Course Notes

Spacecraft Attitude Dynamics

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Part 1:
Attitude dynamics and kinematics

Note for the reader

These short notes are in support of the course "Spacecraft Attitude Dynamics", they are not intended to replace any textbook. Interested readers are encouraged to consult also printed textbooks and archival papers.

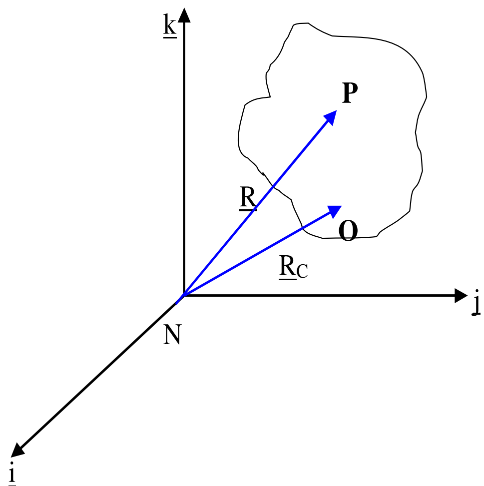
Franco Bernelli Zazzera

Index

Fundamental quantities

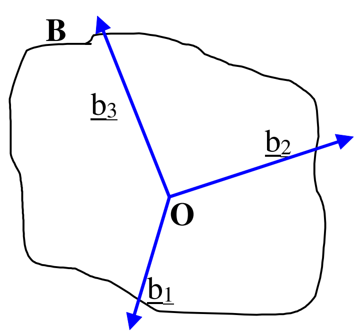
Inertial and body fixed frames

Here we will consider **N** = [$\underline{i}$:$\underline{j}$:$\underline{k}$] to be a fixed inertial frame, where $\underline{i}$, $\underline{j}$, $\underline{k}$ are unit vectors that form a basis of an orthonormal frame



The position of the centre of mass **O** of the rigid body $\underline{R}_C$ and a position vector $\underline{R}$ of an infinitesimal mass **P** of the rigid body can be expressed in the inertial frame, such that, $\underline{R} = x'\underline{\imath} + y'\underline{\jmath} + z'\underline{k}$.

We now introduce a body frame **B** = [$\underline{b}_1$: $\underline{b}_2$: $\underline{b}_3$] an orthonormal frame fixed in the body whose origin is positioned at the centre of mass O



In this way the orientation of a rigid body can be expressed as the relative attitude between the body fixed frame and an inertially fixed frame. Moreover, the attitude of the rigid body $A_{B/N}$ is defined as the relative orientation between the body fixed frame **B** and the inertial frame **N.** Since the orbital and attitude dynamics are generally de-coupled, in the analysis of a spacecraft's attitude we can ignore the translational component between the two frames and consider just the relative orientation such that:

$$\mathbf{B} = A_{B/N}\mathbf{N}$$

$A_{B/N}$ is an orthonormal matrix with the following properties:

$$A_{B/N}A_{B/N}^T = I_{3\times 3}, det(A_{B/N}) = 1$$

The attitude matrix is also known as the Direct Cosine Matrix (DCM).
In addition, we denote the instantaneous angular velocity of each frame as $\underset{\sim}{\omega}_N = \omega_1 {}'\underset{\sim}{\imath} + \omega_2 {}'\underset{\sim}{\jmath} + \omega_3 {}'\underset{\sim}{k}$and $\underset{\sim}{\omega}_B = \omega_1 \underset{\sim}{b}_1 + \omega_2 \underset{\sim}{b}_2 + \omega_3 \underset{\sim}{b}_3$expressed in the basis of there repective frames. The angular velocity of the frame **N** can be expressed in the body frame as $\underset{\sim}{\omega}_N^B = A_{B/N}\underset{\sim}{\omega}_N$
The the relative angular velocity $\underset{\sim}{\omega}_{B/N}$ expressed in body fixed coordinates associated with the relative attitude matrix $A_{B/N}$ is defined by

$$\underset{\sim}{\omega}_{B/N} = \underset{\sim}{\omega}_B - \underset{\sim}{\omega}_N^B = \underset{\sim}{\omega}_B - A_{B/N}\underset{\sim}{\omega}_N$$

Since in this case *N* is a fixed inertial frame then $\underset{\sim}{\omega}_N = \underset{\sim}{0}$ and therefore $\underset{\sim}{\omega}_{B/N} = \underset{\sim}{\omega}_B$.
Note that later in the course we will need to define the attitude between the body frame and a moving frame.

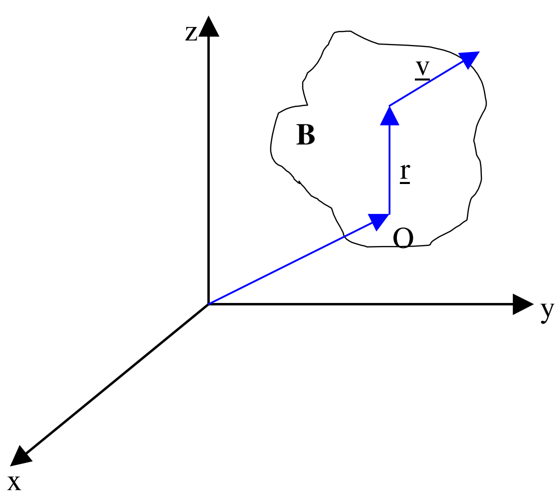
One of the theorems that is used throughout this course is the Transport Theorem. It relates the derivative of an arbitrary vector $\vec{x}$ taken with respect to one orthonormal frame, for example **N**, to a derivative taken with respect to another frame **B** where $\underset{\sim}{\omega}_{B/N}$ is the relative angular velocity vector between the frames. The Transport Theorem is summarize in the equation:

$$\left[\frac{d\vec{x}}{dt}\right]_N = \left[\frac{d\vec{x}}{dt}\right]_B + \underset{\sim}{\omega}_{B/N} \times \vec{x}$$

From here on when referiong to the angular velocity of the body frame with respect to the inertial frame expressed in body fixed coordinates we will simply write $\underset{\sim}{\omega} = \underset{\sim}{\omega}_{B/N}$.

Angular momentum of a rigid body

Let us start by defining the physical quantities that are fundamental and recurrent in the rotational motion of rigid spacecraft. These quantities are the angular momentum and the kinetic energy. The angular momentum of a rigid body "B" is defined referring to a reference frame attached to the rigid body and moving with it, with origin O:



Considering an infinitesimal point mass, it is defined as:

$$d\underline{h}_o = \underline{r} \wedge \underline{v \cdot} dm$$

Therefore, for the whole rigid body "B":

$$\underline{h}_o = \int_B \underline{r} \wedge \underline{v} \cdot dm$$

The velocity is expressed as:

$$\underline{v} = \underline{v}_o + \underline{\omega} \wedge \underline{r} + \underline{v}_{iB}$$

where $\underline{v}_o$ is the velocity of the origin of the reference system attached to the rigid body, $\underline{\omega}$ is the angular velocity of the rigid body and $\underline{v}_{iB}$ is the relative velocity between any point of the body and the origin of the reference system. For a rigid body, this relative velocity is always zero.
Evaluating the integral yields:

$$\underline{h}_o = \int_B \underline{r} \wedge \left(\underline{v}_o + \underline{\omega} \wedge \underline{r}\right) dm = -\underline{v}_o \wedge \int_B \underline{r} dm + \int_B \underline{r} \wedge \left(\underline{\omega} \wedge \underline{r}\right) dm$$

We must now evaluate the vector cross products:

$$\underline{\omega} = \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} \qquad \underline{r} = \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}$$

$$\left(\underline{\omega} \wedge \underline{r}\right) = \begin{Bmatrix} \omega_y z - \omega_z y \\ \omega_z x - \omega_x z \\ \omega_x y - \omega_y x \end{Bmatrix}$$

$$\underline{r} \wedge \left(\underline{\omega} \wedge \underline{r}\right) = \begin{Bmatrix} \omega_x y^2 - \omega_y xy + \omega_x z^2 - \omega_z xz \\ \omega_y z^2 - \omega_z yz + \omega_y x^2 - \omega_x xy \\ \omega_z x^2 - \omega_x xz + \omega_z y^2 - \omega_y yz \end{Bmatrix}$$

The first term of the integral can be expressed as a function of the first moment of mass, that is a measure of the distance between the origin of the reference system and the center of mass, while integrating the terms of the second vector term, since angular velocity is constant at any point of the rigid body, we can form the so called inertia matrix and write the angular momentum as:

$$\underline{h}_o = -\underline{v}_o \wedge S_o + I\underline{\omega}$$

where S_o is the first moment of mass and I is the inertia matrix that is expressed as:

$$S_o = \int_B \underline{r} dm$$

$$I = \begin{bmatrix} I_{xx} = \int_B (y^2 + z^2)dm & I_{xy} = \int_B -xydm & I_{xz} = \int_B -xzdm \\ I_{yx} = \int_B -yxdm & I_{yy} = \int_B (x^2 + z^2)dm & I_{yz} = \int_B -yzdm \\ I_{zx} = \int_B -zxdm & I_{zy} = \int_B -zydm & I_{zz} = \int_B (x^2 + y^2)dm \end{bmatrix}$$

Inertia matrix

The inertia matrix is symmetric and the terms along the diagonal are called inertia moments, while the off-diagonal terms are called inertia products. Inertia moments are always positive, while there is no general rule for the inertia products. The inertia matrix, by its definition, must respect precise constraints among its elements. It is possible to define 6 triangular inequalities that relate the inertia moments, which represent constraints on the admissible values of inertia moments. These constraints are on the sum of two inertia moments and on the difference between two inertia moments, for example:

$$I_{xx} + I_{yy} = \int (x^2 + y^2 + 2z^2)dm \geq \int (x^2 + y^2)dm = I_{zz}$$
$$I_{xx} - I_{yy} = \int (y^2 - x^2)dm \leq \int (x^2 + y^2)dm = I_{zz}$$

The remaining inequalities are obtained by simple rotation of the labels x, y and z.
There exist also 3 inequalities that relate the inertia products:

$$I_{xx} = \int (y^2 + z^2)dm = \int (y + z)^2 dm - \int 2zydm = A + 2I_{zy}$$
$$A \geq 0 \qquad I_{xx} \geq 0$$
$$I_{xx} \geq 2I_{zy}$$

Again, remaining inequalities are obtained by simple rotation of the labels x, y and z. Overall, 9 constraints relate the elements of the inertia matrix.

Rotation kinetic energy of a rigid body

Kinetic energy is defined as:

$$2T = \int_B \underline{v} \cdot \underline{v} dm = \int_B (\underline{v}_o + \underline{\omega} \wedge \underline{r}) \cdot (\underline{v}_o + \underline{\omega} \wedge \underline{r}) dm$$
$$= \int_B \left(\underline{v}_o \cdot \underline{v}_o + 2\underline{v}_o \cdot (\underline{\omega} \wedge \underline{r}) + (\underline{\omega} \wedge \underline{r}) \cdot (\underline{\omega} \wedge \underline{r}) \right) dm$$

The first term represents the kinetic energy due to the linear motion of the rigid body, not of interest in this case since we study only rotational motion. Therefore, the kinetic energy due to the angular motion is expressed as:

$$2T_{rot} = \int_B \left(2\underline{v}_o \cdot (\underline{\omega} \wedge \underline{r}) + (\underline{\omega} \wedge \underline{r}) \cdot (\underline{\omega} \wedge \underline{r})\right) dm = 2\underline{v}_o \cdot \underline{\omega} \wedge S_o + \int_B (\underline{\omega} \wedge \underline{r}) \cdot (\omega \wedge \underline{r}) dm$$

Expanding the term to be integrated we have:

$$(\underline{\omega} \wedge \underline{r}) \cdot (\underline{\omega} \wedge \underline{r}) = \omega_y^2 z^2 + \omega_z^2 y^2 + \omega_z^2 x^2 + \omega_x^2 z^2 + \omega_x^2 y^2 + \omega_y^2 x^2 - 2\omega_y \omega_z yz - 2\omega_z \omega_x xz - 2\omega_y \omega_x xy$$

That, once integrated, leads to:

$$\int_B (\underline{\omega} \wedge \underline{r}) \cdot (\underline{\omega} \wedge \underline{r}) dm = I_{xx}\omega_x^2 + I_{yy}\omega_y^2 + I_{zz}\omega_z^2 + 2I_{yz}\omega_y\omega_z + 2I_{xz}\omega_z\omega_x + 2I_{xy}\omega_y\omega_x$$

We can then write the kinetic energy using matrix notation, as:

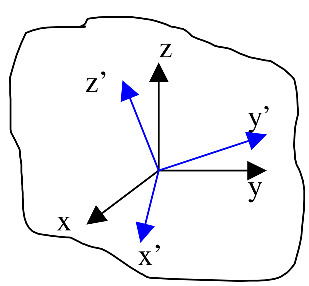
$$2T = 2\underline{v}_o \cdot \underline{\omega} \wedge S_o + \underline{\omega} I \underline{\omega}$$

By choosing the origin O coincident with the center of mass of the system, S_o vanishes and the angular momentum and kinetic energy become:

$$\underline{h} = I \cdot \underline{\omega}$$
$$2T = \underline{\omega} \cdot I \cdot \underline{\omega}$$

Principal axes and principal inertia moments

We can now try to find a further simplification of the expressions of the angular momentum and kinetic energy, trying to simplify the inertia matrix. In order to do so, we must change the reference system, keeping the origin coincident with the center of mass but such that the kinetic energy is a purely quadratic expression in the components of $\underline{\omega}$, thus reducing the inertia matrix to diagonal form.



Any vector can be transformed from one reference system to another by multiplication by the direction cosine matrix. This matrix contains the projections on the second reference frame of the axis defining the first reference frame.

$$\underline{\omega}' = a\underline{\omega}$$

a represents the direction cosines matrix, that is orthogonal ($a^T=a^{-1}$). Applying the multiplication rule to the expression of the kinetic energy leads to:

$$2T = \left(a^T \underline{\omega'}\right) \cdot I \cdot \left(a^T \underline{\omega'}\right)$$

and, in matrix notation[1]:

$$2T = \underline{\omega}'^T a I a^T \underline{\omega'} = \underline{\omega}'^T I' \underline{\omega'}$$

where I' is the inertia matrix in the new reference frame (a I a^T).
Coming back to the problem of finding a rotation capable of transforming the inertia matrix into diagonal form, we can write this problem as:

$$aIa^T = diag\left(I_x', I_y', I_z'\right)$$

This is the standard eigenvalue problem for matrix I. Once the eigenvalue problem is solved, the rotation matrix a is composed by the eigenvectors associated to the eigenvalues of I, that are I_x, I_y and I_z. Each row of rotation matrix a is one eigenvector. Eigenvectors correspond to the direction cosines of the new reference frame, called principal inertia frame, with respect to the starting reference frame. The principal inertia frame is therefore reference frame with origin in the center of mass of the rigid body and such that all inertia products are equal to zero. Calling I_x, I_y, I_z, the principal inertia moments, the angular momentum and kinetic energy become:

$$\underline{h} = I\underline{\omega} = [I_x\omega_x \quad I_y\omega_y \quad I_z\omega_z]^T$$
$$2\mathrm{T} = \underline{\omega} I \underline{\omega} = I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2$$

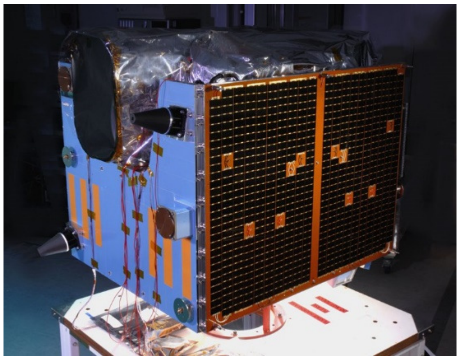
It is pointed out that a plane of symmetry (in terms of mass distribution) includes 2 principal inertia axes, and then the third is uniquely defined as the axis orthogonal to the plane and passing through the center of mass. It is also pointed out that, normally, it is easy to evaluate the inertia matrix in a geometrically meaningful reference. Should this not be a principal axis frame, it can be calculated by solving the eigenvalue problem as described.
The principal inertia matrix includes always the maximum possible and the least possible inertia moment, no other reference can show along the diagonal greater or smaller inertia moments.

Typical values of the principal moments of inertia for a range of spacecraft are given below:

UKube-1 – 3U CubeSat
$I_1 = 0.0109 kgm^2, I_2 = 0.0504 kgm^2, I_3 = 0.055 kgm^2$

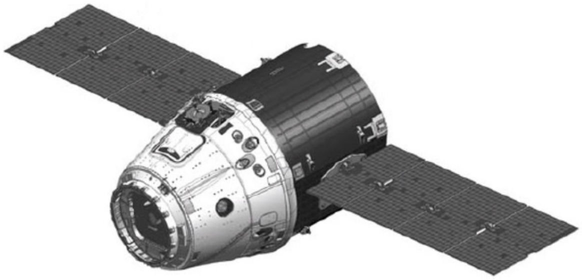
[1] Matrix notation for vector dot product (inner product or scalar product) consists in the following:

$$\underline{v}_1 \cdot \underline{v}_2 = v_1^T v_2$$



Rapid Eye - Micro-spacecraft
$I_1 = 19.5kgm^2, I_2 = 19kgm^2, I_3 = 12.6kgm^2$

Space X 's unmanned 10 tonne spacecraft
$I_1 = 20,000kgm^2, I_2 = 20,000kgm^2, I_3 = 25,000kgm^2$



Or, for example, using a simple cuboid geometry rotating about its centre of mass which is positioned as its centre of geometry then we have:

$$I_d = I_1 = \frac{m}{12}\left(w^2 + h^2\right)$$

$$I_w = I_2 = \frac{m}{12}\left(d^2 + h^2\right)$$

$$I_h = I_3 = \frac{m}{12}\left(w^2 + d^2\right)$$

Inertia ellipsoid, kinetic energy ellipsoid, angular momentum ellipsoid

Assuming that the angular velocity has a constant direction, we can write:

$$\underline{\omega} = \omega\hat{\eta}$$

where η is the unit vector defining the direction of $\underline{\omega}$. The kinetic energy can be written in the form:

$$2T = \underline{\omega} I \underline{\omega} = I_{xx}\omega_x^2 + I_{yy}\omega_y^2 + I_{zz}\omega_z^2 + 2\left(I_{xy}\omega_x\omega_y + I_{yz}\omega_y\omega_z + I_{xz}\omega_x\omega_z\right) = I_\eta\omega_\eta^2$$

The expression of I_η can now be evaluated as:

$$I_\eta = I_{xx}\left(\frac{\omega_x}{\omega}\right)^2 + I_{yy}\left(\frac{\omega_y}{\omega}\right)^2 + I_{zz}\left(\frac{\omega_z}{\omega}\right)^2 + 2\left(I_{xy}\left(\frac{\omega_x}{\omega}\right)\left(\frac{\omega_y}{\omega}\right) + I_{yz}\left(\frac{\omega_y}{\omega}\right)\left(\frac{\omega_z}{\omega}\right) + I_{xz}\left(\frac{\omega_x}{\omega}\right)\left(\frac{\omega_z}{\omega}\right)\right)$$

with:

$$\frac{\omega_x}{\omega} = \frac{\underline{\omega}\,\underline{i}}{\omega} = l_{\eta x}$$
$$\frac{\omega_y}{\omega} = \frac{\underline{\omega}\,\underline{j}}{\omega} = l_{\eta y}$$
$$\frac{\omega_z}{\omega} = \frac{\underline{\omega}\,\underline{k}}{\omega} = l_{\eta z}$$

defined as the direction cosines of unit vector η. In a principal inertia reference system we can write:

$$I_\eta = I_x l_{\eta x}^2 + I_y l_{\eta y}^2 + I_z l_{\eta z}^2$$

that can be transformed into:

$$\frac{l_{\eta x}^2/I_\eta}{1/I_x} + \frac{l_{\eta y}^2/I_\eta}{1/I_y} + \frac{l_{\eta z}^2/I_\eta}{1/I_z} = 1$$

This is an ellipsoid, called inertia ellipsoid, in which the coordinates of each point represent the components of the generic direction η divided by the square root of the inertia moment associated to that direction, while the semi-axes are inversely proportional to the square root of each principal inertia moment.

Semi-axes: $\sqrt{1/I_x}, \sqrt{1/I_y}, \sqrt{1/I_z}$

Point coordinates $l_{\eta x}/\sqrt{I_\eta}\,, l_{\eta y}/\sqrt{I_\eta}\,, l_{\eta z}/\sqrt{I_\eta}$

Representing this ellipsoid in a Cartesian space, in a principal inertia frame, it is possible to calculate the inertia moment associated to any direction. Tracing any direction, the intersection with the ellipsoid occurs at a distance from the origin equal to $1/\sqrt{I_\alpha}$, where $\underline{\alpha}$ is the unit vector of the given direction. In the same way, the intersection of the ellipsoid with the coordinate axes allows to calculate the principal inertia moments.

Now we can write the expression of the kinetic energy as a function of the generic inertia moment in the direction η:

$$2T = I_\eta \omega^2 = I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2$$

This can be written also as:

$$\frac{\omega_x^2}{2T/I_x} + \frac{\omega_y^2}{2T/I_y} + \frac{\omega_z^2}{2T/I_z} = 1$$

that represents another ellipsoid, called kinetic energy ellipsoid, in a space defined by angular velocities. This ellipsoid represents all possible angular velocities compatible with the given kinetic energy T. The intersection of the ellipsoid with any direction is at distance $\sqrt{2T/I_\alpha}$ from the origin. A similar representation holds for the angular momentum:

$$h^2 = \underline{h} \cdot \underline{h} = I_x^2 \omega_x^2 + I_y^2 \omega_y^2 + I_z^2 \omega_z^2$$

that can be written as:

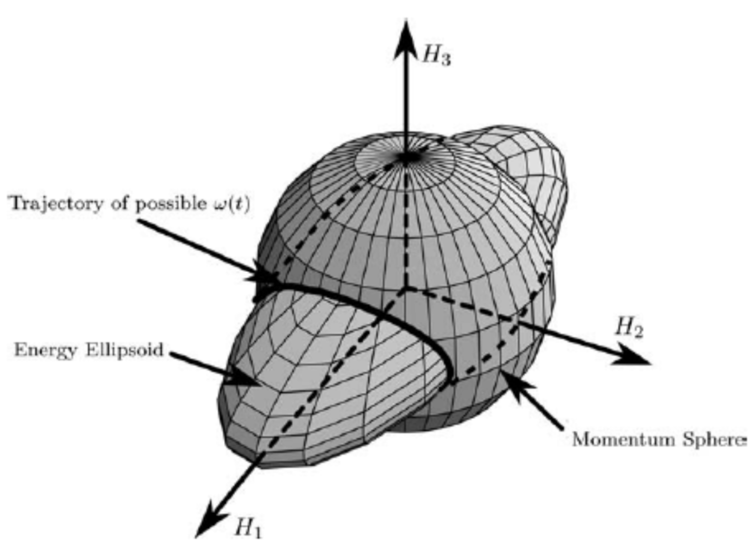
$$\frac{\omega_x^2}{h^2/I_x^2} + \frac{\omega_y^2}{h^2/I_y^2} + \frac{\omega_z^2}{h^2/I_z^2} = 1$$

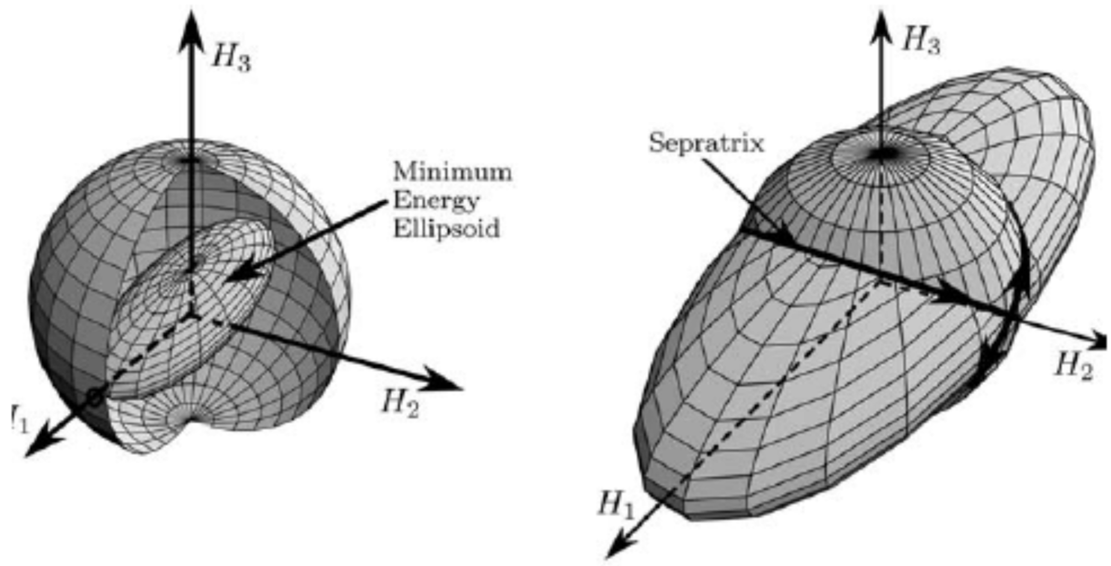
Finally, we have a third ellipsoid, called angular momentum ellipsoid, which represents the compatibility of the angular velocity with the given level of angular momentum.

Geometrical/graphical description of torque-free motion

Polhode and herpolhode

The inertia ellipsoid has no intersection with the kinetic energy ellipsoid, unless the two are coincident, while the angular momentum ellipsoid must intersect the kinetic energy ellipsoid at least in one point in order to represent a real motion. In fact, if no energy dissipation exists and no torque is applied, both kinetic energy and angular momentum must be constant. So, given T and h, angular motion must evolve in such a way that angular velocity vector identifies the intersection of the kinetic energy ellipsoid with the angular momentum ellipsoid. In fact, angular velocity must be compatible with the energy level and with the angular momentum defined by the initial conditions of motion. The intersection of the kinetic energy ellipsoid with the angular momentum ellipsoid generates two lines, called polhodes.





(a) Minimum energy case (b) Intermediate energy case

So far we have looked at the problem in a body frame, not fixed in inertial space. In an inertial frame, we will observe vector $\underline{h}$ as a fixed vector, if no torque is applied. Then, if no dissipation is present, we have:

$$T = \underline{\omega} \cdot \underline{h} = cos\,t$$
$$\underline{h} = cos\,t$$

We can observe that the projection of $\underline{\omega}$ over $\underline{h}$ is constant:

$$\omega_h = \frac{\underline{\omega} \cdot \underline{h}}{|\underline{h}|} = \frac{T}{|\underline{h}|} = cos\,t$$

Now, considering a plane orthogonal to $\underline{h}$, at a distance ω_h from the center of mass, the terminal point of vector $\underline{\omega}$ lies always on this plane due to the conservation of the projection of $\underline{\omega}$ over $\underline{h}$. This plane is called invariant plane. All this is a consequence of conservation of kinetic energy T with time:

$$\frac{d}{dt}(\underline{\omega} \cdot \underline{h}) = \frac{1}{dt}(d\underline{\omega} \cdot \underline{h} + \underline{\omega} \cdot d\underline{h}) = 0$$
$$d\underline{h} = 0$$
$$d\underline{\omega} \cdot \underline{h} = 0 \quad \Rightarrow \quad d\underline{\omega} \perp \underline{h}$$

The line traced by the angular velocity on the invariant plane is in general not closed, it is called herpolhode. The herpolhode is in any case a curve limited in space. In the body axes reference frame the derivative of the angular velocity vector is always tangent to the kinetic energy ellipsoid, while in the inertial space the kinetic energy ellipsoid rolls on the invariant plane as the angular velocity draws the herpolhode.
From the mathematical point of view, the intersection of the two ellipsoids is evaluated as:

$$\frac{\omega_x^2}{2T/I_x}+\frac{\omega_y^2}{2T/I_y}+\frac{\omega_z^2}{2T/I_z}=1$$
$$\frac{\omega_x^2}{h^2/I_x^2}+\frac{\omega_y^2}{h^2/I_y^2}+\frac{\omega_z^2}{h^2/I_z^2}=1$$

Equating the two left hand terms, equal to 1, we find the polhode:

$$\omega_x^2\left[I_x\left(\frac{I_x}{h^2}-\frac{1}{2T}\right)\right]+\omega_y^2\left[I_y\left(\frac{I_y}{h^2}-\frac{1}{2T}\right)\right]+\omega_z^2\left[I_z\left(\frac{I_z}{h^2}-\frac{1}{2T}\right)\right]=0$$

In order to obtain a real solution, the three terms into brackets must show differences in sign. Multiplying each term by h^2 we have three terms like:

$$I-\frac{h^2}{2T}$$

Assuming $I_x > I_y > I_z$ we obtain the conditions for a sign change:

$$I_x>\frac{h^2}{2T} \qquad \text{and} \qquad I_z<\frac{h^2}{2T}$$

It is not necessary to impose conditions on I_y since the change in sign is already guaranteed by the previous conditions. The final constraint is then:

$$I_x>\frac{h^2}{2T}>I_z$$

that guarantees the possibility of having a real rotational motion.

