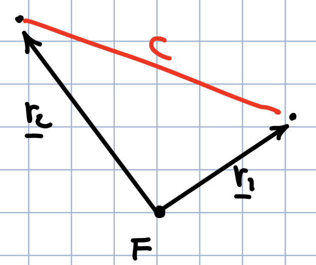
ORBITAL MECHANICS

SUMMARY OF WHAT WE HAVE SEEN LAST TIME

$$\Delta t = t_2 - t_1 = fun(a, c, r_1 + r_2)$$



$$\sqrt{\mu}\,(t_2 - t_1) = a^{3/2}\left[E_2 - E_1 - e(\sin E_2 - \sin E_1)\right] \rightarrow$$ kepler time law initial value problem.

$$E_p = \frac{1}{2}(E_1 + E_2)$$

$$E_m = \frac{1}{2}(E_2 - E_1) > 0$$

$$\cos\xi = e\cos E_p$$

$$\Rightarrow \sqrt{\mu}\,(t_2 - t_1) = a^{3/2}\left(2E_m - 2\cos\xi \sin E_m\right)$$

$$\alpha = \xi + E_m$$

$$\beta = \xi - E_m$$

$$\alpha - \beta = \xi + E_m - \xi + E_m \qquad \alpha - \beta = 2E_m$$

$$\alpha + \beta = \xi + E_m + \xi - E_m \qquad \alpha + \beta = 2\xi$$

$$\sin\alpha - \sin\beta = \sin(\xi + E_m) - \sin(\xi - E_m)$$

$$= \sin\xi \cancel{\cos E_m} + \cos\xi \sin E_m - \sin\xi \cancel{\cos E_m} + \cos\xi \sin E_m$$

$$= 2\cos\xi \sin E_m$$

$$\Rightarrow \sqrt{\mu}\,(t_2 - t_1) = a^{3/2}\,(\alpha - \beta - \sin\alpha + \sin\beta) \rightarrow$$ Lambert Equation
Boundary value problem

Lambert's equation describes the elliptical $\Delta t = t_2 - t_1$ for the case of less than 1 revolution as $\Delta\theta < 2\pi$

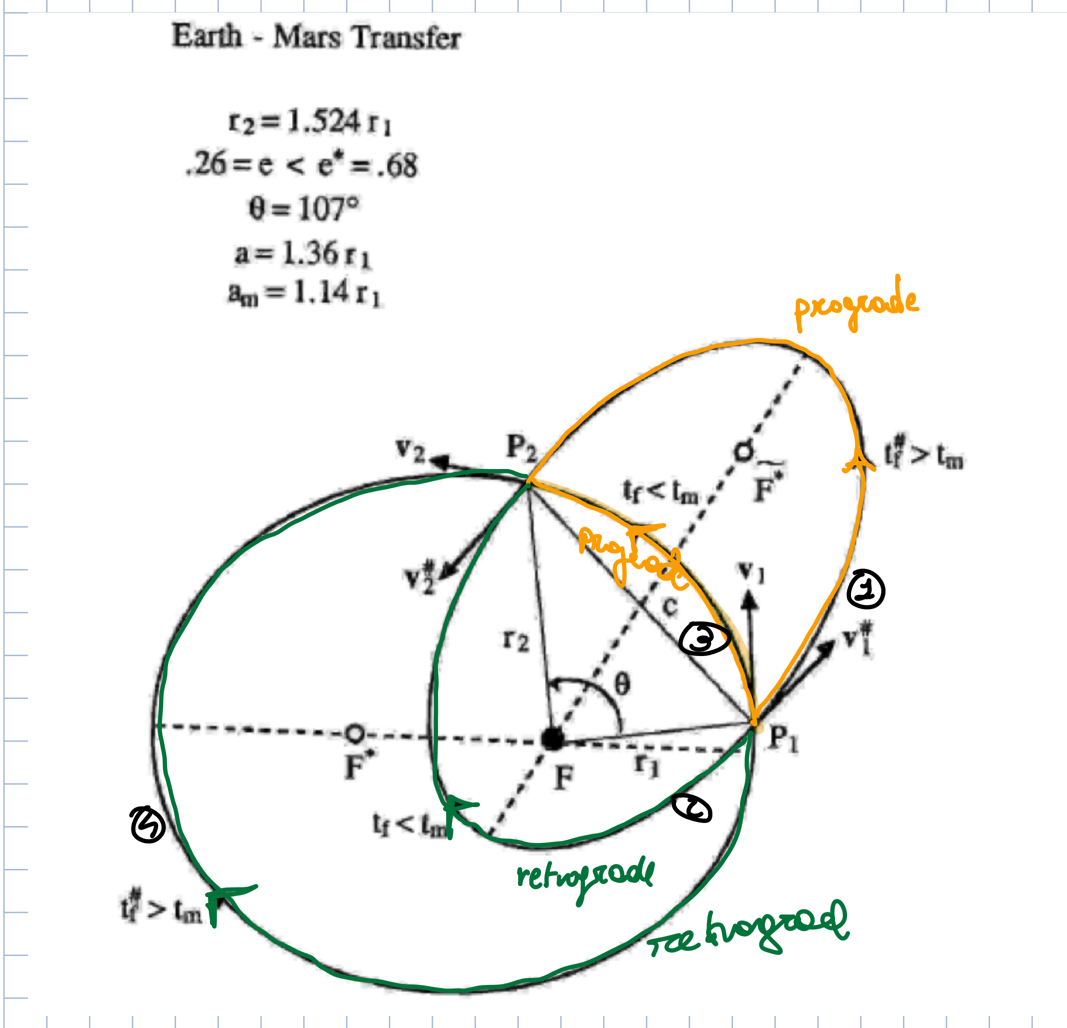
α and β can be written in terms of c, $r_1 + r_2$ and a

