# ORBITAL MECHANICS

We go on expressing $a, e, p$ as a function $r_p, r_a$

$r_p$ (1.45)
$r_a$ (1.46) $\longrightarrow \dfrac{r_p}{r_a} = \dfrac{1-e}{1+e} \longrightarrow r_p(1+e) = r_a(1-e)$

$$(r_p + r_a)e = r_a - r_p \implies \boxed{e = \frac{r_a - r_p}{r_a + r_p}} \quad (1.47)$$

very useful

$$\left.\begin{aligned} r_a - r_p &= \overline{FF'} \quad \text{Distance of foci} \\ r_a + r_p &= 2a \end{aligned}\right\} \implies e = \frac{\overline{FF'}}{2a} \quad (1.48)$$

## SEMI-MAJOR AXIS

$$r_p = a(1-e)$$
$$r_a = a(1+e)$$
$$p = a(1-e^2)$$

$$(1.40) \longrightarrow \boxed{a = \frac{r_p + r_a}{2}} \quad (1.49)$$

## SEMI LATUS RECTUM

$$\left\{\begin{aligned} r_p &= \frac{p}{1+e} \\ e &= \frac{r_a - r_p}{r_a + r_p} \end{aligned}\right. \longrightarrow r_p(1+e) = p \longrightarrow p = r_p\left(1 + \frac{r_a - r_p}{r_a + r_p}\right)$$

$$\boxed{p = \frac{2 r_a r_p}{r_a + r_p}} \quad (1.50)$$

## VELOCITY

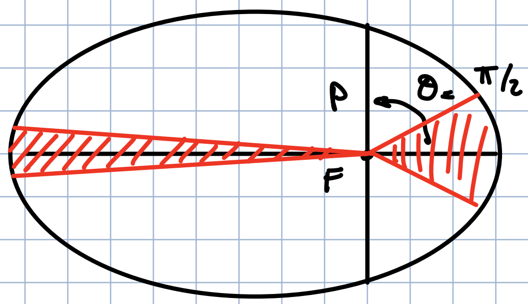
(1.33) (1.34) $v_\theta, v_r$

$$v = \sqrt{v_\theta^2 + v_r^2} \longleftarrow \text{magnitude of the velocity}$$

$$= \sqrt{\frac{\mu^2}{h^2}\left(e^2 \sin^2\theta + (1 + e\cos\theta)^2\right)} \qquad \text{but} \quad p = \frac{h^2}{\mu} \quad (1.38)$$

$$\implies v = \left[\frac{\mu}{p}\left(e^2 \sin^2\theta + (1 + e\cos\theta)^2\right)\right]^{1/2}$$

$$\boxed{v = \left[\frac{\mu}{p}\left(e^2 + 1 + 2e\cos\theta\right)\right]^{1/2}}$$




$p = r\left(\theta = \frac{\pi}{2}\right)$

① $r_p \rightarrow$ $\boxed{v_p = \sqrt{\frac{\mu}{p}}\,(1+e)}$ periapsis maximum velocity (1.52)

② $r_a \rightarrow$ $\boxed{v_a = \sqrt{\frac{\mu}{p}}\,(1-e)}$ apoapsis minimum velocity (1.53)

↳ we do not have any velocity along the radial direction

ORBIT ENERGY AND VELOCITY

$\mathcal{E}$ = CONSTANT TOTAL ENERGY $\forall$ point $\in$ orbit.

① $\mathcal{E}_p = \underbrace{\frac{v_p^2}{2}}_{(1.52)} - \underbrace{\frac{\mu}{r_p}}_{(1.45)} = \frac{\mu}{2p}(1+e)^2 - \frac{\mu(1+e)}{p}$

$$\mathcal{E}_p = \frac{\mu}{2p}\left[(1+e)^2 - 2(1+e)\right] = \frac{\mu}{2p}\left(1 + \cancel{2e} + e^2 - 2 - \cancel{2e}\right)$$

$$= \frac{\mu}{2p}\left(e^2 - 1\right)$$

$$\mathcal{E}_p = \frac{\mu}{2p}\left(e^2-1\right) = -\frac{\mu}{2p}\left(1-e^2\right) = -\frac{\mu^2}{2h^2}\left(1-e^2\right)$$

