ORBITAL MECHANICS

Locus of the eccentricity vectors is a straight line that is normal to the chord

DEMONSTRATION

$$r = \frac{p}{1 + e\cos\theta} \qquad \underline{e} \cdot \underline{r_1} = r_1 e \cos\theta_1$$

$$r(1 + e\cos\theta) = p \rightarrow r + re\cos\theta = p \qquad re\cos\theta = p - r$$

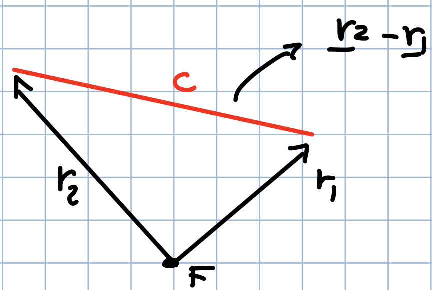
$$\rightarrow \underline{e} \cdot \underline{r_1} = r_1 e \cos\theta_1 = p - r_1$$

$$\underline{e} \cdot \underline{r_2} = p - r_2$$

$$\frac{\underline{e} \cdot \underline{r_1} - \underline{e} \cdot \underline{r_2}}{c} = \frac{p - r_1 - p + r_2}{c}$$

$$- \frac{\underline{e} \cdot (\underline{r_2} - \underline{r_1})}{c} = \frac{r_2 - r_1}{c} \qquad (3.41)$$

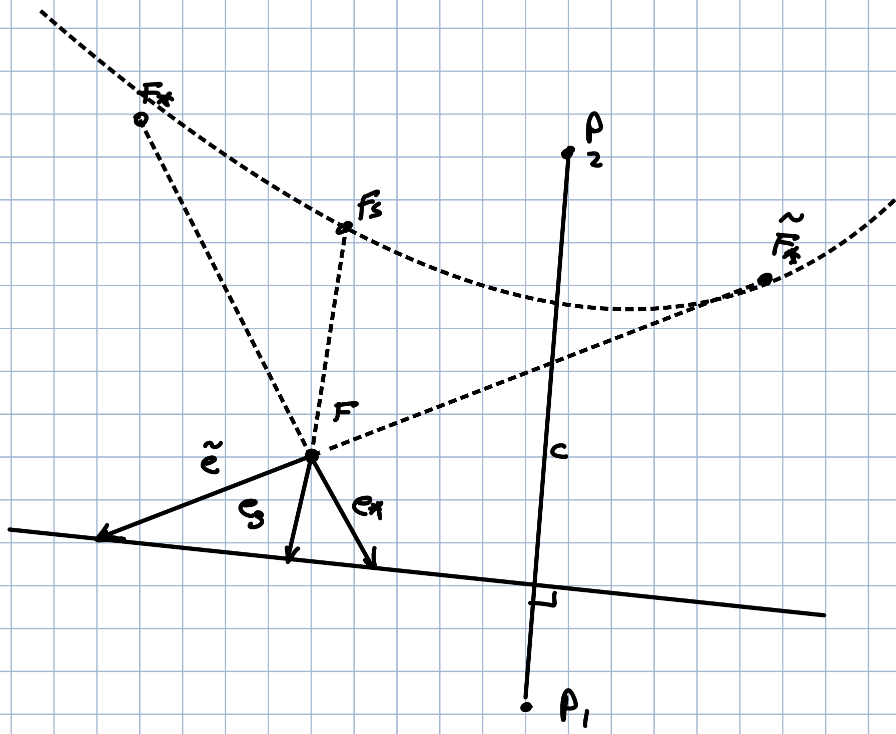
$\underline{r_2} - \underline{r_1}$ parallel to the chord $\rightarrow \frac{\underline{r_2} - \underline{r_1}}{c}$ unit vector in the direction of the chord



$\underline{e}$ has a constant projection along chord $\left(\underline{e} \cdot \frac{\underline{r_2} - \underline{r_1}}{c}\right)$, $\frac{\underline{r_2} - \underline{r_1}}{c} = \hat{c}$

because it is equal to $\frac{r_2 - r_1}{c}$

$\Rightarrow$ locus of the eccentricity vector is a straight line $\perp$ to the chord