Given the result so far obtained, it is difficult to make any consideration on the polhode as a line in space, while it is interesting to analyze its projections onto the three coordinate planes. We have:

$$\frac{I_x\omega_x^2}{2T}+\frac{I_y\omega_y^2}{2T}+\frac{I_z\omega_z^2}{2T}=1$$
$$I_x\omega_x^2+I_y\omega_y^2+I_z\omega_z^2=2T$$

Looking at the projection onto the plane (x-y), we reduce to two dimensions by eliminating the coordinate ω_z:

$$I_z\omega_z^2=2T-I_x\omega_x^2-I_y\omega_y^2$$

Substitution into the polhode equation gives:

$$I_x\omega_x^2\left(\frac{I_x}{h^2}-\frac{1}{2T}+\frac{1}{2T}-\frac{I_z}{h^2}\right)+I_y\omega_y^2\left(\frac{I_y}{h^2}-\frac{I_z}{h^2}\right)+2T\left(\frac{I_z}{h^2}-\frac{1}{2T}\right)=0$$
$$I_x\omega_x^2\left(\frac{I_x}{h^2}-\frac{I_z}{h^2}\right)+I_y\omega_y^2\left(\frac{I_y}{h^2}-\frac{I_z}{h^2}\right)+2T\frac{I_z}{h^2}-1=0$$

$$\omega_x^2\left(\frac{I_x(I_x-I_z)}{h^2}\right)+\omega_y^2\left(\frac{I_y\left(I_y-I_z\right)}{h^2}\right)+\frac{2TI_z-h^2}{h^2}=0$$
$$\omega_x^2\left(\frac{I_x(I_x-I_z)}{h^2-2TI_z}\right)+\omega_y^2\left(\frac{I_y\left(I_y-I_z\right)}{h^2-2TI_z}\right)=1$$

that represents a planar conic curve. Operating in the same way, we have the projections onto the (y-z) plane:

$$\omega_y^2\left(\frac{I_y\left(I_y-I_x\right)}{h^2-2TI_x}\right)+\omega_z^2\left(\frac{I_z(I_z-I_x)}{h^2-2TI_x}\right)=1$$

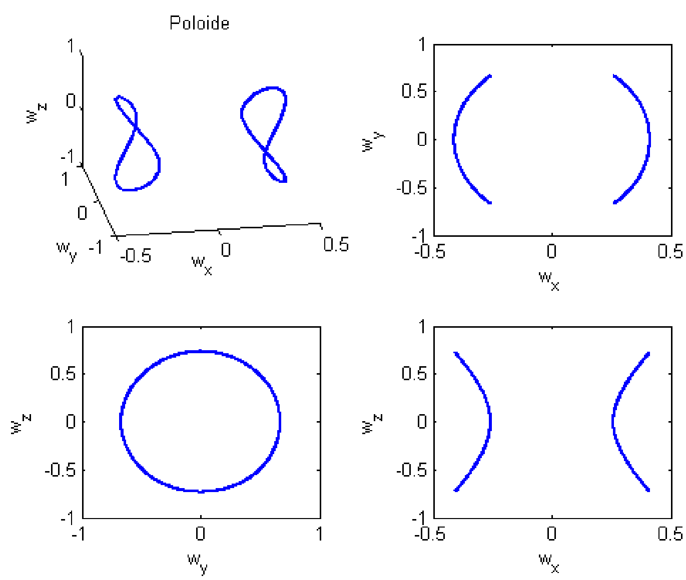
and onto the (x-z) plane:

$$\omega_x^2\left(\frac{I_x\left(I_x-I_y\right)}{h^2-2TI_y}\right)+\omega_z^2\left(\frac{I_z\left(I_z-I_y\right)}{h^2-2TI_y}\right)=1$$

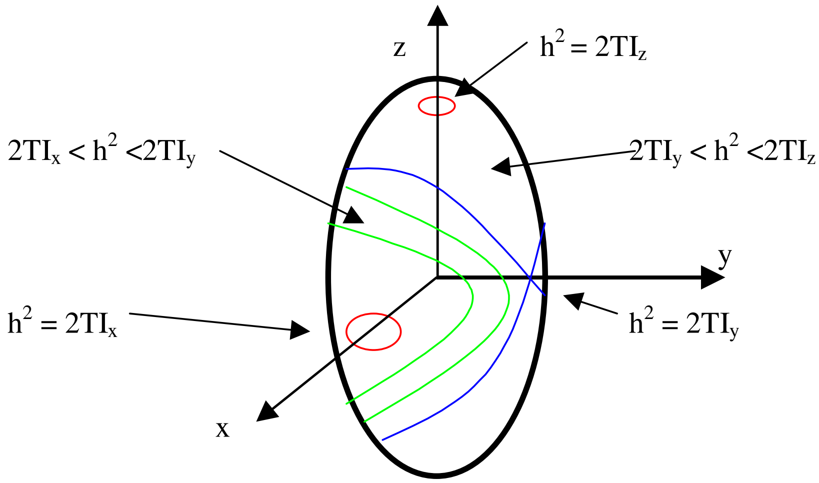
Analyzing the planar conic curves and the signs of all coefficients, we have that:

$$h^2-2TI_z>0$$
$$h^2-2TI_x<0$$
$$h^2-2TI_y?$$

Therefore, we know that the projections on planes (x-y) and (y-z) are ellipses, since coefficients are of the same sign, while projection on plane (x-z) is a hyperbola since the signs of the coefficients are different.



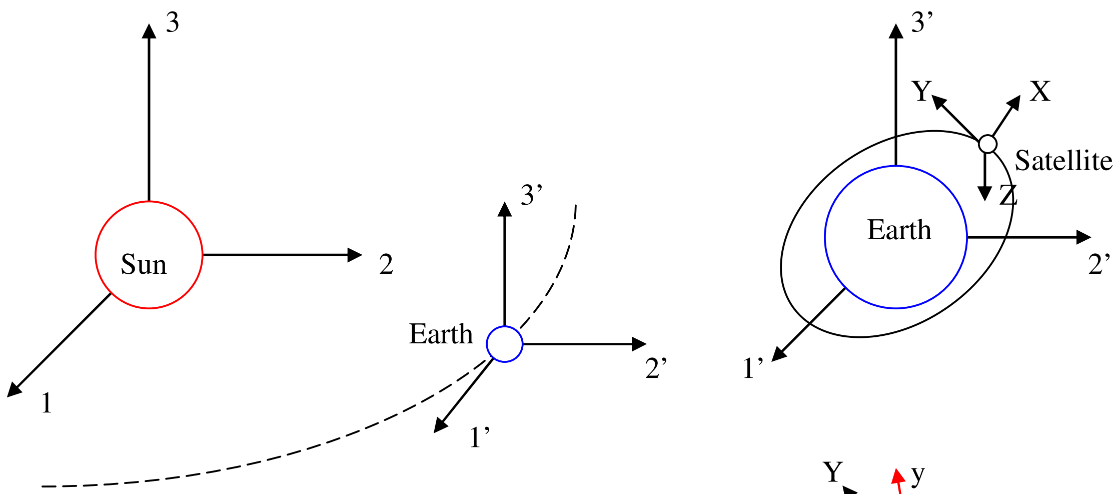
The fact that, looking at the polhode in the (x-z) plane, i.e. looking at it from the y axis, we see a hyperbola can be considered as a sort of instability. In fact, if the angular velocity is slightly shifted from the x-axis or from the z-axis it will remain confined in a region close to the axis, while a slight shift from the y-axis would cause a dramatic departure from the original condition. All this can be seen by looking at the kinetic energy ellipsoid and at the trajectories that are drawn on it by the angular velocity vector. It is possible to divide the ellipsoid into areas corresponding to different initial conditions:

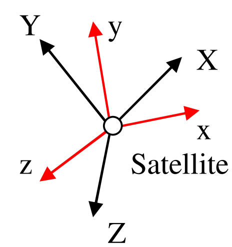


The inequalities hold for a given ellipsoid where the energy level has been fixed.

Attitude parameters

It is always necessary to be able to switch from a reference frame to another reference frame, such as the case of representing the principal inertia frame (body frame) as seen by an inertial frame or by an Earth-centered reference frame or even by a local-vertical-local-horizontal frame.





X is the satellite local vertical direction
Y is the satellite velocity direction
Z is the third direction, orthogonal to X and Y

x,y,z are the satellites' principal inertia axes

To define the orientation of one frame with respect to another, three parameters are the minimal set required. Often redundant parameters are used, i.e. more than three, either in order to improve the physical insight into the transformation or to simplify some numerical analysis.

Direction cosines

The attitude of the rigid-body $A_{B/N}$ is defined as the relative orientation between the body fixed frame **B** and the inertial frame **N** defined explcitly as:

$$\mathbf{B}=A_{B/N}\mathbf{N}$$

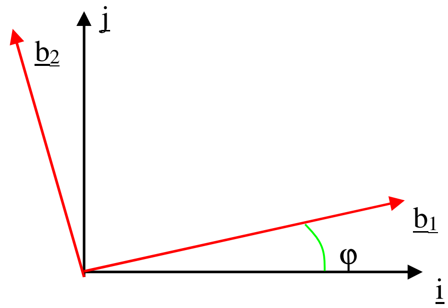
$A_{B/N}$ is an orthonormal matrix with the following properties:

$$A_{B/N}A_{B/N}^{T} = I_{3\times3}, det(A_{B/N}) = 1$$

Each axis of the reference frame that we target is defined in the current reference frame by the three components of its unit direction vector. Calling u, v and w the unit vectors of the new reference frame, assembling the components of the three axes we obtain the direction cosines matrix:

$$A_{B/N} = \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$$

in which each row represents one axis. This matrix allows switching from the representation of a vector in one reference to another reference. As example, let consider a planar rotation:



$$A_{B/N} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The formal expression of the unit vector in the new reference ($\underset{\sim}{a}_B$) is obtained by multiplying the original unit vector ($\underset{\sim}{a}_N$) by the direction cosine matrix A:

$$\underset{\sim}{a}_B = A_{B/N}\underset{\sim}{a}_N$$

For the inverse rotation it is necessary to write the direction cosine matrix of axes 1,2,3 in the reference *u,v,w*; the result is the transpose of matrix A:

$$\underset{\sim}{a}_N = A_{B/N}^T\underset{\sim}{a}_B$$

This transformation does not affect the magnitude of the vectors and the relative angles between vectors. This attitude representation includes 9 parameters, 6 more than the minimal set, but it must be considered that the three axes are mutually orthogonal. So, there must be six constraints among direction cosines, three expressing mutual orthogonality and three expressing the invariability of magnitude of the unit vectors:

$$\begin{cases} |\underline{u}| = u_1^2 + u_2^2 + u_3^2 = 1 \\ |\underline{v}| = v_1^2 + v_2^2 + v_3^2 = 1 \\ |\underline{w}| = w_1^2 + w_2^2 + w_3^2 = 1 \end{cases} \qquad \text{Magnitude}$$

$$\begin{cases} \underline{u} \perp \underline{v} \quad \Rightarrow \quad \underline{u} \cdot \underline{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 = 0 \\ \underline{u} \perp \underline{w} \quad \Rightarrow \quad \underline{u} \cdot \underline{w} = u_1 w_1 + u_2 w_2 + u_3 w_3 = 0 \\ \underline{v} \perp \underline{w} \quad \Rightarrow \quad \underline{v} \cdot \underline{w} = v_1 w_1 + v_2 w_2 + v_3 w_3 = 0 \end{cases} \qquad \text{Orthogonality conditions}$$

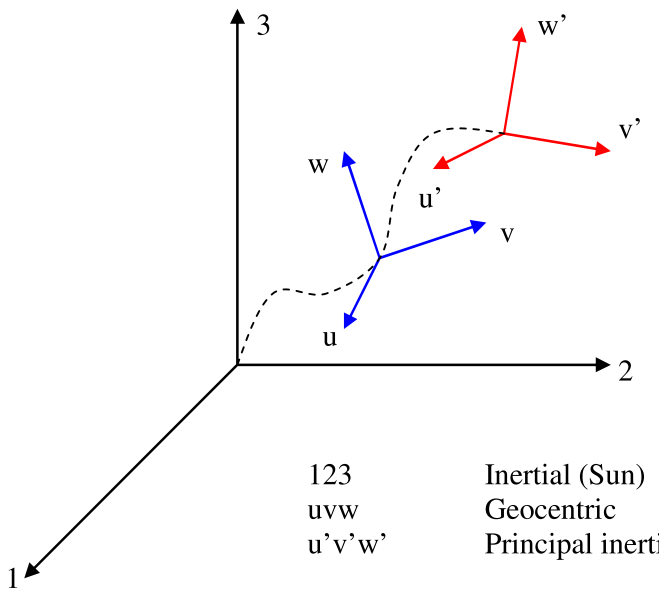
The six constraints among the direction cosines lead to the following property of the direction cosines matrix:

$$AA^T = I$$

Then, A is orthogonal so that the following holds:

$$A^T = A^{-1}$$

We can now see how to treat a sequence of two rotations:



123	Inertial (Sun)
uvw	Geocentric
u'v'w'	Principal inertia

The generic rotation of vector a from the reference 123 into reference uvw is expressed as:

$$a_{uvw} = A \cdot a_{123}$$

In the same way, the second rotation from reference uvw to reference u'v'w' is expressed as:

$$a_{u'v'w'} = A' \cdot a_{uvw}$$

Then, the overall rotation from reference 123 to reference u'v'w' becomes:

$$a_{u'v'w'} = A'' \cdot a_{123} = A' \cdot a_{uvw} = A'A \cdot a_{123}$$

Concluding, in order to compose a sequence of two rotations it is sufficient to build the overall direction cosine matrix as the product of the two individual rotations (direction cosine matrix), taken in the reversed order compared to the sequence of rotations.

$$A'' = A'A$$

Generalizing the concept, lets say we have two frames **L** and **N** and a vector expressed with respect to each frame. Then their orientation is related by:

$$\underset{\sim}{a}_L = A_{L/N} \cdot \underset{\sim}{a}_N$$

A second rotation could be performed in the same way between a frame **L** and a frame **B**:

$$\underset{\sim}{a}_B = A_{B/L} \cdot \underset{\sim}{a}_L$$

So, to rotate a vector expressed in the frame **N** to **B** we can write:

$$\underset{\sim}{a}_B = A_{B/L} A_{L/N} \underset{\sim}{a}_N$$

If we have two different moving frames expressed with respect to the same inertial frame we can write:

$$A_{B/N} = A_{B/L} A_{L/N} \Leftrightarrow A_{B/L} = A_{B/N} A_{L/N}^T$$

The relative rotation between two frames is often called the rotation error such that $A_{B/L} = A_e$, $A_{B/N}$ is the spaceraft body frame with respect to this frame while $A_{L/N} = A_d$ usually corresponds to some desired reference frame. If the spacecraft attitude matches the desired reference frame, then $A_e = I$ which corresponds to a zero error.

Euler axis and angle

From matrix algebra it is known that real, orthogonal matrices have one unit eigenvalue, to which we associate the eigenvector $\underline{e}$:

$$A\underline{e} = \underline{e} \qquad \text{(the eigenvalue is 1)}$$

Therefore, vector $\underline{e}$ does not change due to the rotation represented by matrix A. This is possible only if the rotation occurs around axis $\underline{e}$. This axis is called Euler axis, and the rotation amplitude around this axis is called Euler angle. We can now try to relate the direction cosines matrix A with vector $\underline{e}$. If we consider elementary rotations around each coordinate axis 1, 2 and 3, we can notice that:

$$A_3(\phi) = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \underline{e} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
$$A_2(\phi) = \begin{bmatrix} \cos\phi & 0 & \text{-}\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{bmatrix} \qquad \underline{e} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$
$$A_1(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \qquad \underline{e} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

In each case, the trace of matrix A is the same and equals:

$$tr(A) = 1 + 2\,cos\,\phi$$

Then it is possible to infer the relation between Euler angle and rotation matrix A:

$$cos\,\phi = \frac{1}{2}(tr(A) - 1)$$

Given matrix A it is possible to evaluate $\underline{e}$ and φ with simple operations.

Matrix A has the following form in relation to components of the Euler axis $\underline{e}$ and angle φ:

$$A = \begin{bmatrix} \cos\phi + e_1^2(1-\cos\phi) & e_1e_2(1-\cos\phi) + e_3\sin\phi & e_1e_3(1-\cos\phi) - e_2\sin\phi \\ e_1e_2(1-\cos\phi) - e_3\sin\phi & \cos\phi + e_2^2(1-\cos\phi) & e_2e_3(1-\cos\phi) + e_1\sin\phi \\ e_1e_3(1-\cos\phi) + e_2\sin\phi & e_2e_3(1-\cos\phi) - e_1\sin\phi & \cos\phi + e_3^2(1-\cos\phi) \end{bmatrix}$$

or:

$$A = I\cos\phi + (1-\cos\,\phi)\underline{ee}^T - \sin\phi[e\wedge]$$

where:

$$[e\wedge] = \begin{bmatrix} 0 & -e_3 & e_2 \\ e_3 & 0 & -e_1 \\ -e_2 & e_1 & 0 \end{bmatrix}$$

represents cross product matrix operator.

If we know the attitude parameters expressed as Euler axis and angle, it is therefore straightforward to evaluate the direction cosine matrix A. The inverse operation is also possible, given the rotation matrix A evaluate Euler axis and angle, in fact:

$$\phi = cos^{-1}\left[\frac{1}{2}(tr(A)-1)\right]$$

$$\begin{cases} e_1 = \dfrac{(A_{23}-A_{32})}{2sin\phi} \\ e_2 = \dfrac{(A_{31}-A_{13})}{2sin\phi} \\ e_3 = \dfrac{(A_{12}-A_{21})}{2sin\phi} \end{cases}$$

It is remarked that the direct transformation, from Euler axis/angle to rotation matrix A, is always possible, while the inverse transformation, from rotation matrix A to Euler axis/angle, is not always defined. In fact, when $\sin\varphi$ becomes zero the Euler axis is undetermined. This occurs when $\varphi = n\,\pi$, that physically means that the axis is not uniquely determined, i.e., it is possible to reach the final configuration through at least two rotations of same amplitude but different axis. When the inverse transformation is not determined, the condition is known as a singularity in the attitude parameterization.

It is also remarked that the inverse trigonometric function arcos returns two possible values of the Euler angle ($\pm\phi$) that in turn provide two different axes ($\pm\underline{e}$). This is actually different from the singularity condition, since we are always looking at the same axis, once from the positive and once from the negative direction, and to each condition we associate the same rotation angle with opposite sign. This is physically the same condition.

Euler axis/angle parameters are useful since they are only 4 parameters (the constraint condition is associated to the fact that $\underline{e}$ is a unit vector), but unfortunately it is not possible to infer a relation for sequence of rotations.

Quaternion

A quaternion is a vector of four parameters, linked to the Euler axis/angle parameters through the relation:

$$\begin{cases} q_1 = e_1 sin\dfrac{\phi}{2} \\ q_2 = e_2 sin\dfrac{\phi}{2} \\ q_3 = e_3 sin\dfrac{\phi}{2} \\ q_4 = cos\dfrac{\phi}{2} \end{cases}$$

It can be noticed that the quaternion is normalized:

$$q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1$$

In some cases it is convenient to divide the quaternion into a vector part $\underline{q}$ and a scalar part q_4:

$$\underline{q} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} \quad , \quad q_4$$

The direct transformation, given the quaternion evaluate the direction cosine matrix A, leads to:

$$A = \begin{bmatrix} q_1^2 - q_2^2 - q_3^2 + q_4^2 & 2(q_1q_2 + q_3q_4) & 2(q_1q_3 - q_2q_4) \\ 2(q_1q_2 - q_3q_4) & -q_1^2 + q_2^2 - q_3^2 + q_4^2 & 2(q_2q_3 + q_1q_4) \\ 2(q_1q_3 + q_2q_4) & 2(q_2q_3 - q_1q_4) & -q_1^2 - q_2^2 + q_3^2 + q_4^2 \end{bmatrix}$$

or:

$$A = \left(q_4^2 - \underline{q}^T\underline{q}\right) I + 2\underline{qq}^T - 2\mathrm{q}_4[q \wedge]$$

where:

$$[q \wedge] = \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix}$$

The inverse transformation, from matrix A to quaternion, is represented as:

$$\begin{cases} q_1 = \dfrac{1}{4q_4}(A_{23} - A_{32}) \\ q_2 = \dfrac{1}{4q_4}(A_{31} - A_{13}) \\ q_3 = \dfrac{1}{4q_4}(A_{12} - A_{21}) \\ q_4 = \pm\dfrac{1}{2}(1 + A_{11} + A_{22} + A_{33})^{\frac{1}{2}} \end{cases}$$

The sign ambiguity has the same explanation given for the Euler axis/angle case. We are always looking at the same axis, once from the positive and once from the negative direction, and to each condition we associate the same rotation angle with opposite sign. This is physically the same condition. Quaternions have no singular condition even in the inverse transformation. In fact, should q_4 become zero it is always possible to evaluate one of the other components of the quaternion, which will be surely different from zero due to the normalization constraint, and then evaluate the remaining three components. The inverse transformation has then the following three further sets of alternative equations:

$$
\begin{aligned}
q_1^2 &= \pm\frac{1}{2}\sqrt{1 + A_{11} - A_{22} - A_{33}} \\
q_2^2 &= \frac{1}{4\mathrm{q}_1^2}(A_{12} + A_{21}) \\
q_3^2 &= \frac{1}{4\mathrm{q}_1^2}(A_{13} + A_{31}) \\
q_4^2 &= \frac{1}{4\mathrm{q}_1^2}(A_{23} - A_{32})
\end{aligned}
$$

$$
\begin{aligned}
q_2^3 &= \pm\frac{1}{2}\sqrt{1 - A_{11} + A_{22} - A_{33}} \\
q_1^3 &= \frac{1}{4\mathrm{q}_2^3}(A_{12} + A_{21}) \\
q_3^3 &= \frac{1}{4\mathrm{q}_2^3}(A_{23} + A_{32}) \\
q_4^3 &= \frac{1}{4\mathrm{q}_2^3}(A_{31} - A_{13})
\end{aligned}
$$

$$
\begin{aligned}
q_3^4 &= \pm\frac{1}{2}\sqrt{1 - A_{11} - A_{22} + A_{33}} \\
q_1^4 &= \frac{1}{4\mathrm{q}_3^4}(A_{13} + A_{31}) \\
q_2^4 &= \frac{1}{4\mathrm{q}_3^4}(A_{23} + A_{32}) \\
q_4^4 &= \frac{1}{4\mathrm{q}_3^4}(A_{12} - A_{21})
\end{aligned}
$$

It is even possible to select the set of equations to adopt on the basis of the maximization of the square root term, for a reduced truncation error in the following operations. Unfortunately, quaternions have no physical meaning and therefore their use is not intuitive.

Adopting quaternions, it is possible to express a sequence of two consecutive rotations, combining quaternion components in the reversed order of rotations. Given a first rotation represented by quaternion q, a second rotation represented by quaternion q', the overall rotation is represented by quaternion q" computed as $q^{''} = q \otimes q^{'}$:

$$
q" = \begin{bmatrix} q_4^{'} & q_3^{'} & -q_2^{'} & q_1^{'} \\ -q_3^{'} & q_4^{'} & q_1^{'} & q_2^{'} \\ q_2^{'} & -q_1^{'} & q_4^{'} & q_3^{'} \\ -q_1^{'} & -q_2^{'} & -q_3^{'} & q_4^{'} \end{bmatrix} q
$$

Recalled the relative attitude error $A_e = A_{B/N}A_d^T$ this is equivalent to $\hat{q}_e = \hat{q}_d^T \otimes \hat{q}_{B/N}$ which is explicitly:

$$\begin{bmatrix} q_{1e} \\ q_{2e} \\ q_{3e} \\ q_{4e} \end{bmatrix} = \begin{bmatrix} q_{4c} & q_{3c} & -q_{2c} & -q_{1c} \\ -q_{3c} & q_{4c} & q_{1c} & -q_{2c} \\ q_{2c} & -q_{1c} & q_{4c} & -q_{3c} \\ q_{1c} & q_{2c} & q_{3c} & q_{4c} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}$$

Gibbs vector

Gibbs vector is a minimal attitude parameterization, since it includes only three parameters. It can also be seen as a non-normalized quaternion, and can be inferred from Euler axis/angle:

$$\begin{cases} g_1 = \dfrac{q_1}{q_4} = e_1 \, tan \dfrac{\phi}{2} \\ g_2 = \dfrac{q_2}{q_4} = e_2 \, tan \dfrac{\phi}{2} \\ g_3 = \dfrac{q_3}{q_4} = e_3 \, tan \dfrac{\phi}{2} \end{cases}$$

Also in this case it is possible to represent the direct transformation from Gibbs vector to rotation matrix A:

$$A = \frac{1}{1 + g_1^2 + g_2^2 + g_3^2} \begin{bmatrix} 1 + g_1^2 - g_2^2 - g_3^2 & 2(g_1 g_2 + g_3) & 2(g_1 g_3 - g_2) \\ 2(g_1 g_2 - g_3) & 1 - g_1^2 + g_2^2 - g_3^2 & 2(g_2 g_3 + g_1) \\ 2(g_1 g_3 + g_2) & 2(g_2 g_3 - g_1) & 1 - g_1^2 - g_2^2 + g_3^2 \end{bmatrix}$$

or:

$$A = \frac{\left(1 - \underline{g}^2\right) I + 2 \underline{g} \underline{g}^T - 2[g \wedge]}{\left(1 + \underline{g}^2\right)}$$

where:

$$[g \wedge] = \begin{bmatrix} 0 & -g_3 & g_2 \\ g_3 & 0 & -g_1 \\ -g_2 & g_1 & 0 \end{bmatrix}$$

The inverse transformation is given by:

$$\begin{cases} g_1 = \dfrac{A_{23} - A_{32}}{1 + A_{11} + A_{22} + A_{33}} \\ g_2 = \dfrac{A_{31} - A_{13}}{1 + A_{11} + A_{22} + A_{33}} \\ g_3 = \dfrac{A_{12} - A_{21}}{1 + A_{11} + A_{22} + A_{33}} \end{cases}$$

The inverse transformation becomes singular when $\varphi = (2n + 1)\,\pi$.

Two consecutive rotations (first rotation expressed by vector g, second rotation represented by vector g’) can be merged into a single Gibbs vector g” with the following rule:

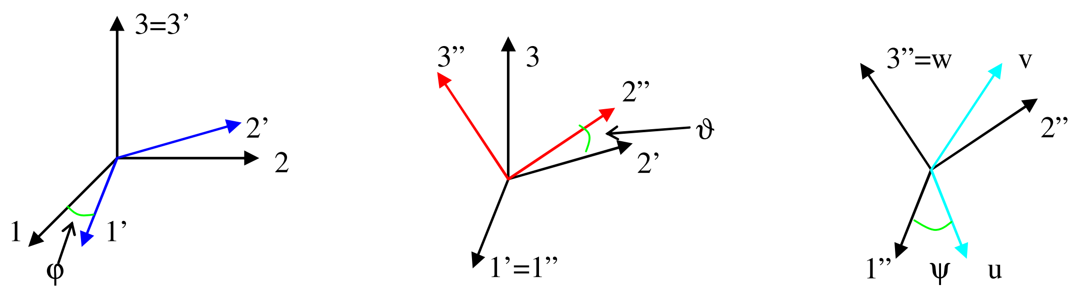
$$g'' = \frac{\underline{g} + \underline{g}' - \underline{g}' \wedge \underline{g}}{1 - \underline{g} \cdot \underline{g}'}$$

Also this parameterization has a poor physical meaning and furthermore it has singularity configuration in the inverse transformation, therefore part of the advantages of having only three parameters are lost.

Euler angles

The attitude can be represented by means of three Euler angles, which are a minimal representation and have a clear physical interpretation. The concept is based on the fact that it is always possible to make two orthogonal frames overlap by appropriate rotation of one of them three times around its reference axes.

Let us make an example to overlap systems 1,2,3 and u,v,w.
We start by rotating by angle φ around axis 3, that brings the triad in the new configuration 1',2',3'.
Now rotate by angle ϑ around axis 1', that brings the triad in the configuration 1",2",3".
Finally, rotate by angle ψ around axis 3" to superimpose the triad to the reference u,v,w.



