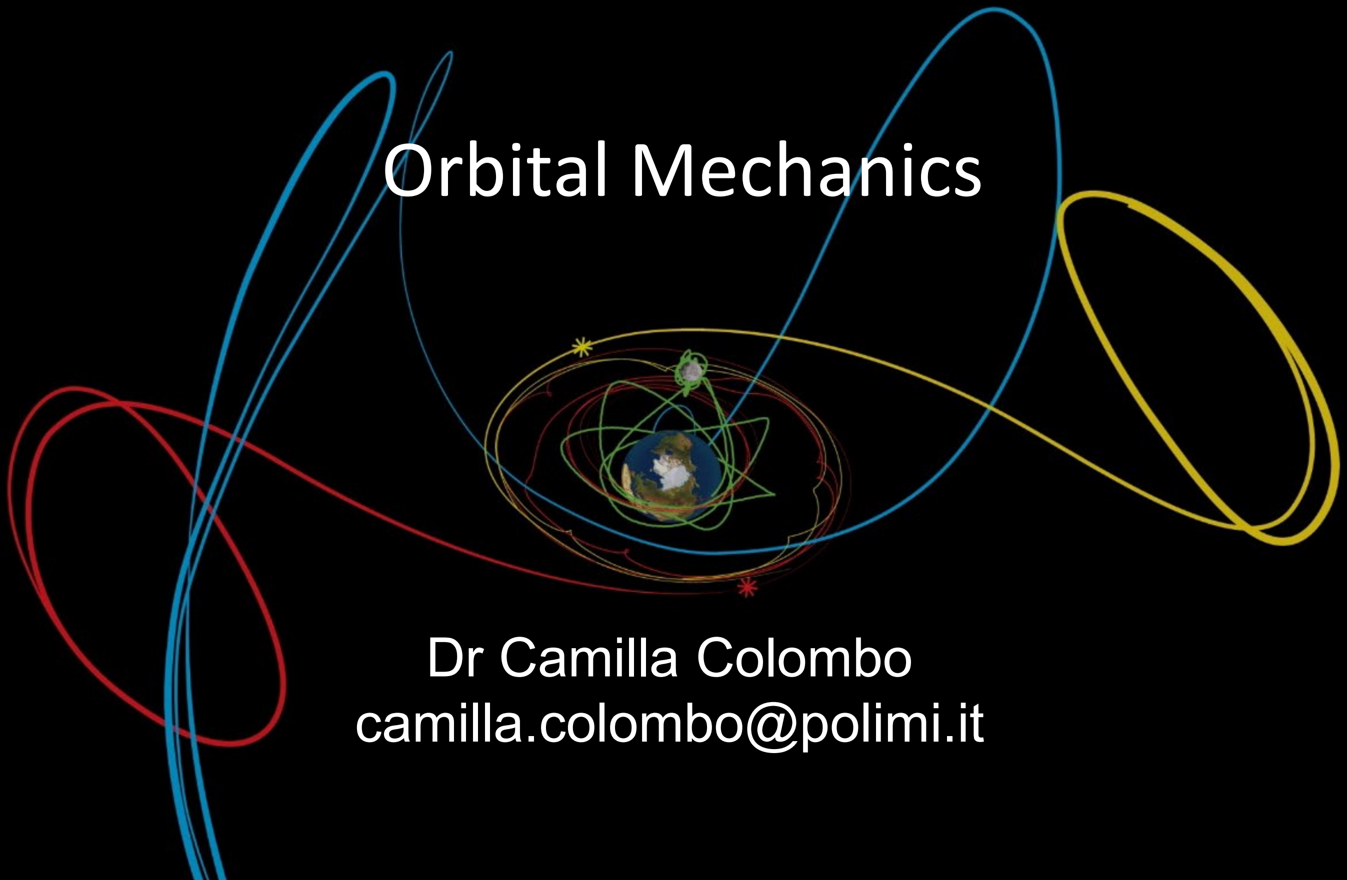
Orbital Mechanics



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6. THREE BODY PROBLEM

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- R. H. Battin, An Introduction to the Mathematics and Methods of Astrodynamics, Revised Edition, AIAA Educational Series, Reston, 1999.
- Avanzini G., Colasurdo G., "SpacE Exploration and Development Systems, Astrodynamics", Politecnico di Torino, 2016, II edition, Chapter 6
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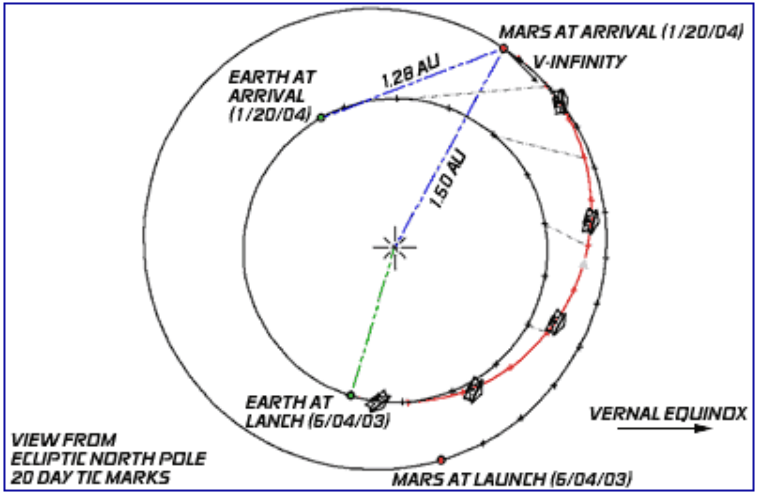
Recall: interplanetary missions

Use of 'patch-conics' method as an approximate analysis technique to estimate important mission parameters

- Δv
- transfer time
- swing-by characteristics

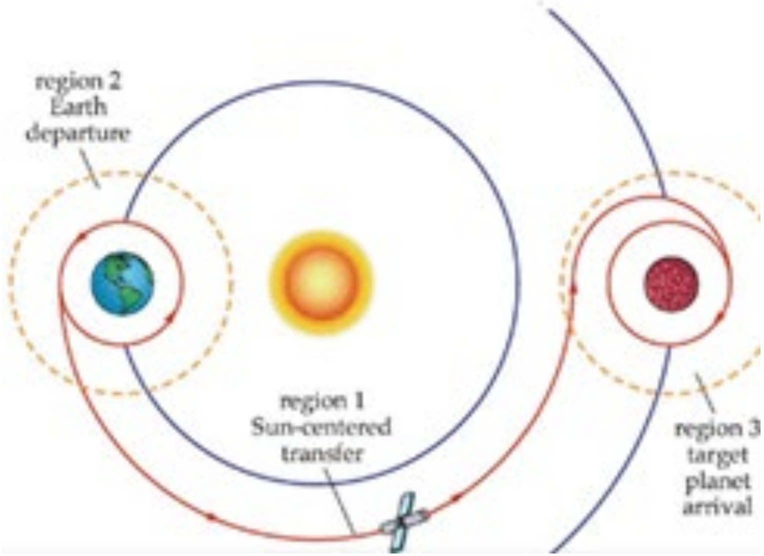
Elliptical transfer to Mars (Sun-centred)



- Sphere-of-influence (SOI) determines region of planet/Sun-centred motion
- Hyperbolic excess v_∞ at Earth escape results in Sun-centred ellipse

Recall: patched conic method

- Calculate transfer from Earth to target as a series of 2-body problems
- Split transfer into 3 phases: planet centred (depart/arrive) and Sun-centred
- Patch together planet/Sun-centred conic section orbits at sphere of influence



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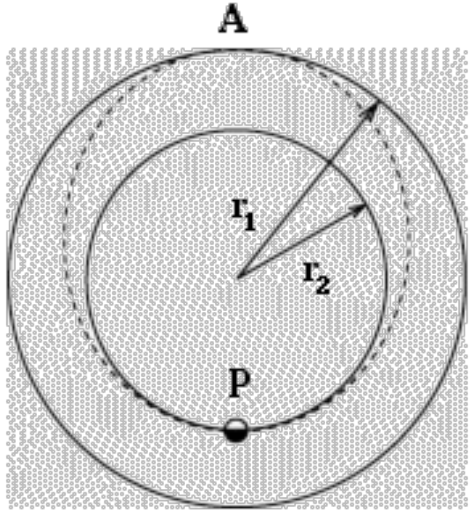
Recall: Earth escape to transfer ellipse

- Consider Earth escape onto the start of a minimum energy Hohmann transfer
- Hyperbolic Earth escape with excess velocity v_∞ to inject onto transfer ellipse
- Escape burn from Earth-centred parking orbit (radius r_{park}) sized for Δv_1

$$\Delta v_1 = \sqrt{\frac{2\mu_{Sun}}{r_1} - \frac{2\mu_{Sun}}{r_1 + r_2}} - \sqrt{\frac{\mu_{Sun}}{r_1}}$$

$$v_\infty = \Delta v_1$$

$$\Delta v_{Escape} = \sqrt{\frac{2\mu_{Earth}}{r_{park}} + v_\infty{}^2} - \sqrt{\frac{\mu_{Earth}}{r_{park}}}$$



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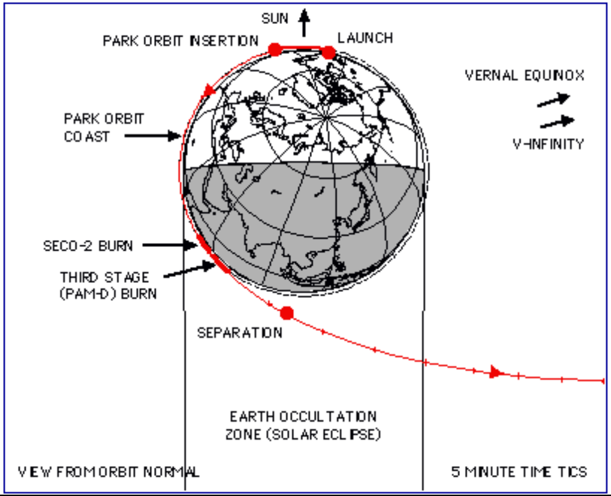
Recall Example: trans-Mars injection

- Hyperbolic Earth departure to a minimum energy Hohmann transfer to Mars
- Start from 300 km LEO Earth-centred parking orbit then apply escape Δv
- Find total Δv for parabolic escape then Δv_1 is 6.1 km/s (much less efficient)

$$\Delta v_1 = \sqrt{\frac{2\mu_{Sun}}{r_1} - \frac{2\mu_{Sun}}{r_1 + r_2}} - \sqrt{\frac{\mu_{Sun}}{r_1}} = 2.93 \text{ km/s}$$

$$\Delta v_{Escape} = \sqrt{\frac{2\mu_{Earth}}{r_{park}} + \Delta v_1^{\,2}} - \sqrt{\frac{\mu_{Earth}}{r_{park}}} = 3.59 \text{ km/s}$$



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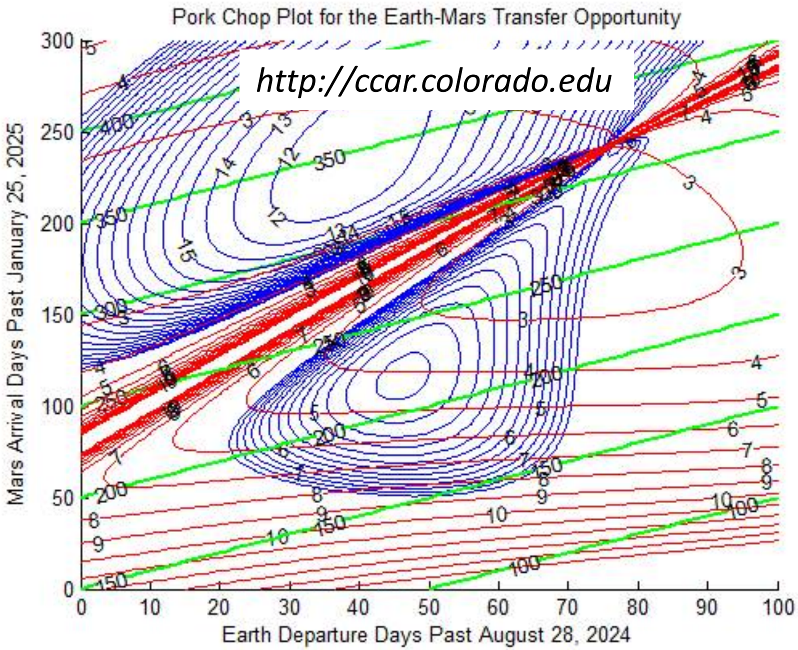
Recall: Interplanetary transfers

- Hohmann is an approximation to real transfers
- Maximum Δ*v* available from launcher determines departure launch window



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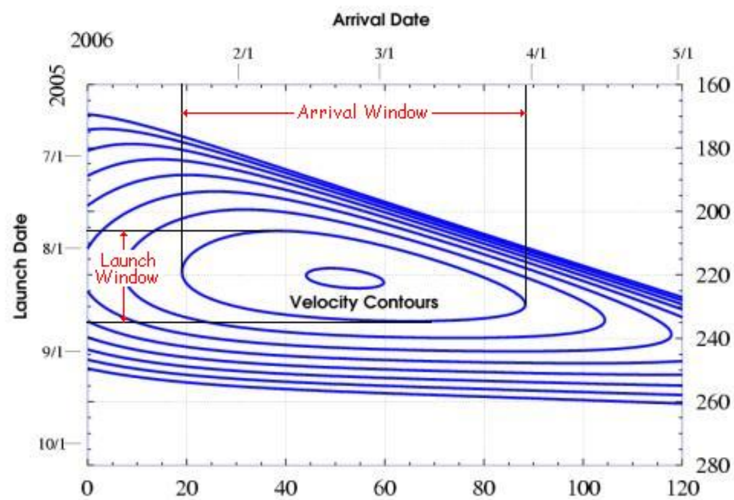

Δv contours for Earth-Mars transfer

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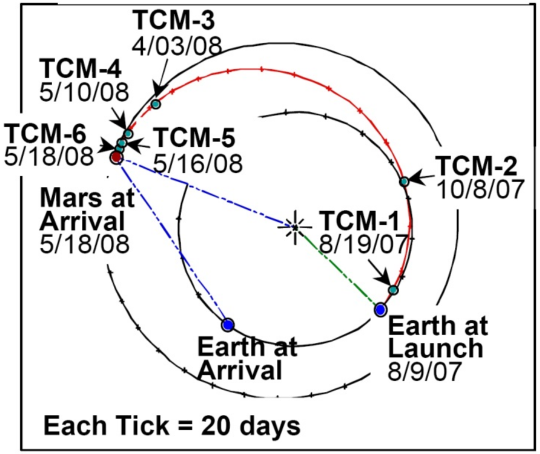
Recall: Interplanetary transfers

- Plan trajectory correction manoeuvres (TCM), particularly near arrival date

TCMs for Earth-Mars transfer



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Astronomical observatory spacecraft

These can be found in a variety of orbits

- Hubble → LEO
- Integral, XMM Newton → HEO
- James Webb Space Telescope (2nd generation Hubble Space Telescope), SOHO, Herschel, Planck, Gaia → Lagrange point L_2

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The three-body problem

We can not use patched conic method for mission to the moon because SOI/moon too big almost 1/6 of moon-Earth distance → valid for mission to other moons of planet.