F_s particular pole that give us e_s

$\Rightarrow$ e_s is the eccentricity of the ellipse with the minimum eccentricity

$\underline{e_s}$ // to $\underline{c}$

$\underline{e_s} \triangleq$ foundamental ellipse

$0 < e_s < 1$

$e_{Hyp} = \dfrac{1}{e_s}$ the eccentricity of the hyperbolic locus of the focii

The minimum eccentricity ellipse is characterised by

$$\boxed{e_s = \frac{r_2 - r_1}{c}} \qquad (3.42)$$

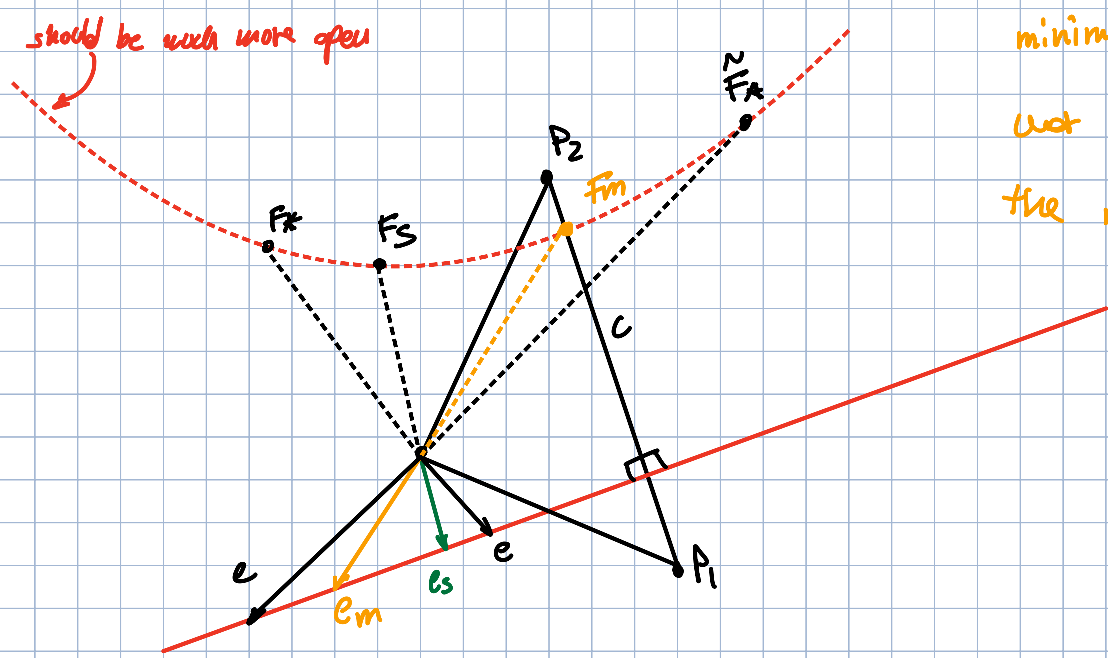
The projection of $\underline{e_s}$ along the chord is $\dfrac{r_2 - r_1}{c}$

moreover $\dfrac{1}{e_s}$ = eccentricity of the hyperbolic locus of the focii

MINIMUM ENERGY ELLIPSE $e_s < 1 \Rightarrow \dfrac{1}{e_s} > 1$ HYPERBOLIC LOCUS OF FOCII

(3.43)

$e_s \triangleq$ foundamental ellipse $\Rightarrow$ semi major axis is perpendicular to the chord



minimum a_m ellipse is not the same as the minimum eccentricity ellipse.

e_s = fundamental ellipse

a_m = minimum semi-major axis (energy) ellipse

$$\varepsilon = -\frac{\mu}{2a}$$

THE LAMBERT THEOREM

Δt : time to transfer between P_1 and P_2

$$\Delta t = t_2 - t_1 = \mathrm{fun}(a, c, r_1 + r_2) \qquad (3.44)$$

The time required to traverse an elliptic arc between specified end points r_1 and r_2 "depends only on a of the ellipse, and on the two properties of the space triangle, namely c and $r_1 + r_2$".

