POLITECNICO
MILANO 1863

Spacecraft Attitude Dynamics

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Euler Equations

Fundamental Properties

$$\underset{\_}{h} = I\underset{\_}{\omega}$$

$$T = \frac{1}{2}\underset{\_}{\omega} \cdot I\underset{\_}{\omega}$$

If the body fixed frame is chosen to coincide with the principle axes then:

$$\underset{\_}{h} = [I_x\omega_x \quad I_y\omega_y \quad I_z\omega_z]^T$$

$$T = \frac{1}{2}(I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2)$$

Euler Equations

Vedi di fare un bel ripasso di Meccanica Aerospaziale e di Automatica.

$$\underline{h} = I\underline{\omega}$$

Recall the Transport Theorem:

Derivative of the angular momentum in a reference frame of the body.

$$\underbrace{\frac{{}^{N}d}{dt}}_{\text{Inertial frame}}\underline{x} = \underbrace{\frac{{}^{B}d}{dt}}_{\text{Body frame not inertial frame.}}\underline{x} + \omega \times \underline{x}$$

Euler Equations for a rigid-body:

$$\boxed{I\frac{d\underline{\omega}}{dt} = I\underline{\omega} \times \underline{\omega} + \underline{M}}$$

$${}^{N}\frac{d\underline{h}}{dt} = \underline{M} \rightarrow {}^{B}\frac{d\underline{h}}{dt} + \underline{\omega} \wedge \underline{h} = \underline{M}$$

are not parallel

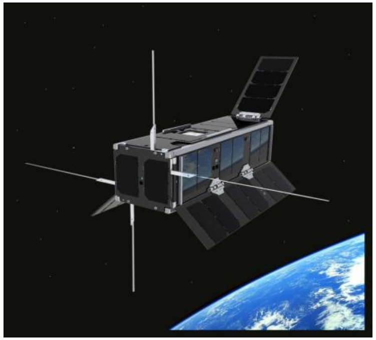
$${}^{B}\frac{d\underline{h}}{dt} = \underline{h} \wedge \underline{\omega} + \underline{M} \rightarrow \underline{\underline{I}}\,{}^{B}\frac{d\underline{\omega}}{dt} = (\underline{\underline{I}}\,\underline{\omega}) \wedge \underline{\omega} + \underline{M}$$

Assuming we are in the principal axis then:

$$\boxed{\begin{aligned}
\dot{\omega}_x &= \frac{I_y - I_z}{I_x}\omega_y\omega_z + \frac{M_x}{I_x} \\
\dot{\omega}_y &= \frac{I_z - I_x}{I_y}\omega_x\omega_z + \frac{M_y}{I_y} \\
\dot{\omega}_z &= \frac{I_x - I_y}{I_z}\omega_y\omega_x + \frac{M_z}{I_z}
\end{aligned}}$$

→ Has no exact solution. It is necessary to execute a numerical integration. It is important to have a reference solution in order to check that the numerical integration is able to arrive at the final reference solution.

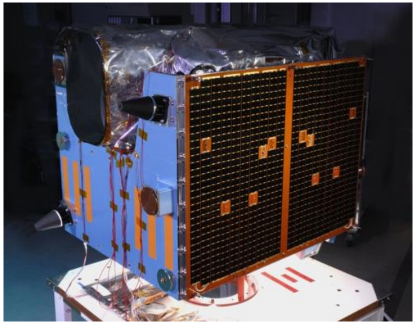
Principal moments of inertia (kg m²) → Varies a lot in Range.