A variety of mission analysis work takes place in circumstances where the patch-conics method does not provide a good approximation, e. g.:

- Trajectory design within the Earth-Moon system (Apollo, Surveyor, Lunar Orbiter, SMART-1, etc.)
- Orbital flight about the moons of the major planets (Europa, Enceledus, etc.)
- Trajectory design within the Pluto/Charon system

These cases are examples of the **3-body problem**

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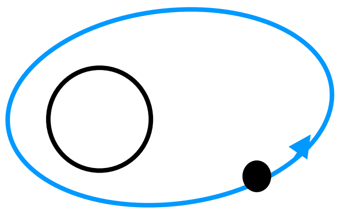
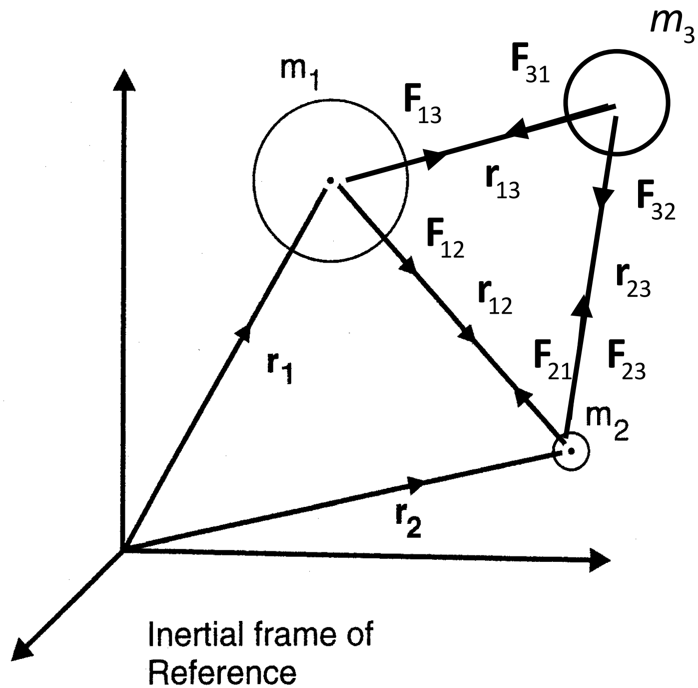
The three-body problem

We have a progression of astrodynamical problem types

- Two-body problem
- Restricted two-body problem ($m_1 >> m_2$, and $\mu = G \cdot m_1$)



- Three-body problem
- …
- n-body problem

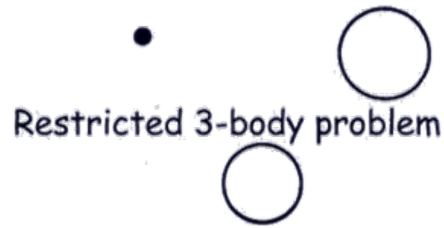


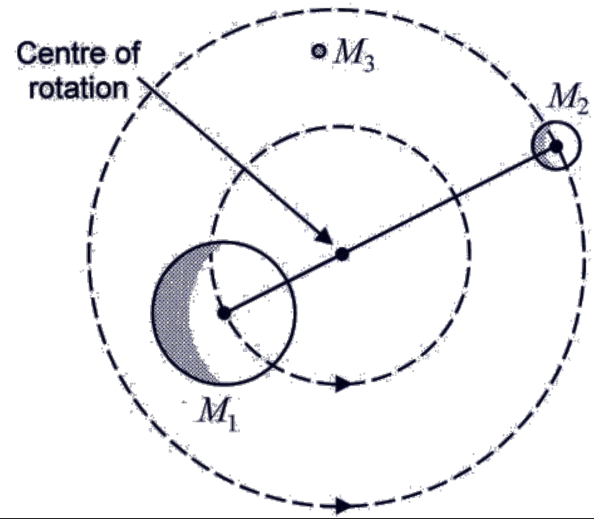
The circular restricted three-body problem

Easier to write dynamics of 3 body problem wrt a new reference frame → no more inertial reference frame → About center of rotation / baricenter of the system

Considering the interaction of three bodies only

- Three-body problem
- Restricted three-body problem ($m_1 >> m_3$ and $m_2 >> m_3$)
- Circular restricted-three body problem (CRTBP) ↳ only for 3 body. (m_1 and m_2 on circular orbits)





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Study of the CRTBP in the rotating frame

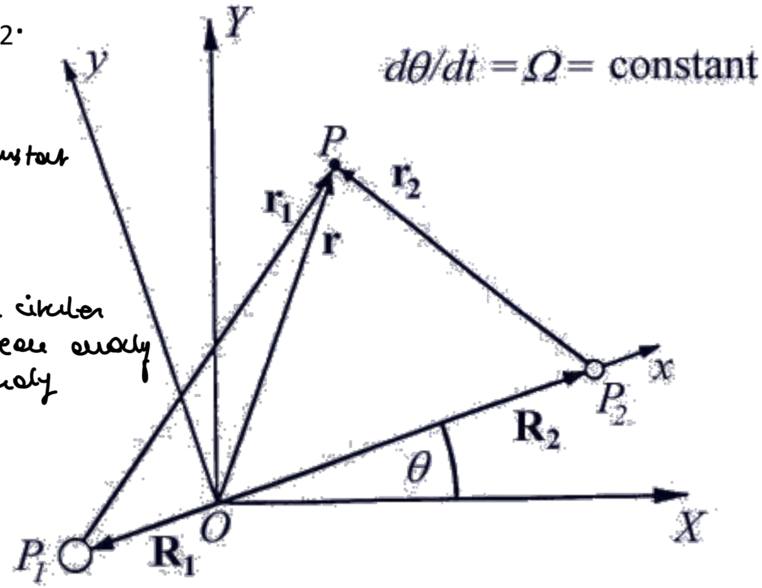
How to write the dynamics in this reference frame?
Origin at the center of mass of Moon and Earth + XY always in the plane Moon – Earth.

- Set up inertial (X, Y, Z) and rotating (x, y, z) reference frames (Z = z)
- We wish to develop the equations of motion of negligible mass m_3 (point P in the diagram).
- Point P represents a spacecraft, under the gravitational influence of the very much larger masses P_1 and P_2.

P_1 and P_2 → remain constant in modulus because $|\overline{P_1P_2}| = \text{constant}$

$\frac{d\theta}{dt} = \dot{\theta} = \text{constant} = \Omega$

↳ Due to the fact that for a circular orbit Moon – Earth the mean anomaly is equal to the true anomaly



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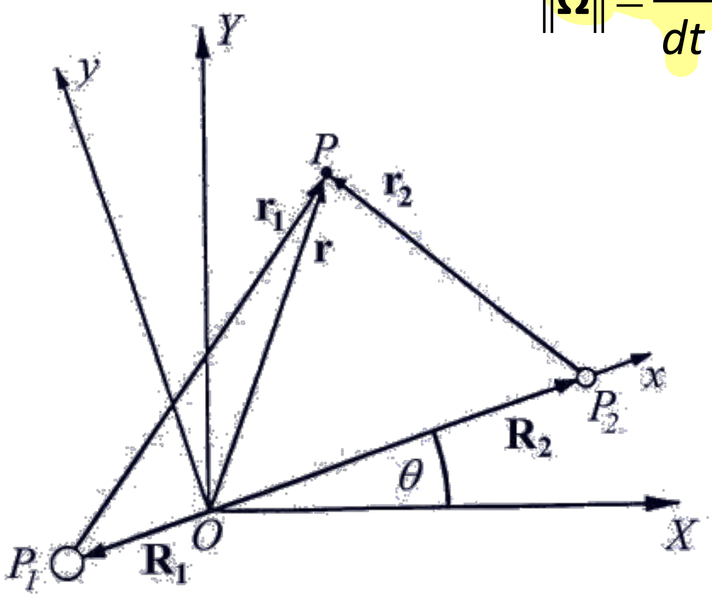
Study of the CRTBP in the rotating frame

System (x, y, z) rotates with constant angular speed with respect to the inertial frame:

$$\|\boldsymbol{\Omega}\| = \frac{d\theta}{dt} = \text{constant}$$



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Study of the CRTBP in the rotating frame

In the inertial frame, Newton's second law applies

$$\mathbf{F} = m\mathbf{a}$$

where ***a*** is the inertial (absolute) acceleration $\mathbf{a} = \dfrac{\mathbf{F}}{m}$

We need to derive the acceleration in the rotating frame

Study of the CRTBP in the rotating frame

Let's concentrate on x, y components:

In the rotating frame

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

i Unit vector (versor) in the rotating frame (along x axis)

j Unit vector (versor) in the rotating frame (along y axis)

z Unit vector (versor) in the rotating frame (along z axis)

↳ equal for inertial or rotating frame

Therefore the **absolute velocity** is

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{x}\mathbf{i} + x\frac{d\mathbf{i}}{dt} + \dot{y}\mathbf{j} + y\frac{d\mathbf{j}}{dt} + \dot{z}\mathbf{k} =$$

$$= \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k} + \left(x\frac{d\mathbf{i}}{dt} + y\frac{d\mathbf{j}}{dt} \right) =$$

$$= \frac{\delta \mathbf{r}}{\delta t} + (\boldsymbol{\Omega} \times \mathbf{r})$$

↳ Derivative only of the components

$$\underline{\Omega} = \begin{Bmatrix} 0 \\ 0 \\ \Omega \end{Bmatrix}$$

↳ only along z

Notation: if $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$

then $\dfrac{\delta \cdot}{\delta t}$ means $\dfrac{\delta \mathbf{c}}{\delta t} = \dot{c}_1\mathbf{i} + \dot{c}_2\mathbf{j} + \dot{c}_3\mathbf{k}$

$\boldsymbol{\Omega}$ Constant rotational speed of rotating frame wrt inertial frame

Study of the CRTBP in the rotating frame

The **absolute acceleration** is:

$$\mathbf{a}=\frac{d\mathbf{v}}{dt}=\frac{\delta\mathbf{v}}{\delta t}+\left(\boldsymbol{\Omega}\times\mathbf{v}\right)$$

$$=\frac{\delta}{\delta t}\left(\frac{\delta\mathbf{r}}{\delta t}+\left(\boldsymbol{\Omega}\times\mathbf{r}\right)\right)+\boldsymbol{\Omega}\times\left(\frac{\delta\mathbf{r}}{\delta t}+\left(\boldsymbol{\Omega}\times\mathbf{r}\right)\right)$$

$$=\frac{\delta^2\mathbf{r}}{\delta t^2}+\left(\dot{\boldsymbol{\Omega}}\times\mathbf{r}\right)+\boldsymbol{\Omega}\times\frac{\delta\mathbf{r}}{\delta t}+\boldsymbol{\Omega}\times\frac{\delta\mathbf{r}}{\delta t}+\boldsymbol{\Omega}\times\left(\boldsymbol{\Omega}\times\mathbf{r}\right)$$

$$\mathbf{a}=\frac{\delta^2\mathbf{r}}{\delta t^2}+\left(\dot{\boldsymbol{\Omega}}\times\mathbf{r}\right)+\boldsymbol{\Omega}\times\left(\boldsymbol{\Omega}\times\mathbf{r}\right)+\left(2\boldsymbol{\Omega}\times\frac{\delta\mathbf{r}}{\delta t}\right)$$

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Study of the CRTBP in the rotating frame