$$s = \frac{c + r_1 + r_2}{2}$$ semiperimeter of the space triangle

$$\sin\frac{\alpha}{2} = \left(\frac{s}{2a}\right)^{1/2}$$

$$\sin\frac{\beta}{2} = \left(\frac{s-c}{2a}\right)^{1/2}$$

example



There are 4 arcs

- 2 arcs $\Delta\theta < \pi$
- 2 arcs $\Delta\theta > \pi$

These four arcs correspond to the quadrant ambiguity of α and β

$\Delta t_1 + \Delta t_2 = T(a)$
↓
period of the ellipse

$\Delta t_3 + \Delta t_4 = T(a)$

↳ with the proper sign

We can see that a little bit better with the following figure

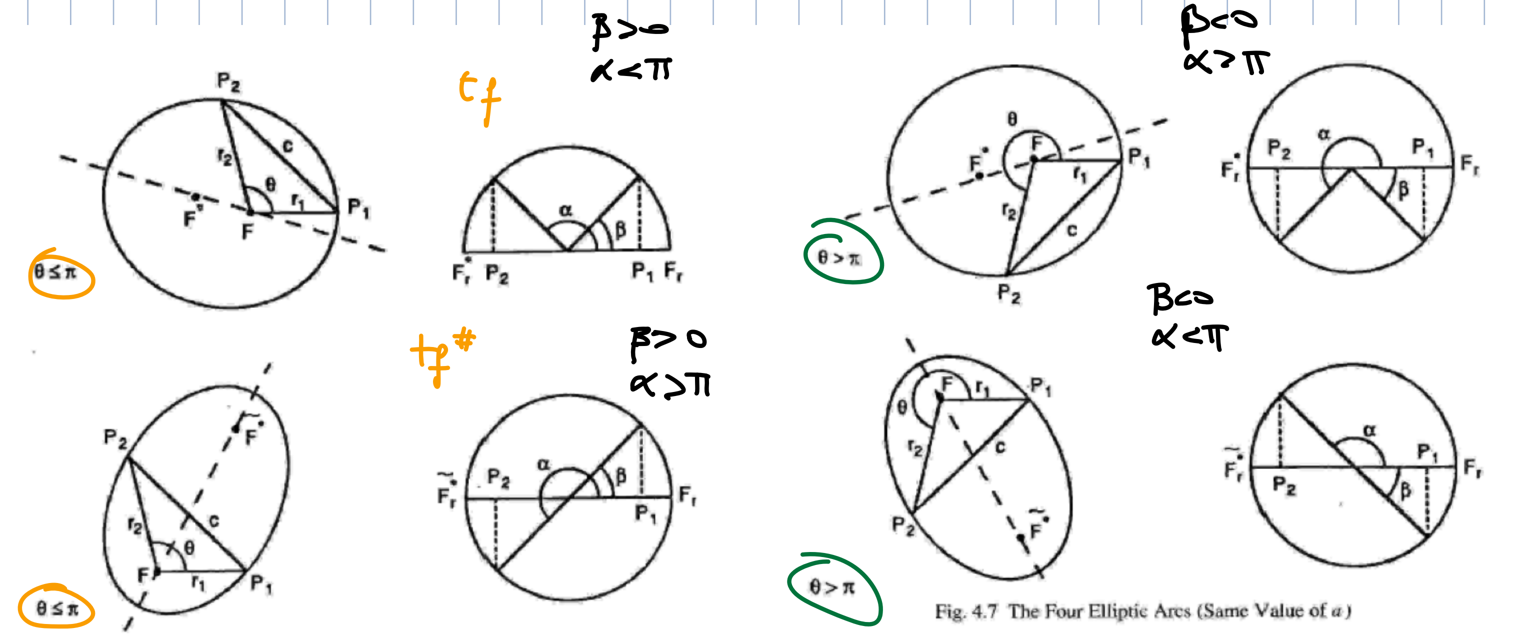


Fig. 4.7 The Four Elliptic Arcs (Same Value of a)