The direction cosine matrix can be written by adopting the rule of consecutive rotations. In fact, it is possible to combine the elementary rotation matrices since each rotation is around one of the coordinate axes:

$$A_{313}(\phi, \vartheta, \psi) = A_3(\psi) \cdot A_1(\vartheta) \cdot A_3(\phi)$$

where:

$$A_3(\psi) = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$A_1(\vartheta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\vartheta & \sin\vartheta \\ 0 & -\sin\vartheta & \cos\vartheta \end{bmatrix}$$
$$A_3(\phi) = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The labels 1,2,3 of the rotation matrices do not indicate in this case the axis of rotation, but rather the rotation type, i.e., the family of axes around which the triad is rotating. The overall rotation matrix is labeled according to the rotation sequence and the rotation angles associated to each individual rotation:

$$A_{313}(\phi,\vartheta,\psi)$$

In this case it is fundamental to remind that matrix products must be made in the reversed order as compared to the rotations. It is also possible to superimpose the two triads by a different set of rotation axes, and in this case the values of the rotation angles will also change radically.
This is true even if only the last rotation is changed, all angles will change their magnitude. Matrix A_{313} is then written as:

$$A_{313}=\begin{bmatrix} \cos\psi\cos\phi-\sin\psi\sin\phi\cos\vartheta & \cos\psi\sin\phi+\sin\psi\cos\phi\cos\vartheta & \sin\psi\sin\vartheta \\ -\sin\psi\cos\phi-\cos\psi\sin\phi\cos\vartheta & -\sin\psi\sin\phi+\cos\psi\cos\phi\cos\vartheta & \cos\psi\sin\vartheta \\ \sin\phi\sin\vartheta & -\cos\phi\sin\vartheta & \cos\vartheta \end{bmatrix}$$

It can be noticed that the inverse transform, from matrix A to Euler angles, in this case takes the form:

$$\begin{cases} \vartheta = \cos^{-1}(A_{33}) \\ \phi = -\tan^{-1}\left(\dfrac{A_{31}}{A_{32}}\right) \\ \psi = \tan^{-1}\left(\dfrac{A_{13}}{A_{23}}\right) \end{cases}$$

In this case we notice a singularity when $\sin\vartheta$ tends to zero. This means that the first and last rotations are actually around the same physical direction in space, therefore it is not possible to distinguish them individually, but rather only the sum or difference of the two.

With Euler angles we have a different rotation matrix A for each combination of rotation axes. Two consecutive rotations cannot be around the same axis, they would become one single rotation, therefore the possible sequences are 12, out of which 6 have all indexes different and 6 are characterized by the same first and third index:

312,213,123,321,231,132	all different indexes
313,323,212,232,131,121	first and third index equal

The different sets of Euler angles are singular for different values of the second rotation angle ϑ, depending on the fact that the indexes are all different or the first and third index coincide. When all indexes are different, the singularity condition is $\vartheta = (2n+1)\pi/2$, while for equal first and third index the singularity occurs when $\vartheta = n\pi$.
Here follow the symbolic expressions of the direction cosine matrix for the 12 possible sequences, assuming that the first rotation is always represented by angle φ, the second by angle ϑ and the third by angle ψ.

$$A_{123}=\begin{bmatrix} \cos\psi\cos\vartheta & \cos\psi\sin\vartheta\sin\phi+\sin\psi\cos\phi & -\cos\psi\sin\vartheta\cos\phi+\sin\psi\sin\phi \\ -\sin\psi\cos\vartheta & -\sin\psi\sin\vartheta\sin\phi+\cos\psi\cos\phi & \sin\psi\sin\vartheta\cos\phi+\cos\psi\sin\phi \\ \sin\vartheta & -\cos\vartheta\sin\phi & \cos\vartheta\cos\phi \end{bmatrix}$$

$$A_{132}=\begin{bmatrix} \cos\psi\cos\vartheta & \cos\psi\sin\vartheta\cos\phi+\sin\psi\sin\phi & \cos\psi\sin\vartheta\sin\phi-\sin\psi\cos\phi \\ -\sin\vartheta & \cos\phi\cos\vartheta & \cos\vartheta\sin\phi \\ \sin\psi\cos\vartheta & \sin\psi\sin\vartheta\cos\phi-\cos\psi\sin\phi & \sin\psi\sin\vartheta\sin\phi+\cos\psi\cos\phi \end{bmatrix}$$

$$A_{231} = \begin{bmatrix} \cos\vartheta\cos\phi & \sin\vartheta & -\cos\vartheta\sin\phi \\ -\cos\psi\sin\vartheta\cos\phi + \sin\psi\sin\phi & \cos\psi\cos\vartheta & \cos\psi\sin\vartheta\sin\phi + \sin\psi\cos\phi \\ \sin\psi\sin\vartheta\cos\phi + \cos\psi\sin\phi & -\sin\psi\cos\vartheta & -\sin\psi\sin\vartheta\sin\phi + \cos\psi\cos\phi \end{bmatrix}$$

$$A_{213} = \begin{bmatrix} \cos\psi\cos\phi + \sin\psi\sin\phi\sin\vartheta & \sin\psi\cos\vartheta & -\cos\psi\sin\phi + \sin\psi\cos\phi\sin\vartheta \\ -\sin\psi\cos\phi + \cos\psi\sin\phi\sin\vartheta & \cos\psi\cos\vartheta & \sin\psi\sin\phi + \cos\psi\cos\phi\sin\vartheta \\ \sin\phi\cos\vartheta & -\sin\vartheta & \cos\phi\cos\vartheta \end{bmatrix}$$

$$A_{312} = \begin{bmatrix} \cos\psi\cos\phi - \sin\psi\sin\phi\sin\vartheta & \cos\psi\sin\phi + \sin\psi\cos\phi\sin\vartheta & -\sin\psi\cos\vartheta \\ -\sin\phi\cos\vartheta & \cos\phi\cos\vartheta & \sin\vartheta \\ \sin\psi\cos\phi + \cos\psi\sin\phi\sin\vartheta & \sin\psi\sin\phi - \cos\psi\cos\phi\sin\vartheta & \cos\vartheta\cos\psi \end{bmatrix}$$

$$A_{321} = \begin{bmatrix} \cos\phi\cos\vartheta & \cos\vartheta\sin\phi & -\sin\vartheta \\ -\cos\psi\sin\phi + \sin\psi\cos\phi\sin\vartheta & \cos\psi\cos\phi + \sin\psi\sin\phi\sin\vartheta & \sin\psi\cos\vartheta \\ \sin\psi\sin\phi + \cos\psi\cos\phi\sin\vartheta & -\sin\psi\cos\phi + \cos\psi\sin\phi\sin\vartheta & \cos\psi\cos\vartheta \end{bmatrix}$$

$$A_{121} = \begin{bmatrix} \cos\vartheta & \sin\phi\sin\vartheta & -\cos\phi\sin\vartheta \\ \sin\psi\sin\vartheta & \cos\psi\cos\phi - \sin\psi\sin\phi\cos\vartheta & \cos\psi\sin\phi + \sin\psi\cos\phi\cos\vartheta \\ \cos\psi\sin\vartheta & -\sin\psi\cos\phi - \cos\psi\sin\phi\cos\vartheta & -\sin\psi\sin\phi + \cos\psi\cos\phi\cos\vartheta \end{bmatrix}$$

$$A_{131} = \begin{bmatrix} \cos\vartheta & \cos\phi\sin\vartheta & \sin\phi\sin\vartheta \\ -\cos\psi\sin\vartheta & -\sin\psi\sin\phi + \cos\psi\cos\phi\cos\vartheta & \sin\psi\cos\phi + \cos\psi\sin\phi\cos\vartheta \\ \sin\psi\sin\vartheta & -\cos\psi\sin\phi - \sin\psi\cos\phi\cos\vartheta & \cos\psi\cos\phi - \sin\psi\sin\phi\cos\vartheta \end{bmatrix}$$

$$A_{212} = \begin{bmatrix} \cos\psi\cos\phi - \sin\psi\sin\phi\cos\vartheta & \sin\psi\sin\vartheta & -\cos\psi\sin\phi - \sin\psi\cos\phi\cos\vartheta \\ \sin\phi\sin\vartheta & \cos\vartheta & \cos\phi\sin\vartheta \\ \sin\psi\cos\phi + \cos\psi\sin\phi\cos\vartheta & -\cos\psi\sin\vartheta & -\sin\psi\sin\phi + \cos\psi\cos\phi\cos\vartheta \end{bmatrix}$$

$$A_{232} = \begin{bmatrix} -\sin\psi\sin\phi + \cos\psi\cos\phi\cos\vartheta & \cos\psi\sin\vartheta & -\sin\psi\cos\phi - \cos\psi\sin\phi\cos\vartheta \\ -\cos\phi\sin\vartheta & \cos\vartheta & \sin\phi\sin\vartheta \\ \cos\psi\sin\phi + \sin\psi\cos\phi\cos\vartheta & \sin\psi\sin\vartheta & \cos\psi\cos\phi - \sin\psi\sin\phi\cos\vartheta \end{bmatrix}$$

$$A_{313} = \begin{bmatrix} \cos\psi\cos\phi - \sin\psi\sin\phi\cos\vartheta & \cos\psi\sin\phi + \sin\psi\cos\phi\cos\vartheta & \sin\psi\sin\vartheta \\ -\sin\psi\cos\phi - \cos\psi\sin\phi\cos\vartheta & -\sin\psi\sin\phi + \cos\psi\cos\phi\cos\vartheta & \cos\psi\sin\vartheta \\ \sin\phi\sin\vartheta & -\cos\phi\sin\vartheta & \cos\vartheta \end{bmatrix}$$

$$A_{323} = \begin{bmatrix} -\sin\psi\sin\phi + \cos\psi\cos\phi\cos\vartheta & \sin\psi\cos\phi + \cos\psi\sin\phi\cos\vartheta & -\cos\psi\sin\vartheta \\ -\cos\psi\sin\phi - \sin\psi\cos\phi\cos\vartheta & \cos\psi\cos\phi - \sin\psi\sin\phi\cos\vartheta & \sin\psi\sin\vartheta \\ \cos\phi\sin\vartheta & \sin\phi\sin\vartheta & \cos\vartheta \end{bmatrix}$$

If angles are small, i.e., up to 10°- 15°, we can assume cos x =1, sin x = x, x*x =0 (with x in radians), therefore the dependence on the order of rotations is lost and we can write:

$$A_{312}(\phi,\vartheta,\psi) = \begin{bmatrix} 1 & \phi & -\psi \\ -\phi & 1 & \vartheta \\ \psi & -\vartheta & 1 \end{bmatrix} = A_{321}(\phi,\psi,\vartheta) = A_{213}(\psi,\vartheta,\phi) = \ldots.$$

It is reminded that in any case the correspondence between axis and angle must be kept. Matrix A can also be written as:

$$A = I - [angles\ \wedge]$$

If the first and last indexes are coincident, due to loss of importance of the order of rotations, we are always in the condition that the first and last rotations are almost around the same physical direction in space, so it is like having only two rotations:

$$A_{313}(\phi,\vartheta,\psi) = \begin{bmatrix} 1 & \phi+\psi & 0 \\ -\phi-\psi & 1 & \vartheta \\ 0 & -\vartheta & 1 \end{bmatrix}$$

For small angles, quaternion can be simplified and directly connected to Euler angles when the three indexes are different. For the sequence 312 we can write:

$$\begin{cases} q_1 = \frac{1}{2}\vartheta \\ q_2 = \frac{1}{2}\psi \\ q_3 = \frac{1}{2}\phi \\ q_4 = 1 \end{cases}$$

Attitude parameterization adopting Euler angles has a clear physical meaning but is numerically time consuming. Moreover, the rule of consecutive rotations, although existing, is really hard to implement due to its complexity, therefore it will not be illustrated.

Attitude kinematics

Direction cosines

We now want to define how attitude parameters change with time, and how they are related to the angular velocity expressed in body frame.
We begin by considering the direction cosines matrix:

$$\frac{dA}{dt} = f(\omega_u, \omega_v, \omega_w)$$

If matrix A is known at time t, then it is possible to evaluate the matrix at time t+Δt adopting the rule of consecutive rotations:

$$A(t + \Delta t) = A' A(t)$$

where A' represents the rotation in the interval Δt. Express matrix A' as a function of Euler axis and angle in the interval Δt:

$$A' = I \cos\phi + (1 - \cos\phi)\underline{e}\underline{e}^T - \sin\phi[\underline{e}\wedge]$$

To define the derivative we must consider short time intervals, therefore φ will be small:

$$A' = I - \phi[\underline{e}\wedge]$$

Euler axis for small rotations will be coincident with the angular velocity vector direction, so that:

$$\phi = \omega\Delta t \qquad \underline{\omega} = \omega\underline{e}$$

$$\phi[\underline{e}\wedge] = \omega\Delta t \begin{bmatrix} 0 & -e_w & e_v \\ e_w & 0 & -e_u \\ -e_v & e_u & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\omega_w & \omega_v \\ \omega_w & 0 & -\omega_u \\ -\omega_v & \omega_u & 0 \end{bmatrix} \Delta t = [\underline{\omega}\wedge]\Delta t$$

Therefore we can write:

$$A' = I - [\underline{\omega}\wedge]\Delta t$$
$$A(t + \Delta t) = A(t) - \Delta t[\underline{\omega}\wedge]A(t)$$

Applying the definition of derivative, we have:

$$\frac{dA}{dt} = \lim_{\Delta t \to 0} \frac{A(t + \Delta t) - A(t)}{\Delta t} = \lim_{\Delta t \to 0} -\frac{\Delta t[\underline{\omega}\wedge]A(t)}{\Delta t} = -[\underline{\omega}\wedge]A(t)$$

Then, knowing ω we can integrate dA/dt in order to update matrix A. Since integration is done numerically, after some steps matrix A will no longer be orthogonal. It is therefore necessary to orthogonalize matrix A after a certain number of integration steps.

This operation is long and complex, if done exactly, so that approximate solutions to the orthogonalization are normally used. In this way, approximate solutions can be evaluated with a higher frequency (at every integration step) since the numerical process is fast enough. It can be demonstrated that, starting from an initial guess $A_0(t)$ of the direction cosines matrix, not orthogonal due to numerical errors, the recursive formula

$$A_{k+1}(t)= A_k(t)*3/2- A_k(t)*A_k^T(t)*A_k(t)/2$$

converges rapidly, with increasing k, to the exact value of A. In a first order approximation it is possible to adopt a single step iteration

$$A(t)= A_0(t)*3/2- A_0(t)*A_0^T(t)*A_0(t)/2$$

The kinematic equation of the form $\dot{A} = -[\underline{\omega} \wedge]A$ has the following properties

$$A[\underline{\omega} \wedge]A^T = [A\underline{\omega} \wedge]$$
$$\underline{\omega} \wedge \underline{x} = [\underline{\omega} \wedge]\underline{x}$$
$$tr([\underline{\omega} \wedge]A) = -\underline{\omega}^T(A - A^T)^V$$

where ^{V} denotes the inverse cross-product map, i.e., it extracts from a matrix the off-diagonal elements.

Quaternions

The procedure adopted to derive the kinematic relation for direction cosines can now be repeated to analyze the quaternion kinematics. We can assume that quaternion at time t + Δt is given by the composition of quaternion at time t and a quaternion representing the rotation in the interval Δt:

$$q(t + \Delta t) = \begin{bmatrix} q_4' & q_3' & -q_2' & q_1' \\ -q_3' & q_4' & q_1' & q_2' \\ q_2' & -q_1' & q_4' & q_3' \\ -q_1' & -q_2' & -q_3' & q_4' \end{bmatrix} q(t) \qquad \text{with} \qquad \begin{cases} q_1' = e_u \sin\dfrac{\phi}{2} \\ q_2' = e_v \sin\dfrac{\phi}{2} \\ q_3' = e_w \sin\dfrac{\phi}{2} \\ q_4' = \cos\dfrac{\phi}{2} \end{cases}$$

Then:

$$q(t + \Delta t) = \left\{ I\cos\frac{\phi}{2} + \begin{bmatrix} 0 & e_w & -e_v & e_u \\ -e_w & 0 & e_u & e_v \\ e_v & -e_u & 0 & e_w \\ -e_u & -e_v & -e_w & 0 \end{bmatrix} \sin\frac{\phi}{2} \right\} q(t)$$

For short intervals Δt we can approximate:

$$\phi = \omega\Delta t \qquad \cos\frac{\phi}{2} = 1 \qquad \sin\frac{\phi}{2} = \frac{\phi}{2} = \frac{\omega\Delta t}{2}$$

and evaluate e_u, e_v, e_w, as a function of $\underline{\omega} = \omega\, \underline{e}$ and therefore write:

$$q(t + \Delta t) = \left[I + \frac{1}{2}\Omega\Delta t\right] q(t)$$

where:

$$\Omega = \begin{bmatrix} 0 & \omega_w & -\omega_v & \omega_u \\ -\omega_w & 0 & \omega_u & \omega_v \\ \omega_v & -\omega_u & 0 & \omega_w \\ -\omega_u & -\omega_v & -\omega_w & 0 \end{bmatrix}$$

Taking the limit for t→0:

$$\frac{dq}{dt} = \lim_{\Delta t \to 0} \frac{q(t + \Delta t) - q(t)}{\Delta t} = \frac{1}{2}\Omega q(t)$$

An alternative way to derive the quaternion kinematics is to substitute

$$A = \begin{bmatrix} q_1^2 - q_2^2 - q_3^2 + q_4^2 & 2(q_1q_2 + q_3q_4) & 2(q_1q_3 - q_2q_4) \\ 2(q_1q_2 - q_3q_4) & -q_1^2 + q_2^2 - q_3^2 + q_4^2 & 2(q_2q_3 + q_1q_4) \\ 2(q_1q_3 + q_2q_4) & 2(q_2q_3 - q_1q_4) & -q_1^2 - q_2^2 + q_3^2 + q_4^2 \end{bmatrix}$$

into

$$\frac{dA}{dt}A^T = -[\underline{\omega} \wedge]$$

Which yields

$$\begin{aligned} \omega_1 &= 2(\dot{q}_1q_4 + \dot{q}_2q_3 - \dot{q}_3q_2 - \dot{q}_4q_1) \\ \omega_2 &= 2(\dot{q}_2q_4 + \dot{q}_3q_1 - \dot{q}_1q_3 - \dot{q}_4q_2) \\ \omega_3 &= 2(\dot{q}_3q_4 + \dot{q}_1q_2 - \dot{q}_2q_1 - \dot{q}_4q_3) \end{aligned}$$

Also, differentiating the constraint, we have

$$0 = 2(\dot{q}_1q_1 + \dot{q}_2q_2 + \dot{q}_3q_3 + \dot{q}_4q_4)$$

And rearranging gives

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix} = \frac{1}{2}\begin{pmatrix} 0 & \omega_3 & -\omega_2 & \omega_1 \\ -\omega_3 & 0 & \omega_1 & \omega_2 \\ \omega_2 & -\omega_1 & 0 & \omega_3 \\ -\omega_1 & -\omega_2 & -\omega_3 & 0 \end{pmatrix}\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}$$

Therefore, also in this case, knowing the angular velocity we can integrate the derivative of the quaternion to propagate attitude. As for the direction cosines, there is a problem of normalization, but since q is a vector its normalization is trivial.

Gibbs vector

It is not possible to evaluate the derivative of the Euler axis/angle parameterization, due to the lack of rule for consecutive rotations. For the Gibbs vector this can instead be evaluated, with the usual procedure already adopted for quaternion and direction cosines:

$$g(t+\Delta t) = \frac{\underline{g}(t) + \underline{g}' - \underline{g}' \wedge \underline{g}(t)}{1 - \underline{g}(t) \cdot \underline{g}'}$$

where:

$$\underline{g}' = \underline{e}\tan\frac{\phi}{2} = \xrightarrow{\text{small } \Delta t} = \frac{1}{2}\underline{\omega}\Delta t$$

The derivative is then:

$$\frac{dg}{dt} = \frac{1}{2}\left[\underline{\omega} - \underline{\omega} \wedge \underline{g}(t) + \left(\underline{g}(t) \cdot \underline{\omega}\right)\underline{g}(t)\right]$$

Euler angles

Euler angles have no convenient rule for combining two consecutive rotations, however it is possible to calculate the derivatives of the angles for each sequence of rotations. Let consider the sequence 313 (φ,ϑ,ψ):

If we have variation of angle φ alone, then rotation will be about axis 3, so that we can write:

$$\underline{\omega} = \dot{\phi}\underline{3}$$

If we have variation of ϑ alone, then rotation will be about axis 1’, so that:

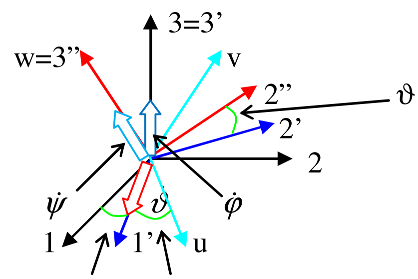
$$\underline{\omega} = \dot{\vartheta}\underline{1}'$$

Again, if we have variation of angle ψ alone, rotation will be about axis w, so that:

$$\underline{\omega} = \dot{\psi}\underline{w}$$

The three contributions are independent one from the other, therefore by simple combination of the three contributions we obtain:

$$\underline{\omega} = \dot{\phi}\underline{3} + \dot{\vartheta}\underline{1}' + \dot{\psi}\underline{w}$$



Taking the projections of $\underline{\omega}$ on axes u,v,w :

$$\begin{cases} \omega_u = \underline{\omega} \cdot \underline{u} = \dot{\phi}\underline{3} \cdot \underline{u} + \dot{\vartheta}\underline{1}' \cdot \underline{u} + \dot{\psi}\underline{w} \cdot \underline{u} = \dot{\phi}\underline{3} \cdot \underline{u} + \dot{\vartheta}\underline{1}' \cdot \underline{u} \\ \omega_v = \underline{\omega} \cdot \underline{v} = \dot{\phi}\underline{3} \cdot \underline{v} + \dot{\vartheta}\underline{1}' \cdot \underline{v} + \dot{\psi}\underline{w} \cdot \underline{v} = \dot{\phi}\underline{3} \cdot \underline{v} + \dot{\vartheta}\underline{1}' \cdot \underline{v} \\ \omega_w = \underline{\omega} \cdot \underline{w} = \dot{\phi}\underline{3} \cdot \underline{w} + \dot{\vartheta}\underline{1}' \cdot \underline{w} + \dot{\psi}\underline{w} \cdot \underline{w} \end{cases}$$

Inner products $\underline{3}$ $\underline{u}$, $\underline{3}$ $\underline{v}$, $\underline{3}$ $\underline{w}$ represent the third column of matrix A_{313}, while inner products $\underline{1}$'$\underline{u}$, $\underline{1}$'$\underline{v}$, $\underline{1}$'$\underline{w}$ represent the first column of matrix A_{313} if we consider $\varphi = 0$, in fact the first column represent axis 1, that is coincident with axis 1' in case $\varphi = 0$. Finally, $\underline{w}$ $\underline{w}$ = 1 since $\underline{w}$ is a unit vector.

We then have:

$$\begin{cases} \omega_u = \dot{\phi} \sin\vartheta \sin\psi + \dot{\vartheta} \cos\psi \\ \omega_v = \dot{\phi} \sin\vartheta \cos\psi - \dot{\vartheta} \sin\psi \\ \omega_w = \dot{\phi} \cos\vartheta + \dot{\psi} \end{cases}$$

from which the kinematic relations are obtained as:

$$\begin{cases} \dot{\phi} = \dfrac{(\omega_u \sin\psi + \omega_v \cos\psi)}{\sin\vartheta} \\ \dot{\vartheta} = \omega_u \cos\psi - \omega_v \sin\psi \\ \dot{\psi} = \omega_w - (\omega_u \sin\psi + \omega_v \cos\psi)\dfrac{\cos\vartheta}{\sin\vartheta} \end{cases}$$

where angles φ,ϑ,ψ are taken at time t.

Note that an alternative derivation of the kinematics can be performed by substituting the parametrized matrix into the kinematic equations in DCM form, for example substituting

$$A_{313} = \begin{bmatrix} \cos\psi\cos\phi - \sin\psi\sin\phi\cos\vartheta & \cos\psi\sin\phi + \sin\psi\cos\phi\cos\vartheta & \sin\psi\sin\vartheta \\ -\sin\psi\cos\phi - \cos\psi\sin\phi\cos\vartheta & -\sin\psi\sin\phi + \cos\psi\cos\phi\cos\vartheta & \cos\psi\sin\vartheta \\ \sin\phi\sin\vartheta & -\cos\phi\sin\vartheta & \cos\vartheta \end{bmatrix}$$

Into

$$\dot{A}_{B/N} A_{B/N}^T = -[\omega \wedge]$$

yields the same result.