Noting that $\dot{\boldsymbol{\Omega}} = \mathbf{0}$ for the Circular restricted thee body problem:

$$\mathbf{a} = \frac{\delta^2 \mathbf{r}}{\delta t^2} + \left(\dot{\boldsymbol{\Omega}} \times \mathbf{r}\right) + \boldsymbol{\Omega} \times \left(\boldsymbol{\Omega} \times \mathbf{r}\right) + \left(2\boldsymbol{\Omega} \times \frac{\delta \mathbf{r}}{\delta t}\right)$$

$$\mathbf{a} = \frac{\delta^2 \mathbf{r}}{\delta t^2} + \boldsymbol{\Omega} \times \left(\boldsymbol{\Omega} \times \mathbf{r}\right) + \left(2\boldsymbol{\Omega} \times \frac{\delta \mathbf{r}}{\delta t}\right)$$

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Statement of gravitational forces acting on P

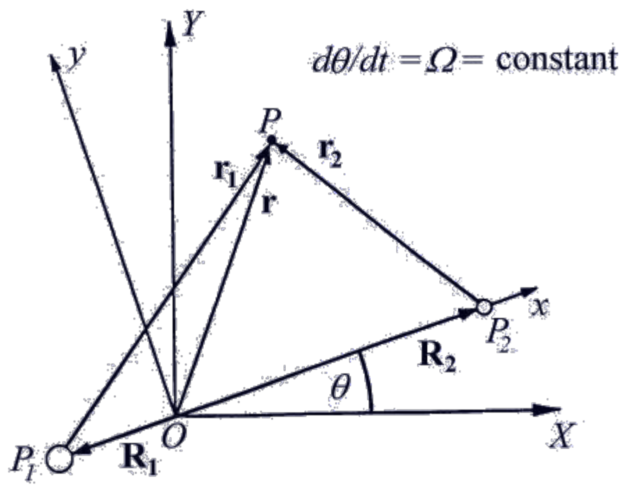
But also: $\mathbf{a} = \dfrac{\mathbf{F}}{m}$

with **F** the gravitational force $\mathbf{F} = -G\dfrac{m_1 m}{r_1^{\;3}}\mathbf{r}_1 - G\dfrac{m_2 m}{r_2^{\;3}}\mathbf{r}_2$

and $\mathbf{r}_1 = \left(x + \mu R\right)\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$$\mathbf{r}_2 = \left(x - \left(1 - \mu\right)R\right)\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

$$\mu = \frac{m_2}{m_1 + m_2}$$



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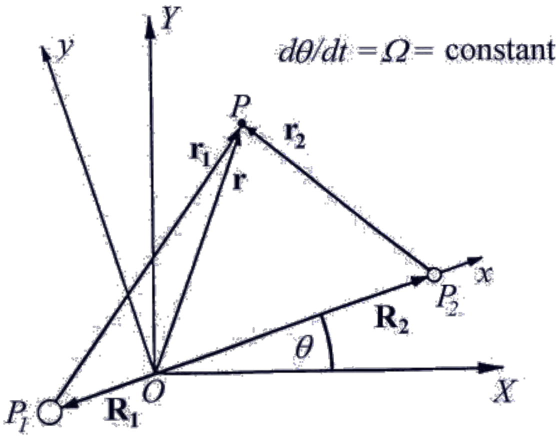
Study of the CRTBP in the rotating frame

Therefore, the **equations of motions** expressed in the rotating frame are

$$\boxed{\frac{\delta^2\mathbf{r}}{\delta t^2} = -G\left(\frac{m_1}{r_1^{\,3}}\mathbf{r}_1 + \frac{m_2}{r_2^{\,3}}\mathbf{r}_2\right) \underbrace{-\,\boldsymbol{\Omega}\times\left(\boldsymbol{\Omega}\times\mathbf{r}\right)}_{\text{centrifugal acceleration}} \underbrace{-\,2\boldsymbol{\Omega}\times\frac{\delta\mathbf{r}}{\delta t}}_{\text{Coriolis acceleration}}}$$



Jacobi integral

→ We can find only the potential 3bP energy → we need to do a numerical integration
Either way.

The first two terms on the RHS can be shown to be the gradient of a scalar function, i.e.

$$\nabla U = G\left(\frac{m_1}{r_1^{\,3}}\mathbf{r}_1 + \frac{m_2}{r_2^{\,3}}\mathbf{r}_2\right) + \mathbf{\Omega}\times\left(\mathbf{\Omega}\times\mathbf{r}\right)$$

If we define a three-body potential energy

$$U = \underbrace{-G\left(\frac{m_1}{r_1} + \frac{m_2}{r_2}\right)}_{\text{Gravitational contribution}} \underbrace{-\frac{1}{2}\Omega^2\left(x^2 + y^2\right)}_{\text{Rotating contribution}}$$

The equations of motion become

$$\boxed{\frac{\delta^2\mathbf{r}}{\delta t^2} = -\nabla U - 2\mathbf{\Omega}\times\frac{\delta\mathbf{r}}{\delta t}}$$

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Jacobi integral

Take dot product with $\dfrac{\delta \mathbf{r}}{\delta t}$, to give (Recall calculating the total mechanical energy)

$$\frac{\delta \mathbf{r}}{\delta t} \circ \frac{\delta^2 \mathbf{r}}{\delta t^2} = -\frac{\delta \mathbf{r}}{\delta t} \circ \nabla U - 2\frac{\delta \mathbf{r}}{\delta t} \circ \left(\mathbf{\Omega} \times \frac{\delta \mathbf{r}}{\delta t} \right)$$

But

- $-2\dfrac{\delta \mathbf{r}}{\delta t} \circ \left(\mathbf{\Omega} \times \dfrac{\delta \mathbf{r}}{\delta t} \right) = 0$ as $\dfrac{\delta \mathbf{r}}{\delta t}$ is perpendicular to $\left(\mathbf{\Omega} \times \dfrac{\delta \mathbf{r}}{\delta t} \right)$ hence their dot product is zero
- $\dfrac{\delta \mathbf{r}}{\delta t} \circ \dfrac{\delta^2 \mathbf{r}}{\delta t^2} = \mathbf{v} \circ \dot{\mathbf{v}} = \dfrac{1}{2}\dfrac{d}{dt} v^2$ where v is the speed in the rotating frame

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Jacobi integral

- $$\frac{\delta \mathbf{r}}{\delta t} \circ \nabla U = \frac{\delta x}{\delta t} \circ \frac{\delta U}{\delta x} + \frac{\delta y}{\delta t} \circ \frac{\delta U}{\delta y} + \frac{\delta z}{\delta t} \circ \frac{\delta U}{\delta z} = \frac{dU}{dt}$$

Therefore $$\frac{\delta \mathbf{r}}{\delta t} \circ \frac{\delta^2 \mathbf{r}}{\delta t^2} = -\frac{\delta \mathbf{r}}{\delta t} \circ \nabla U - 2\frac{\delta \mathbf{r}}{\delta t} \circ \left(\mathbf{\Omega} \times \frac{\delta \mathbf{r}}{\delta t} \right)$$ becomes

$$\frac{1}{2}\frac{d}{dt}v^2 = -\frac{dU}{dt} \quad \Rightarrow \quad \frac{1}{2}\frac{d}{dt}v^2 + \frac{dU}{dt} = 0$$

gravitational and rotational frame

Giving $$v^2 + 2U = -C$$

↳ can be used to understand the region of space I can reach with a given potential

$$\underbrace{\frac{1}{2}v^2}_{\text{kinetic energy}} + \underbrace{U}_{\text{potential energy}} = \underbrace{E}_{\text{total energy}} \qquad -\frac{1}{2}C = E$$

specific for the unit of mass

Integral of motion

↳ Important because it is a scalar function that is kept constant during the motion.

C = Jacobi constant or Jacobi integral

Jacobi integral

U is a function only on distances. I can plot it as shown below.

But $$U = -G\left(\frac{m_1}{r_1} + \frac{m_2}{r_2}\right) - \frac{1}{2}\Omega^2\left(x^2 + y^2\right)$$

Therefore the **Jacobi integral in the rotating frame** (cf energy integral for the restricted 2-body problem) is

$$v^2 + 2U = -C$$

$$\boxed{v^2 - 2G\left(\frac{m_1}{r_1} + \frac{m_2}{r_2}\right) - \Omega^2\left(x^2 + y^2\right) = -C}$$

- This can be used as an independent check on computational solutions of the motion of the spacecraft P.
- It can be used to bound the motion of the mass

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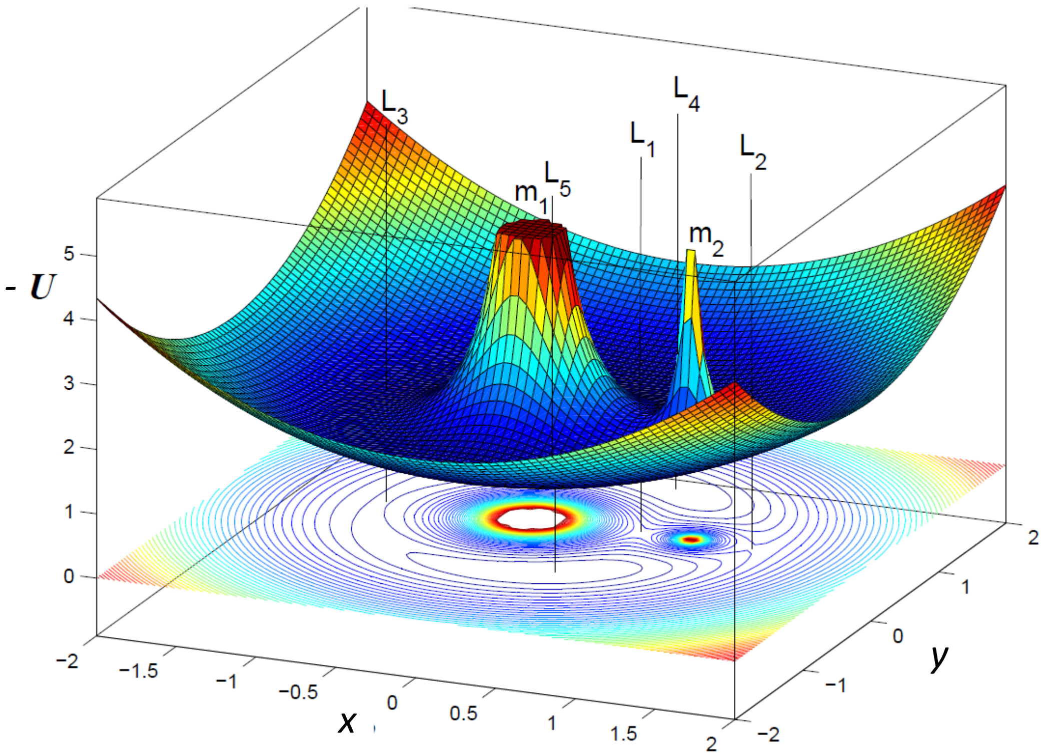
Jacobi integral

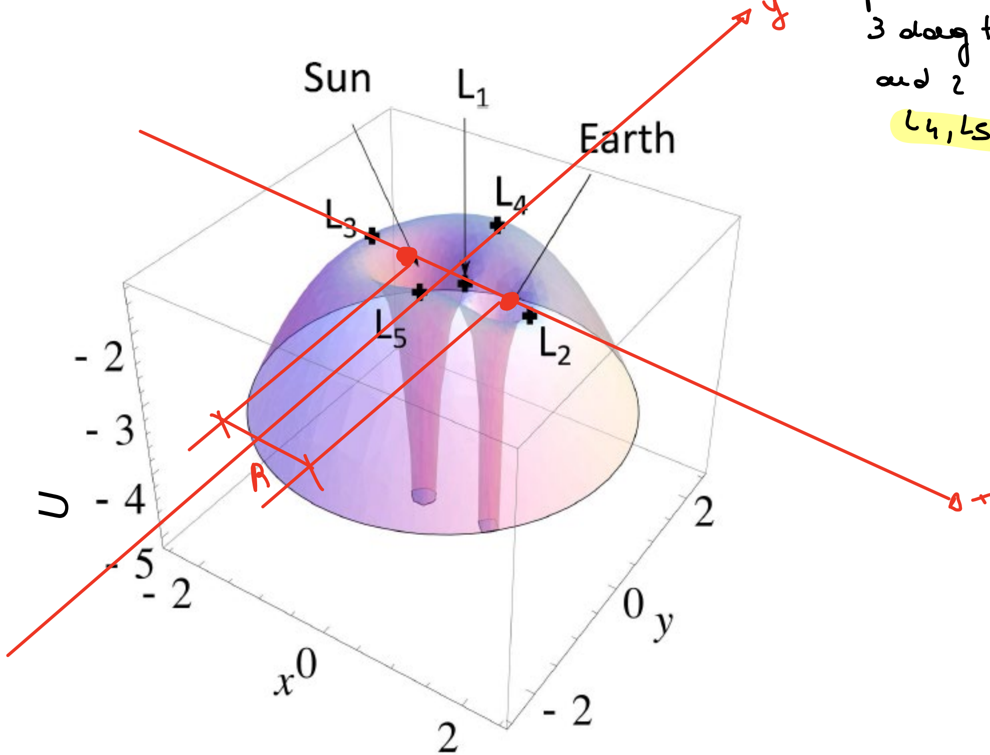


Three–body potential (-U) in the x, y plane

➢ *Colsurdo, Avanzini*

Lagrangian points

we can Identify 5 equilibrium point.
3 along the x axis → L1, L2, L3
and 2 not on the x axis
L4, L5



