ORBITAL MECHANICS

$$\underline{e} = \frac{\dot{\underline{r}} \wedge \underline{h}}{\mu} - \frac{\underline{r}}{r} \qquad (1.22)$$

The line defined by the eccentricity vector $\underline{e}$ is called the apse line.

$\underline{e}$ = costant is constant.

$\underline{r}(t), \underline{v}(t), \underline{h}(t) \Rightarrow$ (eq 1.22) function $(\underline{r}(t), \underline{v}(t), \underline{h}(t))$ = CONSTANT

→ This is a useful test to verify the correctness of the simulation.

Eq (1.21) $\cdot \underline{r}$

$$\frac{\underline{r} \cdot \underline{r}}{r} + \underline{e} \cdot \underline{r} = \frac{\underline{r} \cdot (\dot{\underline{r}} \wedge \underline{h})}{\mu} \qquad (1.23)$$

Recall : $\underline{a} \cdot (\underline{b} \wedge \underline{c}) = (\underline{a} \wedge \underline{b}) \cdot \underline{c}$

$$\underline{r} \cdot (\dot{\underline{r}} \wedge \underline{h}) = \underbrace{(\underline{r} \wedge \dot{\underline{r}})}_{\underline{h}} \cdot \underline{h} = \underline{h} \cdot \underline{h} = h^2 \qquad (1.24)$$

Substituting (1.24) in (1.23)

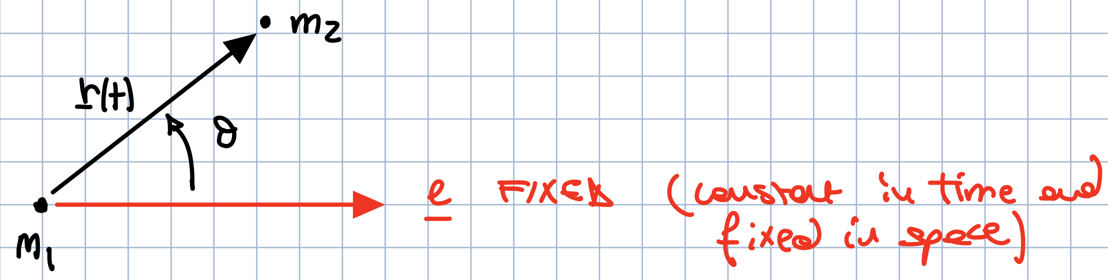
$$\frac{r^2}{r} + \underline{e} \cdot \underline{r} = \frac{h^2}{\mu} \qquad (1.25)$$

→ there is no more the variable of time. there are not (˙) anymore

$\underline{r} \cdot \underline{e} = r e \cos\theta$

e = eccentricity $= \|\underline{e}\|$

θ = True anomaly → angle between $\underline{e}$ (fixed) and the variable $\underline{r}(t)$



Calculate θ using the right-hand side rule.

NOTE θ may be called ν, f

Eq (1.25) rewrite it as

$$r + re\cos\theta = \frac{h^2}{\mu} \qquad \boxed{r = \frac{h^2/\mu}{1 + e\cos\theta}} \qquad (1.26)$$

We have found the close form solution of the equation.

(1.26) orbit equation : path of m_2 around m_1 (relative to m_1)

h, e, μ are constant

$\underline{e}$ is a vector $\Rightarrow$ $e = \|\underline{e}\| \geq 0$

$$\theta(t) \longrightarrow r(\theta(t)) = \frac{h^2/\mu}{1 + e\cos\theta}$$

Eq (1.26) is an equation describing a conic section.

conic section $\begin{cases} \text{parabola} \\ \text{hyperbola} \\ \text{ellipse} \\ \text{circular} \end{cases}$

(1.26) → mathematical demonstration of Kepler's first law

Keplerian orbit = two body problem

To integrate Eq (1.8) $\ddot{\underline{r}} + \frac{\mu \underline{r}}{r^3} = 0$ we need 6 constant

