POLITECNICO
MILANO 1863

Spacecraft Attitude Dynamics

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Attitude parameters

Attitude dynamics and kinematics

Dynamics

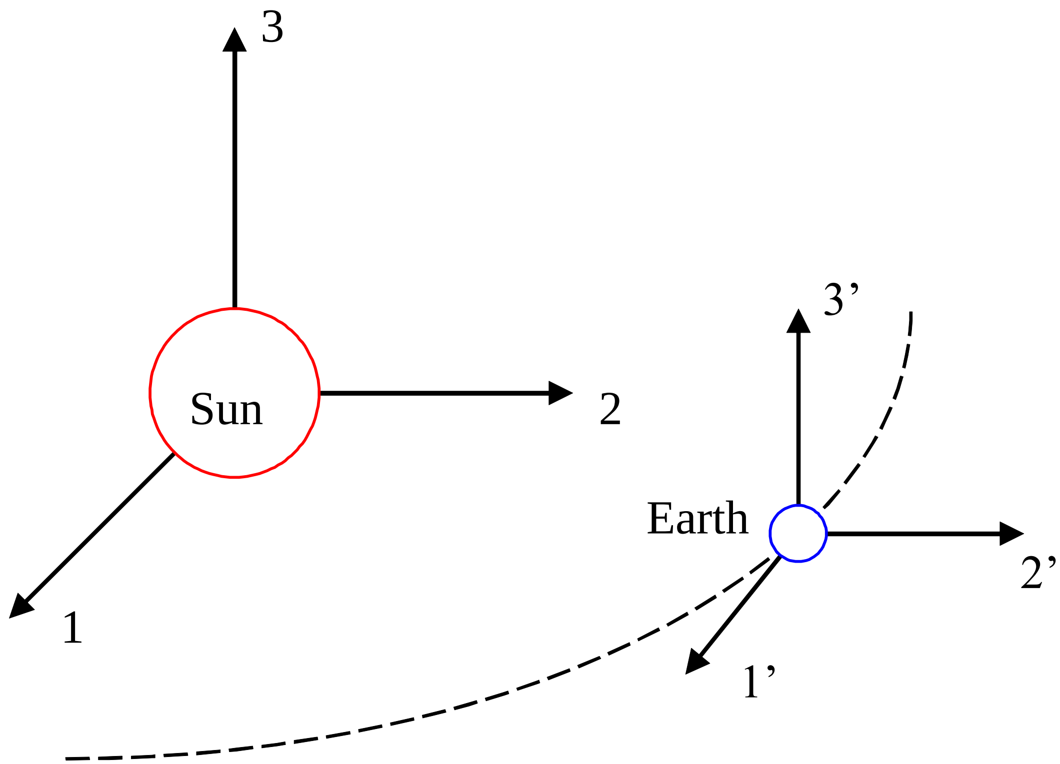
Kinematics

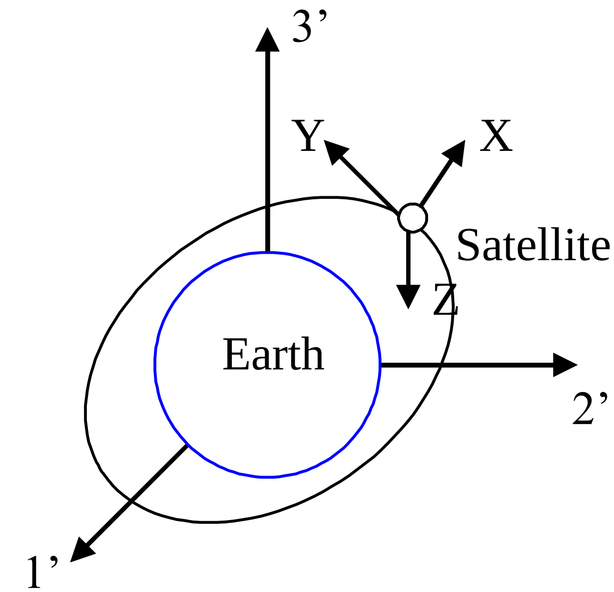
$$
\begin{cases}
\dot{\omega}_x = \dfrac{(I_y - I_z)}{I_x}\omega_z\omega_y + \dfrac{M_x}{I_x} \\
\dot{\omega}_y = \dfrac{(I_z - I_x)}{I_y}\omega_x\omega_z + \dfrac{M_y}{I_y} \\
\dot{\omega}_z = \dfrac{(I_x - I_y)}{I_z}\omega_y\omega_x + \dfrac{M_z}{I_z}
\end{cases}
\longrightarrow
$$