Rotating and gravitational potential energy in the CR3BP

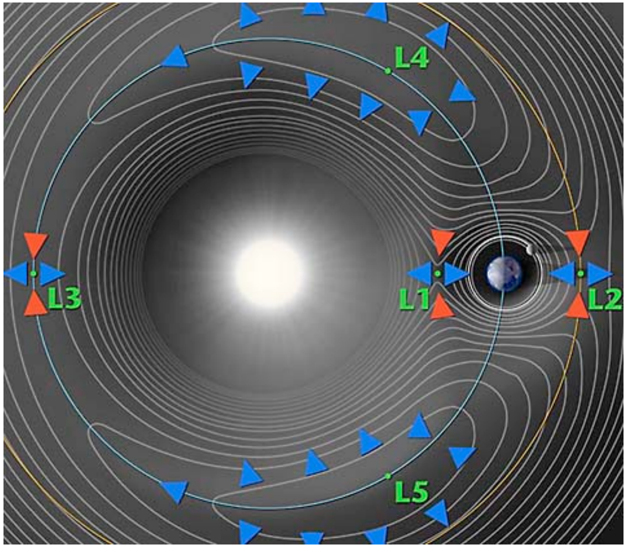
➢ *Figure from Soldini*

In any 3 body problem there are always these 5 equilibrium point

Lagrangian points

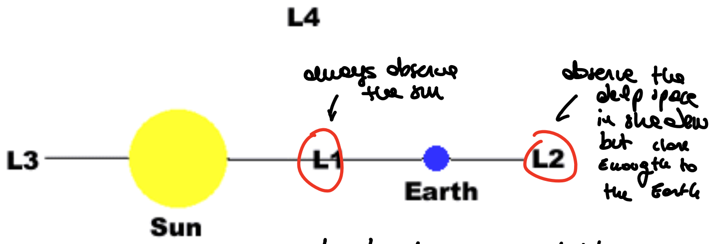
- Three-body problem cannot be solved in closed form (too few conserved quantities)
- 5 Lagrange points where spacecraft is fixed relative to rotating Sun-Earth line

  ↳ important



3-body potential field

Sun-Earth rotating reference frame



L1, L2, L3 are unstable equilibrium point so it is better to rotate around it.

At point L4 and L5 they are stable → there are a lot of asteroid orbiting at those point for Sun and Jupiter reference frame.

Lagrangian points

- Given two massive bodies in circular orbits around their common centre of mass, it can be shown that there are five positions in space, the Lagrangian libration points, where a third body of comparatively negligible mass would maintain its position relative to the two massive bodies.
- As seen in the frame which rotates with the same period as the two co-orbiting bodies, the gravitational fields of two massive bodies combined with the centrifugal force are in balance at the Lagrangian points.

For a static equilibrium, it is necessary that the following equations hold:

$$\boxed{\dot{x} = \ddot{x} = \dot{y} = \ddot{y} = \dot{z} = \ddot{z} = 0}$$

Lagrangian points

$$\frac{\delta^2 \mathbf{r}}{\delta t^2} = -\nabla U - 2\mathbf{\Omega} \times \frac{\delta \mathbf{r}}{\delta t} \quad \Rightarrow \quad \frac{\delta^2 \mathbf{r}}{\delta t^2} + 2\mathbf{\Omega} \times \frac{\delta \mathbf{r}}{\delta t} = -\nabla U$$

Can be written component wise as

$$\begin{Bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{Bmatrix} + 2\begin{Bmatrix} 0 \\ 0 \\ \Omega_z \end{Bmatrix} \times \begin{Bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{Bmatrix} = -\begin{Bmatrix} \dfrac{\partial U}{\partial x} \\ \dfrac{\partial U}{\partial y} \\ \dfrac{\partial U}{\partial z} \end{Bmatrix}$$

(we are here writing only the three equations for the acceleration but there are also 3 ODE on the velocity:

$$\frac{dx}{dt} = v_x, \frac{dy}{dt} = v_y, \frac{dz}{dt} = v_z$$

Lagrangian points

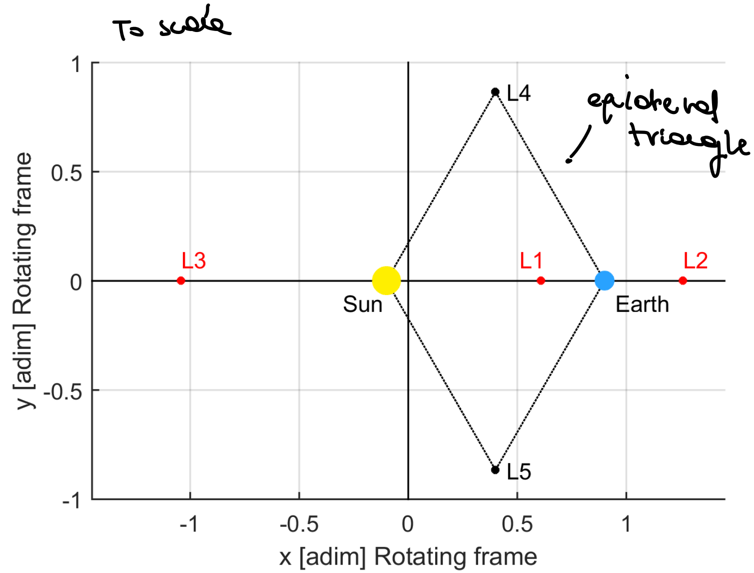
$$\begin{cases} \ddot{x} - 2\Omega_z \dot{y} = -\dfrac{\partial U}{\partial x} \\ \ddot{y} + 2\Omega_z \dot{x} = -\dfrac{\partial U}{\partial y} \\ \ddot{z} = -\dfrac{\partial U}{\partial z} \end{cases}$$

Setting $\dot{x} = \ddot{x} = \dot{y} = \ddot{y} = \dot{z} = \ddot{z} = 0$

$$\begin{cases} 0 = -\dfrac{\partial U}{\partial x} \\ 0 = -\dfrac{\partial U}{\partial y} \\ 0 = -\dfrac{\partial U}{\partial z} \end{cases}$$

Equilibrium: Lagrangian points

Lagrangian points



Using the equation of the three-body potential

$$U = -G\left(\frac{m_1}{r_1} + \frac{m_2}{r_2}\right) - \frac{1}{2}\Omega^2\left(x^2 + y^2\right)$$

We can compute the partial derivatives and set the equilibrium conditions.

All the equilibrium points lie in the x, y plane

- L_1, L_2, L_3: 3 collinear equilibrium points on the x-axis (unstable equilibria)
- L_4, L_5: 2 equilateral equilibrium points (stable equilibria)

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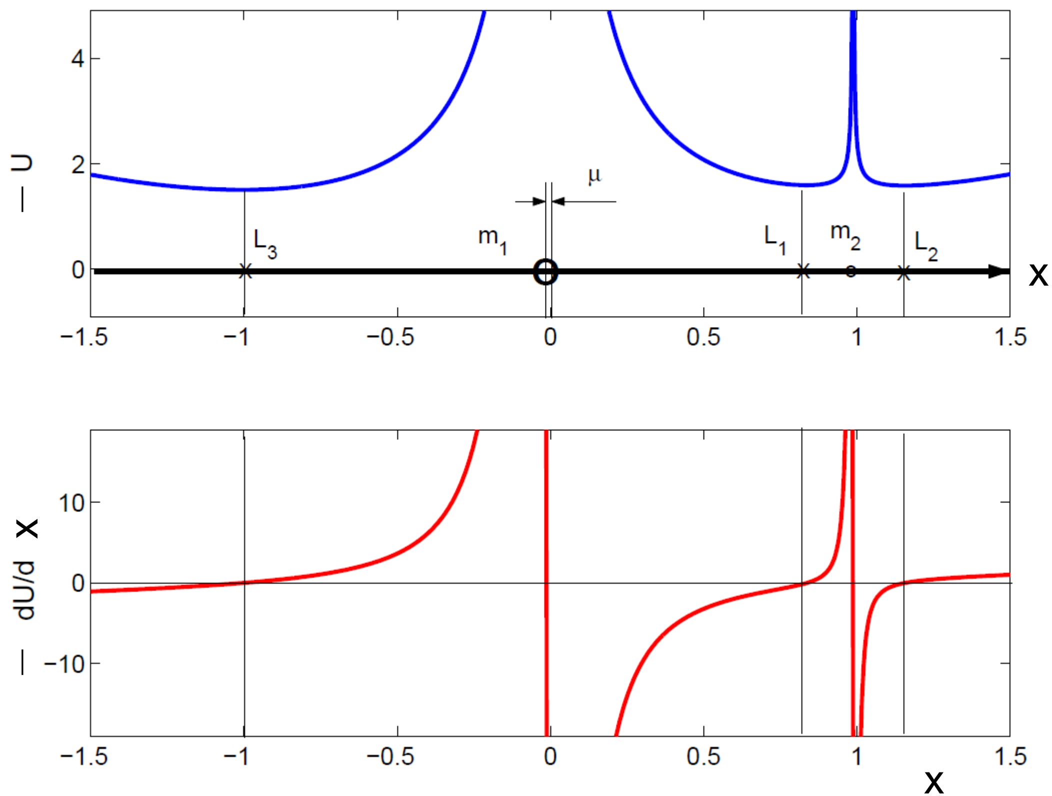
Lagrangian points



Position of the collinear points along

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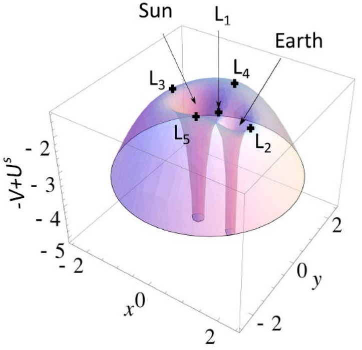
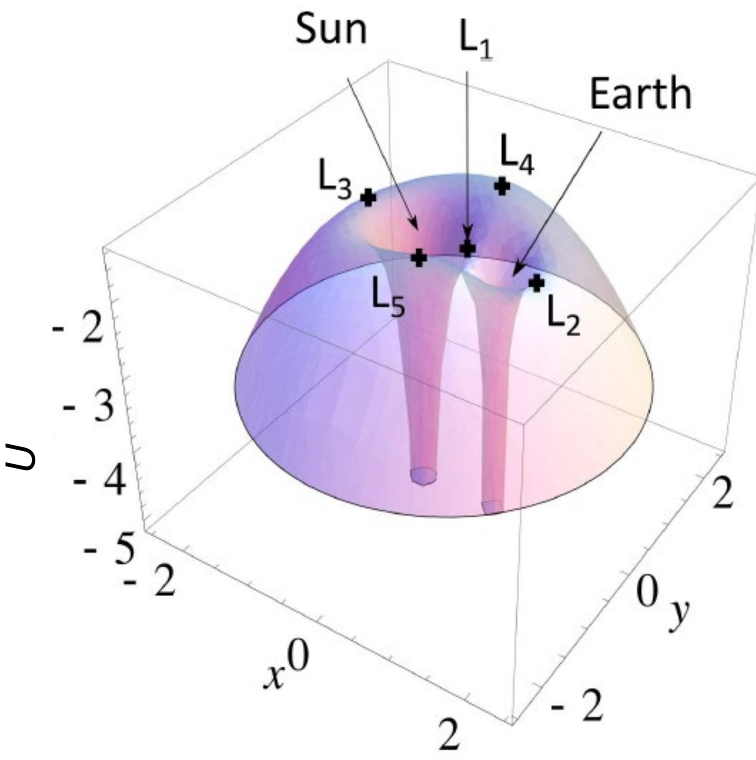
Zero velocity curves or Hill surfaces

The Zero Velocity Curves define regions in space where the spacecraft is allowed to move.

- They have the advantage of giving a qualitative picture of the motion of the spacecraft under the influence of the two primaries.
- Indeed, by fixing the initial state of the spacecraft, the energy of the spacecraft remain constant as there
- The ZVC is obtained from the intersection of the energy of the spacecraft (constant) with the potential energy



Zero velocity curves or Hill surfaces



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Zero velocity curves or Hill surfaces

- Critical Jacobi constant $C^* = -2U$ or critical energy $E^* = 0 + U$ at the limit of $v^2=0$

  ↳ It has only gravitational or rotatory energy → space craft has no more kinetic energy. → Zero velocity curves.

- $U = -G\left(\frac{m_1}{r_1} + \frac{m_2}{r_2}\right) - \frac{1}{2}\Omega^2\left(x^2 + y^2\right)$ is a negative quantity therefore

$$C^* > 0, \quad \mathrm{E}^* < 0$$

$$\left[\begin{array}{l} C = -v^2 - 2U \\ E = \frac{1}{2}v^2 + U \end{array}\right]$$

Allowed regions: $C \le C^*, \ or \ E \ge E^*$ as $v^2 \ge 0$

Forbidden regions: $C > C^*, \ or \ E < E^*$ as $v^2 < 0$ (no motion is possible)

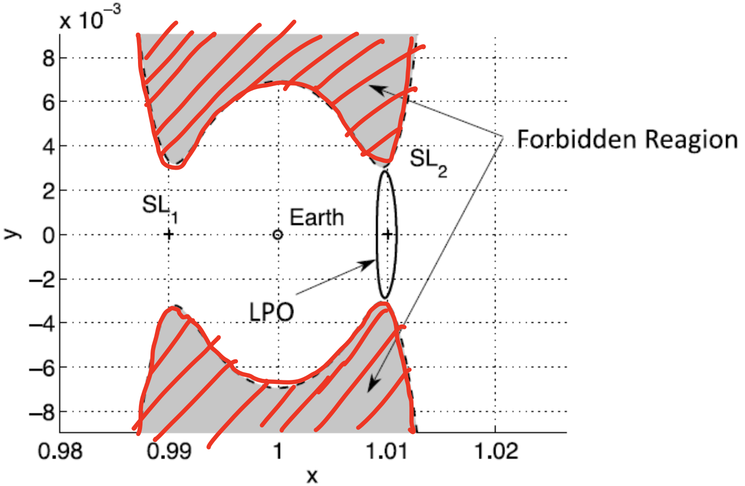
Zero velocity curves or Hill surfaces

If we set $v = 0$, then the Jacobi integral defines surfaces which form the boundary of the region where the motion of the spacecraft P is permitted (the region where $v^2 \geq 0$).