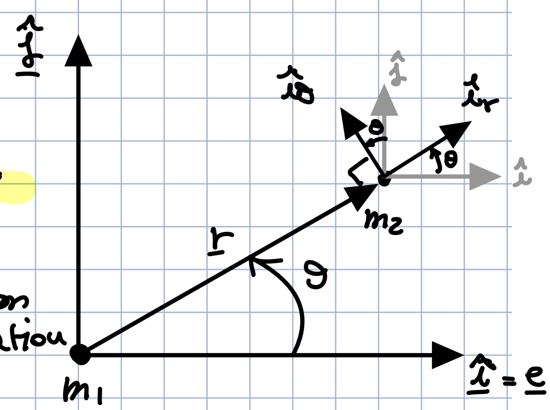
$\underline{h}$ 3 constant
$\underline{e}$ 3 constant
} but $\underline{h}$ and $\underline{e}$ are correlated

$\underline{e} \cdot \underline{h} = 0$ (1.27) so $6-1=5$ constant

we need another one constant to fully define the motion by integrating Eq (1.8).

ORBIT KINEMATICS

Let us take a cartesian reference frame



$\hat{i}_r, \hat{i}_\theta, \hat{i}_h$ RADIAL-TRANSVERSAL-OUT OF PLANE REFERENCE FRAME

↳ is not fixed follows mass m_2 and change orientation based on θ.

$\underline{r} = r \hat{i}_r$

$$\dot{\underline{r}} = \dot{r} \hat{i}_r + r \frac{d\hat{i}_r}{dt} \quad (1.29)$$

$$\begin{cases} \hat{i}_r = \cos\theta \, \hat{i} + \sin\theta \, \hat{j} \\ \hat{i}_\theta = -\sin\theta \, \hat{i} + \cos\theta \, \hat{j} \end{cases} \quad (1.30)$$

$$\begin{cases} \frac{d\hat{i}_r}{dt} = \dot{\theta} \hat{i}_\theta \\ \frac{d\hat{i}_\theta}{dt} = -\dot{\theta} \hat{i}_r \end{cases} \quad (2.31)$$

$$\underline{a} = \ddot{\underline{r}} = \ddot{r} \hat{i}_r + \dot{r}\dot{\theta} \hat{i}_\theta + \dot{r}\dot{\theta} \hat{i}_\theta + r\ddot{\theta} \hat{i}_\theta - r\dot{\theta}^2 \hat{i}_r$$
$$= (\ddot{r} - r\dot{\theta}^2) \hat{i}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta}) \hat{i}_\theta \quad (1.32)$$

NOTE $\underline{r} \cdot \dot{\underline{r}} = \underline{r} \cdot \underline{v} = \underline{r} \cdot (\dot{r} \hat{i}_r + r\dot{\theta} \hat{i}_\theta) = r\dot{r}$

$\underline{v} = v_r \hat{i}_r + v_\theta \hat{i}_\theta$ $\quad v_r = \dot{r}$

$v_\theta = r\dot{\theta}$

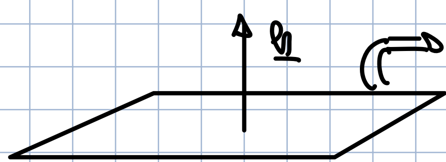
SUMMARY

(A) $\dot{\underline{r}} \cdot Eq(1.8) = 0 \Rightarrow \frac{\dot{r}^2}{2} - \frac{\mu}{r} = \varepsilon$ CONSTANT SCALAR

$$\frac{d\varepsilon}{dt} = 0$$

(B) $\underline{r} \wedge Eq(1.8) = 0 \Rightarrow \underline{h} = \underline{r} \wedge \dot{\underline{r}}$ = CONSTANT

$$\frac{d\underline{h}}{dt} = 0$$



$\Rightarrow$ plane orbit

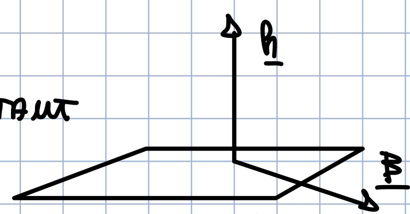
$h = r^2 \dot{\theta}$

$\frac{dA}{dt} = \frac{h}{2}$ Kepler's second law

(C) $Eq(1.8) \wedge \underline{h} \Rightarrow \frac{d\underline{B}}{dt} = 0$

$\dot{\underline{r}} \wedge \underline{h} - \mu \frac{\underline{r}}{r} = \underline{B}$ = CONSTANT