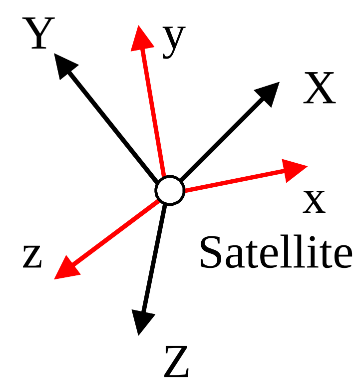
orientation of one frame with respect to another

Dynamics → Kinematics

Attitude parameters





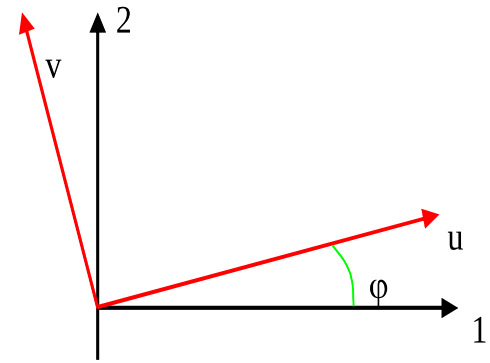


X is the satellite local vertical direction
Y is the satellite velocity direction
Z is the third direction, orthogonal to X and Y

x,y,z are the satellites’ principal inertia axes

Direction cosines

$$A = \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$$



$$A = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

the vector in the new reference (a_{uvw}) is obtained by multiplying the original vector (a_{123}) by the direction cosine matrix A