- Transfer time must satisfy transfer's equation
- FF^* can change as long as r_1+r_2, c, a remain unchanged as long we can identify the same set of α and β
- changing FF^* alters the geometry of the transfer orbit but if α, β function of (a, c, r_1+r_2) remain unchanged, the transfer time will not change.

By compare the Kepler's equation with Lambert's equations

$$\sqrt{\mu}(t_2-t_1) = a^{3/2}\left[E_2 - E_1 - e(\sin E_2 - \sin E_1)\right]$$

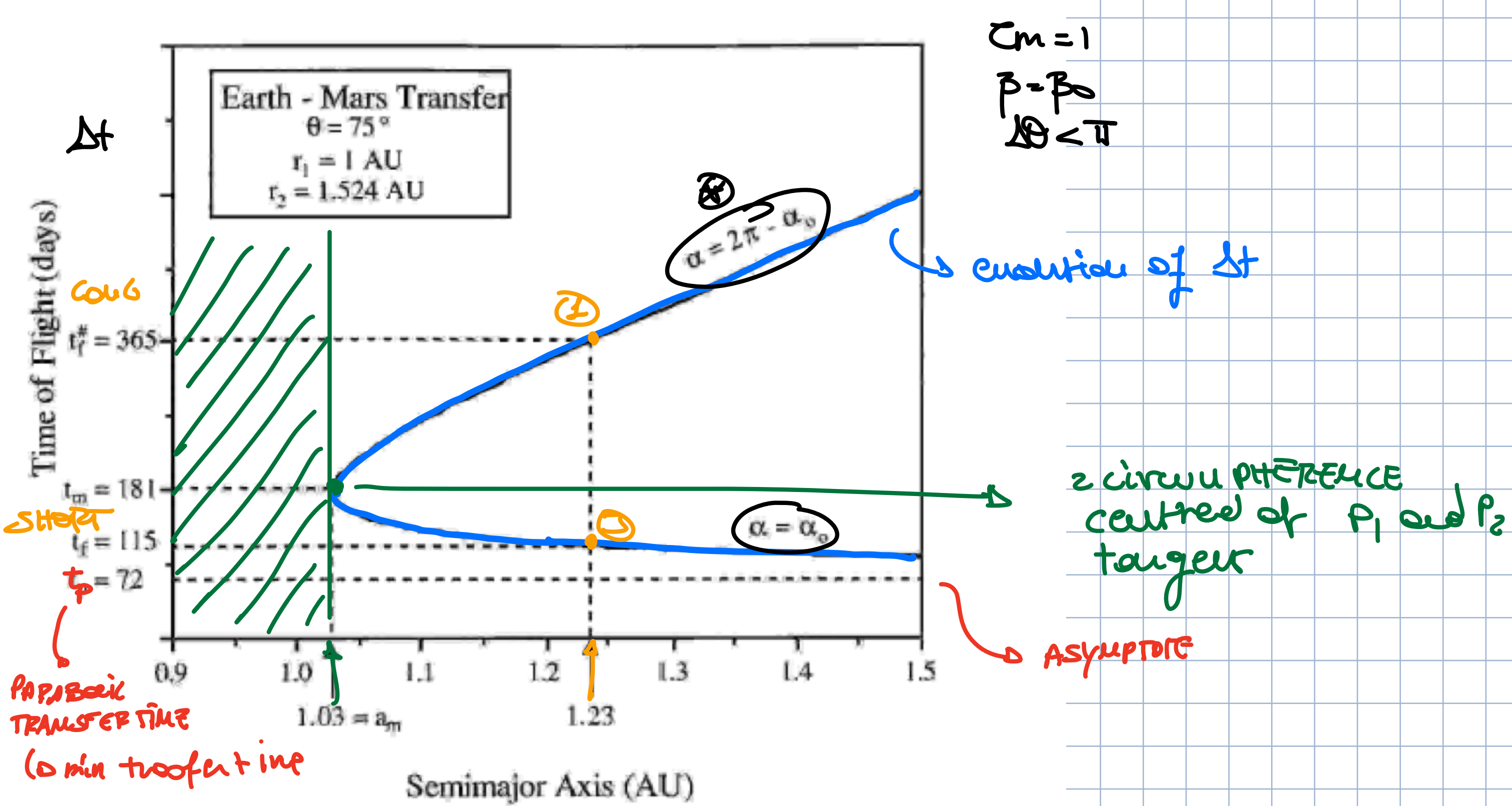
$$\sqrt{\mu}(t_2-t_1) = a^{3/2}\left[\alpha - \beta - (\sin\alpha - \sin\beta)\right]$$