UKube-1 – 3U CubeSat

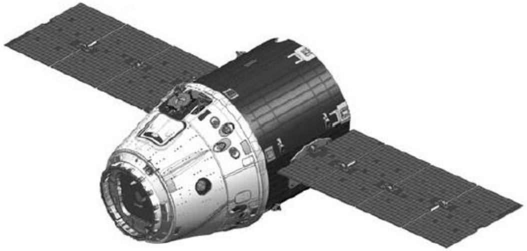
$$I_1 = 0.0109kgm^2, I_2 = 0.0504kgm^2, I_3 = 0.055kgm^2$$

The same perturbation has a higher effect on a small satellite than a big satellite. Small satellite will have higher rate of angular velocity. This kind of dependencies influence the type of integration scheme it should be used.



Rapid Eye - Micro-spacecraft

$$I_1 = 19.5kgm^2, I_2 = 19kgm^2, I_3 = 12.6kgm^2$$



Space X 's unmanned 10 tonne spacecraft

$$I_1 = 20{,}000kgm^2, I_2 = 20{,}000kgm^2, I_3 = 25{,}000kgm^2$$

Conservation shown in coordinate form

$$\underline{h} = [I_x\omega_x \quad I_y\omega_y \quad I_z\omega_z]^T \qquad T = \frac{1}{2}(I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2)$$

Using the chain rule:

$$\frac{d(\square)}{dt} = \frac{d(\square)}{d\omega_x}\frac{d\omega_x}{dt} + \frac{d(\square)}{d\omega_y}\frac{d\omega_y}{dt} + \frac{d(\square)}{d\omega_z}\frac{d\omega_z}{dt}$$

①
$$\begin{cases} \dot{\omega}_x = \dfrac{I_y - I_z}{I_x}\omega_y\omega_z + \cancel{\dfrac{M_x}{I_x}} \\ \dot{\omega}_y = \dfrac{I_z - I_x}{I_y}\omega_x\omega_z + \cancel{\dfrac{M_y}{I_y}} \\ \dot{\omega}_z = \dfrac{I_x - I_y}{I_z}\omega_y\omega_x + \cancel{\dfrac{M_z}{I_z}} \end{cases}$$

$$\frac{d(T)}{dt} = \frac{d(T)}{d\omega_x}\frac{d\omega_x}{dt} + \frac{d(T)}{d\omega_y}\frac{d\omega_y}{dt} + \frac{d(T)}{d\omega_z}\frac{d\omega_z}{dt}$$

$$\boxed{\frac{dT}{dt} = 0}$$

→ It is true if $\underline{M} = 0$ no Torque should be applied.

$$\boxed{\frac{d(\underline{h}\cdot\underline{h})}{dt} = 0}$$

Using equation ① and the chain rule we can demonstrate/verify that the kinetic energy is constant.

Exact Solution for a symmetric spacecraft

$I_x = I_y = I$ Cilindrical Body

How to check if we have integrate correctly the equation of motion?

$$\dot{\omega}_x = \frac{I_y - I_z}{I_x}\omega_y\omega_z$$
$$\dot{\omega}_y = \frac{I_z - I_x}{I_y}\omega_x\omega_z$$
$$\dot{\omega}_z = \frac{I_x - I_y}{I_z}\omega_y\omega_x \rightarrow \phi \quad \omega_z = \omega_{z0}$$

This two can be calculated separately from the third

$$T = \frac{1}{2}(I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2)$$

$$\underline{h} = [I_x\omega_x \quad I_y\omega_y \quad I_z\omega_z]^T$$

$$\rightarrow \begin{cases} \dot{\omega}_x = -\lambda\omega_y \\ \dot{\omega}_y = +\lambda\omega_x \end{cases}$$

It is possible to integrate $\Rightarrow$ Exact solution in analytical terms

Exact Solution is:

$$\begin{cases} \omega_x = \omega_{x0}\cos(\lambda t) - \omega_{y0}\sin(\lambda t) \\ \omega_y = \omega_{x0}\sin(\lambda t) + \omega_{y0}\cos(\lambda t) \\ \omega_z = \omega_{z0} \end{cases}$$

$\Rightarrow$ How to use this solution to check the correct use of the integration scheme?