Lambert, 1761

Demonstrated by Lagrange in 1778

NOTE Δt does not depend on e, it depends on a like

$T(a)$, $\varepsilon(a)$ $\Delta t(a, c, r_1 + r_2)$

Demonstration

Let's use Kepler equation to describe Δt

$$\sqrt{\mu}\,(t_2 - t_1) = a^{3/2}\left[E_2 - E_1 - e(\sin E_2 - \sin E_1)\right] \qquad (3.44)$$

E_1, E_2 are unknown eccentric anomalies

we know only the difference between the two anomalies θ_1, θ_2 unknown $\Leftrightarrow E_1, E_2$

we know only $(\Delta\theta = \theta_2 - \theta_1)$

Note Kepler' equation : initial value problem

Lambert equation : boundary value problem

$$E_p = \frac{1}{2}(E_1 + E_2) \qquad (3.46)$$

$$E_m = \frac{1}{2}(E_1 - E_2) > 0 \qquad (3.47)$$

Let's use (2.4) $\quad r = a(1 - e\cos E)$

we get

$$r_1 + r_2 = a(2 - e(\cos E_1 + \cos E_2)) \qquad (3.48)$$

RECALL

$$\cos\alpha + \cos\beta = 2\cos\left[\frac{1}{2}(\alpha+\beta)\right]\cos\left[\frac{1}{2}(\alpha-\beta)\right]$$

in eq (3.48) prostaferesis formulas

$$r_1 + r_2 = a\left(2 - e\left(2\cos\left[\frac{1}{2}(E_1+E_2)\right]\cos\left[\frac{1}{2}(E_2 - E_1)\right]\right)\right)$$

$$r_1 + r_2 = 2a(1 - e\cos E_p \cos E_m) \qquad (3.49)$$

The chord c can be obtained using the cartesian coordinates with the origin @ center of ellipse

$$\begin{cases} x = a\cos E \\ y = b\sin E \\ b = a\sqrt{1-e^2} \end{cases} \qquad (3.50)$$

the chord can be written as $c^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$

$$\begin{matrix} P_1\,(x_1, y_1) \\ P_2\,(x_2, y_2) \end{matrix} \quad \text{cartesian coordinates wrt the center}$$

$$c^2 = a^2 (\cos E_2 - \cos E_1)^2 + (1-e^2)\, a^2 (\sin E_2 - \sin E_1)^2$$

<u>Recall</u> $\cos\alpha - \cos\beta = 2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$ prostaferesin formula

$$\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

$$c^2 = a^2 (2\sin E_p \sin E_m)^2 + (1-e)^2 a^2 (2\cos E_p \sin E_m)^2$$

$$c^2 = a^2 4\sin^2 E_m \left[\sin^2 E_p + (1-e^2)\cos^2 E_p\right]$$

$$c^2 = 4a^2 \sin^2 E_m \left[1 - \cancel{\cos^2} E_p + \cancel{\cos^2} E_p - e^2 \cos^2 E_p\right]$$

$$c^2 = 4a^2 \sin^2 E_m \left[1 - e^2 \cos^2 E_p\right]$$ simplest form of c

we can introduce another change of variable

$\cos\xi = e\cos E_p$ → it is allowed because $e < 1$

$$c^2 = 4a^2 \sin^2 E_m \left[1 - \cos^2\xi\right]$$

$$c^2 = 4a^2 \sin^2 E_m \sin^2\xi$$

$$\boxed{c = 2a \sin E_m \sin \xi} \qquad (3.52)$$

Eq (3.49) can be written as

careful change of variable, we are getting rid of e

$$r_1 + r_2 = 2a\left(1 - \overbrace{e\cos E_p}^{\cos\xi}\cos E_m\right)$$

$$\boxed{r_1 + r_2 = 2a\left(1 - \cos\xi \cos E_m\right)} \qquad (3.53)$$

Last change of variables

$$\alpha = \xi + E_m \qquad (3.54)$$

$$\beta = \xi - E_m \qquad (3.55)$$

Let's combine the expression of c and $r_1 + r_2$ to compute

(A) $c + r_1 + r_2$

(B) $r_1 + r_2 - c$

(A) Eq (3.53) + Eq (3.52)