α, β can be interpreted as the eccentric anomalies on the RECTILINEAR ELLIPSE $(e = 1)$ between P_1 and P_2 having the same values of a, c and r_1+r_2

Transfer time

Given a $\Delta\theta$ ($\Delta\theta < \pi$) (i.e. given β_0) the behaviour of t_f (Δt) as a function of a.

$\Delta\theta$ with β_0 fixed $\Rightarrow$ $\tau_m = +1$ sign of $\Delta\theta$



$$\sqrt{\mu}\,\Delta t = \text{fun}\left(a, \alpha(a, r_1+r_2, c), \beta(a, r_1+r_2, c)\right)$$

for a given a this is the Δt that we can find.
Note that for a given a there are two solutions $\Rightarrow$ they correspond to the two orbit transfer that we have seen before $\Rightarrow$ we can identify two different ellipse.

The minimum semimajor axis (minimum energy ellipse) a_m correspond to a transfer time t_m (remember is a Δt)

for any semi-major axis $a > a_m$ we have two possible transfer time

$$t_f < t_m \quad \text{③ example}$$
$$t_f^* > t_m \quad \text{② example}$$

t_m can be identified by Lambert's equation by using a_m

$$a_m = \frac{s}{2} \rightarrow \text{refers to } (3.40)$$

$$\sin\frac{\alpha_m}{2} = \left(\frac{s}{2a_m}\right)^{1/2} \longrightarrow \sin\frac{\alpha_m}{2} = \left(\frac{s\,2}{2\,s}\right)^{1/2} = 1 \qquad \frac{\alpha_m}{2} = \frac{\pi}{2} \rightarrow$$

$$\alpha_m = \pi$$

$$\sin\frac{\beta_m}{2} = \left(\frac{s-c}{2a_m}\right)^{1/2} \longrightarrow \sin\frac{\beta_m}{2} = \left(\frac{s-c}{s}\right)^{1/2}$$

Therefore

$$\sqrt{\mu}\, t_m = a_m^{3/2}\left(\pi - \beta_m - (0 - \sin\beta_m)\right)$$

$$\boxed{\sqrt{\mu}\, t_m = \left(\frac{s^3}{8}\right)^{1/2}\left(\pi - \beta_m + \sin\beta_m\right)} \qquad (3.63)$$

we can find t_m wich is not the MINIMUM TRANSFER TIME but is the transfer time associated to a_m.

Note a_m minimum energy we could not find any transfer with smaller a but we can find transfer with smaller transfer time than t_m.

minimum transfer time obtained as an asymptote.

The lower limit of transfer time for an elliptical orbit

t_p approached with $a \to \infty$ (PARABOLIC TRANSFER TIME)

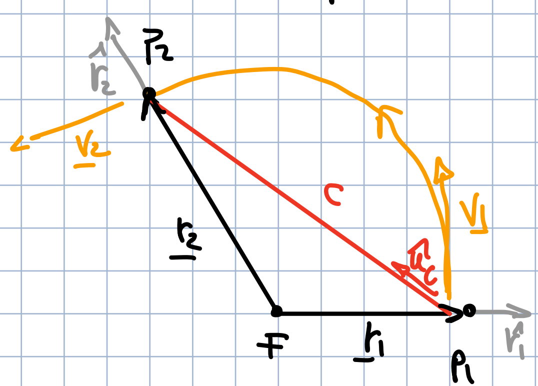
For a given $\underline{r}_1, \underline{r}_2$ (r_1+r_2, c)