Just move to the symmetric spacecraft. In the generic model feed just the number of a symmetric space craft for the I_x and I_y + $\underline{M} = \underline{0}$ no torque applied. The general integration should provide the exact solution cited above. Obviously we do not use the symmetric space craft model because the finding will be automatic.

$$\boxed{\lambda = \frac{(I_z - I)\omega_{z0}}{I}}$$

Equilibrium configurations

$$\dot{\omega}_x = \frac{I_y - I_z}{I_x}\omega_y\omega_z$$
$$\dot{\omega}_y = \frac{I_z - I_x}{I_y}\omega_x\omega_z$$
$$\dot{\omega}_z = \frac{I_x - I_y}{I_z}\omega_y\omega_x$$

NOTE
In the generic case with torque applied the solution will no longer be analytical. Only for one kind of expression of the torque it is possible to compute an analytical solution for the motion of the spacecraft.

Equilibrium Points: $\Rightarrow$ Whenever we have only one component of the angular velocity different from zero we are in a equilibrium point (conservation of all the components) $\Rightarrow$ True only for a case with no torque applied

$$\omega_x = 0, \omega_y = 0, \omega_z = 0$$
$$\omega_x = \omega_x(0), \omega_y = 0, \omega_z = 0$$
$$\omega_x = 0, \omega_y = \omega_y(0), \omega_z = 0$$
$$\omega_x = 0, \omega_y = 0, \omega_z = \omega_z(0)$$

Stability definitions of equilibrium points.

Stability definitions

Consider an autonomous nonlinear dynamical system

$$\dot{\underline{x}} = \underline{f}(\underline{x}), \underline{x}(0) = \underline{x}_0$$

defined on an open set containing the origin, and f is continuous on this open set. Then an equilibrium point x_e is said to be:

perturbation.

state → characteristic variables

1. **Lyapunov stable**, if, for every $\varepsilon > 0$, there exists a $\partial > 0$ such that, if $\|x(0) - x_e\| < \partial$, then for every $t > 0$ we have $\|x(t) - x_e\| < \varepsilon$.
2. The equilibrium of the above system is said to be **asymptotically stable** if it is Lyapunov stable and if $\|x(t) - x_e\| \to 0$ as $t \to \infty$

Equilibrium configurations

$$\dot{\omega}_x = \frac{I_y - I_z}{I_x}\omega_y\omega_z$$
$$\dot{\omega}_y = \frac{I_z - I_x}{I_y}\omega_x\omega_z$$
$$\dot{\omega}_z = \frac{I_x - I_y}{I_z}\omega_y\omega_x$$

Equilibrium Points:

$$\omega_x = 0, \omega_y = 0, \omega_z = 0$$
$$\omega_x = \omega_x(0), \omega_y = 0, \omega_z = 0$$
$$\omega_x = 0, \omega_y = \omega_y(0), \omega_z = 0$$
$$\omega_x = 0, \omega_y = 0, \omega_z = \omega_z(0)$$

→ with $\omega_x \neq 0$ if we give small perturbation to ω_x, ω_y and ω_z the system is Lyapunov stable if ω_x, ω_y and ω_z hover around the initial state. If ω_x, ω_y and ω_z goes back to their initial values before the perturbation with time the system can be classified as asymptotically stable.

→ To remove perturbation it is necessary to have some damping systems, if there are not any kind of damping attitude we can talk only of the Lyapunov stability.