$p = \frac{h^2}{\mu}$

$$\boxed{\mathcal{E}_p = -\frac{1}{2}\frac{\mu^2}{h^2}\left(1-e^2\right)} \quad (1.54)$$

$p = a(1-e^2)$ (1.41)

↳ As a function of the $h$ and $e$ ⟹ clearly the orbital energy $\varepsilon$ is not an indipendent quantity.

$$\Longrightarrow \boxed{\varepsilon_p = -\frac{\mu}{2a}} \Longleftarrow (1.55)$$

It's very important because it gives us $\varepsilon$ in function of the semi major-axis

NOTE

ep (1.53) and ep (1.54) valid for ∀ orbit (elliptical, parabolic, hyperbolic)

From (1.10) and (1.55) $\rightarrow \boxed{\frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}}$ (1.56)

↳ depends on orbit

$a$ defines energy level

$a$ semi major-axis tell us the energy level of the orbit.

(1.56) → $v$

Knowing $\varepsilon$ and $r$ → velocity at any point of the orbit

$$\boxed{v = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a}\right)}} \quad (1.57)$$

## ORBITAL PERIOD OF A CLOSED ORBIT → (valid only for circular and elliptical orbit)

We start from Area of an ellipse $f(a,b)$

$$\boxed{A = \pi a b} \quad (1.58)$$

From here we use Kepler's second law

$dA = \frac{1}{2} h \, dt$ one revolution $\pi a b = \frac{1}{2} h T$

Kepler's second law

$$T = \frac{2\pi a b}{h}$$

$b = a\sqrt{1-e^2}$ (1.44)

$h = \sqrt{\mu p}$ (1.38)

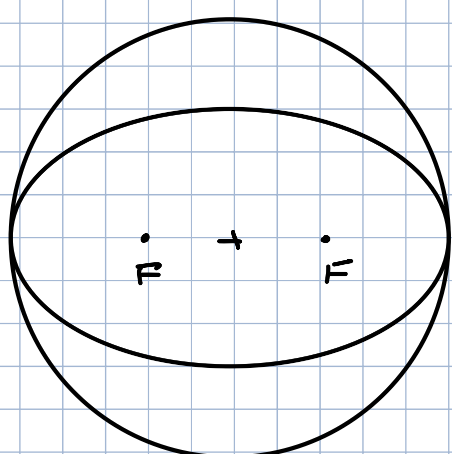
$p = a(1-e^2)$ (1.41)

$$\Rightarrow T = \frac{2\pi e^2 \cancel{\sqrt{1-e^2}}}{\sqrt{\mu a \cancel{(1-e^2)}}}$$

Period of a closed orbit

$$\boxed{T = 2\pi \sqrt{\frac{a^3}{\mu}}} \quad (1.59)$$

it depends only on the semi-major axis (like the total energy)




NOTE

These two orbit have the same energy and orbital period.

## CIRCULAR ORBIT

see characteristic of the elliptical orbit but it is necessary to set $e = \phi$.

$e = \phi$ in eq (1.26) $\boxed{r = \frac{h^2}{\mu} \quad \text{or} \quad r = a}$ (1.60)

$r$ = constant

Eq (1.33) $v_r = \frac{\mu}{h} e \sin\theta \longrightarrow v_r = \dot{r} = \phi$

it is present only the azimuth component of $v$ for circular orbit $v = v_\theta \longrightarrow h$ reduces

$$\boxed{h_{circular} = r v_\theta = r \cdot v} \quad (1.61)$$

$r = \frac{h^2}{\mu} = \frac{r^2 v^2}{\mu}$ $\boxed{v_{circular} = \sqrt{\frac{\mu}{r}}}$ (1.62)

$T = 2\pi \sqrt{\frac{a^3}{\mu}} \longrightarrow T = \frac{\text{circumference}}{\text{speed}} = \frac{2\pi r}{\sqrt{\mu/r}} = 2\pi \sqrt{\frac{r^3}{\mu}}$

$$\boxed{T_c = 2\pi \sqrt{\frac{r^3}{\mu}}} \quad (1.63)$$

$a = r$

eq (1.54) $\varepsilon = -\frac{1}{2} \frac{\mu^2}{h^2} (1 - e^2)$

$$\varepsilon_{circular} = -\frac{1}{2} \frac{\mu^2}{h^2}$$

Eq (1.55) $\varepsilon = -\frac{\mu}{2a}$ $\boxed{\varepsilon_{circular} = -\frac{\mu}{2r}}$ (1.64)

as $r$ gets bigger $\varepsilon_c$ gets smaller.