The magnitude of the velocity vector must be a real number

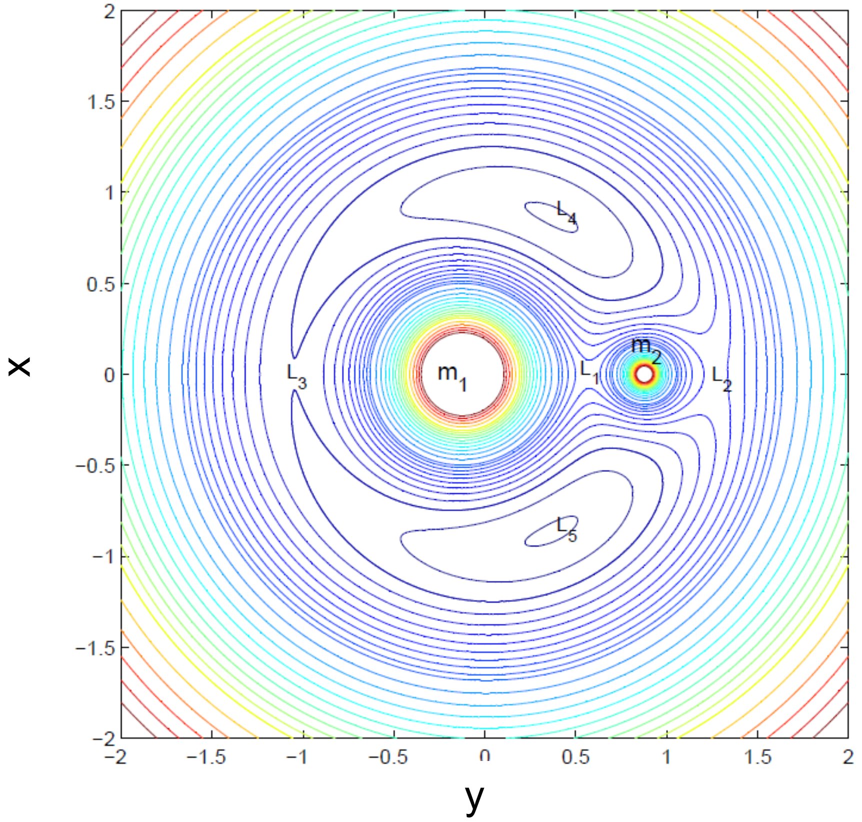
Zero velocity curves or Hill surfaces

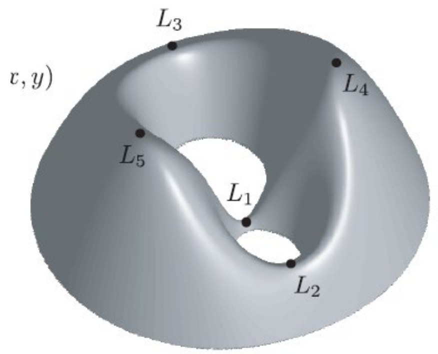
- grey region: forbidden area where the motion of the spacecraft is not physically possible
- white area: region of possible motion.
- Example of Libration Point Orbit (black line) where the spacecraft is placed
- The regions close to L_2, L_1 are called bottle neck regions



Zero velocity curves of a spacecraft in LPO

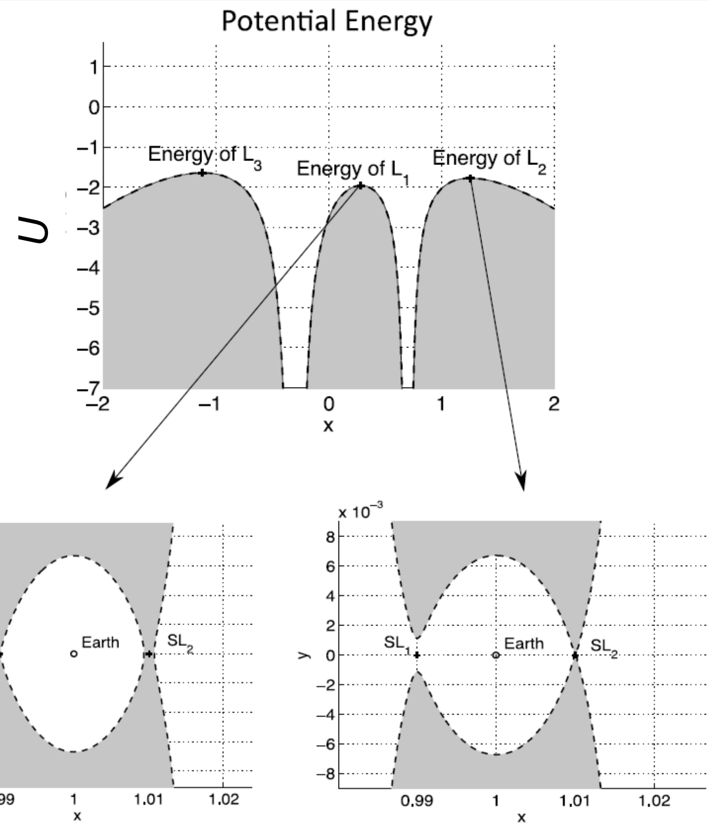
Zero velocity curves or Hill surfaces





Contour line of the three-body potential function in the x-y plane

Zero velocity curves or Hill surfaces



Energy of L_1 ZVCs

Energy of L_2 ZVCs

Zero velocity curves or Hill surfaces

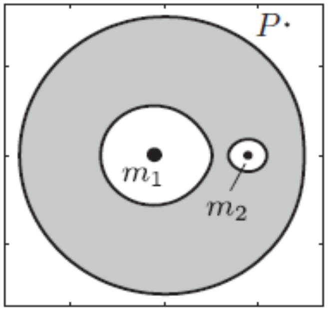
True if I do not do any maneuver in that case I can increase the energy

I can get in the white region by propagating the dynamics.

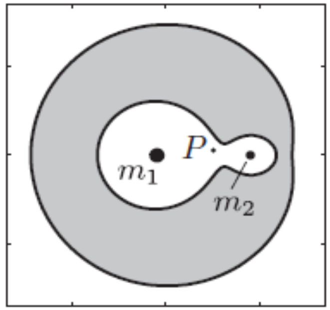
$$E = \frac{1}{2}v^2 + U(x,y,z)$$

$$v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$$

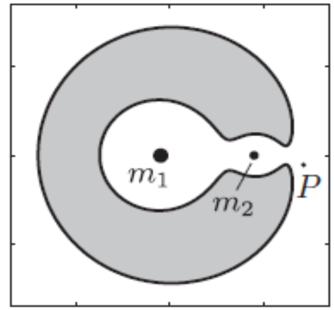
If I have very low energy



Case 1: $E < E_1$

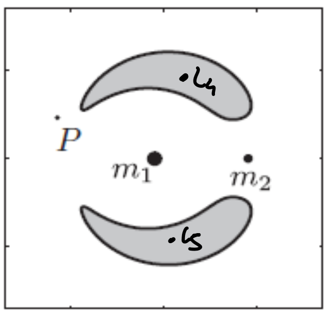


Case 2: $E_1 < E < E_2$

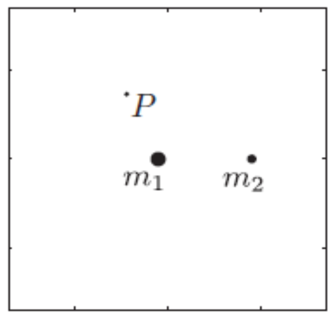


Case 3: $E_2 < E < E_3$

forbidden region around L_4 and L_5



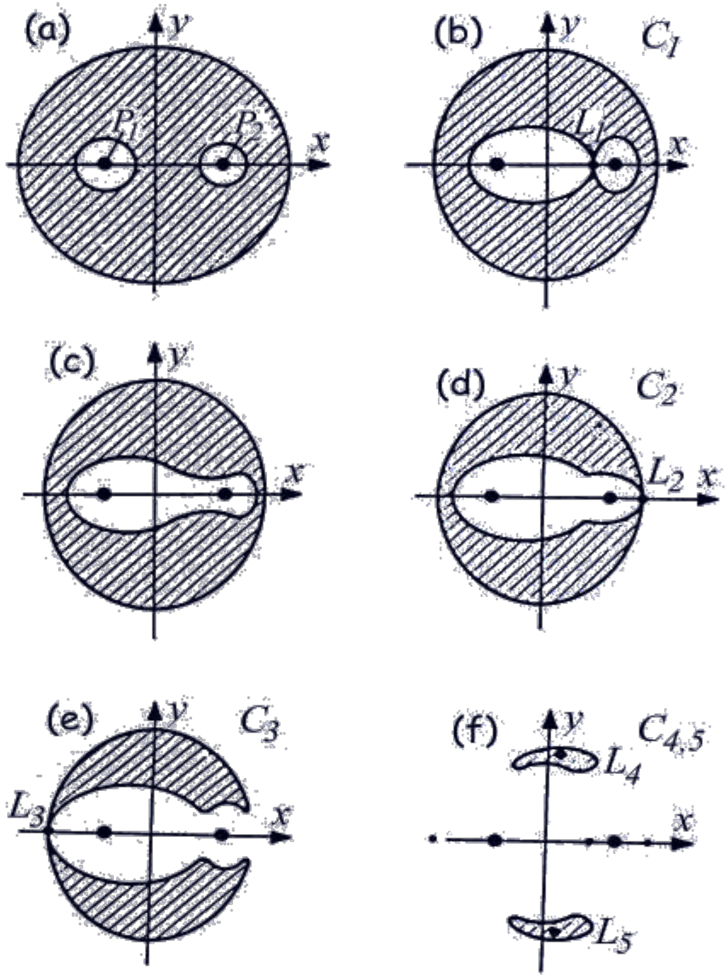
Case 4: $E_3 < E < E_4$



Case 5: $E < E_4$

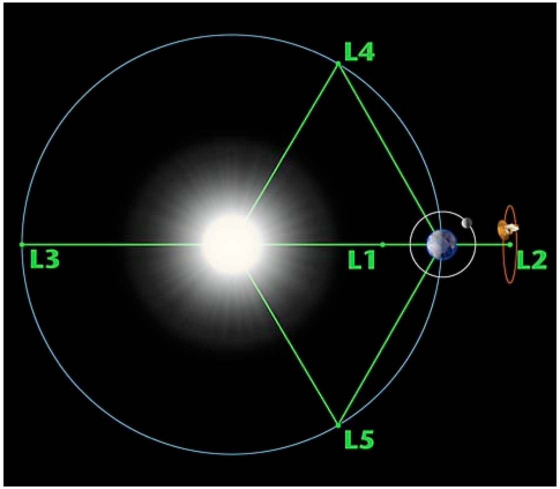
If I have enough energy I can go anywhere I want.