①	a	$\Delta\theta < \pi$ (β_0)	$t_f^* > t_m$	$\Delta t_1 > t_m$
②	a	$\Delta\theta > \pi$		$\Delta t_2 < t_m$
③	a	$\Delta\theta < \pi$ (β_0)	$t_f < t_m$	$\Delta t_3 < t_m$
④	a	$\Delta\theta > \pi$		$\Delta t_4 > t_m$

$$\Delta t_3 + \Delta t_4 = T(a)$$
$$\Delta t_2 + \Delta t_1 = T(a)$$

$\Delta t - a$ for a $\Rightarrow$ 2 solutions with 2 possible Δt

for Δt $\Rightarrow$ 1 solution (one a)



$$\hat{r}_1 = \frac{\underline{r}_1}{r_1} \qquad \hat{r}_2 = \frac{\underline{r}_2}{r_2}$$

$$\hat{u}_c = \frac{\underline{r}_2 - \underline{r}_1}{c}$$

$$\underline{v}_1 = (B+A)\hat{u}_c + (B-A)\hat{r}_1 \qquad A = \left(\frac{\mu}{4a}\right)^{1/2} \operatorname{cotg}\frac{\alpha}{2} \quad B = \left(\frac{\mu}{4a}\right)^{1/2} \operatorname{cotg}\frac{\beta}{2}$$

$$\underline{v}_2 = (B+A)\hat{u}_c - (B-A)\hat{r}_2$$

APPLICATION OF LAMBERT EQUATION → LAMBERT PROBLEM SOLVER

ORBIT DETERMINATION PROBLEM OR TARGETING PROBLEM

Determine the orbit for a given $\underline{r}_1, \underline{r}_2$, $\Delta t = t_2 - t_1$

given $\underline{r}_1, \underline{r}_2, \Delta t \Rightarrow$ find orbit $(a, \underline{v}_1, \underline{v}_2)$

① $\underline{r}_1, \underline{r}_2 \longrightarrow r_1, r_2, r_1 + r_2, c$

② $\Delta\theta$ (user need to select $\varepsilon m = \pm 1$ β_0, $2\pi - \beta_0$)

③ given the space triangle I can calculate t_p (Parabolic transfer time) if $t_2 - t_1 > t_p \to$ I am on an elliptical orbit

if $t_2 - t_1 < t_p \to$ I am on an Hyperbolic orbit

④ Calculate $a_m = \frac{s}{2}$

$$t_m \leftarrow \sqrt{\mu}\, t_m = \left(\frac{s^3}{8}\right)^{1/2}(\pi - \beta_m + \sin\beta_m)$$