Stability Analysis of equilibrium configurations

$\omega_x = C, \omega_y = 0, \omega_z = 0$

Look at the perturbed solution

$$\omega_x = C + \partial\omega_x, \omega_y = 0 + \partial\omega_y, \omega_z = 0 + \partial\omega_z$$

we replace $\omega_x, \omega_y, \omega_z$ with the values above.

$$\dot{\omega}_x = \frac{I_y - I_z}{I_x}\omega_y\omega_z \qquad \partial\dot{\omega}_x = \frac{I_y - I_z}{I_x}\partial\omega_y\partial\omega_z$$

no product of two small values

$$\Rightarrow \quad \partial\dot{\omega}_x = 0$$

$$\dot{\omega}_y = \frac{I_z - I_x}{I_y}\omega_x\omega_z \qquad \partial\dot{\omega}_y = \frac{I_z - I_x}{I_y}(C + \partial\omega_x)\partial\omega_z = \frac{I_z - I_x}{I_y}C\partial\omega_z$$

$$\dot{\omega}_z = \frac{I_x - I_y}{I_z}\omega_y\omega_x \qquad \partial\dot{\omega}_z = \frac{I_x - I_y}{I_z}(C + \partial\omega_x)\partial\omega_y = \frac{I_x - I_y}{I_z}C\partial\omega_y$$

$$\Rightarrow \quad \partial\ddot{\omega}_y = \frac{I_z - I_x}{I_y}C\partial\dot{\omega}_z = \frac{(I_z - I_x)(I_x - I_y)}{I_y I_z}C^2\partial\omega_y$$

$$\partial\ddot{\omega}_y + \frac{(I_x - I_z)(I_x - I_y)}{I_y I_z}C^2\partial\omega_y = \emptyset$$

initial condition of ω_x

Conditions for stability

$$\partial\ddot{\omega}_y + \frac{(I_x - I_z)(I_x - I_y)}{I_y I_z} C_1^2 \partial\omega_y = 0$$ → the same equation of a spring damp system

To have stability we need that the solution does not diverge.

Stable if:

$$\frac{(I_x - I_z)(I_x - I_y)}{\underbrace{I_y I_z}_{>0}} \underbrace{C_1^2}_{>0} > 0$$

$(I_x - I_z)(I_x - I_y) > 0$

>0	>0
or	or
<0	<0

⇓

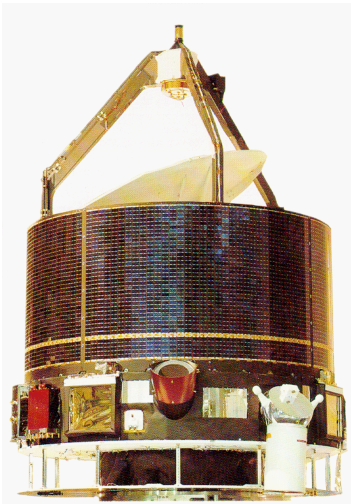
I_x is the biggest or the smallest inertial moment.

To have stability the spin axis of a s/c should be the major or minor inertial axis. It is not stable to spin around the intermediate inertial axis.

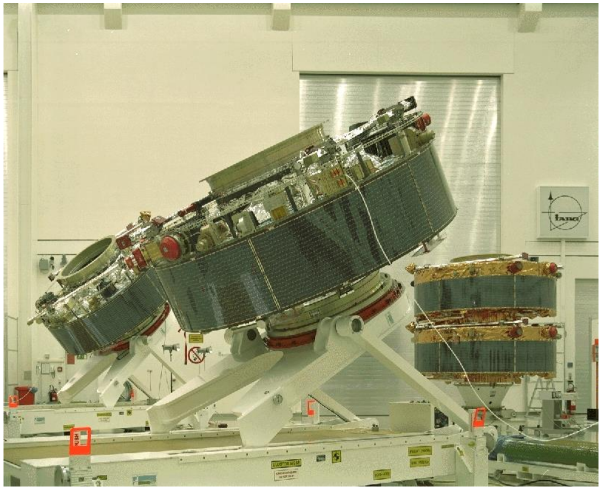
⇒ In reality the major inertia axis is the only one that provides stability → the minor inertia axis it is not always stable.