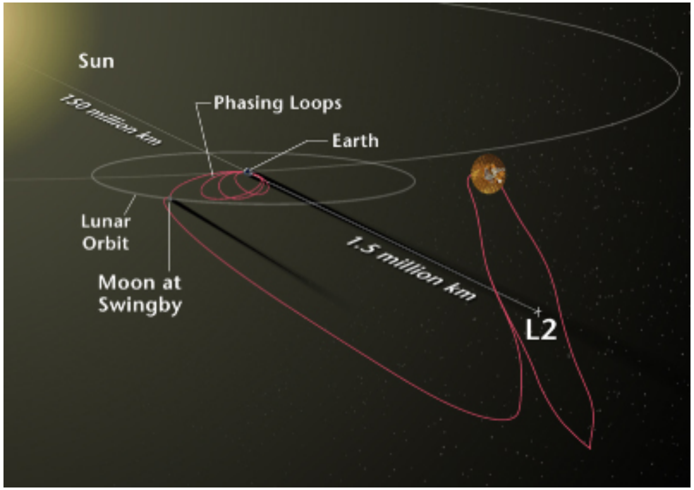
Zero velocity curves or Hill surfaces



Libration points and orbits

- The L_1 and L_2 points have been used for a range of science missions
- L_1 (sunward of Earth) is used for solar physics missions (continuous view)
- L_2 is used for astronomy missions (full sky view, cold for IR telescopes)



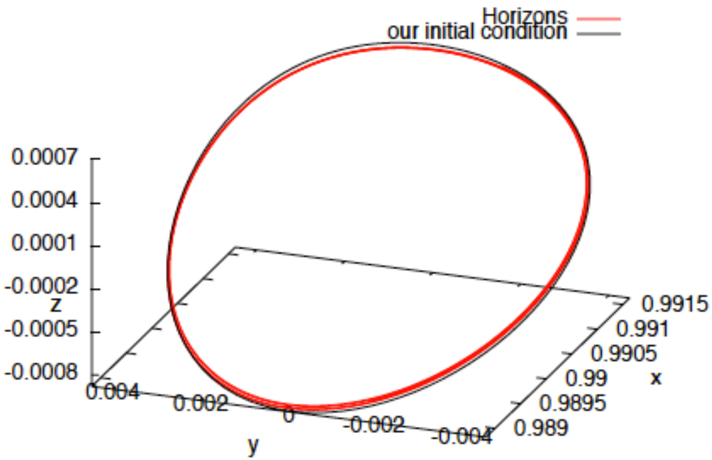


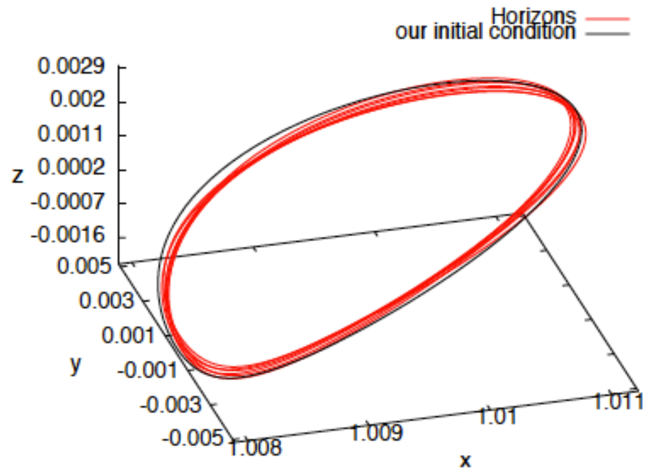
Libration Point Orbits: SOHO and Herschel

SOHO (Expected end 31/12/2016) and **Herschel** (Ended May 2013):

➢ Halo orbit by comparing to the ephemerides by JPL Horizons system

Mission	Orbit	LP	T	C_J
Soho	Halo South	L_1	3.0595858	3.0008294
Herschel	Halo North	L_2	3.0947685	3.0007831





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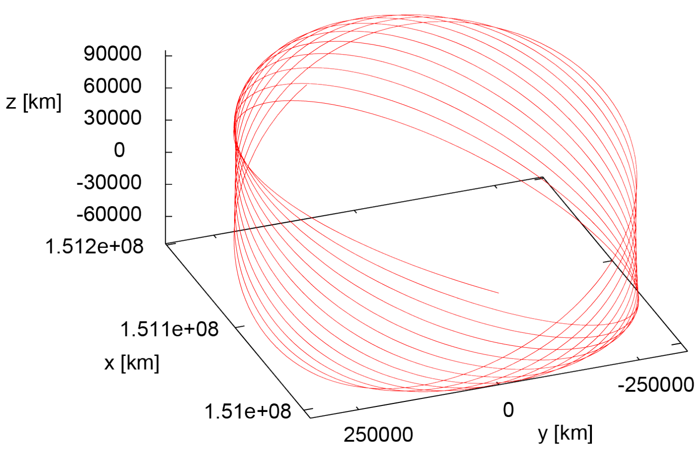
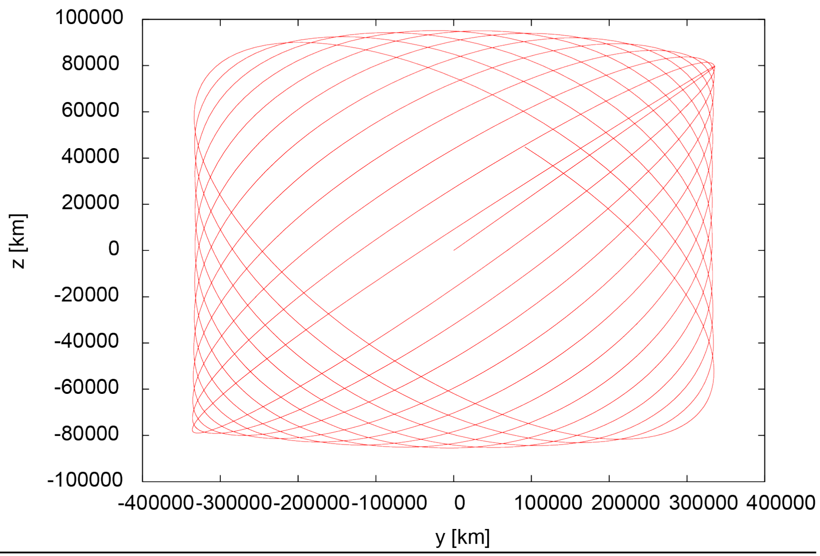
Libration Point Orbits: GAIA

GAIA mission (Near future mission, primary mission end: 2019/2020):

- Lissajous orbit and the corresponding unstable invariant manifold to match the in-plane and out-of-plane amplitudes.





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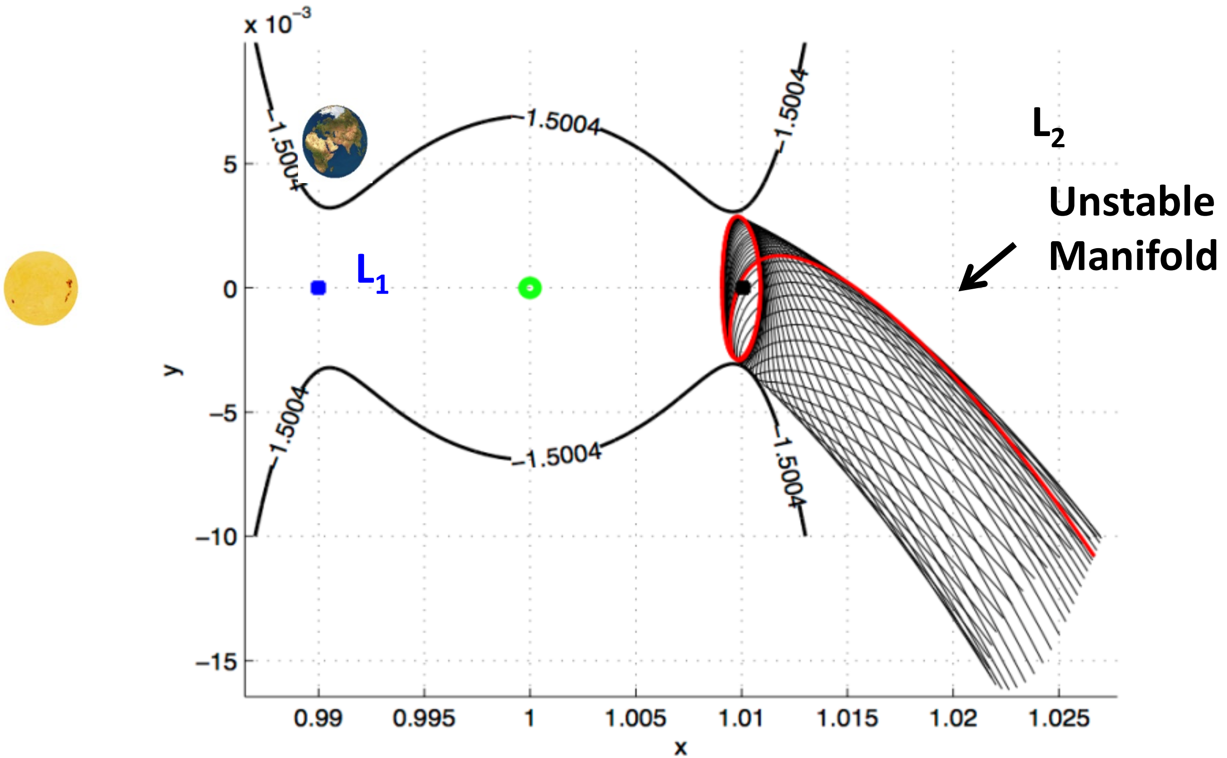
Manifold dynamics



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Not included in exam

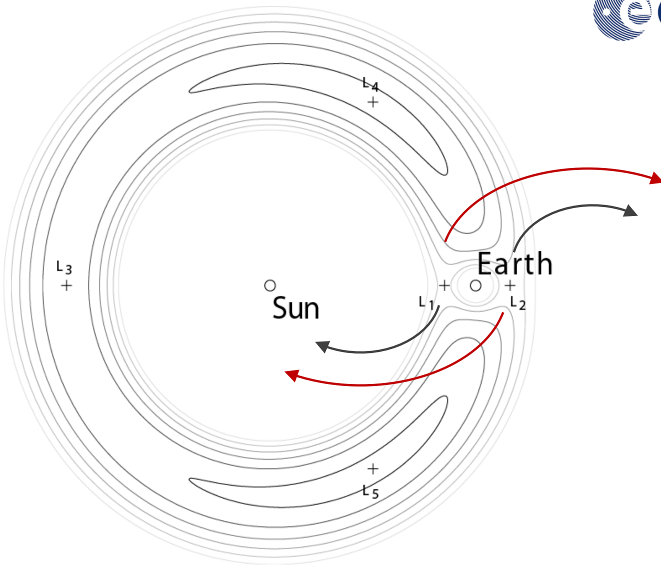
AN APPLICATION

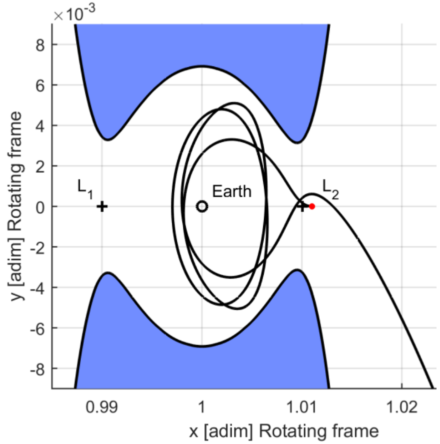
An application: end-of-life disposal (study done for ESA)

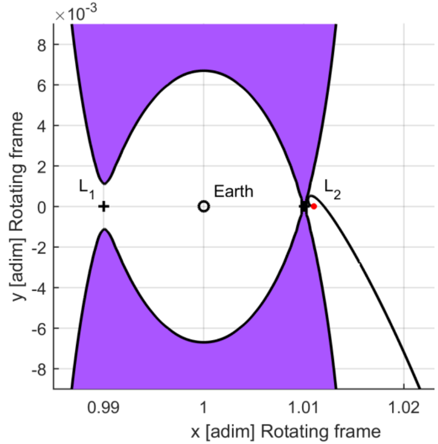
- Interplanetary missions often rely on Earth flybys.
- Rocket upper stage comes back to the Earth.
- No guidelines for Libration Point Orbit (LPO) missions end-of-life.
- Re-entry into Earth vicinity

End-of-life disposal

- Energy of L2 in the CRTBP:
  - $E > E_2$: way opened
  - $E < E_2$: way closed to prevent a re-entry

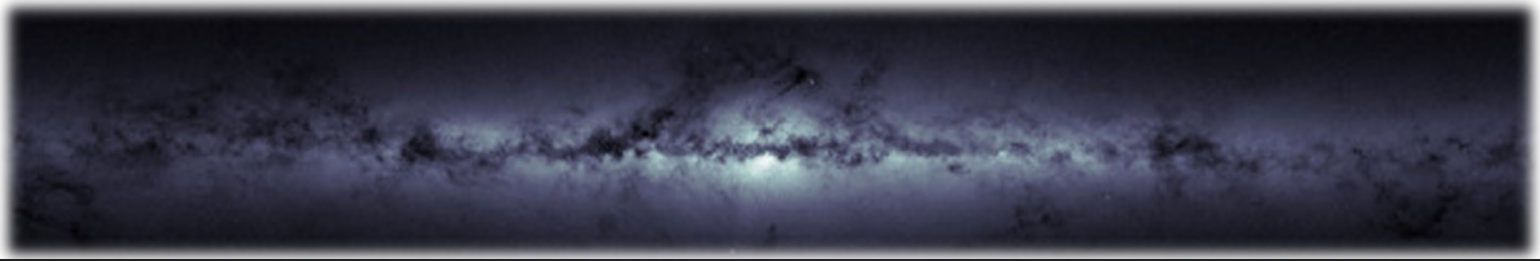






Gaia mission

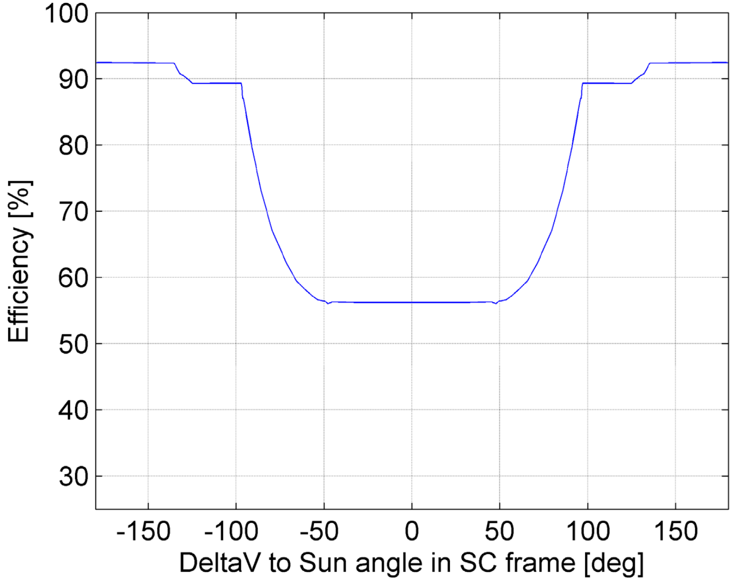
Launch date:	19 December 2013
Mission end:	nominal mission end after 5 years (2018)
Launch vehicle:	Soyuz-Fregat
Launch mass:	2030 kg, including 710 kg of payload, a 920 kg service module, 400 kg of propellant
Mission phase:	Completed first year of science observations
Orbit:	Lissajous-type orbit around L2
Instruments:	Astro (2 identical telescopes and imaging system); BP/RP (Blue and Red Photometers) and RVS (Radial-Velocity Spectrometer)
Partnerships:	Gaia is a fully European mission designed, built and operated by ESA. The Gaia Data Processing and Analysis Consortium (DPAC) will process the raw data to be published in the largest stellar catalogue ever made.