For the other sequences of rotations, we obtain different sets of kinematic equations, adopting the same procedure and considering the composition of appropriate angles and axis of rotation. It is important to notice that in any case we have some singular conditions, for which the derivatives are

infinite, and these singular conditions occur for the same angle values that make singular the inverse transformation of the parameters.
Here follow the sets of kinematic equations for the derivatives of Euler angles for the 12 possible sequences of Euler angles, assuming that the first rotation corresponds to angle φ, the second rotation to angle ϑ and the third rotation to angle ψ. It is assumed that angular velocities ω_u,ω_v, ω_w are those of an arbitrary moving frame **M**=[**u:v:w**] with respect to an inertial frame

$$\text{sequence 123}\begin{cases}\dot{\phi}=\dfrac{(\omega_u\cos\psi-\omega_v\sin\psi)}{\cos\vartheta}\\ \dot{\vartheta}=\omega_v\cos\psi+\omega_u\sin\psi\\ \dot{\psi}=\omega_w-(\omega_u\cos\psi-\omega_v\sin\psi)\dfrac{\sin\vartheta}{\cos\vartheta}\end{cases}$$

$$\text{sequence 132}\begin{cases}\dot{\phi}=\dfrac{(\omega_u\cos\psi+\omega_w\sin\psi)}{\cos\vartheta}\\ \dot{\vartheta}=\omega_w\cos\psi-\omega_u\sin\psi\\ \dot{\psi}=\omega_v+(\omega_u\cos\psi+\omega_w\sin\psi)\dfrac{\sin\vartheta}{\cos\vartheta}\end{cases}$$

$$\text{sequence 231}\begin{cases}\dot{\phi}=\dfrac{(\omega_v\cos\psi-\omega_w\sin\psi)}{\cos\vartheta}\\ \dot{\vartheta}=\omega_w\cos\psi+\omega_v\sin\psi\\ \dot{\psi}=\omega_u-(\omega_v\cos\psi-\omega_w\sin\psi)\dfrac{\sin\vartheta}{\cos\vartheta}\end{cases}$$

$$\text{sequence 213}\begin{cases}\dot{\phi}=\dfrac{(\omega_v\cos\psi+\omega_u\sin\psi)}{\cos\vartheta}\\ \dot{\vartheta}=\omega_u\cos\psi-\omega_v\sin\psi\\ \dot{\psi}=\omega_w+(\omega_v\cos\psi+\omega_u\sin\psi)\dfrac{\sin\vartheta}{\cos\vartheta}\end{cases}$$

$$\text{sequence 312}\begin{cases}\dot{\phi}=\dfrac{(\omega_w\cos\psi-\omega_u\sin\psi)}{\cos\vartheta}\\ \dot{\vartheta}=\omega_u\cos\psi+\omega_w\sin\psi\\ \dot{\psi}=\omega_v-(\omega_w\cos\psi-\omega_u\sin\psi)\dfrac{\sin\vartheta}{\cos\vartheta}\end{cases}$$

$$\text{sequence 321}\begin{cases}\dot{\phi}=\dfrac{(\omega_w\cos\psi+\omega_v\sin\psi)}{\cos\vartheta}\\ \dot{\vartheta}=\omega_v\cos\psi-\omega_w\sin\psi\\ \dot{\psi}=\omega_u+(\omega_w\cos\psi+\omega_v\sin\psi)\dfrac{\sin\vartheta}{\cos\vartheta}\end{cases}$$

$$\text{sequence 121}\begin{cases}\dot{\phi}=\dfrac{(\omega_v\sin\psi+\omega_w\cos\psi)}{\sin\vartheta}\\ \dot{\vartheta}=\omega_v\cos\psi-\omega_w\sin\psi\\ \dot{\psi}=\omega_u-(\omega_v\sin\psi+\omega_w\cos\psi)\dfrac{\cos\vartheta}{\sin\vartheta}\end{cases}$$

$$\text{sequence 131}\begin{cases}\dot{\phi}=\dfrac{-(\omega_v\cos\psi-\omega_w\sin\psi)}{\sin\vartheta}\\ \dot{\vartheta}=\omega_w\cos\psi+\omega_v\sin\psi\\ \dot{\psi}=\omega_u+(\omega_v\cos\psi-\omega_w\sin\psi)\dfrac{\cos\vartheta}{\sin\vartheta}\end{cases}$$

$$\text{sequence 212}\begin{cases}\dot{\phi}=\dfrac{-(\omega_w\cos\psi-\omega_u\sin\psi)}{\sin\vartheta}\\ \dot{\vartheta}=\omega_u\cos\psi+\omega_w\sin\psi\\ \dot{\psi}=\omega_v+(\omega_w\cos\psi-\omega_u\sin\psi)\dfrac{\cos\vartheta}{\sin\vartheta}\end{cases}$$

$$\text{sequence 232}\begin{cases}\dot{\phi}=\dfrac{(\omega_w\sin\psi+\omega_u\cos\psi)}{\sin\vartheta}\\ \dot{\vartheta}=\omega_w\cos\psi-\omega_u\sin\psi\\ \dot{\psi}=\omega_v-(\omega_w\sin\psi+\omega_u\cos\psi)\dfrac{\cos\vartheta}{\sin\vartheta}\end{cases}$$

$$\text{sequence 313}\begin{cases}\dot{\phi}=\dfrac{(\omega_u\sin\psi+\omega_v\cos\psi)}{\sin\vartheta}\\ \dot{\vartheta}=\omega_u\cos\psi-\omega_v\sin\psi\\ \dot{\psi}=\omega_w-(\omega_u\sin\psi+\omega_v\cos\psi)\dfrac{\cos\vartheta}{\sin\vartheta}\end{cases}$$

$$\text{sequence 323}\begin{cases}\dot{\phi}=\dfrac{-(\omega_u\cos\psi-\omega_v\sin\psi)}{\sin\vartheta}\\ \dot{\vartheta}=\omega_v\cos\psi+\omega_u\sin\psi\\ \dot{\psi}=\omega_w+(\omega_u\cos\psi-\omega_v\sin\psi)\dfrac{\cos\vartheta}{\sin\vartheta}\end{cases}$$

Euler angles are integrable, this meaning that:

$$\phi=\phi_0+\int_0^t\dot{\phi}dt$$
$$\vartheta=\vartheta_0+\int_0^t\dot{\vartheta}dt$$
$$\psi=\psi_0+\int_0^t\dot{\psi}dt$$

This is not true for angular velocities, that when integrated do not provide angles, since angular velocity is not a fixed vector. To verify this statement, we recall the integrability condition:

$$dz=Adx+Bdy\quad\rightarrow\quad\frac{\partial A}{\partial y}=\frac{\partial B}{\partial x}$$

$$\omega_u=\dot{\phi}\,sin\,\vartheta\,sin\,\psi+\dot{\vartheta}\,cos\,\psi\quad\rightarrow\quad\omega_u dt=sin\,\vartheta\,sin\,\psi\,d\phi+cos\,\psi\,d\vartheta=Ad\phi+Bd\vartheta$$

$$\left.\begin{matrix}\frac{\partial A}{\partial\vartheta}=sin\,\psi\,cos\,\vartheta\\ \frac{\partial B}{\partial\phi}=0\end{matrix}\right\}\neq\qquad\text{not integrable}$$

The same integrability condition verified for Euler angles:

$$\dot{\phi}=\omega_u\frac{sin\,\psi}{sin\,\vartheta}+\omega_v\frac{cos\,\psi}{sin\,\vartheta}\quad\rightarrow\quad d\phi=\frac{sin\,\psi}{sin\,\vartheta}\omega_u dt+\frac{cos\,\psi}{sin\,\vartheta}\omega_v dt=A\omega_u dt+B\omega_v dt$$

$$\frac{\partial A}{\partial(\omega_v \mathrm{dt})} = \frac{\partial\left(\frac{\sin\psi}{\sin\vartheta}\right)}{\partial(\omega_v \mathrm{dt})} = \frac{1}{\sin^2\vartheta}\left(\cos\psi \frac{\partial\psi}{\partial(\omega_v \mathrm{dt})}\sin\vartheta - \sin\psi\cos\vartheta \frac{\partial\vartheta}{\partial(\omega_v \mathrm{dt})}\right)$$

$$\frac{\partial\psi}{\partial(\omega_v \mathrm{dt})} = \frac{\partial\dot{\psi}}{\partial\omega_v} = -\frac{\cos\psi\cos\vartheta}{\sin\vartheta}$$

$$\frac{\partial\vartheta}{\partial(\omega_v \mathrm{dt})} = \frac{\partial\dot{\vartheta}}{\partial\omega_v} = -\sin\psi$$

therefore:

$$\frac{\partial A}{\partial(\omega_v \mathrm{dt})} = \frac{1}{\sin^2\vartheta}(\sin^2\psi\cos\vartheta - \cos^2\psi\sin\vartheta)$$

In similar way we can evaluate $\frac{\partial B}{\partial(\omega_u dt)}$ and check that it has the same expression, this demonstrates integrability of Euler angles.

Euler equations

Euler equations are the analytic form adopted to describe the attitude motion of a rigid body. We start by writing the fundamental equation:

$$\frac{d\underline{h}}{dt} = \underline{M}$$

In body axes the angular momentum vector is:

$$\underline{h} = \begin{Bmatrix} I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z \\ I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z \\ I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z \end{Bmatrix}$$

Writing the fundamental equation in body axes we have:

$$\underline{\dot{h}} + \underline{\omega} \wedge \underline{h} = \underline{M}$$

and expanding the expression of the angular momentum we obtain a system of first order differential equations, nonlinear and strongly coupled:

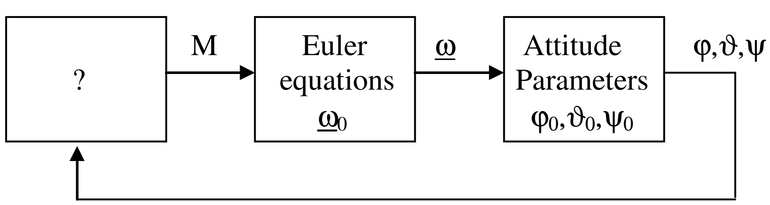
$$\begin{cases} I_{xx}\dot{\omega}_x + I_{xy}\dot{\omega}_y + I_{xz}\dot{\omega}_z - I_{xy}\omega_x\omega_z - I_{yy}\omega_y\omega_z - I_{yz}\omega_z^2 + I_{xz}\omega_x\omega_y + I_{zz}\omega_z\omega_y + I_{yz}\omega_y^2 = M_x \\ I_{yx}\dot{\omega}_x + I_{yy}\dot{\omega}_y + I_{yz}\dot{\omega}_z - I_{yz}\omega_x\omega_y - I_{zz}\omega_x\omega_z - I_{xz}\omega_x^2 + I_{xx}\omega_x\omega_z + I_{xy}\omega_z\omega_y + I_{xz}\omega_z^2 = M_y \\ I_{zx}\dot{\omega}_x + I_{zy}\dot{\omega}_y + I_{zz}\dot{\omega}_z - I_{xx}\omega_x\omega_y - I_{xz}\omega_y\omega_z - I_{xy}\omega_y^2 + I_{yy}\omega_x\omega_y + I_{yz}\omega_z\omega_x + I_{xy}\omega_x^2 = M_z \end{cases}$$

If we refer the dynamics to the principal inertia axes, we can simplify the system dynamics since inertia products will vanish:

$$\begin{cases} I_x\dot{\omega}_x + \left(I_z - I_y\right)\omega_z\omega_y = M_x \\ I_y\dot{\omega}_y + (I_x - I_z)\omega_x\omega_z = M_y \\ I_z\dot{\omega}_z + \left(I_y - I_x\right)\omega_y\omega_x = M_z \end{cases}$$

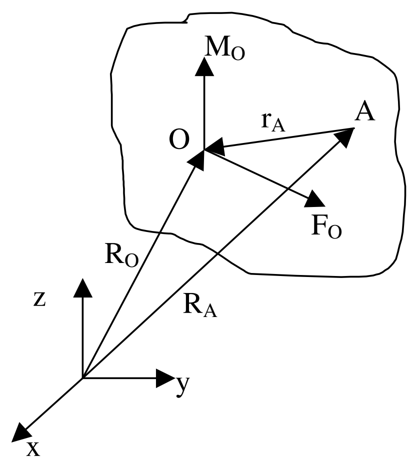
In these equations, if the coupling term vanishes then the rotation around one axis will be driven only by the torque around the same axis.

Rotational dynamics of a satellite is then described by joining Euler equations and an attitude parameterization.



On a satellite we have torques influencing directly the dynamics (Euler equations) and induce changes in the components of angular velocity. Adopting one attitude parameterization we can, knowing the

angular velocity, calculate the attitude parameters that indicate the orientation of the satellite in space and that allow to evaluate the position-dependent torques. In Euler equations it is therefore required to assign initial conditions for angular velocity, while in attitude parameterization we must assign the initial satellite attitude.
If the reference system is not centered in the system center of mass, in deriving the rotational dynamics equations we must also consider the linear motion of the origin of the reference frame. With reference to the following figure, assuming O is the center of mass and A the origin of the reference system, assuming that we define with M_o and F_o the torque and force referred to the system center of mass, we have



$$\frac{d\underline{h_O}}{\mathrm{dt}} = \underline{M_O} \quad ; \quad \underline{F_O} = \mathrm{m}\, \underline{a_O}$$

$$\underline{h_A} = \underline{h_O} + \mathrm{m}\, \underline{r_A} \wedge \underline{\dot{r}_A} \quad ; \quad \underline{M_A} = \underline{M_O} + \underline{r_A} \wedge \underline{F_O} \quad ; \quad \underline{a_A} = \underline{a_O} - \underline{\ddot{r}_A}$$

that is

$$\underline{h_O} = \underline{h_A} - \mathrm{m}\, \underline{r_A} \wedge \underline{\dot{r}_A} \quad ; \quad \underline{M_O} = \underline{M_A} - \underline{r_A} \wedge \underline{F_O} \quad ; \quad \underline{a_O} = \underline{a_A} + \underline{\ddot{r}_A}$$

Substituting these expressions in the equation of the angular momentum derivative, we have

$$\underline{M_O} = \frac{d\underline{h_O}}{\mathrm{dt}} \quad \rightarrow \quad \underline{M_A} - \underline{r_A} \wedge \underline{F_O} = \frac{d}{\mathrm{dt}}\left(\underline{h_A} - \mathrm{m}\, \underline{r_A} \wedge \underline{\dot{r}_A}\right)$$

and, since $\underline{F_O} = \mathrm{m}\, \underline{a_O}$

$$\underline{M_A} - \underline{r_A} \wedge \mathrm{m}\, \left(\underline{a_A} + \underline{\ddot{r}_A}\right) = \frac{d}{\mathrm{dt}}\left(\underline{h_A} - \mathrm{m}\, \underline{r_A} \wedge \underline{\dot{r}_A}\right)$$

For a rigid body we have $\underline{\dot{r}_A} = 0$, $\underline{\ddot{r}_A} = 0$, therefore $\underline{h_O} = \underline{h_A}$, $\underline{a_O} = \underline{a_A}$ and

$$\underline{M_A} = \frac{d\underline{h_A}}{\mathrm{dt}} + \underline{r_A} \wedge \mathrm{m}\, \underline{a_A} = \frac{d\underline{h_O}}{\mathrm{dt}} + \underline{S_A} \wedge \underline{a_A}$$

where $\underline{S_A}$ stands for the static moment of the rigid body with respect to point A. If the reference axes are parallel to the principal inertia axes, then we can write

$$\begin{cases} I_x\dot{\omega}_x + \left(I_z - I_y\right)\omega_z\omega_y + \left(S_y a_z - S_z a_y\right) = M_x \\ I_y\dot{\omega}_y + \left(I_x - I_z\right)\omega_x\omega_z + \left(S_z a_x - S_x a_z\right) = M_y \\ I_z\dot{\omega}_z + \left(I_y - I_x\right)\omega_y\omega_x + \left(S_x a_y - S_y a_x\right) = M_z \end{cases}$$

It is pointed out that the same result would have been obtained by considering the equation of the derivative of the angular momentum vector, expressed in the general form $\underline{h}_o = -\underline{v}_o \wedge S_o + I\underline{\omega}$, with no additional assumptions on the reference frame (fixed or with origin in the system center of mass).

Even in the simplest case, assuming as reference system the principal inertia system, Euler equations are difficult to integrate and therefore numerical integrations is adopted. In some simplified cases we can obtain an analytical solution of the equations, to be used to verify the numerical integration results.

Torque-free motion of a simple spin satellite

Assume the satellite is axial-symmetric. Having two equal principal inertia moments, one equation is decoupled from the other two.

Assume that $I_x = I_y$:

$$\begin{cases} I_x\dot{\omega}_x + (I_z - I_y)\omega_z\omega_y = M_x \\ I_y\dot{\omega}_y + (I_x - I_z)\omega_z\omega_x = M_y \\ I_z\dot{\omega}_z = M_z \end{cases}$$

Assume that no external torque is applied, then we can study the torque-free motion knowing the initial conditions:

$$\begin{cases} I_x\dot{\omega}_x + (I_z - I_y)\omega_z\omega_y = 0 \\ I_y\dot{\omega}_y + (I_x - I_z)\omega_z\omega_x = 0 \\ I_z\dot{\omega}_z = 0 \end{cases}$$

The last equation leads immediately to:

$$\omega_z = \text{const} = \omega_{\text{z0}}$$

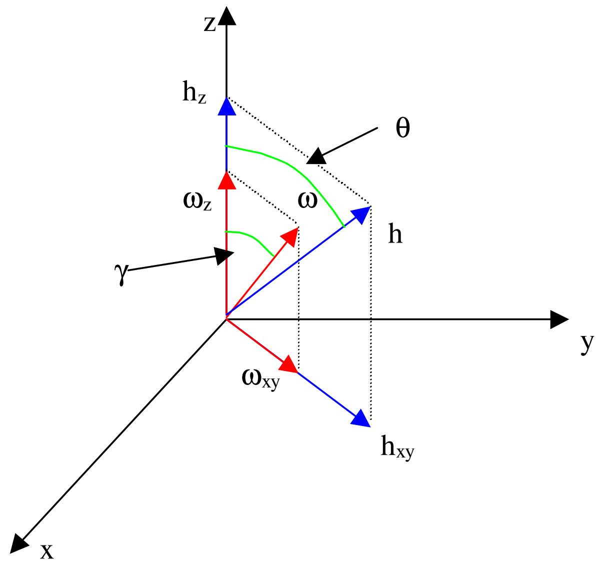
Furthermore, since no torque is applied, angular momentum and kinetic energy are constant. Combining the first two equations, the first multiplied by ω_x and the second multiplied by ω_y, we obtain:

$$\begin{aligned} &I_x\dot{\omega}_x\omega_x + I_x\dot{\omega}_y\omega_y = 0 \\ &\dot{\omega}_x\omega_x + \dot{\omega}_y\omega_y = 0 \\ &\frac{d}{\text{dt}}(\omega_x^2 + \omega_y^2) = 0 \\ &(\omega_x^2 + \omega_y^2) = \text{const} \end{aligned}$$

The magnitude of $\underline{\omega}$ is then constant. The same result would have been obtained by considering the constant kinetic energy. To have the magnitude constant, both the x and y components of the angular velocity must be constant or the projection of the angular velocity on the x-y plane is on a circle. In fact we know that the angular momentum is constant, therefore:

$$\begin{aligned} &\underline{h} = \text{const} \\ &\underline{h} = I_x\left(\omega_x\underline{i} + \omega_y\underline{j}\right) + I_z\omega_z\underline{k} = h_{\text{xy}} + h_z \end{aligned}$$

h_z is constant, while h_{xy} is parallel to ω_{xy} (component on x-y plane) but not constant. Therefore vectors $\underline{h}$ and $\underline{\omega}$ describe cones with axis coincident with axis z if we look at the motion from a body axes reference. If we look at the motion from an inertial reference, $\underline{h}$ will be fixed and $\underline{\omega}$ and the unit vector z describe cones with axes coincident with vector $\underline{h}$. Furthermore, $\underline{\omega}$ and $\underline{h}$ are constantly on a plane intersecting axis z. In case the only non-zero component of the angular velocity is the z component, then motion is referred to as simple spin satellite motion.



We can now solve the first two equations. Define:

$$\lambda = \frac{(I_z - I_x)\omega_z}{I_x}$$

as a characteristic constant of the rigid body, the equations become:

$$\begin{cases} \dot{\omega}_x + \lambda\omega_y = 0 \\ \dot{\omega}_y - \lambda\omega_x = 0 \end{cases}$$

Deriving the first and substituting the value of $\dot{\omega}_y$ from the second equation, we have:

$$\ddot{\omega}_x + \lambda^2\omega_x = 0$$

This is the typical equation of an harmonic system that has the following solution:

$$\omega_x = a\cos(\lambda t) + b\sin(\lambda t)$$
$$\dot{\omega}_x = -a\lambda\sin(\lambda t) + b\lambda\cos(\lambda t)$$

with initial conditions:

$$\omega_x(0) = \omega_{x0}$$
$$\dot{\omega}_x(0) = \dot{\omega}_{x0}$$
$$\omega_y(0) = \omega_{y0}$$
$$\dot{\omega}_y(0) = \dot{\omega}_{y0}$$

We then have:

$$\omega_x = \omega_{x0}\cos(\lambda t) + \frac{\dot{\omega}_{x0}}{\lambda}\sin(\lambda t)$$

From the second equation we obtain:

$$\omega_y = -\frac{\dot{\omega}_x}{\lambda}$$

so that:

$$\omega_x = \omega_{x0}\cos(\lambda t) - \omega_{y0}\sin(\lambda t)$$
$$\omega_y = \omega_{x0}\sin(\lambda t) + \omega_{y0}\cos(\lambda t)$$

In order to demonstrate that ω_{xy} describes a circle, we can write it as a complex number:

$$\omega_{xy} = \omega_x + i\omega_y = \omega_{x0}[\cos(\lambda t) + i\sin(\lambda t)] - \omega_{y0}[\sin(\lambda t) - i\cos(\lambda t)]$$

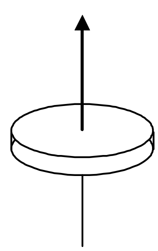
or:

$$\omega_{xy} = \omega_{x0}[cos(\lambda t) + i\, sin(\lambda t)] + i\omega_{y0}[cos(\lambda t) + i\, sin(\lambda t)]$$
$$\omega_{xy} = \left(\omega_{x0} + i\omega_{y0}\right)[cos(\lambda t) + i\, sin(\lambda t)]$$

that is a harmonic motion on a plane, with magnitude given by the initial values of $\underline{\omega}$ and constant angular velocity equal to λ.

At this point we must analyze the case in which I_z is the maximum inertia moment and the case in which I_z is the minimum inertia moment:

I_z maximum $\lambda > 0$



I_z minimum $\lambda < 0$

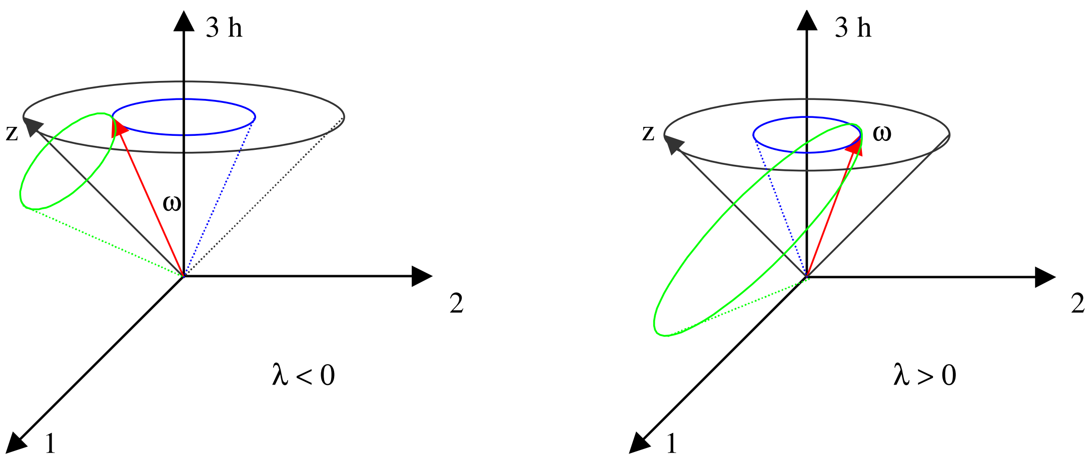
Now we can look at the angles that $\underline{h}$ and $\underline{\omega}$ form with the z axis:

$$tan\,\gamma = \frac{\omega_{xy}}{\omega_z}$$
$$tan\,\vartheta = \frac{h_{xy}}{h_z} = \frac{I_x\omega_{xy}}{I_z\omega_z} = \frac{I_x}{I_z}tan\,\gamma$$

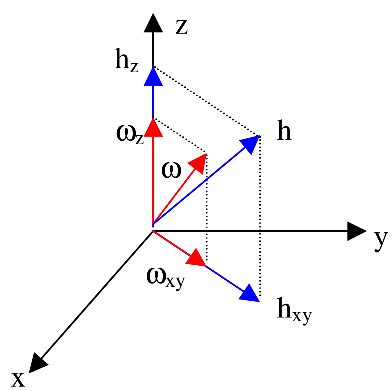
If I_z is the maximum inertia moment, then $\lambda > 0$ and $\vartheta < \gamma$, while if I_z is the minimum inertia moment, $\lambda < 0$ and $\vartheta > \gamma$.

So far we have seen the motion in the body axes frame, and analyzed the motion of the angular momentum vector around the z body axis.
In reality vector $\underline{h}$ is fixed in an inertial frame, so if we allow one of the inertial frame axis to be coincident with $\underline{h}$ we will see the body axis rotating around it, along with the angular velocity vector.

The z axis draws a cone with axis coincident with $\underline{h}$, with aperture ϑ, while the angular velocity draws a cone with axis coincident with z, with aperture γ. Moreover we recall that $\underline{\omega}$ draws also a cone with axis coincident with $\underline{h}$. We then have two cones, one fixed ($\underline{\omega}$ around $\underline{h}$) called "space cone", and one that revolves on it ($\underline{\omega}$ around z), called "body cone". The two cones revolve one outside the other if λ is negative, while they revolve one inside the other if λ is positive.

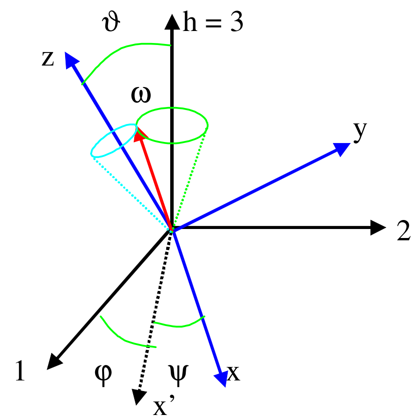


We can now apply one attitude representation to the axial symmetric satellite:



$$\lambda = \frac{(I_z - I_x)\omega_z}{I_x}$$

Adopting the Euler angles with sequence 313, we can represent the inertial reference with the third axis coincident with $\underline{h}$.



$$\begin{cases} \omega_x = \dot{\phi} \sin\vartheta \sin\psi \\ \omega_y = \dot{\phi} \sin\vartheta \cos\psi \\ \omega_z = \dot{\phi} \cos\vartheta + \dot{\psi} \end{cases}$$

We notice that ϑ is constant due to the satellite symmetry, and axis z draws a cone of amplitude ϑ around axis h.

Taking time derivatives, we have:

$$\begin{cases} \dot{\omega}_x = \ddot{\phi} \sin\vartheta \sin\psi + \dot{\phi}\dot{\psi} \cos\psi \sin\vartheta \\ \dot{\omega}_y = \ddot{\phi} \sin\vartheta \cos\psi - \dot{\phi}\dot{\psi} \sin\psi \sin\vartheta \\ \dot{\omega}_z = \ddot{\phi} \cos\vartheta + \ddot{\psi} \end{cases}$$

Replacing in the Euler equations we represent the dynamics as a function of φ,ϑ,ψ and recalling that:

$$\dot{\omega}_z = 0$$
$$\omega_x \dot{\omega}_x + \omega_y \dot{\omega}_y = 0$$

from the second equation we obtain:

$$\ddot{\phi}\dot{\phi} \sin^2\vartheta \sin^2\psi + \dot{\phi}^2\dot{\psi} \sin^2\vartheta \sin\psi \cos\psi + \ddot{\phi}\dot{\phi} \sin^2\vartheta \cos^2\psi - \dot{\phi}^2\dot{\psi} \sin^2\vartheta \sin\psi \cos\psi =$$
$$\ddot{\phi}\dot{\phi} \sin^2\vartheta (\sin^2\psi + \cos^2\psi) = \ddot{\phi}\dot{\phi} \sin^2\vartheta = 0$$

Since ϑ must be constant the following must hold:

$$\ddot{\phi}\dot{\phi} = 0 \quad \Rightarrow \quad \dot{\phi} = cos\,t$$

From the first equation we now have:

$$\ddot{\psi} = 0 \quad \Rightarrow \quad \dot{\psi} = cos\,t$$

We now have the following expressions:

$$\begin{cases} \psi = \psi_0 + \dot{\psi}_0 t \\ \phi = \phi_0 + \dot{\phi}_0 t \\ \vartheta = \vartheta_0 \end{cases}$$

We must now calculate the initial values of the derivatives of φ e ψ, and to do so we make use of the first two Euler equations:

$$\ddot{\phi}\, sin\,\vartheta\, sin\,\psi + \dot{\phi}\dot{\psi}\, cos\,\psi\, sin\,\vartheta + \lambda\dot{\phi}\, sin\,\vartheta\, cos\,\psi = 0$$
$$\ddot{\phi}\, sin\,\vartheta\, cos\,\psi - \dot{\phi}\dot{\psi}\, sin\,\psi\, sin\,\vartheta - \lambda\dot{\phi}\, sin\,\vartheta\, sin\,\psi = 0$$

Since the second time derivative of φ is zero we have:

$$\dot{\psi} = -\lambda$$

$$\dot{\phi} = \frac{\omega_z+\lambda}{cos\,\vartheta} \qquad \begin{cases} I_x\dot{\omega}_x + (I_z - I_y)\omega_z\omega_y = 0 \\ I_y\dot{\omega}_y + (I_x - I_z)\omega_x\omega_z = 0 \\ I_z\dot{\omega}_z + (I_y - I_x)\omega_y\omega_x = 0 \end{cases}$$

In our case $I_z < I_x$ and therefore the time derivative of ψ is positive.