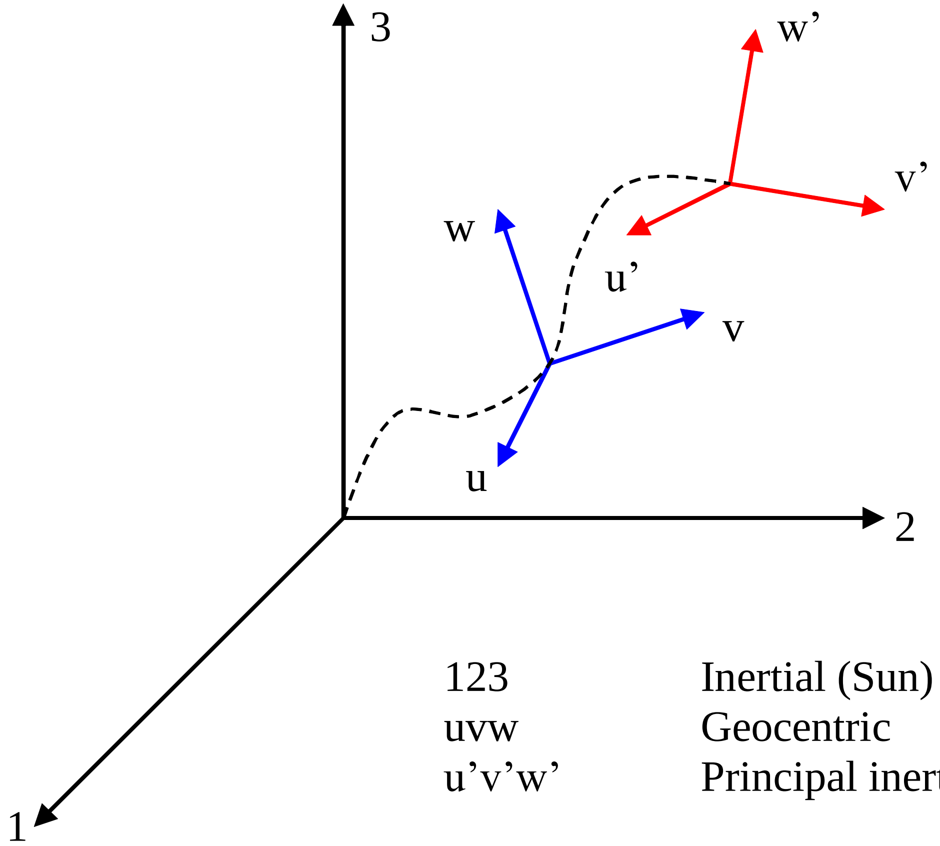
$$a_{uvw} = A \cdot a_{123}$$

$$a_{123} = A^T \cdot a_{uvw}$$

$$AA^T = I$$

$$A^T = A^{-1}$$

Direction cosines



123	Inertial (Sun)
uvw	Geocentric
u'v'w'	Principal inertia

$$a_{uvw} = A \cdot a_{123}$$

$$a_{u'v'w'} = A' \cdot a_{uvw}$$

$$a_{u'v'w'} = A" \cdot a_{123} = A' \cdot a_{uvw} = A'A \cdot a_{123}$$

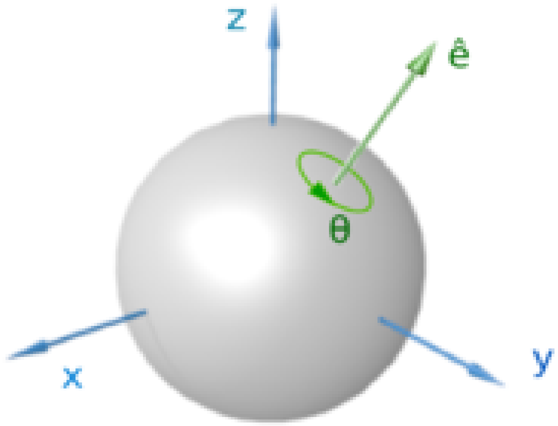
$$A" = A'A$$

Euler axis / angle

Euler’s rotation theorem -> Any single rotation can be represented by a vector (eigenvector) that remains fixed during that rotation and a simple rotation around that vector by an angle θ (eigen-angle)

$$A\underline{e} = \underline{e}$$

$$\underline{\omega} = \dot{\theta}\underline{e}$$



This because orthogonal matrices have one unit eigenvalue.

Now try to relate the direction cosines matrix A with vector $\underline{e}$.

Euler axis / angle

$$A_3(\phi) = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \underline{e} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$A_2(\phi) = \begin{bmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{bmatrix} \qquad \underline{e} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$A_1(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \qquad \underline{e} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \quad tr(A) = 1 + 2\cos\phi$$

$$\downarrow$$

$$\cos\phi = \frac{1}{2}\left(tr(A) - 1\right)$$

$$A = I\cos\phi + (1 - \cos\phi)\underline{e}\underline{e}^T - \sin\phi[e\wedge]$$

$$[e\wedge] = \begin{bmatrix} 0 & -e_3 & e_2 \\ e_3 & 0 & -e_1 \\ -e_2 & e_1 & 0 \end{bmatrix}$$

Euler axis / angle

$$A = \begin{bmatrix} cos\phi + e_1^2(1-cos\phi) & e_1e_2(1-cos\phi) + e_3 sin\phi & e_1e_3(1-cos\phi) - e_2 sin\phi \\ e_1e_2(1-cos\phi) - e_3 sin\phi & cos\phi + e_2^2(1-cos\phi) & e_2e_3(1-cos\phi) + e_1 sin\phi \\ e_1e_3(1-cos\phi) + e_2 sin\phi & e_2e_3(1-cos\phi) - e_1 sin\phi & cos\phi + e_3^2(1-cos\phi) \end{bmatrix}$$

$$\phi = cos^{-1}\left[\frac{1}{2}(tr(A)-1)\right]$$

$$\begin{cases} e_1 = \dfrac{(A_{23}-A_{32})}{2sin\phi} \\ e_2 = \dfrac{(A_{31}-A_{13})}{2sin\phi} \\ e_3 = \dfrac{(A_{12}-A_{21})}{2sin\phi} \end{cases}$$