Gaia disposal constraints and requirements

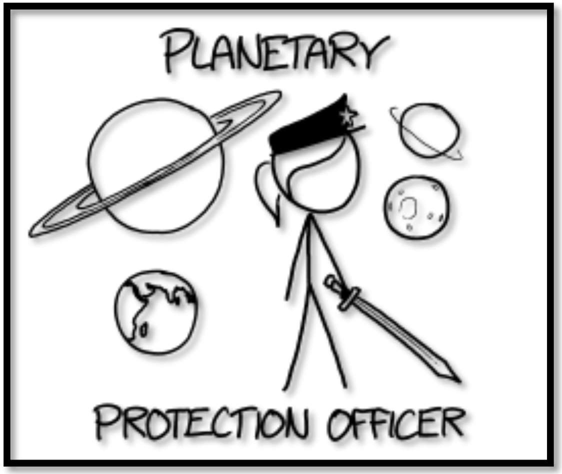
Constraints

- Time window: 01/07/2019 - 31/12/2020
- Maximum 6 months between two disposal manoeuvers
- Thrust efficiency from ESA



Requirement

The spacecraft should not return to the Earth for at least 100 years after the disposal epoch

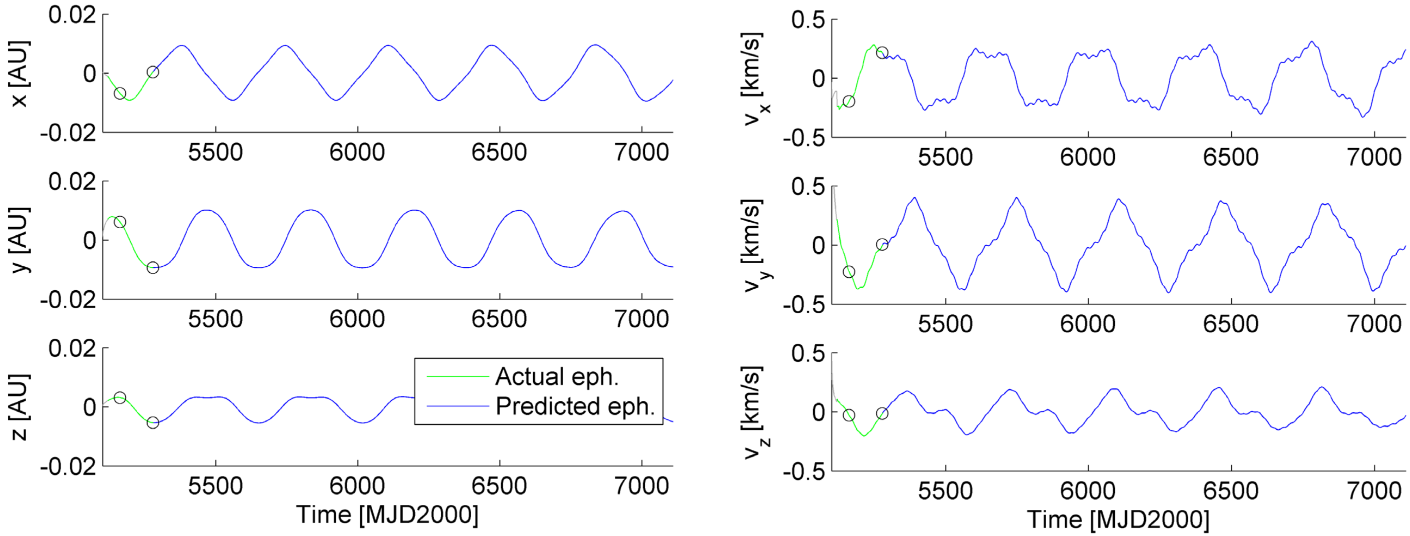


The Gaia mission

Gaia ephemerides

- Gaia actual trajectory, provided by ESA, until 17/06/2014 (green)
- Trajectory predicted thereafter by NASA Horizon system (blue)

(Earth centric J2000)



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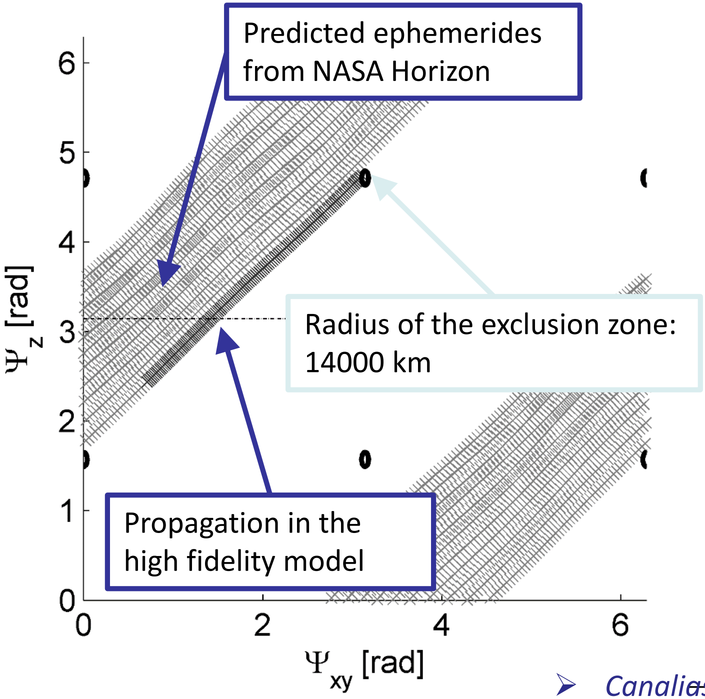
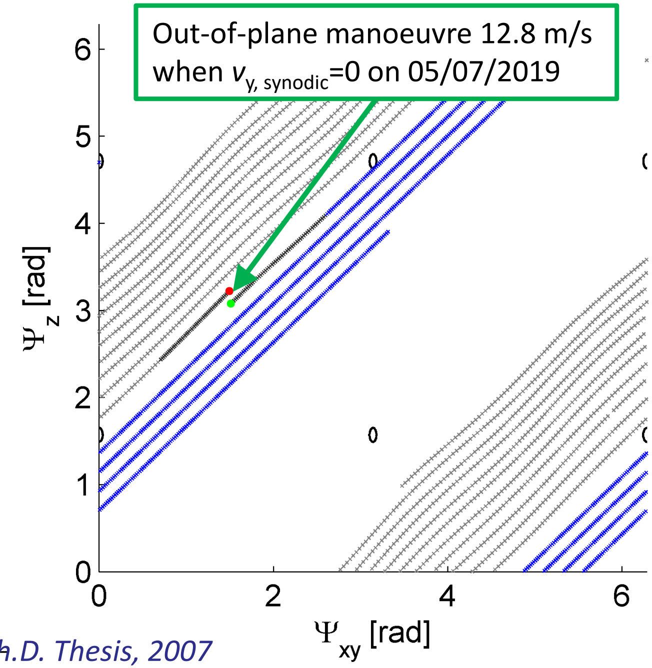
The Gaia mission

Gaia eclipse avoidance manoeuvre

The nominal trajectory will have an **eclipse** on 16/8/2019 (7167.459 MJD2000)

Eclipse studied on the effective phase plane





➢ *Canalias E., Ph.D. Thesis, 2007*

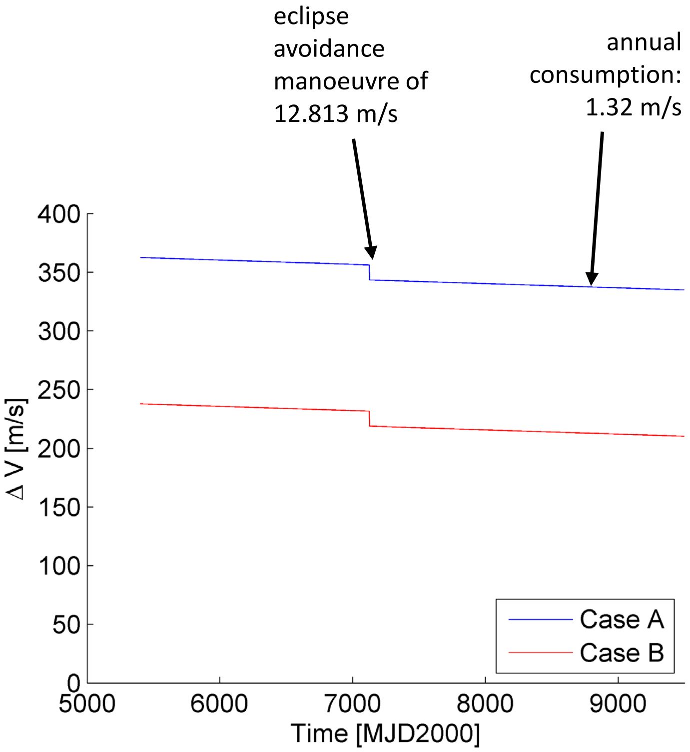
The Gaia mission

Gaia disposal constraints and requirements

Available Δv_{TOT} at 06/07/2019

- **Case A**: 343.52 m/s (propellant mass: 185 kg)
- **Case B**: 218.74 m/s (propellant mass: 115 kg)

in both cases, 20% of Δv_{TOT} is kept for correction manoeuvres.



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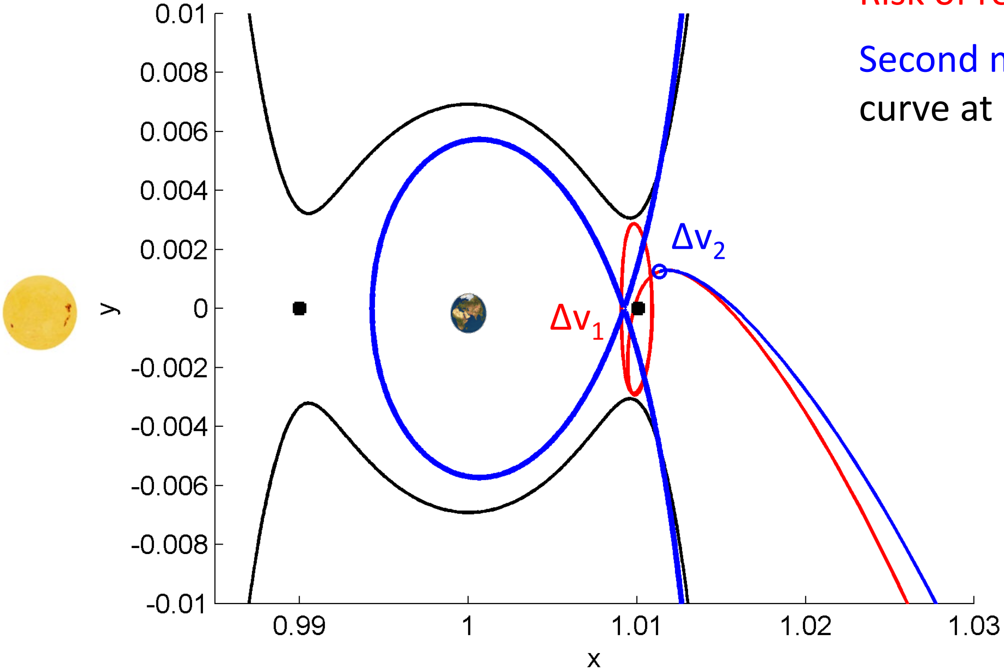
Heliocentric disposal



First manoeuvre to inject the spacecraft into the unstable trajectory leaving the LPO

Risk of returning to Earth through L_2

Second manoeuvre to close the Hill's curve at L_2

Heliocentric disposal

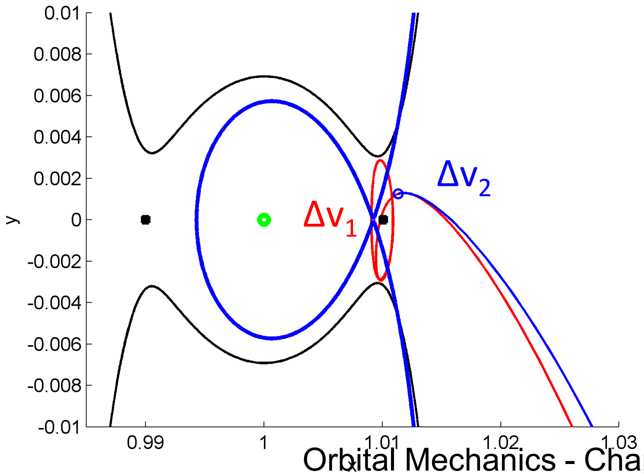
Analytical procedure in the CR3BP

- First small manoeuvre

$$J\left(\mathbf{s}_{syn}\right)=-\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)+x^2+y^2+2\left(\left(1-\beta\right)\frac{1-\mu}{r_S}+\frac{\mu}{r_E}\right) \longrightarrow \left(v^2\right)^- = x^2+y^2+2\left(\left(1-\beta\right)\frac{1-\mu}{r_S}+\frac{\mu}{r_E}\right)-J$$

- Second manoeuvre: Δ*v* to decrease the energy (increase *J*) tangent to spacecraft velocity

$$J_{L_2}=x_{L_2}^2+2\left(\left(1-\beta\right)\frac{1-\mu}{x_{L2}+\mu}+\frac{\mu}{x_{L_2}+\mu-1}\right) \longrightarrow \left(v^2\right)^+ = x^2+y^2+2\left(\left(1-\beta\right)\frac{1-\mu}{r_S}+\frac{\mu}{r_E}\right)-J_{L_2}$$



$$\boxed{\Delta v_{\text{closure @ }J_2} = v^- - v^+}$$

$$\Delta v_{\text{closure @ }FR} = v^- = \sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}$$

Heliocentric disposal

Design in high-fidelity dynamics model

J, J_{L2} computed in **osculating** synodic Sun – Earth + Moon system (RTBP)

Cost function

$$\phi = k \log \frac{J_{L_2}}{J_{\text{leg 2, min}}} + P$$

Penalty factor to solutions that**:**

- have distance from Earth less than the LPO
- Δv_2 lower than the one required to close the zero-velocity curves @ L_2
- are in the forbidden regions

Find solutions where the **minimum J** in the monitored points is **higher than J at L_2**.