$\underline{B} \cdot \underline{h} = 0 \Rightarrow \underline{B} \perp \underline{h}$



$\underline{B} \in$ orbital plane

$\underline{e} = \frac{\underline{B}}{\mu}$ $\quad \frac{d\underline{e}}{dt} = 0$ $\quad \underline{e}$ CONSTANT $\Rightarrow$ APSE LINE FIXED

(D) $\underline{e} + \frac{\underline{r}}{r} = \frac{\dot{\underline{r}} \wedge \underline{h}}{\mu}$ (1.22)

$Eq\ (1.22) \cdot \underline{r} \Rightarrow r = \frac{h^2/\mu}{1 + e\cos\theta}$ ORBIT EQUATION CONIC SECTION Kepler's first law

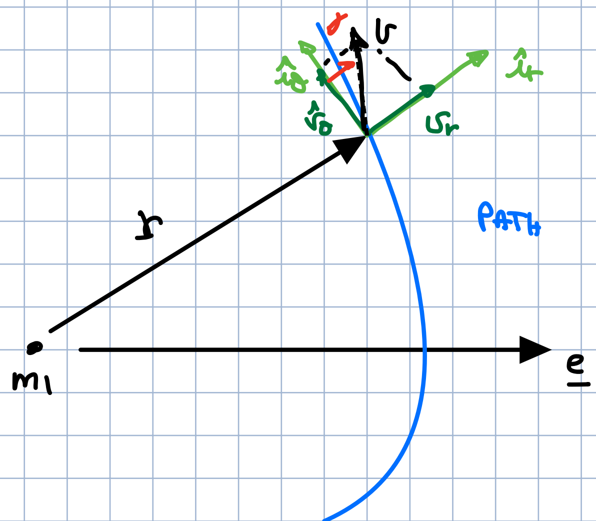
5 CONSTANT OF INTEGRATION $\underset{(3)}{\underline{e}}, \underset{(3)}{\underline{h}}$ but $\underset{(-1)}{\underline{e} \cdot \underline{h} = 0}$ $= 5$

$\underline{r}, \underline{v}$ $\{x, y, z, v_x, v_y, v_z\}$ 6 COMPONENT

⇒ We are missing a constant ε = CONSTANT But it can be derived from the constant angular momentum so it can not be use as the 6th constant

ORBIT FORMULAS

radial and transversal component of velocity.



$\underline{v}$ // path

$\underline{v} = \dot{\underline{r}}$

γ = Flight path angle

angle between v and v_θ

From (1.28) $\quad \underline{v} = \dot{\underline{r}} = \dot{r}\,\hat{i}_r + r\dot{\theta}\,\hat{i}_\theta \quad r(\theta(t))$

$$v_r = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{dr}{d\theta}\dot{\theta} = \frac{dr}{d\theta}\frac{h}{r^2}$$ RECALL eq (1.18) $h = r^2\dot{\theta}$

$$\frac{dr}{d\theta} = -\frac{h^2}{\mu}\frac{-e\sin\theta}{(1+e\cos\theta)^2} = \frac{h^2}{\mu}\frac{e\sin\theta}{(1+e\cos\theta)^2}$$ RECALL eq (1.26)

$$r(\theta) = \frac{h^2/\mu}{1+e\cos\theta}$$

$$\Rightarrow v_r = \frac{h^2}{\mu}\frac{e\sin\theta}{(1+e\cos\theta)^2}\,\frac{h}{r^2}$$ simplify r^2

$$v_r = \frac{\cancel{h^2}}{\cancel{\mu}}\frac{e\sin\theta}{\cancel{(1+e\cos\theta)^2}}\,\cancel{h}\,\frac{\cancel{(1+e\cos\theta)^2}}{h^{\cancel{4}\,3}/\mu^{\cancel{2}}} = \frac{\mu}{h}e\sin\theta$$

$$\boxed{v_r = \frac{\mu}{h}e\sin\theta} \quad (1.33)$$

Azimuth / trasversal component of $\underline{v}$ (v_θ)