when $sin\phi$=0 the Euler axis is undetermined

No rule for consecutive rotations

Quaternion

$$\begin{cases} q_1 = e_1 sin\dfrac{\phi}{2} \\ q_2 = e_2 sin\dfrac{\phi}{2} \\ q_3 = e_3 sin\dfrac{\phi}{2} \\ q_4 = cos\dfrac{\phi}{2} \end{cases} \longrightarrow$$

$$|\underline{e}|^2 = 1$$

$$q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1$$

vector part $\underline{q}$ and a scalar part q_4

$$\hat{q} = [\underline{q} \ ; \ q_4]$$

$$\underline{q} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} \quad , \quad q_4$$

Quaternion

$\hat{q} \rightarrow A$

$$A = \begin{bmatrix} q_1^2 - q_2^2 - q_3^2 + q_4^2 & 2(q_1 q_2 + q_3 q_4) & 2(q_1 q_3 - q_2 q_4) \\ 2(q_1 q_2 - q_3 q_4) & -q_1^2 + q_2^2 - q_3^2 + q_4^2 & 2(q_2 q_3 + q_1 q_4) \\ 2(q_1 q_3 + q_2 q_4) & 2(q_2 q_3 - q_1 q_4) & -q_1^2 - q_2^2 + q_3^2 + q_4^2 \end{bmatrix}$$

$$A = \left(q_4^2 - \underline{q}^T \underline{q}\right) I + 2\underline{q}\underline{q}^T - 2q_4[q \wedge]$$

$$[q \wedge] = \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix}$$

$A \rightarrow \hat{q}$

$$\begin{cases} q_1 = \dfrac{1}{4q_4}(A_{23} - A_{32}) \\ q_2 = \dfrac{1}{4q_4}(A_{31} - A_{13}) \\ q_3 = \dfrac{1}{4q_4}(A_{12} - A_{21}) \\ q_4 = \pm\dfrac{1}{2}(1 + A_{11} + A_{22} + A_{33})^{\frac{1}{2}} \end{cases}$$

Quaternion

Alternative inverse mapping

$$q_1^2 = \pm\frac{1}{2}\sqrt{1 + A_{11} - A_{22} - A_{33}}$$
$$q_2^2 = \frac{1}{4\mathrm{q}_1^2}(A_{12} + A_{21})$$
$$q_3^2 = \frac{1}{4\mathrm{q}_1^2}(A_{13} + A_{31})$$
$$q_4^2 = \frac{1}{4\mathrm{q}_1^2}(A_{23} - A_{32})$$

$$q_2^3 = \pm\frac{1}{2}\sqrt{1 - A_{11} + A_{22} - A_{33}}$$
$$q_1^3 = \frac{1}{4\mathrm{q}_2^3}(A_{12} + A_{21})$$
$$q_3^3 = \frac{1}{4\mathrm{q}_2^3}(A_{23} + A_{32})$$
$$q_4^3 = \frac{1}{4\mathrm{q}_2^3}(A_{31} - A_{13})$$

$$q_3^4 = \pm\frac{1}{2}\sqrt{1 - A_{11} - A_{22} + A_{33}}$$
$$q_1^4 = \frac{1}{4\mathrm{q}_3^4}(A_{13} + A_{31})$$
$$q_2^4 = \frac{1}{4\mathrm{q}_3^4}(A_{23} + A_{32})$$
$$q_4^4 = \frac{1}{4\mathrm{q}_3^4}(A_{12} - A_{21})$$

