141E-02	-4.897E-02	-3.125E-02	9.021E-02	2.813E-01	4.871E-01	0.000E+00
6	2.239E-02	-5.741E-03	-3.267E-02	-7.381E-03	8.949E-02	2.436E-01	4.219E-01
7	8.322E-03	9.116E-03	-9.342E-03	-2.072E-02	5.409E-03	8.431E-02	2.109E-01
8	-1.465E-03	5.971E-03	2.868E-03	-8.848E-03	-1.244E-02	1.218E-02	7.734E-02
9	-2.821E-03	7.334E-04	3.632E-03	4.592E-05	-7.238E-03	-6.806E-03	1.551E-02
10	-1.043E-03	-1.112E-03	1.132E-03	1.995E-03	-1.116E-03	-5.514E-03	-3.020E-03
11	1.854E-04	-7.381E-04	-3.077E-04	9.871E-04	9.660E-04	-1.478E-03	-3.998E-03
12	3.541E-04	-9.271E-05	-4.282E-04	3.091E-05	7.255E-04	3.571E-04	-1.468E-03
13	1.306E-04	1.372E-04	-1.392E-04	-2.163E-04	1.477E-04	4.844E-04	1.720E-05
n	m						
	7	8	9	10	11	12	13
0	0.000E+00						
1	0.000E+00						
2	0.000E+00						
3	0.000E+00						
4	0.000E+00						
5	0.000E+00						
6	0.000E+00						
7	3.654E-01	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00
8	1.827E-01	3.164E-01	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00
9	6.985E-02	1.582E-01	2.740E-01	0.000E+00	0.000E+00	0.000E+00	0.000E+00
10	1.683E-02	6.245E-02	1.370E-01	2.373E-01	0.000E+00	0.000E+00	0.000E+00
11	-5.151E-04	1.695E-02	5.546E-02	1.187E-01	2.055E-01	0.000E+00	0.000E+00
12	-2.766E-03	1.106E-03	1.638E-02	4.901E-02	1.028E-01	1.780E-01	0.000E+00
13	-1.298E-03	-1.806E-03	2.115E-03	1.543E-02	4.316E-02	8.899E-02	1.541E-01

C.2 $\frac{\partial P^{n,m}}{\partial \theta}$ at $\theta = 30^\circ$

n	m						
	0	1	2	3	4	5	6
0	0.000E+00						
1	-8.660E-01	5.000E-01	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00
2	-8.660E-01	-5.000E-01	8.660E-01	0.000E+00	0.000E+00	0.000E+00	0.000E+00
3	-1.299E-01	-7.250E-01	-2.165E-01	1.125E+00	0.000E+00	0.000E+00	0.000E+00
4	3.093E-01	-2.857E-01	-5.567E-01	2.220E-16	1.299E+00	0.000E+00	0.000E+00
5	2.449E-01	9.673E-02	-3.067E-01	-4.063E-01	1.624E-01	1.406E+00	0.000E+00
6	3.444E-02	1.600E-01	-8.201E-03	-2.813E-01	-2.805E-01	2.813E-01	1.461E+00
7	-6.381E-02	6.132E-02	9.283E-02	-5.753E-02	-2.405E-01	-1.785E-01	3.654E-01
8	-4.777E-02	-1.663E-02	5.640E-02	4.688E-02	-7.744E-02	-1.969E-01	-9.743E-02
9	-6.600E-03	-2.864E-02	3.873E-03	4.351E-02	1.731E-02	-8.169E-02	-1.558E-01
10	1.112E-02	-1.083E-02	-1.465E-02	1.127E-02	3.051E-02	-8.163E-04	-7.777E-02
11	8.119E-03	2.651E-03	-9.239E-03	-6.018E-03	1.258E-02	1.972E-02	-1.127E-02
12	1.113E-03	4.656E-03	-8.035E-04	-6.476E-03	-1.181E-03	1.130E-02	1.151E-02
13	-1.784E-03	1.749E-03	2.219E-03	-1.852E-03	-4.018E-03	1.261E-03	9.144E-03
n	m						
n	7	8	9	10	11	12	13
0	0.000E+00						
1	0.000E+00						
2	0.000E+00						
3	0.000E+00						
4	0.000E+00						
5	0.000E+00						
6	0.000E+00						
7	1.477E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00
8	4.219E-01	1.461E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00
9	-3.412E-02	4.567E-01	1.424E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00
10	-1.194E-01	1.442E-02	4.746E-01	1.370E+00	0.000E+00	0.000E+00	0.000E+00
11	-6.988E-02	-8.808E-02	5.085E-02	4.795E-01	1.305E+00	0.000E+00	0.000E+00
12	-1.671E-02	-6.042E-02	-6.191E-02	7.744E-02	4.746E-01	1.233E+00	0.000E+00
13	5.589E-03	-1.891E-02	-5.071E-02	-4.042E-02	9.611E-02	4.624E-01	1.157E+00