Solution of Euler equations in the phase plane $(\dot{\boldsymbol{\omega}}, \boldsymbol{\omega})$

Reconsider the Euler equations:

$$\begin{cases} I_x\dot{\omega}_x + (I_z - I_y)\omega_z\omega_y = 0 \\ I_y\dot{\omega}_y + (I_x - I_z)\omega_x\omega_z = 0 \\ I_z\dot{\omega}_z + (I_y - I_x)\omega_y\omega_x = 0 \end{cases}$$

If we assume that the external torques are zero, then angular momentum and kinetic energy will be constant:

$$2\mathrm{T} = \mathrm{const}$$
$$h^2 = \mathrm{const}$$

We now look for a method to decouple and solve the three equations, making use of the conservation laws just mentioned. Starting from the first equation, taking its time derivative, we have:

$$I_x\ddot{\omega}_x + (I_z - I_y)\dot{\omega}_z\omega_y + (I_z - I_y)\omega_z\dot{\omega}_y = 0$$

From the second and third equations we can compute the time derivative of the angular velocity components, and substitute it in the above equation:

$$I_x\ddot{\omega}_x + (I_z - I_y)\left(\frac{I_x - I_y}{I_z}\right)\omega_y^2\omega_x + (I_z - I_y)\left(\frac{I_z - I_x}{I_y}\right)\omega_z^2\omega_x = 0$$

Now we evaluate:

$$h^2 - 2TI_z = I_x^2\omega_x^2 + I_y^2\omega_y^2 - I_xI_z\omega_x^2 - I_yI_z\omega_y^2$$

so that:

$$\omega_y^2(I_y - I_z)I_y = h^2 - 2TI_z - \omega_x^2(I_x - I_z)I_x$$
$$\omega_y^2(I_y - I_z) = \frac{h^2 - 2TI_z - \omega_x^2(I_x - I_z)I_x}{I_y}$$

Substituting in the previous equation, we get to:

$$I_x\ddot{\omega}_x+\left(\frac{I_x-I_y}{I_z}\right)\frac{\left(2TI_z-h^2\right)}{I_y}\omega_x+\frac{\left(I_x-I_y\right)}{I_z}\frac{\left(I_x-I_z\right)}{I_y}\omega_x^3+\left(I_z-I_y\right)\left(\frac{I_z-I_x}{I_y}\right)\omega_z^2\omega_x=0$$

Now evaluate:

$$h^2-2TI_y=I_z^2\omega_z^2+I_x^2\omega_x^2-I_yI_z\omega_z^2-I_xI_y\omega_x^2$$

so that:

$$\omega_z^2\left(I_z-I_y\right)I_z=h^2-2TI_y+\omega_x^2\left(I_y-I_x\right)I_x$$

$$\omega_z^2\left(I_z-I_y\right)=\frac{h^2-2TI_y+\omega_x^2\left(I_y-I_x\right)I_x}{I_z}$$

Substituting in the previous equation, we get to:

$$I_x\ddot{\omega}_x+\left(\frac{I_x-I_y}{I_z}\right)\frac{\left(2TI_z-h^2\right)}{I_y}\omega_x+\frac{\left(I_x-I_y\right)}{I_z}\frac{\left(I_x-I_z\right)}{I_y}\omega_x^3+\left(\frac{I_z-I_x}{I_y}\right)\frac{\left(h^2-2TI_y\right)}{I_z}\omega_x$$
$$+\frac{\left(I_z-I_x\right)}{I_y}\frac{\left(I_y-I_x\right)}{I_z}\omega_x^3=0$$

or:

$$\ddot{\omega}_x+\left[\frac{\left(I_y-I_x\right)\left(h^2-2TI_z\right)+\left(I_z-I_x\right)\left(h^2-2TI_y\right)}{I_xI_yI_z}\right]\omega_x+\left[\frac{2\left(I_z-I_x\right)\left(I_y-I_x\right)I_x}{I_xI_yI_z}\right]\omega_x^3=0$$

The other two equations are obtained by following the same procedure, simply with a permutation of all the operation indexes:

$$\ddot{\omega}_y+\left[\frac{\left(I_z-I_y\right)\left(h^2-2TI_x\right)+\left(I_x-I_y\right)\left(h^2-2TI_z\right)}{I_xI_yI_z}\right]\omega_y+\left[\frac{2\left(I_z-I_y\right)\left(I_x-I_y\right)I_y}{I_xI_yI_z}\right]\omega_y^3=0$$

$$\ddot{\omega}_z+\left[\frac{\left(I_x-I_z\right)\left(h^2-2TI_y\right)+\left(I_y-I_z\right)\left(h^2-2TI_x\right)}{I_xI_yI_z}\right]\omega_z+\left[\frac{2\left(I_x-I_z\right)\left(I_y-I_z\right)I_z}{I_xI_yI_z}\right]\omega_z^3=0$$

The three equations all have the same structure:

$$\ddot{\omega}+P\omega+Q\omega^3=0$$

Integrating the equation we obtain:

$$\dot{\omega}^2+\mathrm{P}\omega^2+\frac{1}{2}\mathrm{Q}\omega^4=K$$

that, for the x equation, becomes:

$$\dot{\omega}_x^2+P_x\omega_x^2+\frac{1}{2}Q_x\omega_x^4=K_x$$

$$\dot{\omega}_x^2+\omega_x^2\left(P_x+\frac{1}{2}Q_x\omega_x^2\right)=K_x$$

The last expression can be considered as a conic section in the phase plane $(\dot{\omega},\omega)$. We can evaluate the sign of each coefficient.

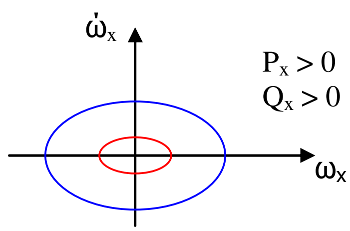
Assuming that $I_z > I_y > I_x$, we have:

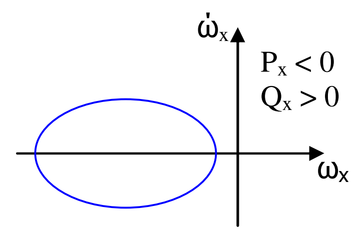
$$I_x < \frac{h^2}{2T} < I_z$$

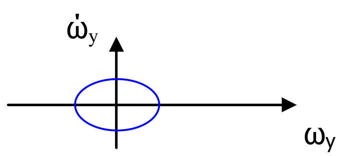
The coefficients have therefore the following signs:

P_x ?	$P_y > 0$	P_z ?
$Q_x > 0$	$Q_y < 0$	$Q_z > 0$

Due to the structure of the coefficients, if the angular velocity component is large enough the Q term will predominate over the P term, while if the angular velocity component is small enough the P term will predominate over the Q term. Then, in the y phase plane if the angular velocity is large the conic section will be a hyperbola, if the angular velocity is small the conic section will be an ellipse. In the x and z phase planes the conic sections are ellipses for large velocities, for small velocities the type of conic section is undefined. This is not really important, since in any case, even for divergent phase plane trace (hyperbola) it is clear that as the angular velocity grows the trace will change into an ellipse. Notice that in the y phase plane (intermediate inertia phase plane), should the velocity grow, we would have a diverging motion. This is not possible due to the conservation of kinetic energy, therefore the conclusion is that in the y phase plane the solution must be such that the angular velocity is small enough to prevent the trace from being a hyperbola.







Attitude stability

We want to understand the stability of the equilibrium points of the Euler equations, as naturally stable equilibria can be exploited for passive stability. Firstly, we need to define the notions of stability we will used in this course. Note that in the following stability definitions and analysis we determine the stability in the vicinity of equilibrium points. For linear systems there is only ever one equilibrium point. Whereas for nonlinear systems there are multiple equilibria. For an autonomous nonlinear dynamical system.

$$\dot{\underset{\sim}{x}} = f(\underset{\sim}{x}),\ \ \underset{\sim}{x}(0) = \underset{\sim}{x}_0$$

An equilibrium point $\underset{\sim}{x}_e$is defined as $0 = f(\underset{\sim}{x}_e)$.

<u>Stability definitions</u>

Consider an autonomous nonlinear dynamical system

$$\dot{\underset{\sim}{x}} = f(\underset{\sim}{x}), \underset{\sim}{x}(0) = \underset{\sim}{x}_0$$

defined on an open set containing the origin, and *f* is continuous on this open set. Then an equilibrium point $x_e$is said to be:

1. **Lyapunov stable**, if, for every $\varepsilon > 0$, there exists a $\partial > 0$such that, if $\|x(0) - x_e\| < \partial$, then for every $t > 0$ we have $\|x(t) - x_e\| < \varepsilon$.

2. The equilibrium of the above system is said to be **asymptotically stable** if it is Lyapunov stable and if $\|x(t) - x_e\| \to 0$as $t \to \infty$

These definitions relate to local stability since $\|x(0) - x_e\| < \partial$ means that it is only stable in a region of the equilibrium point. If ∂ is unbounded then the definitions describe global stability.

First recall how we can deduce the stability for a simple1-DOF linear systems of the form:

$m\ddot{x} + c\dot{x} + kx = 0$

It is possible to use the ansatz solution $x = x_0 e^{\lambda t}$ to determine the stability. Then the characteristic equation is

$$m\lambda^2 + c\lambda + k = 0$$

Which is solved as

$$\lambda = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

It is possible to use the ansatz solution $x = x_0 e^{\lambda t}$ to determine the stability. One immediately sees that if $Re(\lambda) < 0$ then the system is asymptotically stable. If $Re(\lambda) = 0$ then it is marginally stable and for $Re(\lambda) > 0$ it is unstable. $\lambda$is known as the eigenvalue which completely determines the stability of the system. In general, for a time-invariant linear system of the form $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$ where *A* is a *n* x *n* matrix, the stability of the equilibrium point $\underset{\sim}{x} = \underset{\sim}{0}$ can be determined by substituting in the Ansazt solution $\underset{\sim}{x} = \underset{\sim}{x}_0 e^{\lambda t}$. This yields:

$$e^{\lambda t}\lambda\underset{\sim}{x}_0 = e^{\lambda t}A\underset{\sim}{x}_0$$

This can be re-written as:

$$(A - \lambda I)\underset{\sim}{x}_0 = 0$$

then multiplying both sides by the inverse of $(A - \lambda I)$ then we have

$$(A - \lambda I)^{-1}(A - \lambda I)\underset{\sim}{x}_0 = \underset{\sim}{x}_0 = 0$$

Since $\underset{\sim}{x}_0 \neq 0$ in general then the above cannot be true. Therefore, the solution to $(A - \lambda I)\underset{\sim}{x}_0 = 0$ only exists when this inverse does not exist. Equivalently, this condition can be expressed as:

$$det(A - \lambda I) = 0$$

Solving the above expression yields the eigenvalues of the matrix A which in turn indicates the stability of the Linear system.

However, the Euler equations are nonlinear so in order to get information about stability of the equilibrium points we must first linearize about its equilibrium configurations.

To first linearize a nonlinear system of the form $\dot{\underset{\sim}{x}} = f(\underset{\sim}{x}), \underset{\sim}{x}(0) = \underset{\sim}{x}_0$ we define $\underset{\sim}{x} = \underset{\sim}{x}_e + \partial\underset{\sim}{x}$ i.e. $\underset{\sim}{x}$ is displaced infinitesimally from the equilibrium point and we observe how $\partial\underset{\sim}{x}$ behaves. Since the displacement is infinitesimal, we can assume $\partial x_i \partial x_j \approx 0$. In this way our resulting system is linear and the stability can be deduced by considering the eigenvalues. For example, to linearize the Euler equations we simply substitute in $\omega_x = \overline{\omega}_x + \partial\omega_x,\ \omega_y = \overline{\omega}_y + \partial\omega_y,\ \omega_z = \overline{\omega}_z + \partial\omega_z$and cancel all the higher order terms e.g. $\partial\omega_x\partial\omega_y \approx 0$. The following theorem the preservation of the stability properties of equilibrium points after linearization.

Lyapunov's First Stability Theorem

1) If the equilibrium point $\partial\underset{\sim}{x} = 0$ of the linearized system $\partial\dot{\underset{\sim}{x}} = A\partial\underset{\sim}{x}$ is asymptotically stable, then the equilibrium point $\underset{\sim}{x}_e$ of the original nonlinear system, is also asymptotically stable.
2) If the equilibrium point $\partial\underset{\sim}{x} = 0$ of the linearized system $\partial\dot{\underset{\sim}{x}} = A\partial\underset{\sim}{x}$ is unstable, then the equilibrium point $\underset{\sim}{x}_e$ of the original nonlinear system is also unstable.

The main implication of this theorem is that marginal stability of the linearized system does not guarantee marginal stability of the equilibrium point of the original nonlinear system. However, in the case of conserved systems i.e. with no dissipative forces or torques then marginal stability of the linearized system implies marginal stability of the nonlinear system.

At this stage we can discuss attitude stability with time, starting by the analysis of Euler equations in case of torque-free motion:

$$\begin{cases} I_x\dot{\omega}_x + (I_z - I_y)\omega_z\omega_y = 0 \\ I_y\dot{\omega}_y + (I_x - I_z)\omega_x\omega_z = 0 \\ I_z\dot{\omega}_z + (I_y - I_x)\omega_y\omega_x = 0 \end{cases}$$

There is a distinction between torque-free stability and forced-motion stability. The latter depends on the type of torque applied, rarely known in a suitable analytical form, therefore we will mostly concentrate on torque-free stability. To analyze stability the first step consists in evaluating an equilibrium configuration. This is obtained by imposing zero time derivatives of the system states and then solving the corresponding system of equations:

$$\begin{cases}(I_z - I_y)\overline{\omega}_z\overline{\omega}_y = 0 \\ (I_x - I_z)\overline{\omega}_x\overline{\omega}_z = 0 \\ (I_y - I_x)\overline{\omega}_y\overline{\omega}_x = 0\end{cases}$$

The solution is found for one single angular velocity component different from zero. This condition is known as simple spin satellite. Assume, without loss of generality, that the non-zero component is the z component:

$$\overline{\omega}_z \neq 0$$
$$\overline{\omega}_x = \overline{\omega}_y = 0$$

Attitude stability of simple spin satellites

To analyze the stability of this condition, we must first write the linearized equations of motion, considering small perturbations around the equilibrium position:

$$\omega_z^{'} = (\overline{\omega}_z + \omega_z)$$
$$\omega_x^{'} = (\overline{\omega}_x + \omega_x) = \omega_x$$
$$\omega_y^{'} = (\overline{\omega}_y + \omega_y) = \omega_y$$

Angular velocity components are now expressed as the sum of the nominal value plus the small perturbation. In the rest of the analysis, should the assumption of small perturbations hold this will mean that the condition is stable. Rewriting the equations, the product of two perturbations become negligible, so that the linearized equations are:

$$\begin{cases}I_x\dot{\omega}_x + (I_z - I_y)\overline{\omega}_z\omega_y = 0 \\ I_y\dot{\omega}_y + (I_x - I_z)\omega_x\overline{\omega}_z = 0 \\ I_z\dot{\omega}_z = 0\end{cases}$$

The derivative are taken only for the perturbation component of the angular velocity, since the nominal components are constant by definition. The system can now be integrated:

$$\omega_z = \text{const}$$

The perturbation of the z component is therefore constant. For the two remaining equations we can write:

$$\ddot{\omega}_x + \lambda^2\omega_x = 0$$
$$\lambda^2 = \frac{(I_z - I_y)(I_z - I_x)}{I_x I_y}\overline{\omega}_z^2$$

The solution is then of the general form:

$$\omega_x = ae^{i\lambda t}$$

The condition that guarantees a stable motion, harmonic, is $\lambda^2 > 0$, while for $\lambda^2 < 0$ the solution is exponentially diverging, so unstable.
The condition for stability is verified when:

$$\lambda^2 > 0 \quad \Rightarrow \quad I_z > I_y \text{ and } I_z > I_x \quad \text{or} \quad I_z < I_y \text{ and } I_z < I_x$$

that means that I_z must be either the maximum or the minimum inertia moment.

Stability of simple spin satellites with energy dissipation

Now we look at how the two previous stability conditions are changed due to presence of energy dissipation, that is:

$$\dot{T} < 0$$
$$|h| = cos\,t$$

Calling with ω_z the angular velocity, in case the rotation occurs around the maximum inertia axis or the minimum inertia axis we can write:

$$2T_{I_{z\max}} = I_{z_{\max}} \omega^2_{z_{\max}}$$
$$2T_{I_{z\min}} = I_{z_{\min}} \omega^2_{z_{\min}}$$
$$|h| = I_{z_{\max}} \omega_{z_{\max}} = I_{z_{\min}} \omega_{z_{\min}}$$

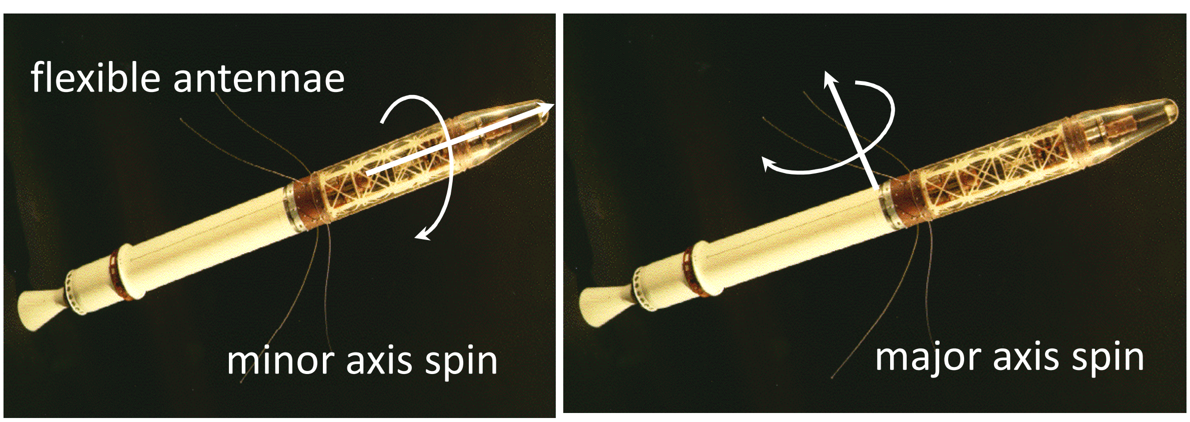
therefore, for conservation of angular momentum, $\omega_{zmax} < \omega_{zmin}$, and substituting this inequality into the expression of kinetic energy we have:

$$TI_{zmax_{I_{zmin}}}$$

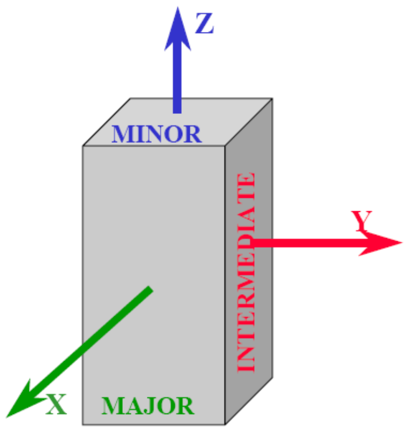
Therefore, in case of energy loss, the satellite will tend to the motion condition that has the minimum energy, rotating around the maximum inertia axis.
The only stable condition, with the usual assumptions on the inertia moments, is then:

$$I_z > I_y > I_x$$

Explorer 1 in the Figure below (first US satellite, 1958) was designed as a minor axis spinner, unexpectedly at the time energy dissipation caused by flexing of the wire antennae on the spaceraft body caused the spacecraft rotation to transition to major axis spin.

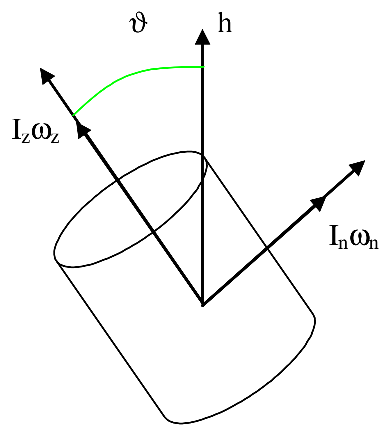


Therefore, we have the Major axis spin rule.



- $I_{xx} > I_{yy} > I_{zz}$
- Major axis spin is stable
- Minor axis spin is stable
- Intermediate axis spin is unstable
- Energy dissipation changes these results
  → Minor axis spin becomes unstable
- This is called the Major-Axis Rule

We can now analyze how the satellite reorients itself from a generic motion condition to the stable one. For simplicity we take the case of axial symmetric satellite.



We take as reference a rotating system, whose plane n-z includes the three vectors ω, z and h. This reference is also principal inertia, and vector h has only 2 components, along the two axes n and z:

$$h^2 = I_z^2\omega_z^2 + I_n^2\omega_n^2$$
$$2T = I_z\omega_z^2 + I_n\omega_n^2$$

The energy loss can be represented as:

$$2\dot{T} = 2I_z\omega_z\dot{\omega}_z + 2I_n\omega_n\dot{\omega}_n < 0$$

In addition, the following holds:

$$2h\dot{h} = 2I_z^2\omega_z\dot{\omega}_z + 2I_n^2\omega_n\dot{\omega}_n = 0 \quad \text{in fact} \qquad \dot{h} = 0$$

so that we have:

$$\omega_n\dot{\omega}_n = -\frac{I_z^2}{I_n^2}\omega_z\dot{\omega}_z$$

and the energy loss is expressed as:

$$\dot{T} = \omega_z\dot{\omega}_z\left(I_z - \frac{I_z^2}{I_n}\right) = \omega_z\dot{\omega}_z\left(\frac{I_z}{I_n}(I_n - I_z)\right) < 0$$

If I_z is the minimum inertia axis, then:

$$\omega_z\dot{\omega}_z < 0$$

$$\begin{array}{ll} \omega_z > 0 & \dot{\omega}_z < 0 \\ \omega_z < 0 & \dot{\omega}_z > 0 \end{array}$$

In both cases, ω_z decreases in amplitude, and ω_n increases, therefore the angular velocity will tend to align itself with the maximum inertia axis. If instead I_z is the maximum inertia moment, then it can be seen that ω_z increases in amplitude, and ω_n decreases, so again the angular velocity will tend to align itself with the maximum inertia axis. We finally have:

$$\tan\vartheta = \frac{I_n\omega_n}{I_z\omega_z}$$

It is possible to repeat the analysis taking the time derivative of angle ϑ. Kinetic energy will vary only until the angular velocity is aligned with the maximum inertia axis, that in this case is coincident with $\underline{h}$. In this final condition all internal dissipation sources must vanish, since loss of energy will no longer be possible without reduction of angular momentum, and this may vary only if external torques are applied to the satellite.

Attitude stability relative to a rotating frame

Previously we have considered the attitude of a spacecraft $A_{B/N}$ defined as the relative orientation between the body fixed frame **B** and an inertially fixed frame **N** such that:

$$\mathbf{B} = A_{B/N}\mathbf{N}$$

However, we may want to describe the attitude of the spacecraft body frame with respect to a moving frame such as the local-vertical-local-horizontal frame.

We can now consider the stability of the satellite taking as parameters the relative rotation of the principal inertia axis relative to a rotating frame. This is the typical case of a simple spin satellite spinning at one rotation per orbit along a circular orbit, for which the stability implies that the principal inertia axis remain aligned with the LVLH frame (local-vertical local-horizontal). For this analysis we must rewrite the equations of motion and then introduce the relative attitude parameters. Calling the LVLH frame **L** we have the relative attitude matrix defined by:

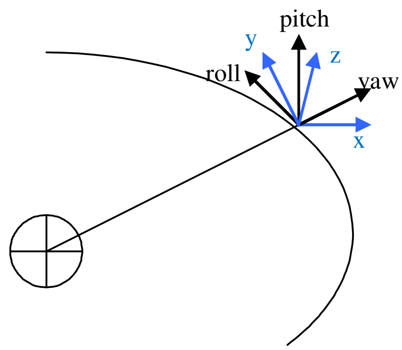
$$\mathbf{B} = A_{B/L}\,\mathbf{L}$$

with the relative angular velocity $\underline{\omega}_{B/L} = \underline{\omega}_B - A_{B/L}\underline{\omega}_L$. Note the main difference with the previous case is that $\underline{\omega}_L \neq \underline{0}$. Since $\underline{\omega}_B = \underline{\omega}_{B/N}$ we can simply substitute $\underline{\omega}_B = \underline{\omega}_{B/L} + A_{B/L}\underline{\omega}_L$ into the Euler equations derived earlier.

The equations are:

$$\begin{cases} I_x\dot{\omega}_x + (I_z - I_y)\omega_z\omega_y = 0 \\ I_y\dot{\omega}_y + (I_x - I_z)\omega_x\omega_z = 0 \\ I_z\dot{\omega}_z + (I_y - I_x)\omega_y\omega_x = 0 \end{cases}$$

We consider a rotation around the z axis of the LVLH frame (the pitch axis), with an angular velocity of magnitude n such that $\underset{\sim}{\omega}_L = [0 \quad 0 \quad n]$, small rotations between the body fixed frame **B** and the LVLH frame **L** represented by a set of Euler angles with three different indexes, to avoid singularity in the nominal condition.



Having considered small rotations, the order in which they are defined is not relevant. Indicating the three small angles as α_x, α_y, α_z, the rotation matrix representing the relative attitude of the satellite in the LVLH frame is:

$$A_{B/L} = \begin{bmatrix} 1 & \alpha_z & -\alpha_y \\ -\alpha_z & 1 & \alpha_x \\ \alpha_y & -\alpha_x & 1 \end{bmatrix}$$

and $\underset{\sim}{\omega}_L = [0 \quad 0 \quad n]$ where n is the angular velocity of the moving frame about the z-axis (typically, one rotation per orbit). Angular velocity becomes:

$$\underset{\sim}{\omega}_B = \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} = \begin{bmatrix} \dot{\alpha}_x \\ \dot{\alpha}_y \\ \dot{\alpha}_z \end{bmatrix} + \begin{bmatrix} 1 & \alpha_z & -\alpha_y \\ -\alpha_z & 1 & \alpha_x \\ \alpha_y & -\alpha_x & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ n \end{bmatrix}$$

Therefore, the time derivatives are:

$$\begin{cases} \dot{\omega}_x = \ddot{\alpha}_x - \dot{\alpha}_y n \\ \dot{\omega}_y = \ddot{\alpha}_y + \dot{\alpha}_x n \\ \dot{\omega}_z = \ddot{\alpha}_z \end{cases}$$

Now we can substitute in the Euler equations:

$$\begin{cases} I_x(\ddot{\alpha}_x - \dot{\alpha}_y n) + (I_z - I_y)(\dot{\alpha}_z + n)(\dot{\alpha}_y + \alpha_x n) = 0 \\ I_y(\ddot{\alpha}_y + \dot{\alpha}_x n) + (I_x - I_z)(\dot{\alpha}_z + n)(\dot{\alpha}_x - \alpha_y n) = 0 \\ I_z\ddot{\alpha}_z + (I_y - I_x)(\dot{\alpha}_x - \alpha_y n)(\dot{\alpha}_y + \alpha_x n) = 0 \end{cases}$$

and expand the products, still neglecting the products of infinitesimal terms:

$$\begin{cases} I_x\ddot{\alpha}_x + n\left(I_z - I_y - I_x\right)\dot{\alpha}_y + n^2\left(I_z - I_y\right)\alpha_x = 0 \\ I_y\ddot{\alpha}_y + n\left(I_x + I_y - I_z\right)\dot{\alpha}_x + n^2(I_z - I_x)\alpha_y = 0 \\ I_z\ddot{\alpha}_z = 0 \end{cases}$$

We then have:

$$\ddot{\alpha}_z = 0 \quad \Rightarrow \quad \dot{\alpha}_z = \text{const} \quad \Rightarrow \quad \alpha_z = \text{const} \cdot t + \text{const}$$

Dividing the first equation by I_x and the second by I_y we get:

$$\begin{cases} \ddot{\alpha}_x + n(K_x - 1)\dot{\alpha}_y + n^2K_x\alpha_x = 0 \\ \ddot{\alpha}_y + n\left(1 - K_y\right)\dot{\alpha}_x + n^2K_y\alpha_y = 0 \end{cases}$$

where the coefficients K_x and K_y are non-dimensional and always in the range between –1 and +1:

$$K_x = \frac{I_z - I_y}{I_x}$$
$$K_y = \frac{I_z - I_x}{I_y}$$

To evaluate the stability, we can take the Laplace transform of the system, and analyze the characteristic polynomial:

$$\begin{bmatrix} (s^2 + K_xn^2) & sn(K_x - 1) \\ sn\left(1 - K_y\right) & \left(s^2 + K_yn^2\right) \end{bmatrix} \begin{Bmatrix} \alpha_x(s) \\ \alpha_y(s) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

The roots of the characteristic polynomial must have non positive real part in order to have a stable system:

$$\begin{aligned} & det[A(s)] = 0 \Rightarrow \\ & (s^2 + n^2K_x)\left(s^2 + n^2K_y\right) - s^2n^2\left(1 - K_y\right)(K_x - 1) = 0 \\ & s^4 + s^2\left(n^2K_x + n^2K_y + n^2\left(1 - K_y\right)(1 - K_x)\right) + n^4K_xK_y = 0 \\ & s^4 + n^2s^2\left(K_x + K_y + 1 - K_x - K_y + K_xK_y\right) + n^4K_xK_y = 0 \\ & s^4 + n^2s^2\left(1 + K_xK_y\right) + n^4K_xK_y = 0 \end{aligned}$$

For stability, s^2 cannot be real or complex, since in this case its square root will correspond to two complex numbers with phase difference equal to π, so one will have positive real part. The only stable condition will then be $s^2<0$, that will give two purely imaginary solutions, with zero real part. The stability conditions are then:

$$\begin{aligned} & b^2 > 4c \qquad a = 1 \\ & b > \sqrt{b^2 - 4c} \end{aligned}$$

Since n^2 is positive, it is possible to solve the characteristic equation for the variable n^2s^2, so the stability conditions are written as:

$$\left(1 + K_xK_y\right)^2 - 4K_xK_y > 0 \quad \rightarrow \quad \left(1 - K_xK_y\right)^2 > 0$$

$$\left(1+K_xK_y\right) > \sqrt{\left(1-K_xK_y\right)^2} \quad \rightarrow \quad K_xK_y > 0$$

The first condition is trivial, so the second is the real stability condition, that is transformed into:

$$\begin{matrix} & & I_z > I_x \\ I_z > I_y & \text{or} & I_z < I_x \\ & & I_z < I_y \end{matrix}$$

We have therefore found the same stability condition that must hold for stability in terms of angular velocity. This because angular velocity components ω_x and ω_y are harmonic, so also angles show a harmonic oscillation. Rotation around axis z is unstable and tends to misalign axis z and the pitch axis.