Quaternion

sequence of two consecutive rotations

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} q_4' & -q_3' & q_2' & q_1' \\ q_3' & q_4' & -q_1' & q_2' \\ -q_2' & q_1' & q_4' & q_3' \\ -q_1' & -q_2' & -q_3' & q_4' \end{bmatrix} \begin{bmatrix} q_1'' \\ q_2'' \\ q_3'' \\ q_4'' \end{bmatrix}$$

$$\hat{q} = \hat{q}'' \otimes \hat{q}' \quad \longleftrightarrow \quad A = A''A'$$

$$\hat{q}^{-1} = [-\underline{q}\ ;\ q_4]$$

Gibbs vector

$$\begin{cases} q_1 = e_1 \sin\frac{\theta}{2} \\ q_2 = e_2 \sin\frac{\theta}{2} \\ q_3 = e_3 \sin\frac{\theta}{2} \\ q_4 = \cos\frac{\theta}{2} \end{cases} \longrightarrow \begin{cases} g_1 = \frac{q_1}{q_4} = e_1 \tan\frac{\theta}{2} \\ g_2 = \frac{q_2}{q_4} = e_2 \tan\frac{\theta}{2} \\ g_3 = \frac{q_3}{q_4} = e_3 \tan\frac{\theta}{2} \end{cases}$$

Singularity at
$\theta = (2n + 1)\,\pi.$

$$A(\underline{g}) = \frac{1}{1 + g_1^2 + g_2^2 + g_3^2} \begin{bmatrix} 1 + g_1^2 - g_2^2 - g_3^2 & 2(g_1 g_2 + g_3) & 2(g_1 g_3 - g_2) \\ 2(g_1 g_2 - g_3) & 1 - g_1^2 + g_2^2 - g_3^2 & 2(g_2 g_3 + g_1) \\ 2(g_1 g_3 + g_2) & 2(g_2 g_3 - g_1) & 1 - g_1^2 - g_2^2 + g_3^2 \end{bmatrix}$$

$$A = \frac{\left(1 - \underline{g}^2\right) I + 2\underline{g}\,\underline{g}^T - 2[g\,\wedge]}{\left(1 + \underline{g}^2\right)}$$

Gibbs vector

Inverse mapping

$$\begin{cases} g_1 = \dfrac{A_{23} - A_{32}}{1 + A_{11} + A_{22} + A_{33}} \\ g_2 = \dfrac{A_{31} - A_{13}}{1 + A_{11} + A_{22} + A_{33}} \\ g_3 = \dfrac{A_{12} - A_{21}}{1 + A_{11} + A_{22} + A_{33}} \end{cases}$$

singular when $\varphi = (2n + 1)\,\pi$

consecutive rotations

$$g'' = \frac{\underline{g} + \underline{g}' - \underline{g}' \wedge \underline{g}}{1 - \underline{g} \cdot \underline{g}'}$$

Euler angles

Any rotation can be de-composed into the multiplication of three trivial rotations

$$A_1(\psi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & \sin\psi \\ 0 & -\sin\psi & \cos\psi \end{bmatrix}$$

rotation by angle ψ around axis 1

$$A_2(\vartheta) = \begin{bmatrix} \cos\vartheta & 0 & -\sin\vartheta \\ 0 & 1 & 0 \\ \sin\vartheta & 0 & \cos\vartheta \end{bmatrix}$$