NOTE

With the same energy a launcher can position one heavy satellite in a low earth-orbit or a smaller and lighter satellite in a elliptical or circular orbit with a bigger semi-latum rectum.

example LEO altitude [150 km 2000 km

- remote sensing
- imaging
- navigation satellites
- telecommunication (internet broadband / large constellation).

example GEO geostationary Earth orbit

s/c remains stationary above same point on Earth.

rotational velocity of Earth $\quad w_E = \dfrac{2\pi}{\text{sidereal day}} \qquad (1.65)$

sidereal day: time it takes to Earth to complete 1 full revolution wrt inertial direction (fixed star)
23h and 56 min.

synodic day (solar day): 24 h time Sun takes full apparent rotation around Earth (noon to noon).

$$\boxed{w_E = \frac{2\pi}{23\,h\ 56\,min}}$$ angular velocity of the Earth.

GEO s/c are used for → - telecoms
- television broadcast

$r_{GEO} = 42164$ km

$$\left.\begin{aligned} v_{GEO} &= \omega_E r_{GEO} \\ v_{GEO} &= \sqrt{\frac{\mu}{r_{GEO}}} \end{aligned}\right\} \quad \omega_E^2 r_{GEO}^2 = \frac{\mu}{r_{GEO}}$$

$$r_{GEO}^3 = \frac{\mu}{\omega_E^2} \qquad \boxed{r_{GEO} = \sqrt[3]{\frac{\mu_E}{\omega_E^2}}}$$

$$\mu_E = G(m_E + m_{s/c}) = G m_E = 398600 \ \frac{km^3}{s^2}$$

$$h_{GEO} = r_{GEO} - R_E = 35786 \ km$$

$$v_{GEO} = \sqrt{\frac{\mu_E}{r_{GEO}}} = 3.075 \ \frac{km}{s}$$

## PARABOLIC TRAJECTORY

$e = 1$

From eq (1.26) $\quad r = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta}$

as $\theta \to \pi$ $\quad 1 + e\cos\theta \to 0$ and $r \to \infty$ OPEN ORBIT

From eq (1.54) $\quad \varepsilon = -\frac{1}{2}\frac{\mu^2}{h^2}(1-e^2)$

$$\boxed{\varepsilon_{PAR} = 0}$$

(1.66)

Eq (1.56) $\quad \frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}$ → Parabolic trajectory $\frac{v^2}{2} - \frac{\mu}{r} = 0$

$$\Rightarrow \quad \boxed{v_{PAR} = \sqrt{\frac{2\mu}{r}}} \qquad (1.67) \quad \forall \text{ point on the parabola.}$$

$r \to \infty$ $\quad \lim_{\substack{r \to \infty \\ \theta \to \pi}} v_{PAR} = 0$

$m_2$ escapes from $m_1$ at $\phi$ relative velocity.

$\Rightarrow$ MINIMUM ESCAPE ENERGY (PATH)

At $r$ from $m_1$ the minimum escape velocity is the one of a parabola

FIRST COSMIC SPEED $\qquad \boxed{v_{cir} = \sqrt{\frac{\mu}{r}}} \qquad (1.68)$

SECOND COSMIC SPEED $\qquad \boxed{\begin{array}{l} v_{esc} = \sqrt{\frac{2\mu}{r}} \\ v_{es} = \sqrt{2}\, v_{cir} \end{array}} \qquad (1.69)$

① $r$ on circular orbit ($v_{cir}$) to escape boost velocity of 41.4%

After escaping from the Earth $v_{par} \xrightarrow[r \to \infty]{} 0$ we are at $\infty$ but with a $\phi$ relative velocity $\rightarrow$ at this point the s/c is in orbit around the Sun with the same velocity of the Earth. Parabolic orbit is the limit between the Earth and the Sun.