To analyze the actual motion of the satellite we must evaluate the system characteristic roots:

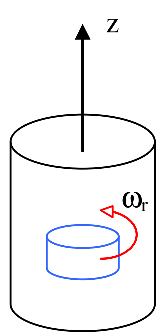
$$s^2 = \frac{-n^2\left(1+K_xK_y\right) \pm n^2\sqrt{\left(1-K_xK_y\right)^2}}{2}$$

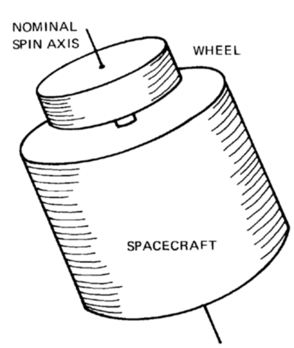
The solution depends upon the nominal angular velocity n, therefore if nominal motion is slow also the oscillations around axes x and y will be slow. The oscillation is not asymptotically stable since the characteristic roots have zero real part, while for asymptotic stability a negative real part is required.

Stability of dual spin satellites

Following the previous analysis, we know that if I_z is the maximum inertia moment then the rotation is stable, and if the nominal angular velocity is high the frequency of oscillation is also high and the amplitude of the resulting oscillations will be small. We would like however to have a satellite whose stability and frequency of oscillation are independent from the axis of rotation and from the nominal value of the angular velocity, so that we could have stable motion even if rotation is around an axis that is not the maximum inertia axis.

In the simple spin satellite the stability is dependent upon $I_z\omega_z$, that is the nominal angular momentum. In order to increase the nominal angular momentum, without varying ω_z, we can introduce an additional element that can rotate relatively to the z axis. This is the dual spin satellite.





Left: Momentum bias design. Right: Dual-Spin spacecraft

The angular momentum becomes:

$$\overline{h} = I_x\omega_x\underline{i} + I_y\omega_y\underline{j} + (I_z\omega_z + I_r\omega_r)\underline{k}$$

where I_x, I_y and I_z include the presence of the rotor and ω_r is the relative velocity of the rotor around the z axis. We can write the Euler equations for this system, simply replacing the term $I_z\omega_z$ with the term $I_z\omega_z+I_r\omega_r$:

$$\begin{cases} I_x\dot{\omega}_x + (I_z - I_y)\omega_z\omega_y + I_r\omega_r\omega_y = M_x \\ I_y\dot{\omega}_y + (I_x - I_z)\omega_x\omega_z - I_r\omega_r\omega_x = M_y \\ I_z\dot{\omega}_z + (I_y - I_x)\omega_y\omega_x + I_r\dot{\omega}_r = M_z \\ I_r\dot{\omega}_r = M_r \end{cases}$$

The fourth equation is now required since we added one degree of freedom to the system. M_r is the relative torque between rotor and satellite.
We can now analyze the stability of the torque-free motion of the dual spin satellite. The equilibrium condition is obtained with the nominal angular velocity of the satellite aligned with axis z, and arbitrary relative velocity of the rotor. Linearizing the equations in this condition, we have:

$$\begin{cases} I_x\dot{\omega}_x + (I_z - I_y)\overline{\omega}_z\omega_y + I_r\overline{\omega}_r\omega_y = 0 \\ I_y\dot{\omega}_y + (I_x - I_z)\omega_x\overline{\omega}_z - I_r\overline{\omega}_r\omega_x = 0 \\ I_z\dot{\omega}_z + I_r\dot{\omega}_r = 0 \\ I_r\dot{\omega}_r = 0 \end{cases}$$

The last two equations are decoupled, that is we can decouple the equations representing rotations around the nominal direction of $\underline{h}$. From these two equations we have ω_z and ω_r constant.
The first two equations are again solved by taking the time derivative of the first and substituting the derivative obtained by the second equation:

$$\dot{\omega}_y = \lambda_y\omega_x$$
$$\ddot{\omega}_x + \lambda_x\lambda_y\omega_x = 0$$

having introduced:

$$\lambda_x = \frac{(I_z - I_y)\overline{\omega}_z + I_r\overline{\omega}_r}{I_x} = K_x\overline{\omega}_z + \frac{I_r}{I_x}\overline{\omega}_r$$
$$\lambda_y = -\frac{(I_x - I_z)\overline{\omega}_z - I_r\overline{\omega}_r}{I_y} = K_y\overline{\omega}_z + \frac{I_r}{I_y}\overline{\omega}_r$$

Stability condition now becomes:

$$\lambda_x\lambda_y > 0$$

Taking the case:

$$\lambda_x > 0$$
$$\lambda_y > 0$$

we have two inequalities whose solution depends on I_r and $\overline{\omega}_r$:

$$\begin{cases}(I_z - I_y)\overline{\omega}_z + I_r\overline{\omega}_r > 0 \\ (I_z - I_x)\overline{\omega}_z + I_r\overline{\omega}_r > 0\end{cases}$$

For given $\overline{\omega}_z$, $\overline{\omega}_r$, I_r we can evaluate the conditions on I_x, I_y, I_z that guarantee stability. This is normally not the best way to look at the problem, since in general we want to analyze stability for given $\overline{\omega}_z$, I_x, I_y, I_z, evaluating the conditions on I_r and $\overline{\omega}_r$. So, normally we evaluate the characteristics of the rotor that guarantee the stability of the overall system. If $\overline{\omega}_z = 0$then the system is stable for any $\overline{\omega}_r$, while if $\overline{\omega}_r = 0$ we have the same result of the simple spin satellite.

The most interesting case is the general case, for which stability holds if:

$$\begin{cases}(I_y - I_z)\overline{\omega}_z < I_r\overline{\omega}_r \\ (I_x - I_z)\overline{\omega}_z < I_r\overline{\omega}_r\end{cases}$$

This system of inequalities has always a set of solutions, since it is enough to solve the inequality with the greatest left side term.

Moreover, there is the condition:

$$\lambda_x < 0$$
$$\lambda_y < 0$$

that would lead to:

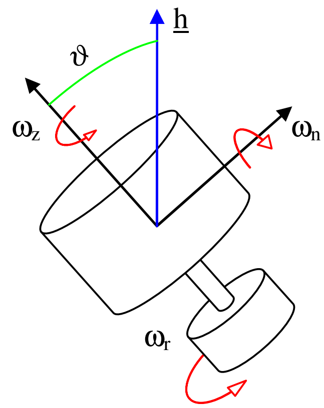
$$\begin{cases}(I_y - I_z)\overline{\omega}_z > I_r\overline{\omega}_r \\ (I_x - I_z)\overline{\omega}_z > I_r\overline{\omega}_r\end{cases}$$

This second set of inequalities are valid only for the case with no energy loss.
The dual spin satellite can then be stable even if rotating around an axis that is not the maximum inertia axis.

Effects of energy loss on axial symmetric dual spin satellite

The system angular momentum is:

$$\underline{h} = (I_z\omega_z + I_r\omega_r)\underline{k} + I_n\omega_n\underline{i}$$
$$h^2 = (I_z\omega_z + I_r\omega_r)^2 + (I_n\omega_n)^2$$
$$2h\dot{h} = 2(I_z\omega_z + I_r\omega_r)(I_z\dot{\omega}_z + I_r\dot{\omega}_r) + 2I_n^2\omega_n\dot{\omega}_n = 0$$



The kinetic energy is:

$$2T = I_z\omega_z^2 + I_r\omega_r^2 + I_n\omega_n^2$$
$$2\dot{T} = 2I_z\omega_z\dot{\omega}_z + 2I_r\omega_r\dot{\omega}_r + 2I_n\omega_n\dot{\omega}_n$$
$$\dot{T} = I_z\omega_z\dot{\omega}_z + I_r\omega_r\dot{\omega}_r + I_n\omega_n\dot{\omega}_n = \dot{T}_s + \dot{T}_r$$

We can evaluate an equivalent angular velocity as:

$$\omega_e = \frac{(I_z\omega_z + I_r\omega_r)}{I_n}$$
$$\Rightarrow I_n\omega_n\dot{\omega}_n = -\omega_e(I_z\dot{\omega}_z + I_r\dot{\omega}_r) \quad \text{since } 2\mathrm{h}\dot{\mathrm{h}} = 0$$
$$\dot{T}_s + \dot{T}_r = I_z\dot{\omega}_z(\omega_z - \omega_e) + I_r\dot{\omega}_r(\omega_r - \omega_e)$$

It is an arbitrary distribution of the energy loss, with the first term relative to the satellite and the second relative to the rotor:

$$\dot{T}_s = I_z\dot{\omega}_z\lambda_z \qquad \text{misalignment of } \underline{\mathrm{h}}$$
$$\dot{T}_r = I_r\dot{\omega}_r\lambda_r \qquad \text{viscous energy loss due to the relative velocity}$$

with:

$$\lambda_z = (\omega_z - \omega_e)$$
$$\lambda_r = (\omega_r - \omega_e)$$

We therefore have:

$$I_n\omega_n\dot{\omega}_n = -\omega_e\left(\frac{\dot{T}_s}{\lambda_z} + \frac{\dot{T}_r}{\lambda_r}\right)$$

Now, evaluate angle ϑ:

$$\sin\vartheta = \frac{h_n}{h}$$

Taking the time derivative:

$$\dot{\vartheta}\cos\vartheta = \frac{I_n\dot{\omega}_n}{h}$$

$$\dot{\vartheta} = \frac{I_n\dot{\omega}_n}{h\cos\vartheta} = \frac{I_n\omega_n\dot{\omega}_n}{h\cos\vartheta\,\omega_n} = \frac{-\omega_e\left(\frac{\dot{T}_s}{\lambda_z} + \frac{\dot{T}_r}{\lambda_r}\right)}{h\cos\vartheta\,\omega_n} =$$
$$= -\frac{I_n\omega_e\left(\frac{\dot{T}_s}{\lambda_z} + \frac{\dot{T}_r}{\lambda_r}\right)}{h\cos\vartheta\, I_n\omega_n} = -\frac{I_n\omega_e\left(\frac{\dot{T}_s}{\lambda_z} + \frac{\dot{T}_r}{\lambda_r}\right)}{h\cos\vartheta\, h_n} =$$

$$= -\frac{I_n \omega_e \left(\frac{\dot{T}_s}{\lambda_z} + \frac{\dot{T}_r}{\lambda_r}\right)}{h\cos\vartheta \mathrm{h}\sin\vartheta} = -\frac{2\mathrm{I}_n \omega_e \left(\frac{\dot{T}_s}{\lambda_z} + \frac{\dot{T}_r}{\lambda_r}\right)}{h^2 \sin 2\vartheta} =$$
$$= -\frac{2\mathrm{I}_n \omega_e}{h^2 \sin 2\vartheta}\left(\frac{\dot{T}_s}{\lambda_z} + \frac{\dot{T}_r}{\lambda_r}\right) = -A\left(\frac{\dot{T}_s}{\lambda_z} + \frac{\dot{T}_r}{\lambda_r}\right)$$

since:

$$0 < \vartheta < \pi/2$$

therefore $\sin(\vartheta) > 0$ and where A > 0.
For stability we should have $\dot{\vartheta} < 0$, that is:

$$\left(\frac{\dot{T}_s}{\lambda_z} + \frac{\dot{T}_r}{\lambda_r}\right) > 0$$

Assuming the contribution of the nominal value of h is given mostly by the presence of the rotor, that is:

$$I_r \omega_r >> I_z \omega_z$$

we then have:

$$\omega_e = \frac{I_r \omega_r}{I_n}$$
$$\lambda_z = \omega_z - \frac{I_r \omega_r}{I_n} \cong -\frac{I_r \omega_r}{I_n}$$
$$\lambda_r = \omega_r - \frac{I_r \omega_r}{I_n}$$

In the case $I_r >> I_n$ (case only theoretical, since this would imply a rotor bigger than the satellite):

$$\begin{cases} \lambda_z < 0 \\ \lambda_r < 0 \end{cases} \quad \rightarrow \quad \sum \frac{\dot{T}}{\lambda} > 0 \quad \rightarrow \quad \text{stable}$$

The more general case is then $I_r < I_n$, for which:

$$\frac{I_n \dot{T}_s}{(I_n \omega_z - I_r \omega_r)} + \frac{I_n \dot{T}_r}{(I_n \omega_r - I_r \omega_r)} > 0$$

This stability condition is transformed into a condition on the energy loss terms:

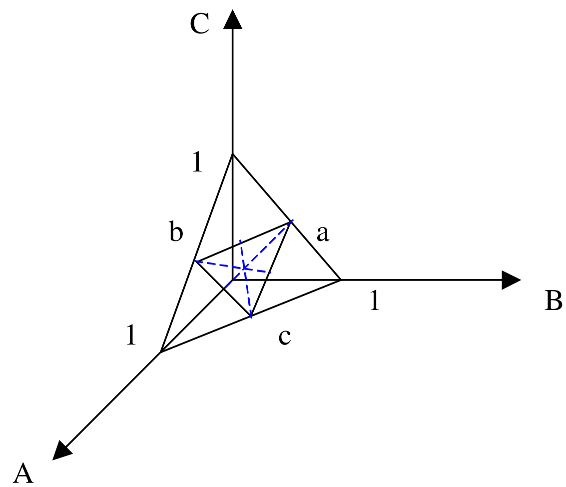
$$\dot{T}_r > \frac{\dot{T}_s (I_n \omega_r - I_r \omega_r)}{(I_r \omega_r - I_n \omega_z)}$$

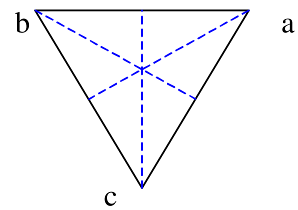
In general, the rotor energy loss can be tuned with some viscous damping mechanism. Notice that now we have a condition that is no longer purely geometrical. In fact, in the case $\dot{\omega}_r = 0$, then the rotor will show no energy loss and $\dot{T}_r$ would be zero, so that even with energy loss in the satellite the system would be unstable. Therefore the rotor should generate some energy loss to ensure stability.

Stability diagrams

We can now take a look at a few diagrams, known as canonical representations, used to evaluate attitude stability of satellites.

Let consider a plane including the three points on the axes at unit distance from the origin, represented by the equation:





$$A + B + C = 1$$

or, defining:

$$A = \frac{I_x}{I_x + I_y + I_z} \quad ; \quad \mathrm{B} = \frac{I_y}{I_x + I_y + I_z} \quad ; \quad \mathrm{C} = \frac{I_z}{I_x + I_y + I_z}$$

$$\frac{I_x}{\sum I} + \frac{I_y}{\sum I} + \frac{I_z}{\sum I} = 1$$

Connecting the three mid points on the sides of the triangle, we form a second triangle (a,b,c) that includes all the possible combinations of inertia moments.

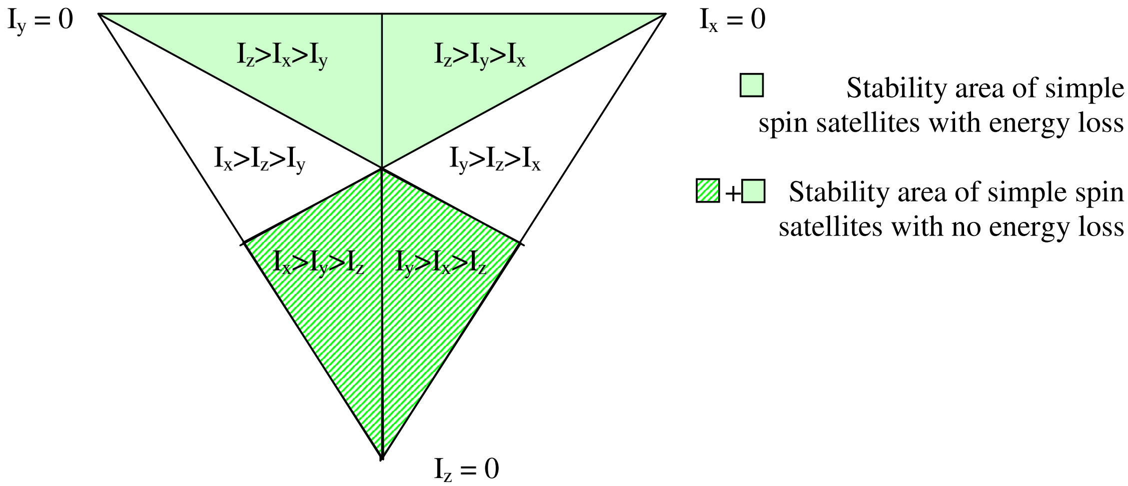
In fact, the following holds:

- In point *a*: $I_x = 0$ and $I_y = I_z$
- In point *b*: $I_y = 0$ and $I_x = I_z$
- In point *c*: $I_z = 0$ and $I_y = I_x$

Along the bisectors two inertia moments are equal:

- Along the bisector from *a*: $I_z = I_y$
- Along the bisector from *b*: $I_z = I_x$
- Along the bisector from *c*: $I_x = I_y$

At the intersection of the three bisectors the three inertia moments are all equal.



Alternatively, through a simple change of variables, it is possible to represent in the same way the plane as a function of K_x, K_y, K_z, in fact:

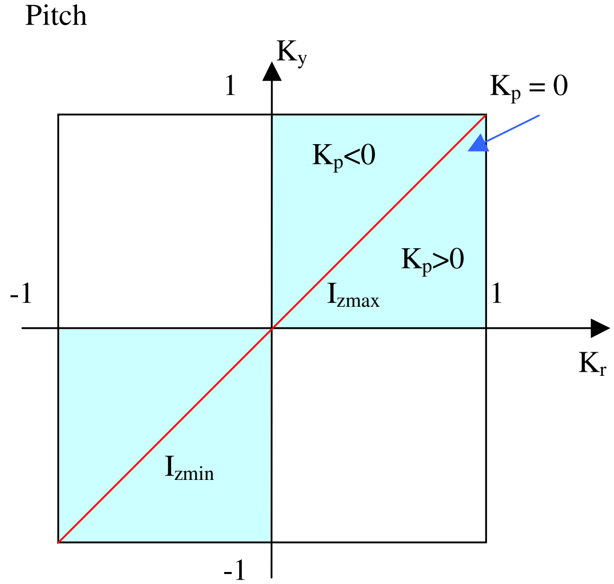
$$K_x = \frac{I_z - I_y}{I_x}$$

$$K_y = \frac{I_z - I_x}{I_y}$$

$$K_z = \frac{I_y - I_x}{I_z}$$

It is then possible to reduce the representation to the K_x – K_y plane. In this case, typically the coefficients are renamed as K_p, K_r, and K_y, where the correspondence with the previously defined K_x, K_y, and K_z coefficients is:

$K_x = K_y$	Yaw
$K_y = K_r$	Roll
$K_z = K_p$	Pitch



It is possible to see that:

$$\begin{cases} I_x K_y = I_z - I_y \Rightarrow I_z = I_x K_y + I_y \\ I_y K_r = I_z - I_x \Rightarrow I_z = I_y K_r + I_x \end{cases}$$
$$\Rightarrow I_x K_y + I_y = I_y K_r + I_x$$

Therefore:

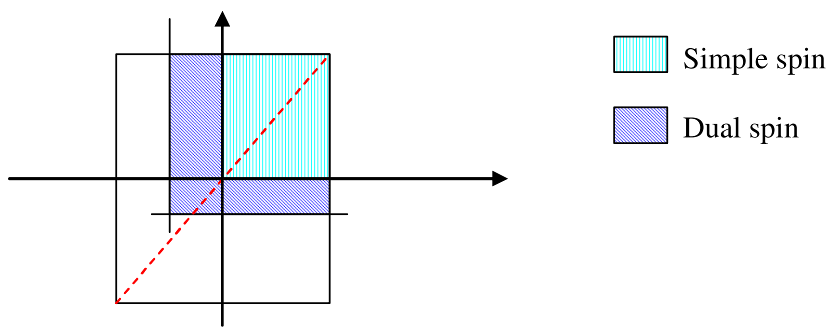
$$(K_y - 1) = \frac{I_y}{I_x}(K_r - 1)$$

If $I_y > I_x$ we then have $K_p > 0$, $(K_r - 1) < 0$, $(K_y - 1) < 0$, so that $(K_y - 1) < (K_r - 1) \quad \rightarrow \quad K_r > K_y$.

The simple spin satellite is stable in the dashed regions of the following figure, while for the dual spin satellite the stability holds if:

$$\lambda_x = K_y \overline{\omega}_z + \frac{I_r}{I_x}\overline{\omega}_r > 0$$
$$\lambda_y = K_r \overline{\omega}_z + \frac{I_r}{I_y}\overline{\omega}_r > 0$$
$$K_y > -\frac{I_r}{I_x}\frac{\overline{\omega}_r}{\overline{\omega}_z}$$
$$K_r > -\frac{I_r}{I_y}\frac{\overline{\omega}_r}{\overline{\omega}_z}$$

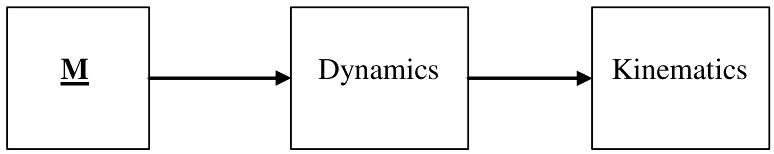
If the satellite angular velocity and the rotor angular velocity have the same sign, the stability region is increased as shown in the figure.



If the angular velocity of satellite and rotor have opposite signs, then the stability region is reduced.

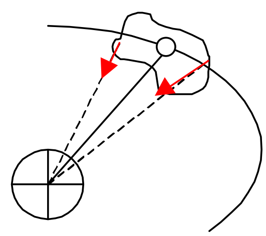
Disturbing torques

Now we can analyze the disturbing torques that act on the satellite:

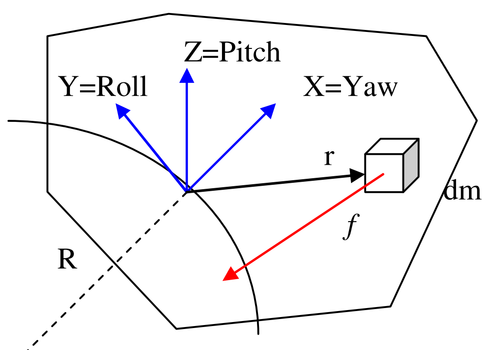


Start with the torque due to gravity gradient:

Gravity gradient torque



The gravity field is not uniform, therefore there could be a torque acting on the satellite. This is mainly true for large satellites, and even if the resulting torque is small the effect can be considerable due to the long time of action.



We can evaluate the torque generated by the elementary force *f*, due to the gravity acting on the elementary mass dm:

$$dM = -\underline{r} \wedge \frac{Gm_t dm}{|R+r|^3}\left(\underline{R} + \underline{r}\right)$$

$$M = -\int_B \underline{r} \wedge \frac{Gm_t}{|R+r|^3}\left(\underline{R} + \underline{r}\right) dm$$

r is much smaller than R, therefore it can be considered as a perturbation:

$$|R+r|^{-3}=\left(R^2+r^2+2\underline{R}\cdot\underline{r}\right)^{-\frac{3}{2}}=R^{-3}\left(1+\frac{r^2}{R^2}+2\frac{\underline{R}\cdot\underline{r}}{R^2}\right)^{-\frac{3}{2}}\cong R^{-3}\left(1+2\frac{\underline{R}\cdot\underline{r}}{R^2}\right)^{-\frac{3}{2}}$$

taking the series expansion at $r=0 \Rightarrow\cong R^{-3}\left(1-\frac{3}{2}(1)^{-\frac{5}{2}}2\frac{\underline{R}\cdot\hat{r}}{R^2}r\right)=R^{-3}\left(1-3\frac{\underline{R}\cdot\underline{r}}{R^2}\right)$

Then:

$$M=-\frac{Gm_t}{R^3}\int_B \underline{r}\wedge\left(1-3\frac{\underline{R}\cdot\underline{r}}{R^2}\right)\left(\underline{R}+\underline{r}\right)dm$$

If the reference system has origin in the center of mass:

$$\int \underline{r}\wedge\left(\underline{r}+\underline{R}\right)dm=\int\left(\underline{r}\wedge\underline{r}\right)+\left(\underline{r}\wedge\underline{R}\right)dm=-\underline{R}\wedge S_0=0$$

so that:

$$M=-\frac{Gm_t}{R^3}\int_B \underline{r}\wedge\left(-3\frac{\underline{R}\cdot\underline{r}}{R^2}\right)\left(\underline{R}+\underline{r}\right)dm=-\frac{Gm_t}{R^5}\int_B \underline{r}\wedge\left(\underline{R}+\underline{r}\right)\left(-3\underline{R}\cdot\underline{r}\right)dm$$

Finally, we get to:

$$M=\frac{3Gm_t}{R^5}\int_B\left(\underline{r}\cdot\underline{R}\right)\left(\underline{r}\wedge\underline{R}\right)dm$$

Taking as reference system the LVLH frame (x-yaw, y-roll, z-pitch), we have:

$$\underline{r}=x\underline{i}+y\underline{j}+z\underline{k}$$
$$\underline{R}=R\underline{i}$$

So that:

$$M=\frac{3Gm_t}{R^5}\int_B Rx\left(z\underline{j}-y\underline{k}\right)Rdm=\frac{3Gm_t}{R^3}\int_B x\left(z\underline{j}-y\underline{k}\right)dm=\frac{3Gm_t}{R^3}\int_B\left(xz\underline{j}-xy\underline{k}\right)dm$$
$$=\frac{3Gm_t}{R^3}\left(-I_{xz}\underline{j}+I_{xy}\underline{k}\right)$$

We therefore have a torque with components in roll and pitch, and in case one principal axis is aligned with the yaw axis then the torque will vanish, since in this case I_{xz} e I_{xy} are zero. Furthermore the torque is inversely proportional to the third power of the distance from the attracting body (Earth).
In order to evaluate the effect of the torque on the satellite dynamics, this has to be evaluated in the principal axis frame and inserted in the Euler equations.
Taking as reference the principal inertia axes, the expression of $\underline{r}$ does not change but:

$$\underline{R}=R\left(c_1\underline{i}+c_2\underline{j}+c_3\underline{k}\right)$$

where c_1, c_2, c_3 are the direction cosines of the radial direction in the principal axes. Then:

$$M=\frac{3Gm_t}{R^3}\int_B\left(xc_1+yc_2+zc_3\right)\begin{pmatrix}yc_3-zc_2\\zc_1-xc_3\\xc_2-yc_1\end{pmatrix}dm$$

Evaluating all the terms under the integral sign, considering that the reference is principal inertia, we have:

$$M = \frac{3Gm_t}{R^3}\int_B \begin{pmatrix} (y^2 - z^2)c_2c_3 \\ (z^2 - x^2)c_1c_3 \\ (x^2 - y^2)c_1c_2 \end{pmatrix} dm = \frac{3Gm_t}{R^3}\begin{Bmatrix} (I_z - I_y)c_2c_3 \\ (I_x - I_z)c_1c_3 \\ (I_y - I_x)c_1c_2 \end{Bmatrix}$$

Therefore if one of the principal axes is aligned with the radial direction the torque is zero because only one of the direction cosines is non-zero.

Stability of simple spin satellites subject to gravity gradient torque

In this case the Euler equations become:

$$\begin{cases} I_x\dot{\omega}_x + (I_z - I_y)\omega_z\omega_y = \dfrac{3Gm_t}{R^3}(I_z - I_y)c_3c_2 \\ I_y\dot{\omega}_y + (I_x - I_z)\omega_x\omega_z = \dfrac{3Gm_t}{R^3}(I_x - I_z)c_1c_3 \\ I_z\dot{\omega}_z + (I_y - I_x)\omega_y\omega_x = \dfrac{3Gm_t}{R^3}(I_y - I_x)c_2c_1 \end{cases}$$

This torque is generated by the same force field that causes the orbital motion, so it is acting continuously, and it depends on the attitude of the satellite (the orientation of the radial direction in the principal inertia frame). We can introduce a set of three small Euler angles to define the relative orientation of the principal axes and the LVLH frame:

$$\begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} = \begin{bmatrix} \dot{\alpha}_x \\ \dot{\alpha}_y \\ \dot{\alpha}_z \end{bmatrix} + \begin{bmatrix} 1 & \alpha_z & -\alpha_y \\ -\alpha_z & 1 & \alpha_x \\ \alpha_y & -\alpha_x & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ n \end{bmatrix}$$
$$\begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = \begin{bmatrix} 1 & \alpha_z & -\alpha_y \\ -\alpha_z & 1 & \alpha_x \\ \alpha_y & -\alpha_x & 1 \end{bmatrix}\begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

In this way we can write the system dynamics as a function of the Euler angles and their derivatives, including the gravity gradient torque that now becomes an integral part of the system dynamics. Assuming that n is equal to one rotation per orbit, then:

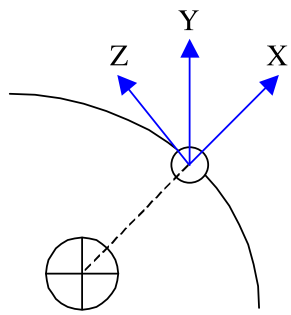
$$n^2 = \frac{V^2}{R^2} = \frac{1}{R^2}\left(\sqrt{\frac{K}{R}}\right)^2 = \frac{K}{R^3} = \frac{Gm_t}{R^3}$$

Therefore, in case of circular orbit, the attitude dynamics including gravity gradient becomes:

$$\begin{cases} I_x\dot{\omega}_x + (I_z - I_y)\omega_z\omega_y = 3n^2(I_z - I_y)c_3c_2 \\ I_y\dot{\omega}_y + (I_x - I_z)\omega_x\omega_z = 3n^2(I_x - I_z)c_1c_3 \\ I_z\dot{\omega}_z + (I_y - I_x)\omega_y\omega_x = 3n^2(I_y - I_x)c_2c_1 \end{cases}$$

where n is the nominal angular velocity along the orbit.