Spin stabilisation

- Simple and low cost method of attitude stabilisation (largely passive)
- Generally not suitable for imaging payloads (but can use a scan platform)
- Poor power efficiency since entire spacecraft body covered with solar cells



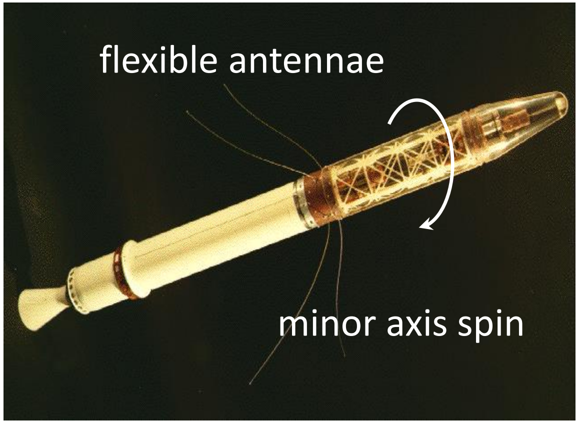
ESA Giotto spacecraft

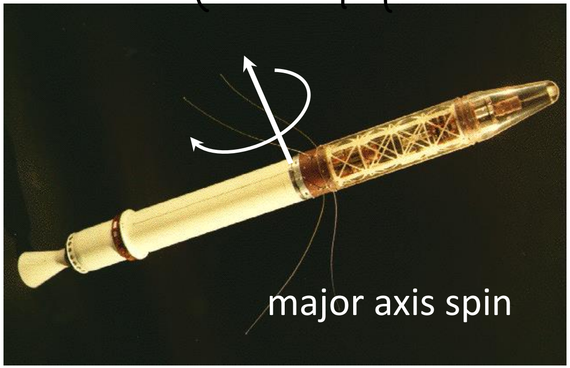


ESA Cluster spacecraft

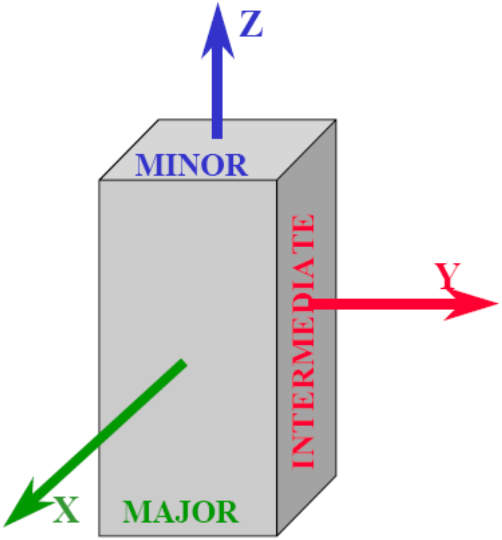
Explorer 1

Explorer 1 in the Figure below (first US satellite, 1958) was designed as a minor axis spinner → Does not provide stability.
because the spacecraft is not infinitely rigid
the flexibility of antennae introduced
perturbation that
was not bounded.





Major axis spin rule



- $I_{xx} > I_{yy} > I_{zz}$
- Major axis spin is stable
- Minor axis spin is stable
- Intermediate axis spin is unstable
- Energy dissipation changes these results
  $\rightarrow$ Minor axis spin becomes unstable
- This is called the Major-Axis Rule

Major axis spin rule

we will reach the minimum value of the energy and that will be the final condition of the rotation.

no object is really rigid expecially an object like spacecraft

energy dissipation, no torque

$\dot{T} < 0$

$|h| = const$ → Because there is no external torque.

we want the same

How we can represent the energy and angular momentum of the spacecraft in this condition