$$r_1 + r_2 + c = 2a\left(\sin E_m \sin\xi + 1 - \cos E_m \cos\xi\right)$$

$$r_1 + r_2 + c = 2a\left(1 - \cos\left(E_m + \xi\right)\right)$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$r_1 + r_2 + c = 2a\left(1 - \cos\alpha\right)$$

$$\boxed{r_1 + r_2 + c = 4a \sin^2\frac{\alpha}{2}} \qquad (3.56)$$

Recall $\sin^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{2}$ $\quad 1-\cos\alpha = 2\sin^2\frac{\alpha}{2}$

(B) Eq (3.53) − Eq (3.52)

$$r_1+r_2-c = 2a\left(1-\cos E_m\cos\xi - \sin E_m \sin\xi\right)$$

$$r_1+r_2-c = 2a\left(1-\cos(\underbrace{E_m-\xi}_{\beta})\right)$$

$$r_1+r_2-c = 2a\left(1-\cos\beta\right)$$

$$\boxed{r_1+r_2-c = 4a\sin^2\frac{\beta}{2}} \qquad (3.57)$$

Let's write Eq (3.45) as function of E_m and ξ

$$\sqrt{\mu}(t_2-t_1) = a^{3/2}\left[E_2-E_1-e(\sin E_2-\sin E_1)\right]$$

Considering that

$$E_2-E_1 = 2E_m \qquad \text{from Eq (3.47)}$$

and recalling $\sin\alpha-\sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

$$\Rightarrow \sin E_2 - \sin E_1 = 2\cos\frac{E_2+E_1}{2}\sin\frac{E_2-E_1}{2} =$$

$$= 2\cos E_p \sin E_m$$

Therefore $\sqrt{\mu}(t_2-t_1) = a^{3/2}\left[2E_m - 2\underbrace{e\cos E_p}_{\cos\xi}\sin E_m\right]$

$$\sqrt{\mu}(t_2-t_1) = a^{3/2}\left[2E_m - 2\cos\xi\sin E_m\right]$$

$$\sqrt{\mu}\,(t_2-t_1) = 2a^{3/2}\left(E_m - \cos\xi \sin E_m\right)$$

but α and β $\quad \alpha = \xi + E_m$

$$\beta = \xi - E_m$$

$$\rightarrow \begin{array}{l} \alpha-\beta = 2E_m \\ \alpha+\beta = 2\xi \end{array} \rightarrow \begin{cases} E_m = \dfrac{\alpha-\beta}{2} & (3.58) \\ \xi = \dfrac{\alpha+\beta}{2} & (3.59) \end{cases}$$

we get

$$\sqrt{\mu}\,(t_2-t_1) = a^{3/2}\left(\alpha-\beta \underbrace{-2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}}_{\sin\alpha-\sin\beta}\right)$$

$$\boxed{\sqrt{\mu}\,(t_2-t_1) = a^{3/2}\left[\alpha-\beta-(\sin\alpha-\sin\beta)\right]} \qquad (3.60)$$

LAMBERT'S EQUATION

Eq (3.60) Describes Δt to go from $P_1 \rightarrow P_2$ on an elliptical orbit. Δt for the case less then 1 revolution $0 \le \Delta\theta \le 2\pi$