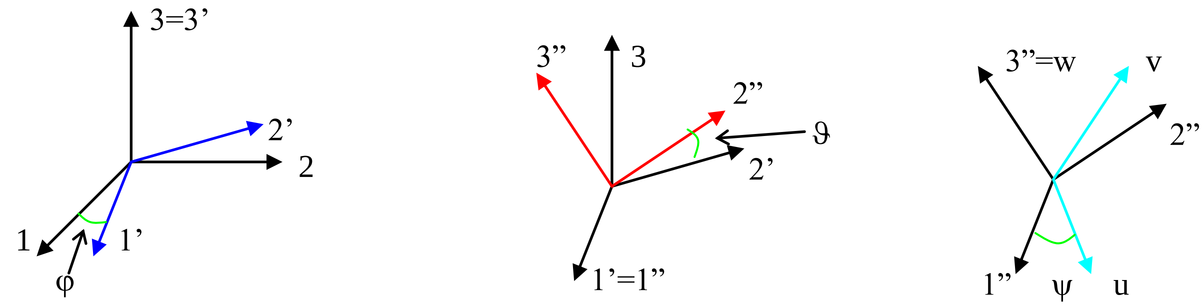
rotation by angle ϑ around axis 2

$$A_3(\phi) = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

rotation by angle ϕ around axis 3

Euler angles

1,2,3 -> u,v,w



$$A_{313}(\phi,\vartheta,\psi) = A_3(\psi)\cdot A_1(\vartheta)\cdot A_3(\phi)$$

$$A_{313} = \begin{bmatrix} \cos\psi\cos\phi - \sin\psi\sin\phi\cos\vartheta & \cos\psi\sin\phi + \sin\psi\cos\phi\cos\vartheta & \sin\psi\sin\vartheta \\ -\sin\psi\cos\phi - \cos\psi\sin\phi\cos\vartheta & -\sin\psi\sin\phi + \cos\psi\cos\phi\cos\vartheta & \cos\psi\sin\vartheta \\ \sin\phi\sin\vartheta & -\cos\phi\sin\vartheta & \cos\vartheta \end{bmatrix}$$

Euler angles

12 possibilities

312,213,123,321,231,132 all different indexes
313,323,212,232,131,121 first and third index equal

$$A_{312} = \begin{bmatrix} \cos\psi\cos\phi - \sin\psi\sin\phi\sin\vartheta & \cos\psi\sin\phi + \sin\psi\cos\phi\sin\vartheta & -\sin\psi\cos\vartheta \\ -\sin\phi\cos\vartheta & \cos\phi\cos\vartheta & \sin\vartheta \\ \sin\psi\cos\phi + \cos\psi\sin\phi\sin\vartheta & \sin\psi\sin\phi - \cos\psi\cos\phi\sin\vartheta & \cos\vartheta\cos\psi \end{bmatrix}$$

No model for consecutive rotations

Euler angles

Inverse mapping

$$A_{313}(\phi,\vartheta,\psi) \longrightarrow \begin{cases} \vartheta = cos^{-1}(A_{33}) \\ \phi = -\,tan^{-1}\left(\dfrac{A_{31}}{A_{32}}\right) \\ \psi = tan^{-1}\left(\dfrac{A_{13}}{A_{23}}\right) \end{cases} \longrightarrow$$

Singularity at
$\vartheta = n\pi$

$$A_{123}(\phi,\vartheta,\psi) = \begin{bmatrix} \cos\psi\cos\vartheta & \cos\psi\sin\vartheta\sin\phi+\sin\psi\cos\phi & -\cos\psi\sin\vartheta\cos\phi+\sin\psi\sin\phi \\ -\sin\psi\cos\vartheta & -\sin\psi\sin\vartheta\sin\phi+\cos\psi\cos\phi & \sin\psi\sin\vartheta\cos\phi+\cos\psi\sin\phi \\ \sin\vartheta & -\cos\vartheta\sin\phi & \cos\vartheta\ \cos\phi \end{bmatrix}$$

$$\searrow \begin{cases} \vartheta = sin^{-1}(A_{11}) \\ \phi = -\,tan^{-1}\left(\dfrac{A_{32}}{A_{33}}\right) \\ \psi = -tan^{-1}\left(\dfrac{A_{21}}{A_{11}}\right) \end{cases} \longrightarrow$$

Singularity at
$\vartheta = (2n+1)\pi/2$

Approximation for small angles

If angles are small, we can assume cos x =1, sin x = x, x*x =0 (with x in radians)

$$A_{312}(\phi,\vartheta,\psi) = \begin{bmatrix} 1 & \phi & -\psi \\ -\phi & 1 & \vartheta \\ \psi & -\vartheta & 1 \end{bmatrix} = A_{321}(\phi,\psi,\vartheta) = A_{213}(\psi,\vartheta,\phi) = \ldots$$

$$A = I - [angles \wedge]$$

$$A_{313}(\phi,\vartheta,\psi) = \begin{bmatrix} 1 & \phi+\psi & 0 \\ -\phi-\psi & 1 & \vartheta \\ 0 & -\vartheta & 1 \end{bmatrix}$$

$$\begin{cases} q_1 = \frac{1}{2}\vartheta \\ q_2 = \frac{1}{2}\psi \\ q_3 = \frac{1}{2}\phi \\ q_4 = 1 \end{cases}$$