Spin around major or minor axis

$$2T_{I\,zmax} = I_{z\,max}\omega_{z\,max}^2 \quad \} \text{ Major inertia axis}$$

$$2T_{I\,zmin} = I_{z\,min}\omega_{z\,min}^2 \quad \} \text{ Minor inertia axis}$$

$$|h| = I_{z\,max}\omega_{z\,max} = I_{z\,min}\omega_{z\,min}$$

is constant

$\omega_{z\,min} > \omega_{z\,max}$

$\omega_{z\,max} = \frac{I_{z\,min}}{I_{z\,max}}\omega_{z\,min}$

conservation of angular momentum

$\omega_{zmax} < \omega_{zmin}$

$$2T_{I\,zmax} = I_{zmax}\frac{I_{zmin}^2}{I_{zmax}^2}\omega_{zmin}^2 = \frac{I_{zmin}^2}{I_{zmax}}\omega_{zmin}^2 = \frac{I_{zmin}}{I_{zmax}}2T_{I\,zmin}$$

<1

$T_{I\,zmax} < T_{I\,zmin}$

$$T_{I\,zmin} > T_{I\,zmax}$$

→ This is the reason why the major axis rule is always true
It is the situation of minimum energy.

Solutions in the phase space

Another form of the Euler equation

Assume the external torques are zero $\Rightarrow$

$$\begin{cases} I_x\dot{\omega}_x + (I_z - I_y)\omega_z\omega_y = 0 \\ I_y\dot{\omega}_y + (I_x - I_z)\omega_x\omega_z = 0 \\ I_z\dot{\omega}_z + (I_y - I_x)\omega_y\omega_x = 0 \end{cases} \Rightarrow \begin{matrix} 2\mathrm{T} = \mathrm{const} \\ h^2 = \mathrm{const} \end{matrix}$$

Time derivative of x equation $\Rightarrow$

$$I_x\ddot{\omega}_x + (I_z - I_y)\dot{\omega}_z\omega_y + (I_z - I_y)\omega_z\dot{\omega}_y = 0$$

Compute $\dot{\omega}_y$ and $\dot{\omega}_z$ from other 2 equations

$$I_x\ddot{\omega}_x + (I_z - I_y)\left(\frac{I_x - I_y}{I_z}\right)\omega_y^2\omega_x + (I_z - I_y)\left(\frac{I_z - I_x}{I_y}\right)\omega_z^2\omega_x = 0$$

Solutions in the phase space

From definitions

$$h^2 - 2\mathrm{TI}_z = I_x^2\omega_x^2 + I_y^2\omega_y^2 - I_xI_z\omega_x^2 - I_yI_z\omega_y^2$$
$$\omega_y^2(I_y - I_z)I_y = h^2 - 2\mathrm{TI}_z - \omega_x^2(I_x - I_z)I_x$$
$$\omega_y^2(I_y - I_z) = \frac{h^2 - 2\mathrm{TI}_z - \omega_x^2(I_x - I_z)I_x}{I_y}$$

$$h^2 - 2\mathrm{TI}_y = I_z^2\omega_z^2 + I_x^2\omega_x^2 - I_yI_z\omega_z^2 - I_xI_y\omega_x^2$$
$$\omega_z^2(I_z - I_y)I_z = h^2 - 2\mathrm{TI}_y + \omega_x^2(I_y - I_x)I_x$$
$$\omega_z^2(I_z - I_y) = \frac{h^2 - 2\mathrm{TI}_y + \omega_x^2(I_y - I_x)I_x}{I_z}$$

Substituting in the previous equation

$$\ddot{\omega}_x + \left[\frac{\overbrace{(I_y - I_x)(h^2 - 2TI_z) + (I_z - I_x)(h^2 - 2TI_y)}^{\text{constant}}}{I_xI_yI_z}\right]\omega_x + \left[\frac{\overbrace{2(I_z - I_x)(I_y - I_x)I_x}^{\text{constant.}}}{I_xI_yI_z}\right]\omega_x^3 = 0$$

↳ lengthy but contains only the second derivative of ω_x that contain a state variable.