$$\alpha_m = \pi$$

$$\sin\left(\frac{\beta_m}{2}\right) = \left(\frac{s-c}{s}\right)^{1/2}$$

⑤ Solve Lambert's equation numerically to find a

$\Delta t < t_m \Rightarrow \alpha_0$ } Look above ⊛

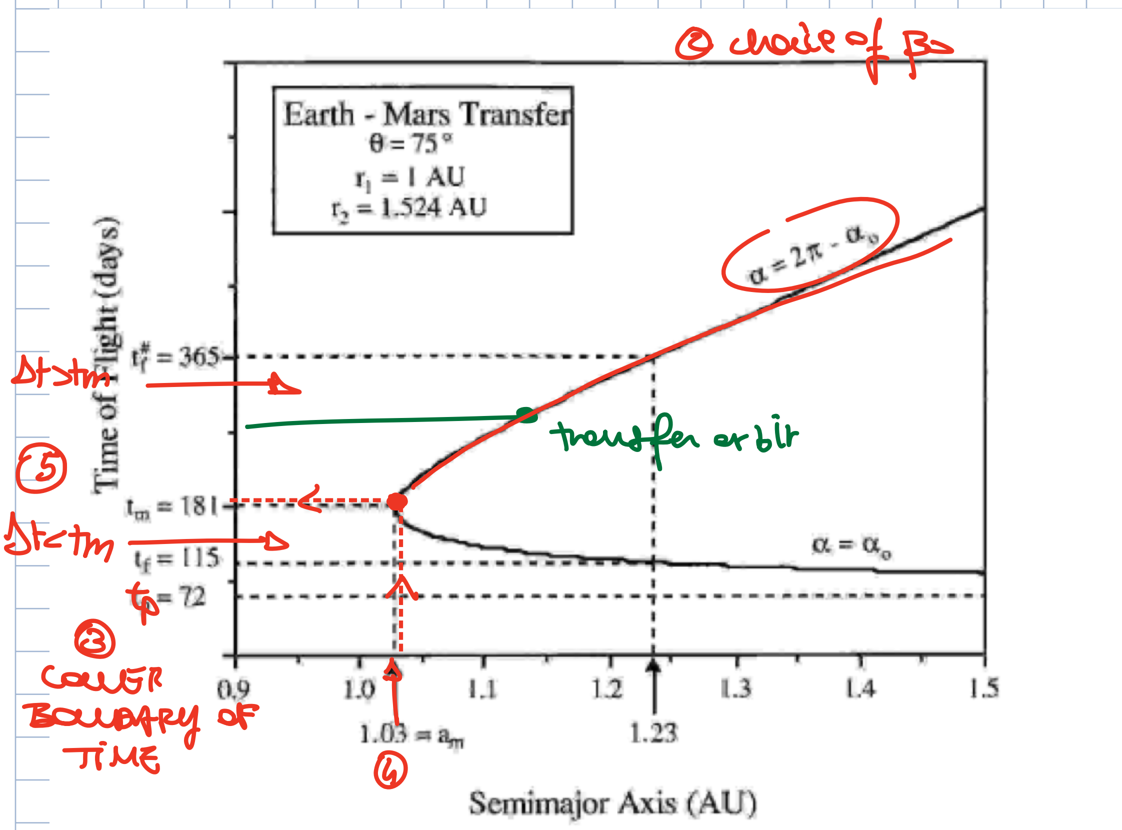
$\Delta t > t_m \Rightarrow 2\pi - \alpha_0$

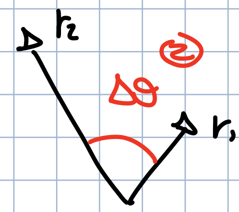
$$\sqrt{\mu}\,\Delta t = a^{3/2}\left[\alpha(a, r_1+r_2, c) - \beta(a, r_1+r_2, c) - (\sin\alpha - \sin\beta)\right]$$

$fun(a, r_1+r_2, c)$

so from this equation I can iteratively solve this equation to find a.

⑥ Known $a \to$ compute $\underline{v}_1$ and $\underline{v}_2$ of the transfer orbit or P_1 and P_2.