In this set of equations we have a twofold coupling between angular velocities and angles. Attitude is represented by the direction cosines c_1, c_2, c_3.
Assuming the principal inertia axes are sufficiently close to the corresponding axes of the LVLH frame, we can write:



$$\begin{Bmatrix}\omega_x\\ \omega_y\\ \omega_z\end{Bmatrix} = \begin{bmatrix}1 & \alpha_z & -\alpha_y\\ -\alpha_z & 1 & \alpha_x\\ \alpha_y & -\alpha_x & 1\end{bmatrix}\begin{Bmatrix}\dot{\alpha}_x\\ \dot{\alpha}_y\\ \dot{\alpha}_z + n\end{Bmatrix}$$

$$\begin{Bmatrix}c_1\\ c_2\\ c_3\end{Bmatrix} = \begin{bmatrix}1 & \alpha_z & -\alpha_y\\ -\alpha_z & 1 & \alpha_x\\ \alpha_y & -\alpha_x & 1\end{bmatrix}\begin{Bmatrix}1\\ 0\\ 0\end{Bmatrix}$$

We can now substitute angular velocities and direction cosines with the corresponding terms depending on angles and their derivatives. Neglecting the terms that include two angles or derivatives (small terms) we get to the linearized form:

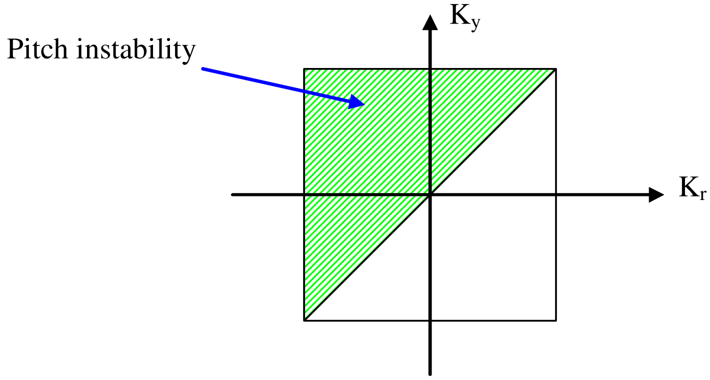
$$\begin{cases} I_x\ddot{\alpha}_x + \left(I_z - I_y - I_x\right)n\dot{\alpha}_y + \left(I_z - I_y\right)n^2\alpha_x = 0\\ I_y\ddot{\alpha}_y + \left(I_x + I_y - I_z\right)n\dot{\alpha}_x + (I_z - I_x)n^2\alpha_y = 3n^2(I_x - I_z)\alpha_y\\ I_z\ddot{\alpha}_z = -3n^2\left(I_y - I_x\right)\alpha_z \end{cases}$$

That can be simplified into:

$$\begin{cases} I_x\ddot{\alpha}_x + \left(I_z - I_y - I_x\right)n\dot{\alpha}_y + \left(I_z - I_y\right)n^2\alpha_x = 0\\ I_y\ddot{\alpha}_y + \left(I_x + I_y - I_z\right)n\dot{\alpha}_x + 4n^2(I_z - I_x)\alpha_y = 0\\ I_z\ddot{\alpha}_z + 3n^2\left(I_y - I_x\right)\alpha_z = 0 \end{cases}$$

The third equation is decoupled from the first two. We can now look at the stability of the system. Starting from the z axis, the stability condition is:

$I_y > I_x$ therefore $K_p > 0 \;\Rightarrow\; K_r > K_y$



Consider now the first two equations. Divide the first by I_x and the second by I_y:

$$\begin{cases} \ddot{\alpha}_x + (K_y - 1)n\dot{\alpha}_y + K_y n^2 \alpha_x = 0 \\ \ddot{\alpha}_y + (1 - K_r)n\dot{\alpha}_x + 4K_r n^2 \alpha_y = 0 \end{cases}$$

The characteristic equation is:

$$\begin{aligned} &(s^2 + n^2 K_y)(s^2 + 4n^2 K_r) - [ns(K_y - 1)][ns(1 - K_r)] = 0 \\ &s^4 + n^2 s^2 (K_y + 4K_r + 1 - K_r - K_y + K_r K_y) + 4n^2 K_r K_y = 0 \\ &s^4 + n^2 s^2 (1 + 3K_r + K_r K_y) + 4n^2 K_r K_y = 0 \end{aligned}$$

The conditions for stability are:

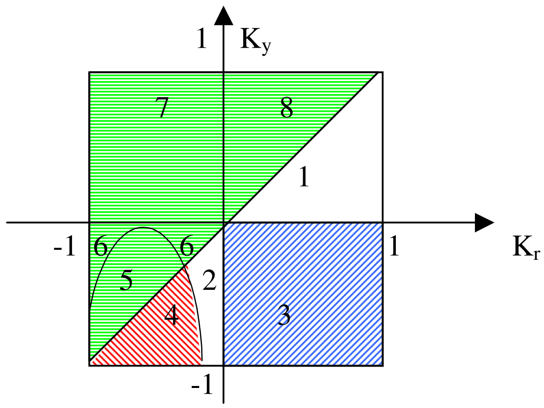
$$\begin{aligned} &x^2 + \mathrm{b}\mathrm{x} + c = 0 \\ &\begin{cases} c > 0 \\ \dfrac{-b \pm \sqrt{b^2 - 4\mathrm{c}}}{2} < 0 \\ b^2 - 4\mathrm{c} > 0 \end{cases} \\ &(1 + 3\mathrm{K}_r + K_r K_y)^2 > 16\mathrm{K}_r K_y \\ &K_r K_y > 0 \end{aligned}$$

That is:

$$\begin{cases} 1 + 3K_r + K_r K_y > 4\sqrt{K_r K_y} \\ K_r K_y > 0 \end{cases}$$

The presence of gravity gradient therefore reduces the stability region. In detail, the first condition can be analyzed by evaluating a few solutions for particular values of coefficient K_y:

$$\begin{aligned} &K_y = 0 \quad K_r = -\frac{1}{3} \\ &K_y = -1 \quad 1 + 2K_r = 4\sqrt{-K_r} \\ &K_y = K_r \quad 1 + 3K + K^2 = -4K \end{aligned}$$



The unmarked regions represent the stable regions. In fact, in region (1) we have:

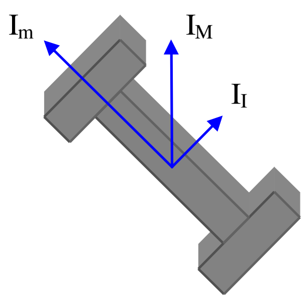
$I_z > I_y > I_x$ $\quad\quad$ $I_p > I_r > I_y$ $\quad$ always

In region (2) we have:

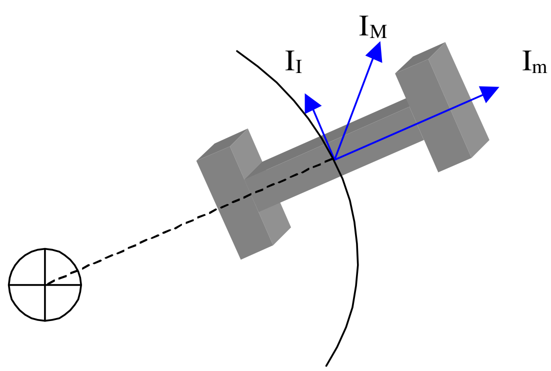
$I_y > I_x > I_z$ $\quad\quad$ $I_r > I_y > I_p$ $\quad$ limited by the additional curve

In the remaining regions we have some kind of instability. In regions (3) and (4) we have yaw and roll instability, in regions (5) and (7) yaw, roll and pitch instability while in regions (6) and (8) only pitch instability. Furthermore in region (8) we have $I_p > I_y > I_r$ while in region (7) $I_y > I_r > I_p$.
Now assume we have an orbiting satellite with three different inertia moments as in the following figure:

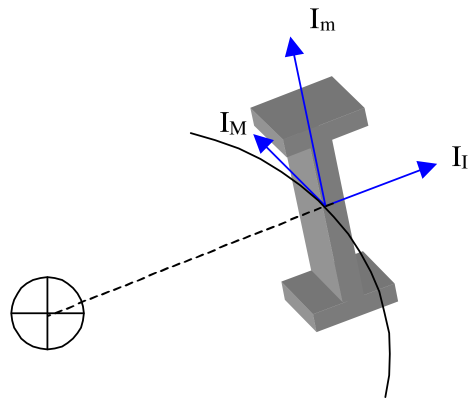
I_m = I minimum
I_M = I maximum
I_I = I intermediate



To have stability identified by the region (1), the satellite must be oriented in orbit as follows:



To have stability identified by the region (2) instead orientation must be:



In order to stabilize a generic configuration, we can have one mass positioned at an appropriate distance from the main body of the satellite, so to modify the inertia moments, but with the limitation due to proper orientation of the satellite, on the contrary we could generate instability.
In this case we could even solve the characteristic equation to evaluate the frequency of natural oscillation of the system. For the pitch equation we have:

$$s^2 + 3K_p n^2 = 0$$
$$\omega_p = n\sqrt{3K_p}$$

For the roll/yaw equations the solution is not trivial, in fact the characteristic equation is:

$$s^4 + n^2 s^2 \left(1 + 3\mathrm{K}_r + K_r K_y\right) + 4\mathrm{n}^4 K_r K_y = 0$$
$$\frac{2}{n^2}\omega_{ry}^2 = -\left(1 + 3K_r + K_r K_y\right) \pm \sqrt{\left(1 + 3K_r + K_r K_y\right)^2 - 16 K_r K_y}$$

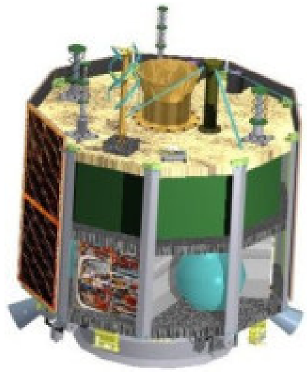
In the special case for which the satellite mass tends to be aligned with the yaw axis, we have $I_p = I_r$ and:

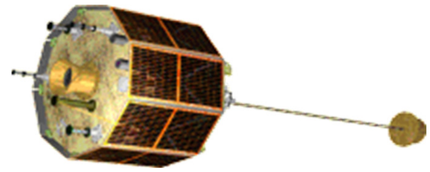
$$I_y \to 0 \Rightarrow K_r \to 1$$
$$K_p \to 1$$
$$K_y = 0$$

The solution of the characteristic equation is then:

$$\frac{2}{n^2}\omega_{ry}^2 = -8$$
$$\omega_{ry} = 2n$$
$$\omega_p = n\sqrt{3}$$

Disturbing torques acting with frequency close to 2n can exist for some geocentric inclined orbits. As example, in two orbit positions symmetric with respect to Earth it is possible to face the same atmospheric conditions and magnetic field conditions, so that resonant motion can be generated.





SSTL UOSAT-12

Gravity gradient torque is exploited for gravity gradient stabilization whereby a boom can be deployed to increase the relative principal moments of inertia. We note that the spacecraft is always stable with configuration (1) where $I_z > I_y > I_x$. In addition, we can observe that with a much larger I_z than the other inertias the gravity gradient torque is much larger for a particular relative attitude. This larger this relative value the more robustness the gravity gradient torque can give relative to other disturbance torques. A gravity gradient boom essentially increases I_z with respect to the other moments of inertia.

Dynamics of dual spin satellites subject to gravity gradient

A dual spin satellite can be subject to gravity gradient. In such case, its dynamics becomes:

$$\begin{cases} I_x\dot{\omega}_x + \left(I_z - I_y\right)\omega_z\omega_y + I_r\omega_r\omega_y = \dfrac{3Gm_t}{R^3}\left(I_z - I_y\right)c_3c_2 \\ I_y\dot{\omega}_y + \left(I_x - I_z\right)\omega_x\omega_z - I_r\omega_r\omega_x = \dfrac{3Gm_t}{R^3}\left(I_x - I_z\right)c_1c_3 \\ I_z\dot{\omega}_z + \left(I_y - I_x\right)\omega_y\omega_x + I_r\dot{\omega}_r = \dfrac{3Gm_t}{R^3}\left(I_y - I_x\right)c_2c_1 \end{cases}$$

The rotor could be aligned with an axis not coincident with any of the principal inertia axes, and in this case the equations of rotational motion are written starting from the expression of the angular momentum:

$$\underline{h} = I\underline{\omega} + I_r\omega_r\underline{r} = \begin{Bmatrix} I_x\omega_x + I_r\omega_r r_x \\ I_y\omega_y + I_r\omega_r r_y \\ I_z\omega_z + I_r\omega_r r_z \end{Bmatrix}$$

where $\underline{r}$ is the direction of the rotor axis.

Adopting this procedure, it is straightforward to extend the case to a general configuration with n rotors having distinct rotation axis.
Assuming the direction $\underline{r}$ is fixed (design parameter), by taking time derivatives we have, for the case of single rotor:

$$\frac{d\underline{h}}{dt} = \underline{M}$$

$$\frac{d\underline{h}}{dt} = \dot{\underline{h}} + \underline{\omega} \wedge \underline{h}$$

$$\begin{cases} I_x\dot{\omega}_x + (I_z - I_y)\omega_z\omega_y + I_r\dot{\omega}_r r_x + I_r\omega_r(\omega_y r_z - \omega_z r_y) = 0 \\ I_y\dot{\omega}_y + (I_x - I_z)\omega_x\omega_z + I_r\dot{\omega}_r r_y + I_r\omega_r(\omega_z r_x - \omega_x r_z) = 0 \\ I_z\dot{\omega}_z + (I_y - I_x)\omega_y\omega_x + I_r\dot{\omega}_r r_z + I_r\omega_r(\omega_x r_y - \omega_y r_x) = 0 \end{cases}$$

We now seek equilibrium configurations characterized by ω_x and ω_y different from zero.

$$\begin{cases} I_r\omega_r\omega_y r_z = 0 \\ -I_r\omega_r\omega_x r_z = 0 \\ (I_y - I_x)\omega_y\omega_x + I_r\omega_r(\omega_x r_y - \omega_y r_x) = 0 \end{cases}$$

Solution of the first two equations leads to $r_z = 0$ (I_r, ω_r, ω_x, ω_y, are all non-zero).
From the last equation we see that the rotor axis must lie on the plane in which we want a non-zero angular velocity. Adding now the gravity gradient:

$$\begin{cases} I_x\dot{\omega}_x + (I_z - I_y)\omega_z\omega_y + I_r\dot{\omega}_r r_x + I_r\omega_r(\omega_y r_z - \omega_z r_y) = \dfrac{3Gm_t}{R^3}(I_z - I_y)c_2c_3 \\ I_y\dot{\omega}_y + (I_x - I_z)\omega_x\omega_z + I_r\dot{\omega}_r r_y + I_r\omega_r(\omega_z r_x - \omega_x r_z) = \dfrac{3Gm_t}{R^3}(I_x - I_z)c_1c_3 \\ I_z\dot{\omega}_z + (I_y - I_x)\omega_y\omega_x + I_r\dot{\omega}_r r_z + I_r\omega_r(\omega_x r_y - \omega_y r_x) = \dfrac{3Gm_t}{R^3}(I_y - I_x)c_2c_1 \end{cases}$$

Consider all the angular velocity components as non-zero, and analyze equilibrium:

$$\begin{cases} (I_z - I_y)\omega_z\omega_y + I_r\omega_r(\omega_y r_z - \omega_z r_y) = \dfrac{3Gm_t}{R^3}(I_z - I_y)c_2c_3 \\ (I_x - I_z)\omega_x\omega_z + I_r\omega_r(\omega_z r_x - \omega_x r_z) = \dfrac{3Gm_t}{R^3}(I_x - I_z)c_1c_3 \\ (I_y - I_x)\omega_y\omega_x + I_r\omega_r(\omega_x r_y - \omega_y r_x) = \dfrac{3Gm_t}{R^3}(I_y - I_x)c_2c_1 \end{cases}$$

This system represents equilibrium in the LVLH frame. Adopting a generic attitude parameterization, non-necessarily linearized since in principle the relative attitude can be a large rotation, we get to three equations that allow evaluating the equilibrium configurations in the LVLH frame.

$$\begin{cases} f_1(\phi, \vartheta, \psi, r_x, r_y, r_z) = 0 \\ f_2(\phi, \vartheta, \psi, r_x, r_y, r_z) = 0 \\ f_3(\phi, \vartheta, \psi, r_x, r_y, r_z) = 0 \end{cases}$$

The solution of the system of equations can lead to equilibrium configurations with the principal axes not aligned with the LVLH frame, with appropriate orientation of the rotor. This allows having Earth pointing satellites whose principal axes are not aligned with the radial direction.

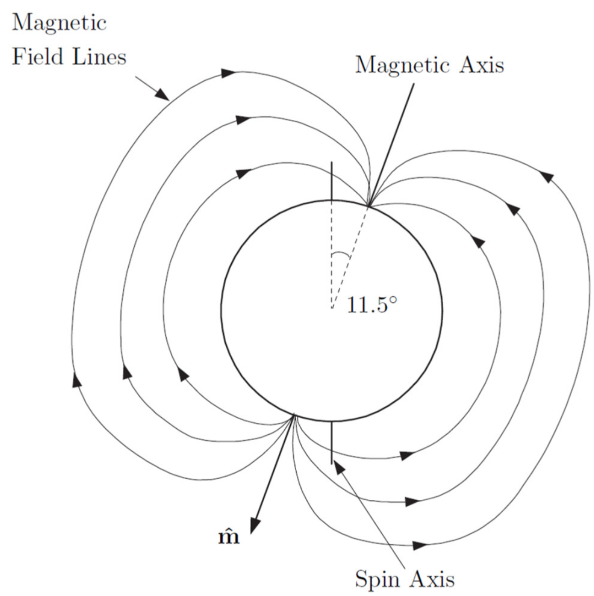
Magnetic torques

The magnetic torque is generated according to the general law $\underline{M} = \underline{m} \wedge \underline{B}$ where $\underline{\mathrm{m}}$ is the residual magnetic induction due to parasitic currents in the satellite and $\underline{\mathrm{B}}$ is the Earth's magnetic field.

Normally, m is an undesired effect, while B is always somehow present. There are models that, given the satellite position, allow evaluating its components. Magnetic torques on a satellite therefore do not depend on its inertia properties but rather on its position and attitude.

Magnetic field model

The Earth's magnetic field is conceptually similar to that generated by a dipole inclined with respect to the Earth's axis by about 11.5 degrees. It is not a fixed field, it slowly rotates and decreases its amplitude in time. The field is more intense at the poles, along the equator its intensity is about half of the polar value, and decreases with the inverse cubic power of the distance to the ground.
A uniform magnetic field would generate sinusoidal iso-intensity curves on the ground, but this is not the case due to perturbations. The magnetic field can be considered similar to a dipole field only above 7000 km from ground. For distances from ground beyond 10 Earth radii (more than the Geosynchronous distance) the magnetic field is deformed by the effect of the solar radiation.



The magnetic field B can be modeled in precise way as the gradient of a scalar potential V, that is B=-∇V, where V is the potential function, normally modeled as a series expansion of spherical harmonics

$$V(r,\theta,\varphi) = R\sum_{n=1}^{k}\left(\frac{R}{r}\right)^{n+1}\sum_{m=0}^{n}\left(g_n^m\cos(m\varphi) + h_n^m\sin(m\varphi)\right)P_n^m(\theta)$$

R is the Earth's equatorial radius (6371.2 km, IGRF standard), g_n^m and h_n^m are called Gaussian (available in tabular form), r, θ and φ are the spherical coordinates of the position of the satellite (distance from the center of the Earth, colatitude and East longitude from Greenwich). Coefficients g and h are evaluated from experimental data and are known up to order 13. These coefficients are subject to time variation. The following table reports as example the coefficients for years 1995 and 2000, expressed in nT.

		IGRF 1995		IGRF 2000	
n	m	g_n^m	h_n^m	g_n^m	h_n^m
1	0	-29682	-	-29615	-
1	1	-1789	5318	-1728	5186
2	0	-2197	-	-2267	-
2	1	3074	-2356	3072	-2478
2	2	1685	-425	1672	-458
3	0	1329	-	1341	-
3	1	-2268	-263	-2290	-227
3	2	1249	302	1253	296
3	3	769	-406	715	-492
4	0	941	-	935	-
4	1	782	262	787	272
4	2	291	-232	251	-232
4	3	-421	98	-405	119
4	4	116	-301	110	-304

		IGRF 1995		IGRF 2000	
n	m	g_n^m	h_n^m	g_n^m	h_n^m
5	0	-210	-	-217	-
5	1	352	44	351	44
5	2	237	157	222	172
5	3	-122	-152	-131	-134
5	4	-167	-64	-169	-40
5	5	-26	99	-12	107
6	0	66	-	72	-
6	1	64	-16	68	-17
6	2	65	77	74	64
6	3	-172	67	-161	65
6	4	2	-57	-5	-61
6	5	17	4	17	1
6	6	-94	28	-91	44

		IGRF 1995		IGRF 2000	
n	m	g_n^m	h_n^m	g_n^m	h_n^m
7	0	78	-	79	-
7	1	-67	-77	-74	-65
7	2	1	-25	0	-24
7	3	29	3	33	6
7	4	4	22	9	24
7	5	8	16	7	15
7	6	10	-23	8	-25
7	7	-2	-3	-2	-6
8	0	24	-	25	-
8	1	4	12	6	12
8	2	-1	-20	-9	-22
8	3	-9	7	-8	8
8	4	-14	-21	-17	-21
8	5	4	12	9	15

		IGRF 1995		IGRF 2000	
n	m	g_n^m	h_n^m	g_n^m	h_n^m
8	6	5	10	7	9
8	7	0	-17	-8	-16
8	8	-7	-10	-7	-3
9	0	4	-	5	-
9	1	9	-19	9	-20
9	2	1	15	3	13
9	3	-12	11	-8	12
9	4	9	-7	6	-6
9	5	-4	-7	-9	-8
9	6	-2	9	-2	9
9	7	7	7	9	4
9	8	0	-8	-4	-8
9	9	-6	1	-8	5

The components of $\underline{B}$ are then

$$B_r = \frac{-\partial V}{\partial r}$$
$$B_\theta = \frac{-1}{r}\frac{\partial V}{\partial \theta}$$
$$B_\varphi = \frac{-1}{r\sin\theta}\frac{\partial V}{\partial \varphi}$$

where B_r represents the radial component, positive outward, B_θ the coelevation component, positive if directed towards south, and B_φ the azimuth component, positive towards east. Coefficients g_n^m and h_n^m are evaluated in the assumption that polynomials P_n^m are normalized according to (Schmidt)

$$\int_0^{\pi}\left[P_n^m(\theta)\right]^2 \sin\theta\, d\theta = \frac{2\left(2-\delta_m^0\right)}{2n+1}$$

where δ_m^0 is the Kronecker delta, δ_i^j=1 if i=j, =0 if i≠j. In alternative, a normalization due to Gauss is possible, according to which

$$P_n^m = S_{\mathrm{n,m}} P^{\mathrm{n,m}}$$

with

$$S_{\mathrm{n,m}} = \left[\frac{(2-\delta_m^0)(n-m)!}{(n+m)!}\right]^{\frac{1}{2}} \frac{(2\mathrm{n}-1)!!}{(n-m)!}$$

Coefficients $S_{n,m}$ are independent from r, θ and φ and can be evaluated from the Gaussian coefficients g_n^m and h_n^m, following the rule:

$$g^{\mathrm{n,m}} = S_{\mathrm{n,m}} g_n^m$$
$$h^{\mathrm{n,m}} = S_{\mathrm{n,m}} h_n^m$$
$$S_{0,0} = 1$$
$$S_{\mathrm{n,0}} = S_{\mathrm{n\text{-}1,0}} \frac{(2\mathrm{n}-1)}{n} \quad \text{for n} \geq 1$$
$$S_{\mathrm{n,m}} = S_{\mathrm{n,m\text{-}1}} \left[\frac{(\delta_m^1+1)(n-m+1)}{(n+m)}\right]^{\frac{1}{2}} \quad \text{for m} \geq 1$$

In a similar way it is possible to get recursive formulas for polynomials $P^{n,m}$

$$P^{0,0} = 1$$
$$P^{\mathrm{n,n}} = \sin\theta\ \mathrm{P}^{n-1,\mathrm{n}-1}$$
$$P^{\mathrm{n,m}} = \cos\theta\ \mathrm{P}^{n-1,\mathrm{m}} - K^{\mathrm{n,m}} P^{n-2,\mathrm{m}}$$

with

$$K^{\mathrm{n,m}} = 0 \quad \text{for n} = 1$$
$$K^{\mathrm{n,m}} = \frac{(\mathrm{n\text{-}1})^2 - m^2}{(2\mathrm{n}-1)(2\mathrm{n}-3)} \quad \text{for n} > 1$$

With these models, the components of magnetic field $\underline{B}$ are

$$B_r = \frac{-\partial V}{\partial r} = \sum_{n=1}^{k}\left(\frac{R}{r}\right)^{n+2}(n+1)\sum_{m=0}^{n}\left(g^{\mathrm{n,m}}\cos(\mathrm{m\varphi}) + h^{\mathrm{n,m}}\sin(\mathrm{m\varphi})\right)P^{\mathrm{n,m}}(\theta)$$
$$B_\theta = \frac{-1}{r}\frac{\partial V}{\partial \theta} = -\sum_{n=1}^{k}\left(\frac{R}{r}\right)^{n+2}\sum_{m=0}^{n}\left(g^{\mathrm{n,m}}\cos(\mathrm{m\varphi}) + h^{\mathrm{n,m}}\sin(\mathrm{m\varphi})\right)\frac{\partial P^{\mathrm{n,m}}(\theta)}{\partial \theta}$$
$$B_\varphi = \frac{-1}{\mathrm{r}\sin\theta}\frac{\partial V}{\partial \varphi} = \frac{-1}{\sin\theta}\sum_{n=1}^{k}\left(\frac{R}{r}\right)^{n+2}\sum_{m=0}^{n} m\left(-g^{\mathrm{n,m}}\sin(\mathrm{m\varphi}) + h^{\mathrm{n,m}}\cos(\mathrm{m\varphi})\right)P^{\mathrm{n,m}}(\theta)$$

In a geocentric inertial reference, the components of $\underline{B}$ are then

$$B_1 = \left(B_r\cos(\delta) + B_\theta\sin(\delta)\right)\cos(\alpha) - B_\varphi\sin(\alpha)$$

$$B_2 = \left(B_r\cos(\delta) + B_\theta\sin(\delta)\right)\sin(\alpha) + B_\varphi\cos(\alpha)$$
$$B_3 = \left(B_r\sin(\delta) - B_\theta\cos(\delta)\right)$$

where δ is the satellite declination (δ=π/2-θ) and α is the right ascension, linked to the longitude φ by the relation $\alpha=\varphi+\alpha_G$, where α_G is the Greenwich meridian right ascension.