Solutions in the phase space

Similarly the other 2 equations

$$\ddot{\omega}_y + \left[\frac{(I_z - I_y)(h^2 - 2\mathrm{TI}_x) + (I_x - I_y)(h^2 - 2\mathrm{TI}_z)}{I_x I_y I_z}\right]\omega_y + \left[\frac{2(I_z - I_y)(I_x - I_y)I_y}{I_x I_y I_z}\right]\omega_y^3 = 0$$

$$\ddot{\omega}_z + \left[\frac{(I_x - I_z)(h^2 - 2\mathrm{TI}_y) + (I_y - I_z)(h^2 - 2\mathrm{TI}_x)}{I_x I_y I_z}\right]\omega_z + \left[\frac{2(I_x - I_z)(I_y - I_z)I_z}{I_x I_y I_z}\right]\omega_z^3 = 0$$

The three equations have the structure $\ddot{\omega} + P\omega + Q\omega^3 = 0$

Integrating the x equation $\dot{\omega}_x^2 + \omega_x^2\left(\underbrace{P_x + \frac{1}{2}Q_x\omega_x^2}\right) = K_x$ → pseudo conic section

↳ because it is not a constant terms.

conic section in the phase plane $(\dot{\omega},\omega)$

Solutions in the phase space

→ We need to understand if the coefficient P and Q are positive or negative → Because this tell us what kind of conic section it is.

Assume $I_z > I_y > I_x$ ⇒ $I_x < \frac{h^2}{2T} < I_z$

P_x ?	$P_y > 0$	P_z ?
$Q_x > 0$	$Q_y < 0$	$Q_z > 0$

but $P_x + \frac{1}{2} Q_x \omega_x^2$ → if $P_x < 0$ for big ω_x →

ω_x big $P_x + \frac{1}{2} Q_x \omega_x^2 > 0$ even if $P_x < 0$

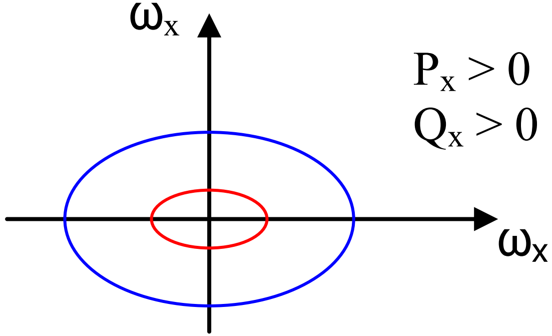
ω_z big $P_z + \frac{1}{2} Q_z \omega_z^2 > 0$ even if $P_z < 0$

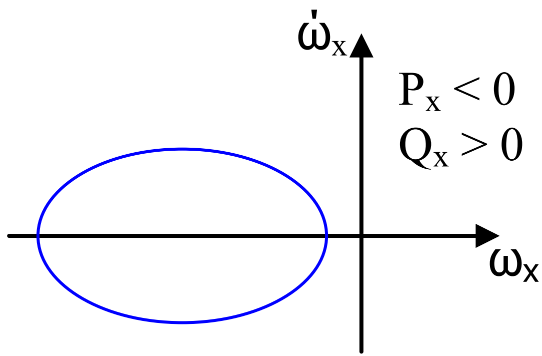
x and z phase planes
- conic sections are ellipses for large velocities
- for small velocities the type of conic section is undefined.

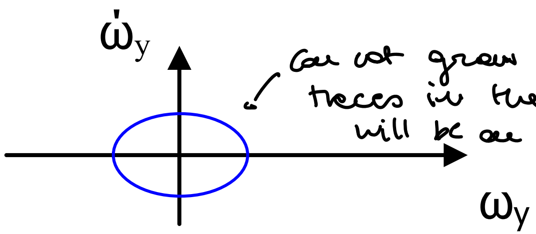
y phase plane
- solution must be such that the angular velocity is small enough to prevent the trace from being a hyperbola → because for an hyperbola the angular velocity grows infinitely → impossible

⇒ the ω_y must be bounded such as to have an ellipse
→ small enough

Solutions in the phase space







Can not grow because the traces in the phase plane will be an hyperbola