From Eq (3.56)

$$r_1 + r_2 + c = 4a\sin^2\left(\frac{\alpha}{2}\right)$$

$$\Rightarrow \sin\frac{\alpha}{2} = \left(\frac{r_1+r_2+c}{4a}\right)^{1/2}$$

semi-perimeter of the space triangle

$$\boxed{\sin\frac{\alpha}{2} = \left(\frac{s}{2a}\right)^{1/2}} \quad (3.61) \quad \text{with} \quad s = \frac{r_1+r_2+c}{2}$$

From eq (3.57)

$$r_1 + r_2 - c = 4a\sin^2\frac{\beta}{2}$$

$$\Rightarrow \quad \sin\frac{\beta}{2} = \left(\frac{r_1+r_2-c}{4a}\right)^{1/2}$$

$$\boxed{\sin\frac{\beta}{2} = \left(\frac{s-c}{2a}\right)^{1/2}} \qquad (3.62)$$

The two equation (3.61) and (3.62) if inserted in the Lambert's equation (3.60) proves the Lambert's theorem

$$(3.61) \longrightarrow \quad \alpha = fun(a, c, r_1+r_2)$$

$$(3.60) \longrightarrow \quad \beta = fun(a, c, r_1+r_2)$$

$$\sqrt{\mu}(t_2-t_1) = a^{3/2}\left[\alpha-\beta-(\sin\alpha-\sin\beta)\right]$$

As Eq (3.61) and (3.62) prove that $\alpha = fun(a, c, r_1+r_2)$

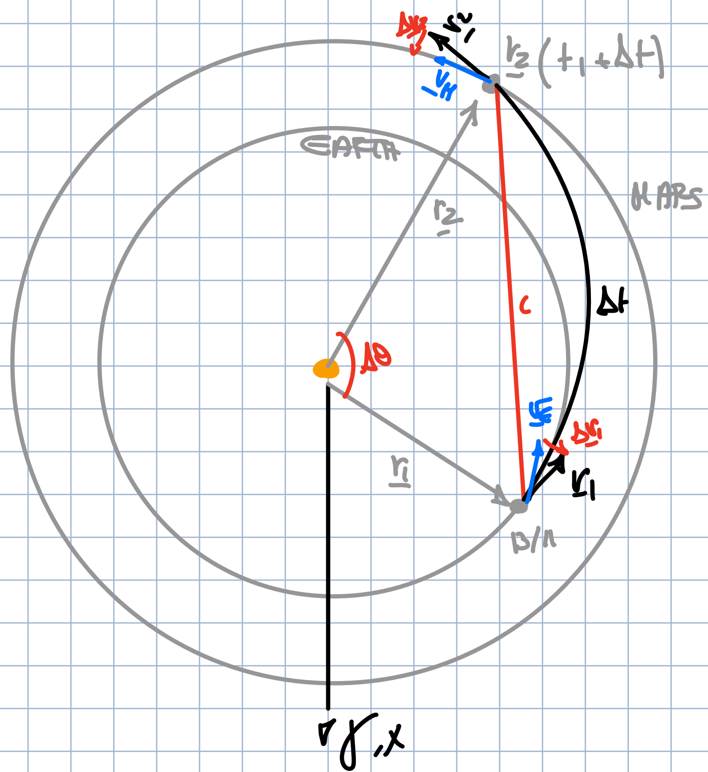
$$\beta = fun(a, c, r_1+r_2)$$

$$\Rightarrow \quad \sqrt{\mu}(t_2-t_1) = a^{3/2}\left[\alpha(a, c, r_1+r_2) - \beta(a, c, r_1+r_2) + \right.$$

$$\left. - (\sin\alpha(a, c, r_1+r_2) - \sin\beta(a, c, r_1+r_2)\right]$$

GIVEN Δt, $\underline{r}_1$, $\underline{r}_2$, ZH

$[a, \underline{v}_1, \underline{v}_2] = \text{lambert}(\Delta t, \underline{r}_1, \underline{r}_2, ZH)$



$\underline{r}_1(t_1)$

$\underline{r}_2(t_1+\Delta t)$

↳ as an input

PORKCHOP PLOT

plots to show for set t_1 set Δt Δv_{TOT}