$v_\theta = r\dot{\theta} = r\frac{h}{r^2} = \frac{h}{r}$ $\quad \dot{\theta}$ = angular velocity of $\underline{r}$

By substituting Eq (1.26)

$$\boxed{v_\theta = \frac{\mu}{h}(1+e\cos\theta)} \qquad (1.34)$$

Eq (1.26) m_2 comes close to m_1 (r smallest) when $\theta=0$
(when $e=0 \Rightarrow r(\theta) = \text{CONSTANT}$)

The closest approach point called PERIAPSIS

$$\boxed{r(\theta=0) = r_p = \frac{h^2/\mu}{1+e}} \qquad (1.35)$$

The periapsis lies on $\underline{e}$

v_r, v_θ AT PERIAPSIS

$$v_r(\theta=0) = 0$$

FOR $0<\theta<\pi$ $\quad \sin\theta>0 \Rightarrow v_r>0$ $\quad r\uparrow$
FOR $\pi<\theta<2\pi$ $\quad \sin\theta<0 \Rightarrow v_r<0$ $\quad r\downarrow$

looking at the evolution of r we can say where the satellite is in the orbit.

the flight path angle γ

$$\tan\gamma = \frac{v_r}{v_\theta} \qquad (1.36)$$

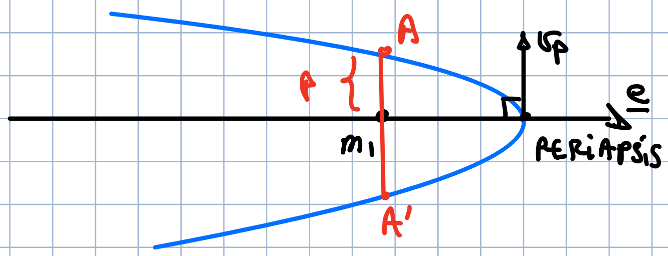
Substituting (1.33), (1.34)

$$\boxed{\tan\gamma = \frac{e\sin\theta}{1+e\cos\theta}} \qquad (1.37)$$

$\gamma>0$ s/c is moving away from r_p $\quad 0<\theta<\pi$

$\gamma<0$ s/c is moving back to r_p $\quad \pi<\theta<2\pi$

orbit is simmetric wrt apse line $\quad \cos\theta = \cos(-\theta)$



We defined a chord ⊥ to e that passes through m_1 (focus)

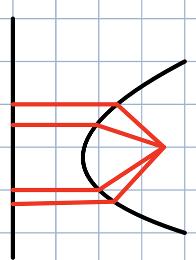
$\overline{A'A}$ = latus rectum

p = semi-latus rectum

$$r\left(\theta = \frac{\pi}{2}\right) = p = \frac{h^2}{\mu} \qquad \boxed{p = \frac{h^2}{\mu}} \qquad (1.38)$$

$p > r_p \rightarrow$ check to see if we are generating a conic.

CONIC SECTION → (intersection between plane and cone)



ratio of these two line is always constant and is equal to e.