Case 1: **all points** along the trajectory $J_{\text{leg 2, min}} = \min\left(J_{\text{leg 2}}\left(\mathbf{s}_{\text{leg 2, syn}}\right)\right)$

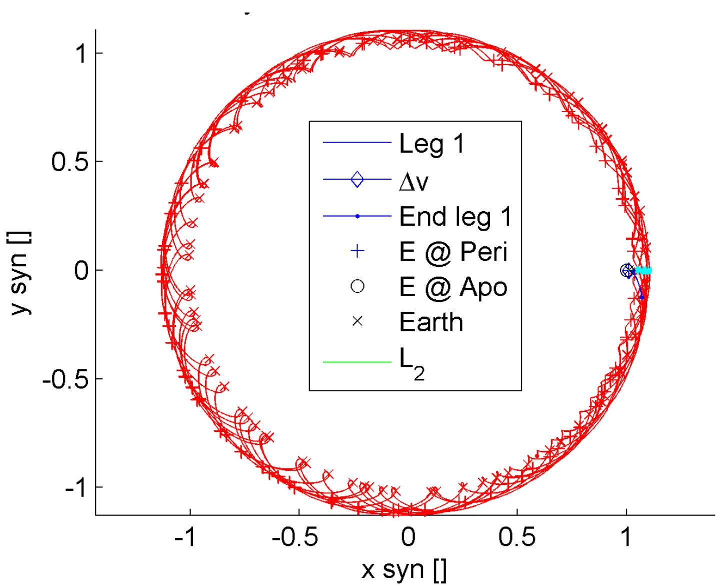
Case 2: only the points at the **close approach** with the Earth $J_{\text{leg 2, min}} = \min\left(J_{\text{leg 2}}\left(\mathbf{s}_{\text{leg 2, syn}} @\text{CA}\right)\right)$

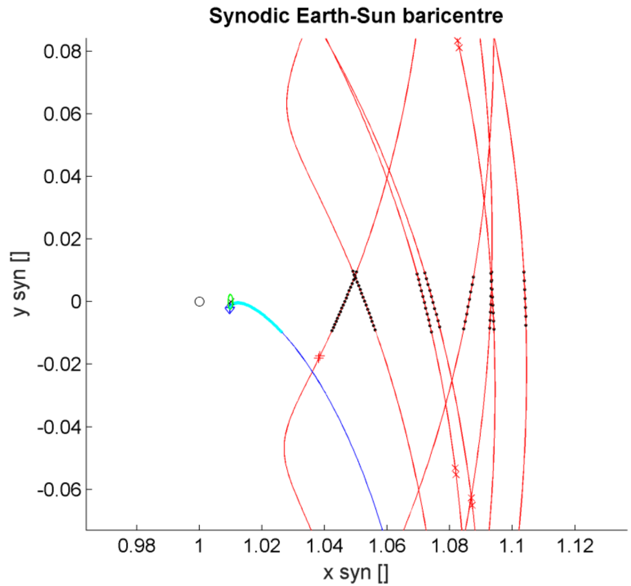
In both case, the time window for the propagation of leg 2 was set equal to **30 years** to reduce the computational time.

Selected trajectory

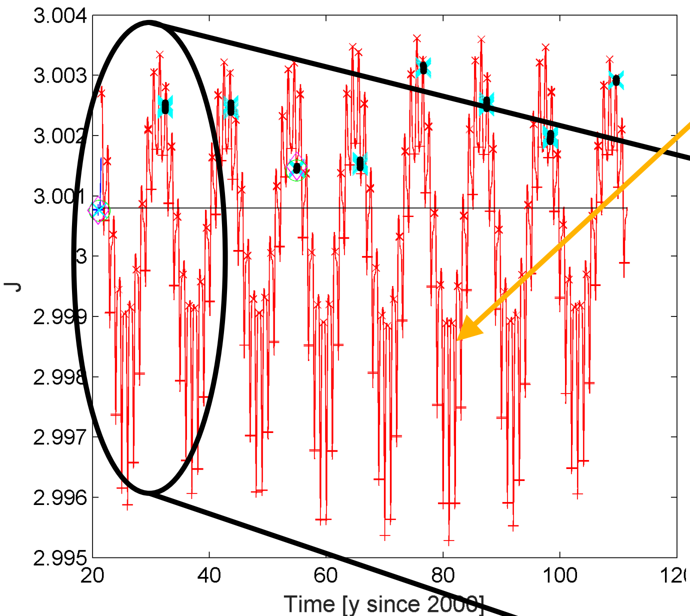
Robust solution

- Gaia always outside the Hill's curves with respect to the L_2 point

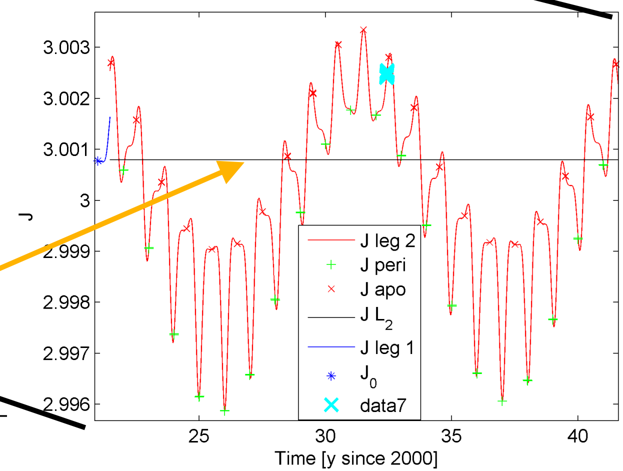




Selected trajectory



High-frequency oscillations in *J* due to rotation of the Earth + Moon around the Sun
(perigees @ local min, apogee @ local max)



Long period oscillations depend on evolution of the trajectory in the Synodic system

Selected trajectory

Sun-centred inertial system

- Disposal manoeuvres moves the spacecraft on an orbit which is far the Earth orbit. Leg 1 = transfer orbit,
  Leg 2 = “final” orbit
- Proposed optimisation is like Maximising Minimum Orbit Interception Distance

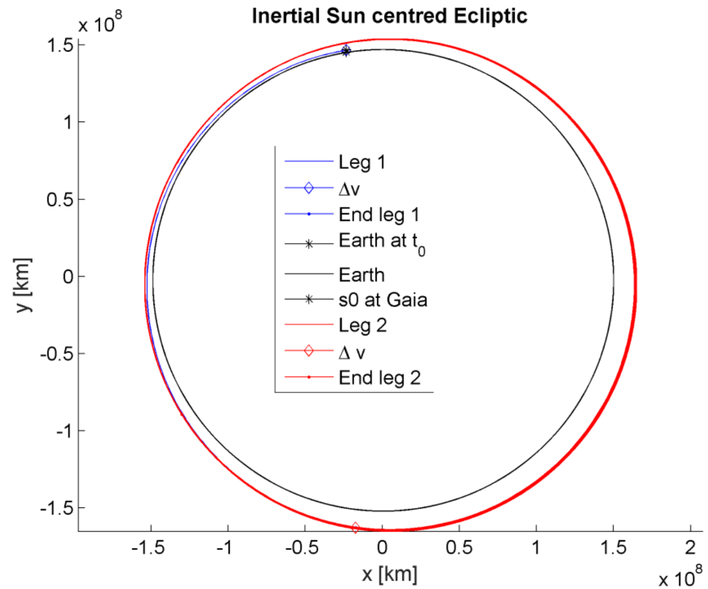
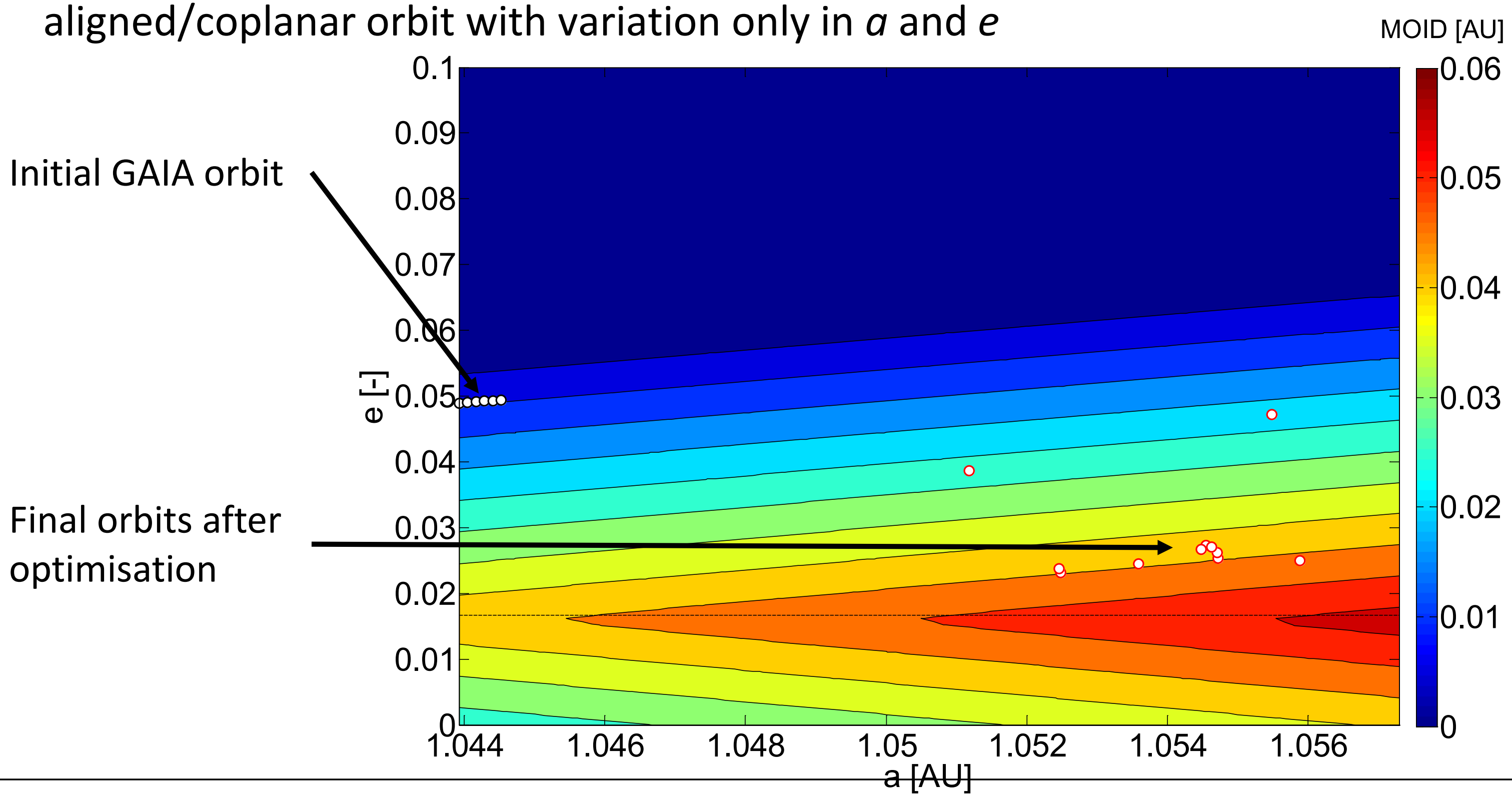


Figure 1. Gaia heliocentric disposal on 7669 MJD2000: trajectory in the Sun-centred ecliptic inertial system.

MOID variation with (*a*,*e*)

Variation of the MOID between the orbit of the Earth around the Sun and an aligned/coplanar orbit with variation only in *a* and *e*



MOID variation with (*a*,*e*)

Variation of the MOID between the orbit of the Earth around the Sun and an aligned/coplanar orbit with variation only in *a* and *e*