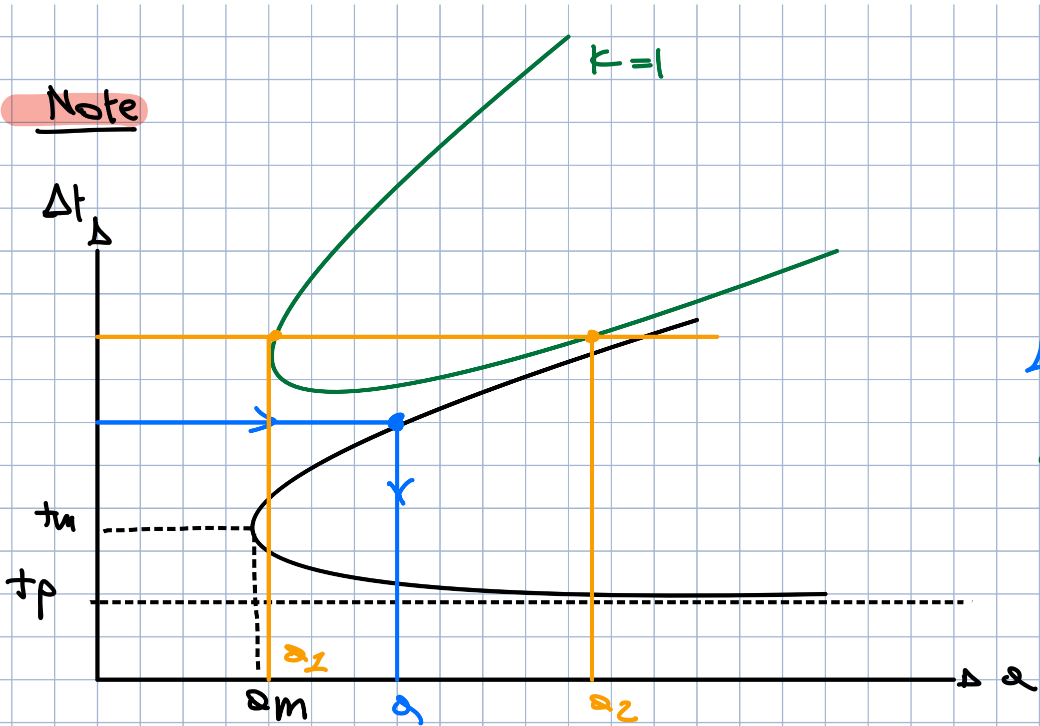




⑧ At this point if we know if we have $\Delta t \gtrless t_m$ we could consider only a branch of the graph.

Now it is possible to invert the relation to find a of the transfer orbit.

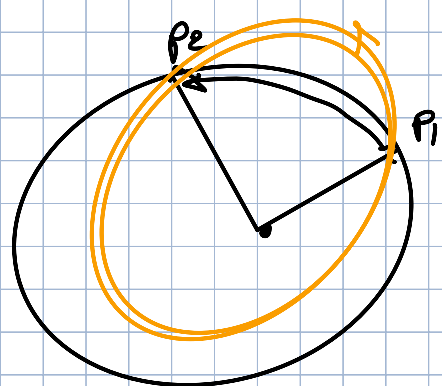
Note



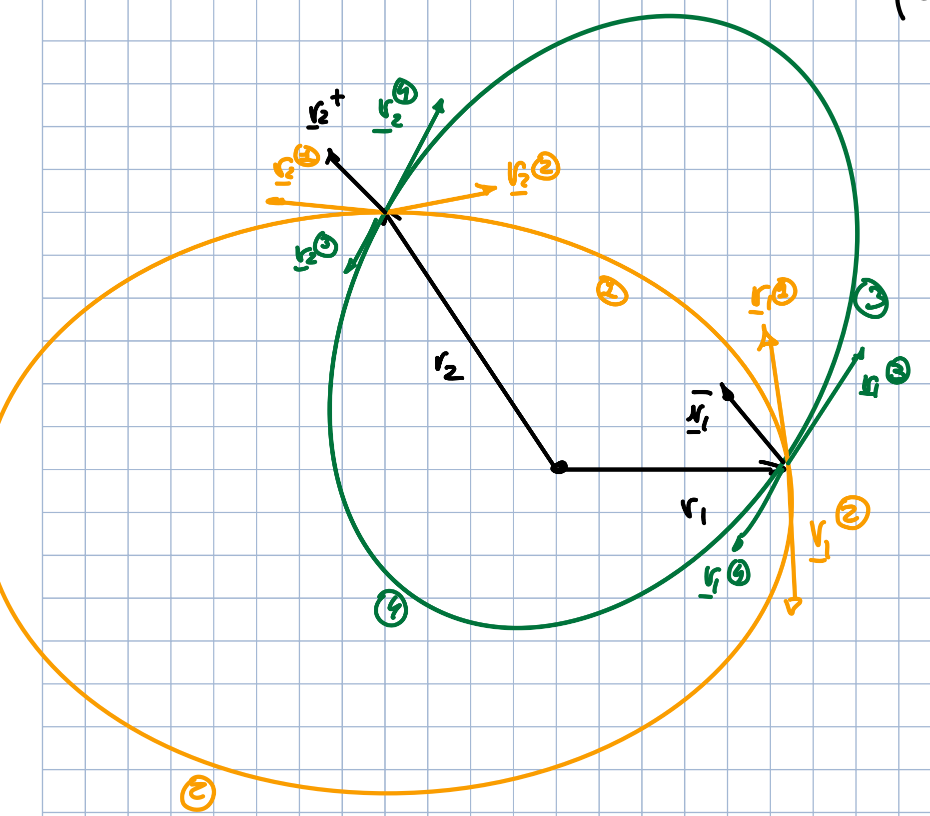
$\Delta\theta < 2\pi$ $\quad k=0$

MULTI-REVOLUTION CASE $\Delta\theta > 2\pi$

For $k \geq 1$ two a exist for a given $\Delta\theta > 2\pi$



THE PORK CHOP PLOT → Particular graph that gives us the total Δv for transfers between planets.