ELLIPTICAL ORBITS

Eq (1.26) $\qquad r = \dfrac{h^2/\mu}{1+e\cos\theta}$

$0 < e < 1 \Rightarrow r(\theta) > 0$ FINITE VALUE

r_p periapsis = minimum r

Maximum value r is apoapsis or apocenter

$$\boxed{r(\theta = \pi) = r_a = \frac{h^2/\mu}{1-e}} \qquad (1.39)$$

$v_r(\theta = \pi) = 0$

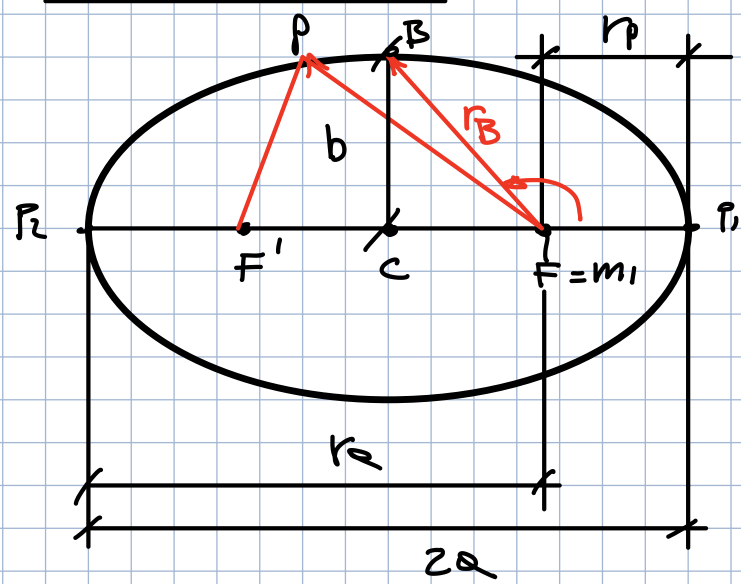
Geometric definition of an ellipse:

Given 2 Focii ellipse is the locus of points where:

$$\boxed{\overline{PF'} + \overline{PF} = 2a}$$

a = semi-major axis

b = semi-minor axis



P_1 : $\overline{P_1F'} + \overline{P_1F} = 2a$

P_2 : $\overline{P_2F'} + \overline{P_2F} = 2a$

$\overline{P_1P_2} = 2a$

$$2a = r_a + r_p \qquad (1.40)$$

$$2a = \frac{h^2}{\mu}\left(\frac{1}{1-e} + \frac{1}{1+e}\right) = \frac{h^2}{\mu}\left(\frac{1+\cancel{e}+1-\cancel{e}}{1-e^2}\right) = \frac{h^2\, 2}{\mu(1-e^2)}$$

$$\boxed{a = \frac{h^2}{\mu}\frac{1}{(1-e^2)}} \quad (1.40) \qquad \boxed{a = \frac{p}{(1-e^2)}}$$

$$p = \frac{h^2}{\mu} \quad \rightarrow \quad \boxed{p = a(1-e^2)} \quad (1.41)$$

$$\boxed{r = \frac{p}{1+e\cos\theta}} \qquad \boxed{r = \frac{a(1-e^2)}{1+e\cos\theta}} \quad (1.42)$$

semi-minor axis b

$$\overline{FB} + \overline{F'B} = 2a$$

$$r_B = \overline{FB} = \overline{F'B} = a \qquad \text{For simmetry}$$

$$b^2 + \overline{CF}^2 = a^2 \qquad \text{Pytagora theorem}$$

$$\overline{CF} = a - r_p \quad \text{and} \quad r_p = \frac{a(1-e^2)}{1+e} = a(1-e)$$

$$\boxed{\overline{CF} = ae} \quad (1.43) \qquad \boxed{r_p = a(1-e)}$$

$$b^2 + a^2e^2 = a^2 \qquad \boxed{b = \sqrt{a^2(1-e^2)}} \quad (1.44) \qquad (1-e^2) < 1 \qquad \sqrt{1-e^2} > 1-e^2$$

$$\Rightarrow \qquad \boxed{r_p < p < b < a}$$

PERIAPSIS AND APOAPSIS

$$r_p = \frac{p}{1+e} \quad \text{OR} \quad r_p = \frac{a(1-e^2)}{1+e} \rightarrow \boxed{r_p = a(1-e)} \quad (1.45)$$

$$r_a = \frac{p}{1-e} \quad \text{OR} \quad r_a = \frac{a(1-e^2)}{1-e} \rightarrow \boxed{r_a = a(1+e)} \quad (1.46)$$