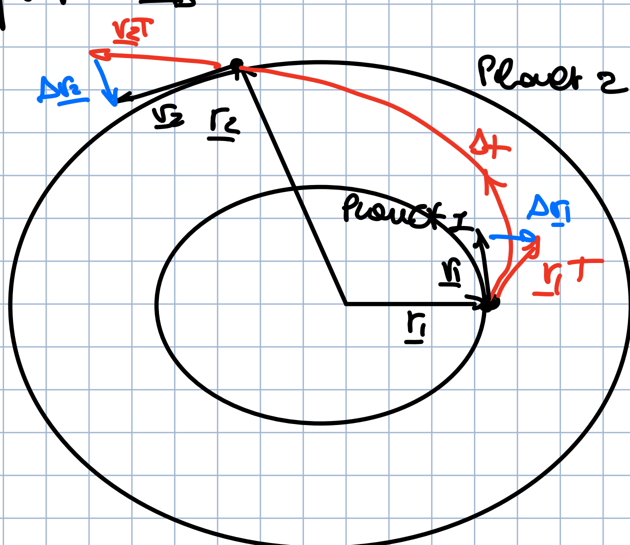
Δt

$\Delta \underline{v} = \Delta \underline{v}_1 + \Delta \underline{v}_2$

$\Delta \underline{v}_1 = \underline{v}_1 - \underline{v}_1^-$

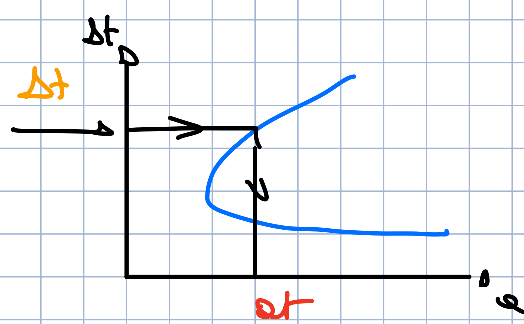
$\Delta \underline{v}_2 = \underline{v}_2^+ - \underline{v}_2$

Pork chop plot →



$\underline{r}_1(t_1) = \underline{r}_1$

$\underline{r}_2(t_1 + \Delta t) = \underline{r}_2$



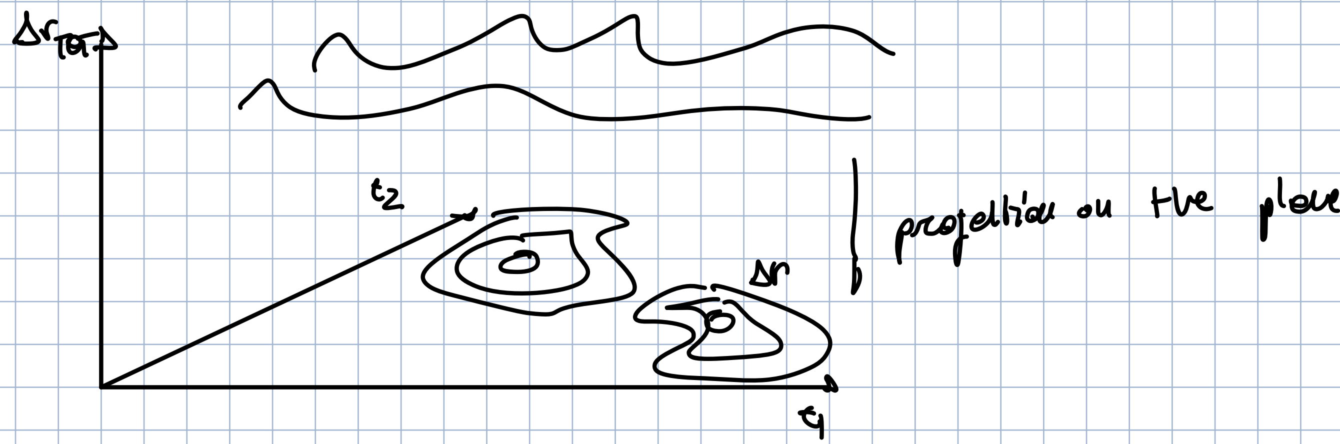
Ephemerides

$\underline{r}_1, \underline{v}_1$ planet 1 function t

$\underline{r}_2, \underline{v}_2$ planet 2 function t

There are two solution with the same Δt, prograde and retrograde → Δv it will be usually smaller for prograde transfer.

Porkchop plot contour plot of the Δv of all possible transfer.



The porkchop plot repeat after the synodic period, the graph repeats when the relative position of the two planet repeats.