Dipole model

As the distance from ground increases, the magnetic field can be more and more approximated to a dipole, obtained by taking only the first term of the series expansion.

$$V(\mathrm{r},\theta,\varphi) = \frac{R^3}{r^2}\left[g_1^0P_1^0(\theta) + (g_1^1\cos(\varphi) + h_1^1\sin(\varphi)P_1^1(\theta)\right] =$$
$$= \frac{1}{r^2}\left[g_1^0R^3\cos(\theta) + g_1^1R^3\cos(\varphi)\sin(\theta) + h_1^1R^3\sin(\varphi)\sin(\theta)\right]$$

We then have:

$$B_r = 2\left(\frac{R}{r}\right)^3\left[g_1^0\cos(\theta) + g_1^1\cos(\varphi)\sin(\theta) + h_1^1\sin(\varphi)\sin(\theta)\right]$$
$$B_\theta = \left(\frac{R}{r}\right)^3\left[g_1^0\sin(\theta) - g_1^1\cos(\varphi)\cos(\theta) - h_1^1\sin(\varphi)\cos(\theta)\right]$$
$$B_\phi = \left(\frac{R}{r}\right)^3\left[g_1^1\sin(\varphi) - h_1^1\cos(\varphi)\right]$$

An alternative form considers the vector $\underline{\mathrm{B}}$ as a dipole of given amplitude and orientation, both evaluated from the previous expression, so that

$$H_0 = \left((g_1^0)^2 + (g_1^1)^2 + \left(h_1^1\right)^2\right)^{1/2}$$
$$\underline{B}(\underline{r}) = \frac{R^3H_0}{r^3}\left[3(\underline{\hat{m}}\cdot\underline{\hat{r}})\underline{\hat{r}} - \underline{\hat{m}}\right] \quad \text{(\^{}indicates unit vector)}$$

where $\underline{\mathrm{r}}$ is the position vector at the point where vector $\underline{\mathrm{B}}$ is evaluated. In the geocentric inertial frame, the components of B can be calculated starting from a unit dipole vector, as

$$\underline{\hat{m}} = \begin{Bmatrix} \sin\theta_m^{'}\cos\alpha_m \\ \sin\theta_m^{'}\sin\alpha_m \\ \cos\theta_m^{'} \end{Bmatrix} \quad \text{with } \alpha_m = \alpha_{G0} + \frac{\mathrm{d}\alpha_G}{\mathrm{dt}}(t - t_0) + \phi_m^{'}$$

where α_{G0} is the right ascension of the Greenwich meridian at the reference time t_0, $d\alpha_G/dt$ the average rotation speed of the Earth, θ'_m and φ'_m give the orientation of the magnetic dipole. Then we have

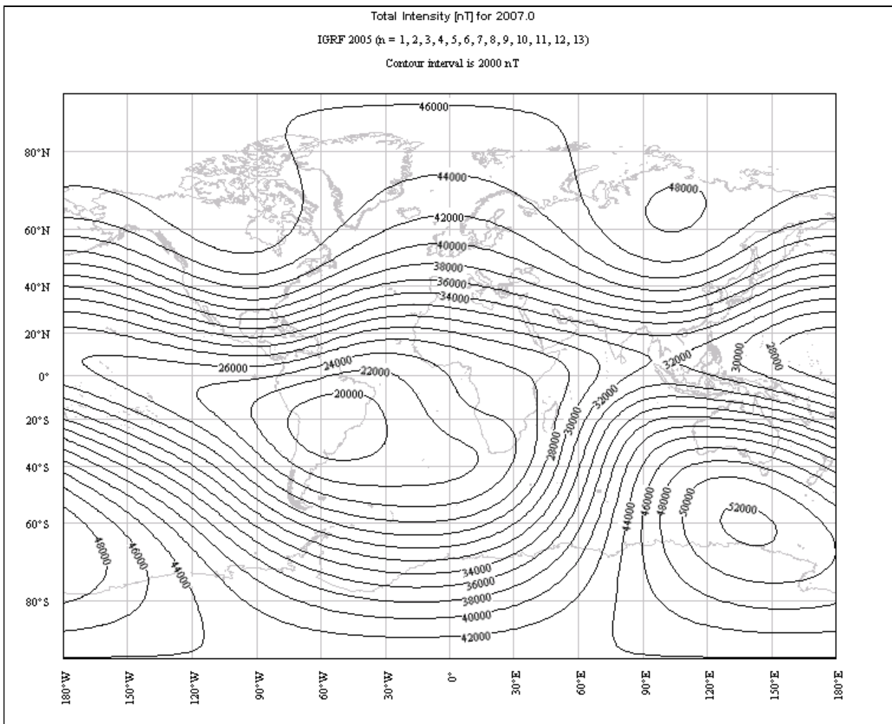
$$\theta_m^{'} = \mathrm{acos}\left(\frac{g_1^0}{H_0}\right) \quad ; \quad \phi_m^{'} = \mathrm{atan}\left(\frac{h_1^1}{g_1^1}\right)$$
$$\underline{\hat{m}}\cdot\underline{\hat{r}} = \hat{r}_x\sin\theta_m^{'}\cos\alpha_m + \hat{r}_y\sin\theta_m^{'}\sin\alpha_m + \hat{r}_z\cos\theta_m^{'}$$

As a consequence, the projection into the geocentric inertial frame gives:

$$B_x = \frac{R^3 H_0}{r^3}\left[3(\underline{\widehat{m}} \cdot \underline{\hat{r}})\hat{r}_x - \sin\theta_m^{'}\cos\alpha_m\right]$$
$$B_y = \frac{R^3 H_0}{r^3}\left[3(\underline{\widehat{m}} \cdot \underline{\hat{r}})\hat{r}_y - \sin\theta_m^{'}\sin\alpha_m\right]$$
$$B_z = \frac{R^3 H_0}{r^3}\left[3(\underline{\widehat{m}} \cdot \underline{\hat{r}})\hat{r}_z - \cos\theta_m^{'}\right]$$

Then to obtain the magnetic field as measured in the body frame we use $\underline{B}_B = A_{B/N}\underline{B}_N$.

Next follow some figures reporting the contour plots of the magnitude of the magnetic field for different altitudes from ground and for different orders of expansion of the magnetic field model. The images have been created using a software developed by the Finnish Meteorological Institute (http://www.ava.fmi.fi/MAGN/igrf/applet.html). It can be noticed that, as height from ground increases, the minimum expansion order that well approximates the true field decreases. At low altitude a fourth order expansion might not be sufficient, while above 20000 km even the dipole model can be acceptable.



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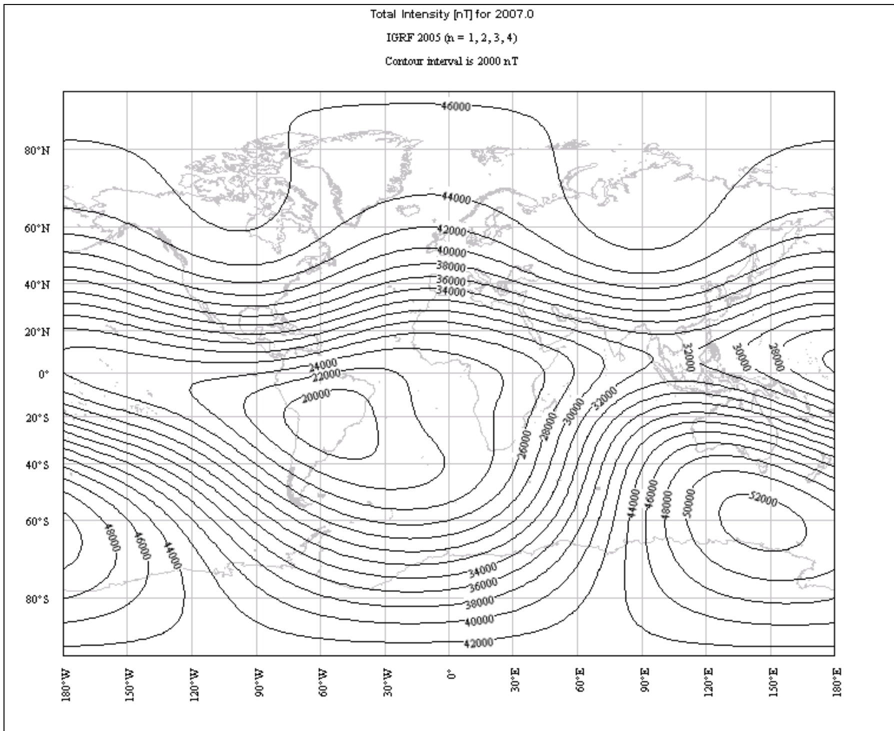
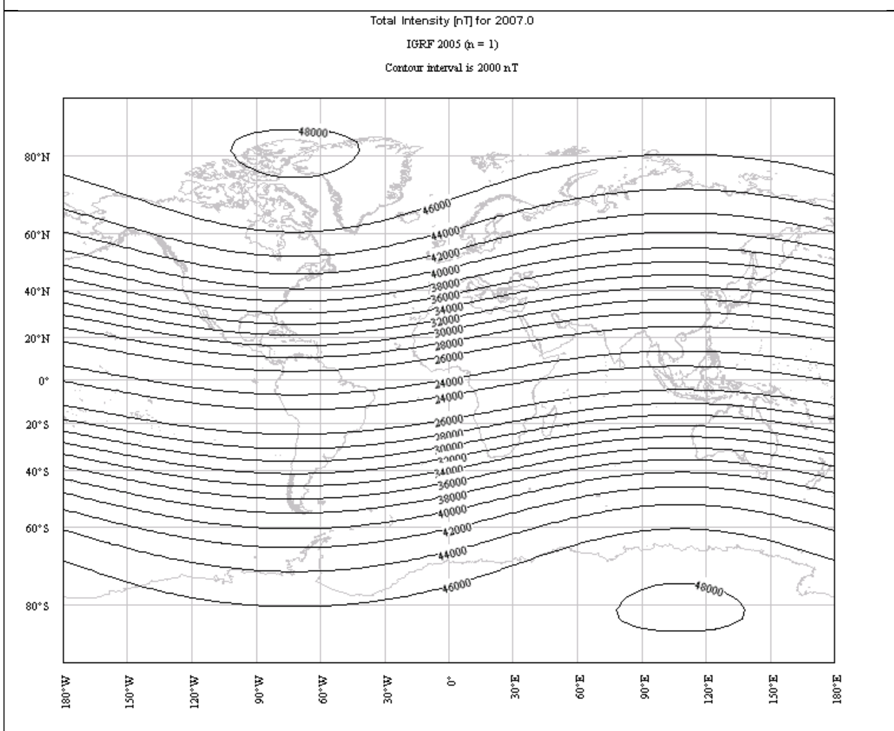
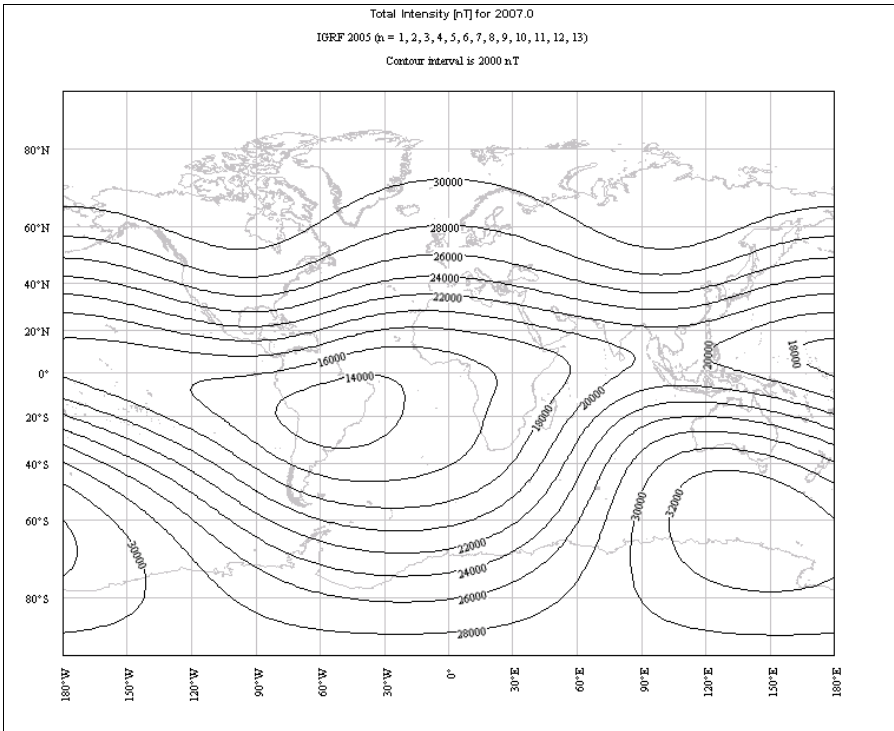
Order 13



Height = 500 km
Order 4



Height = 500 km
Dipole model



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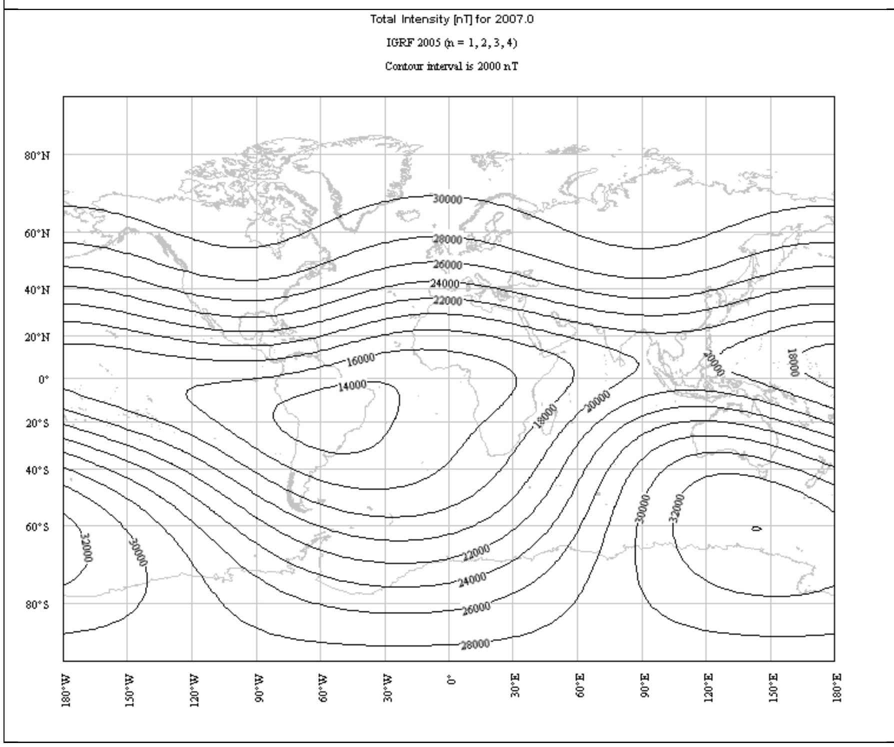
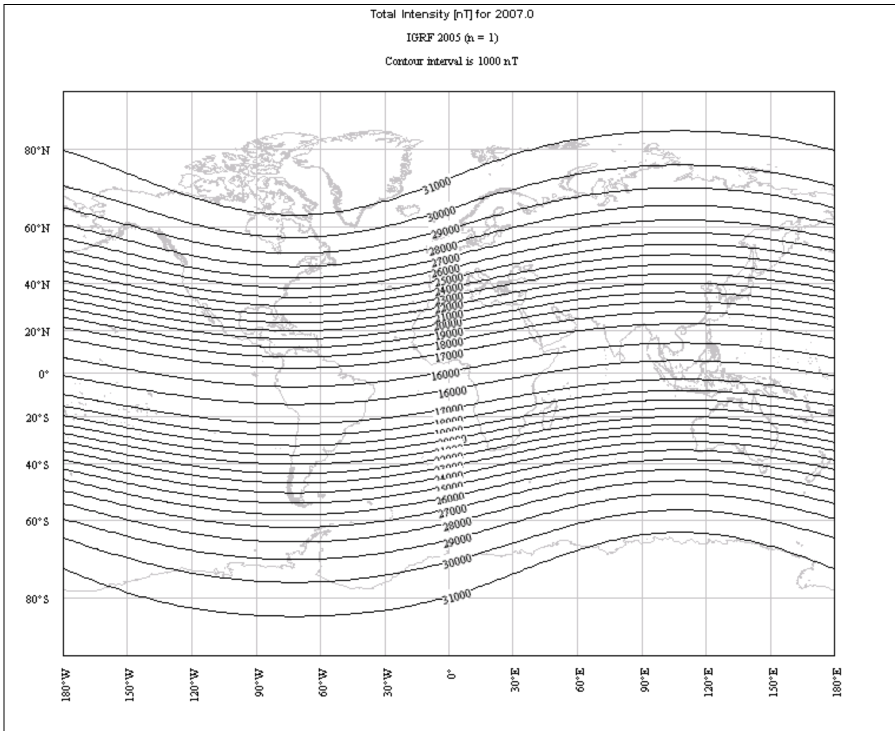
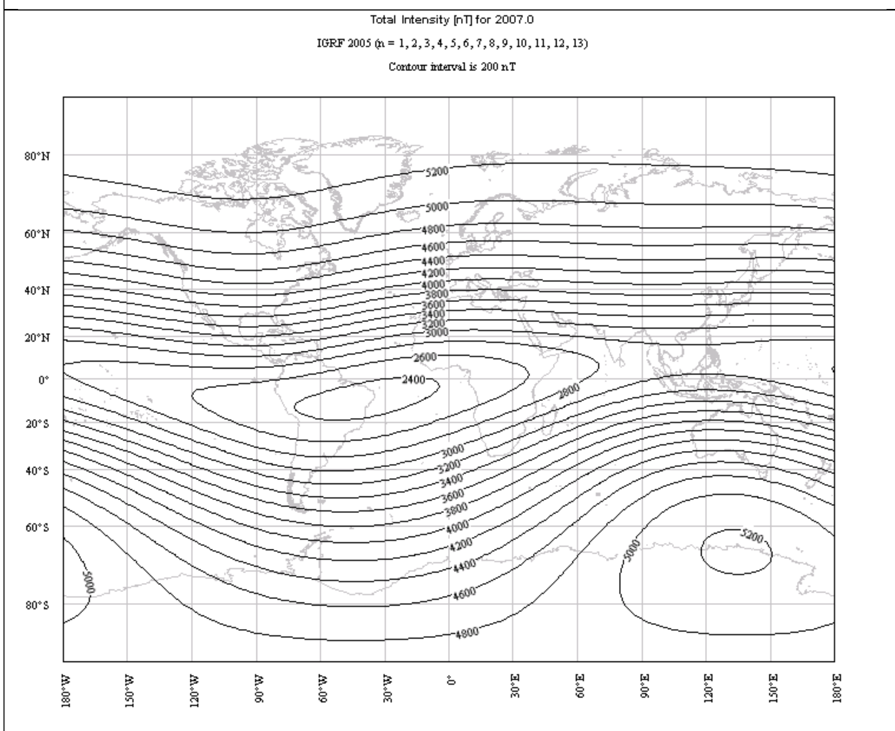
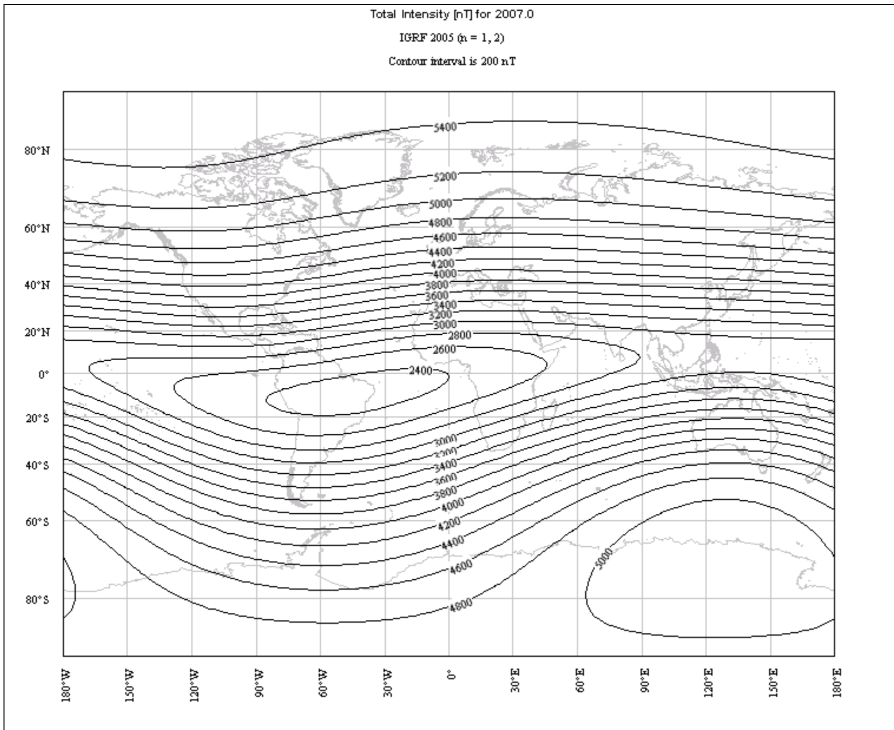
Order 13

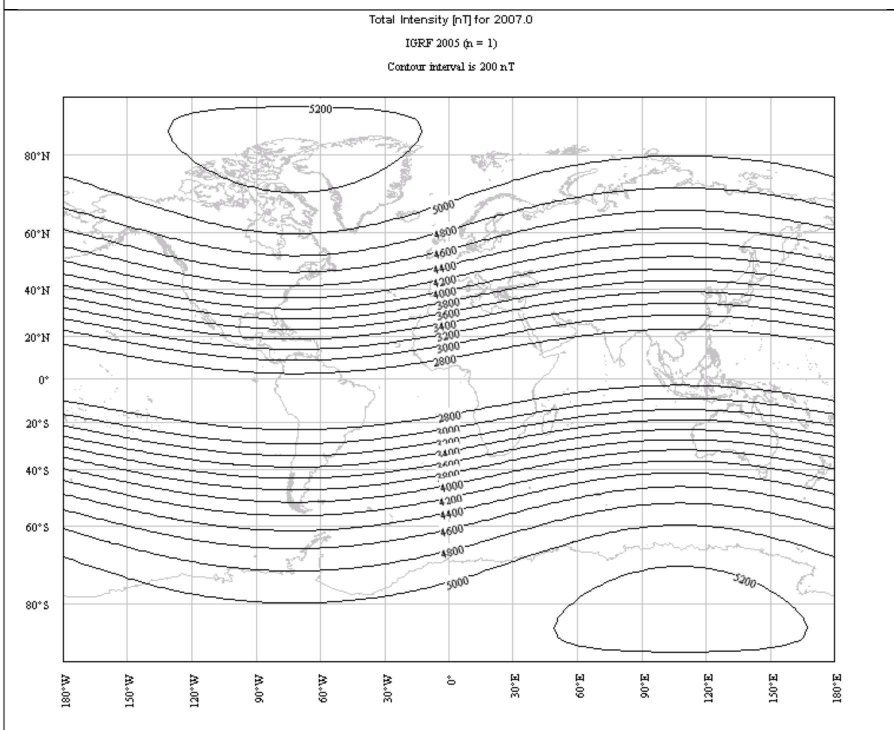


Height = 1500 km
Order 4

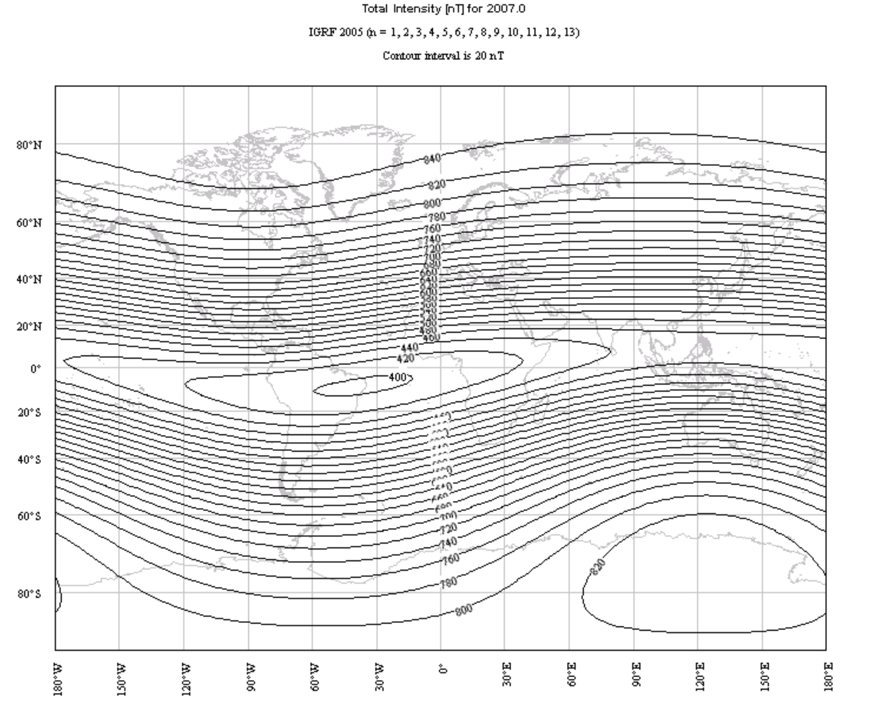
	Height = 1500 km Dipole model
	Height = 8000 km Order 13



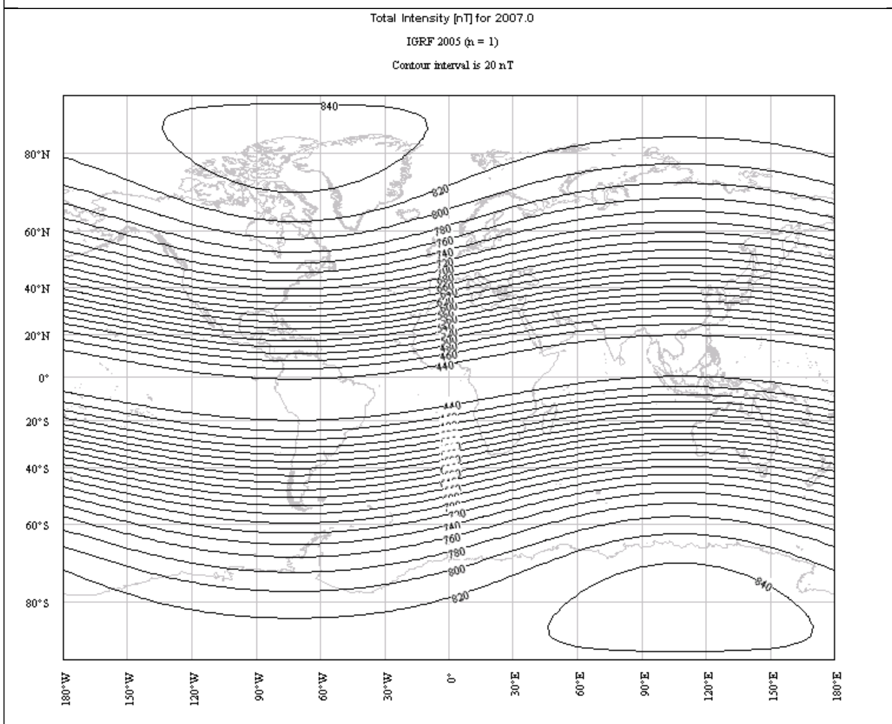
Height = 8000 km
Order 2



Height = 8000 km
Dipole model



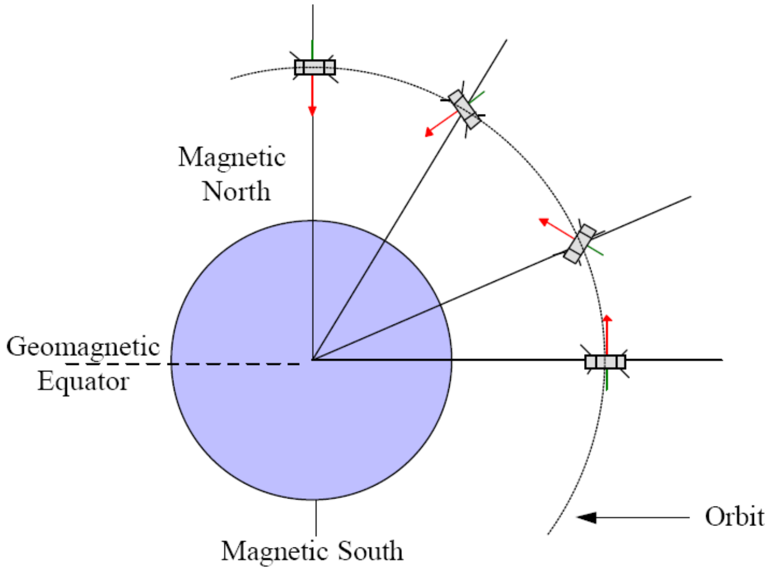
Height = 20000 km
Order 13



Height = 20000 km
Dipole model

Uses and implications of Earth's magnetic field for attitude dynamics and control

A simple low cost passive stabilization uses a permanent magnet for coarse Earth pointing over the polar regions.



The Earth's magnetic field has several implications for attitude determination and control. Firstly, the interaction between local magnetic fields on the spacecraft induced by the flow of current induces a torque on the spacecraft given by the formula:

$$\underline{M} = \underline{m} \wedge \underline{b}_B$$

$\underline{m}$ is the residual magnetic induction due to currents in the satellite. Units of $\underline{m}$ can be stated as $Nm/T = Am^2$. The magnetic torque induced on a spacecraft can be an undesired effect due to parasitic magnetic induction. The parasitic induction is difficult to model in simulation, but we can use an average constant value of $\underline{m} = [0.1 \quad 0.1 \quad 0.1]^T$ which represents quite an extreme case. However, when designing our controls, it ensures that we can test them against a worst case scenario. A more precise estimate of the spacecraft dipole moment $\underline{m}$ can be evaluated adopting the guidelines indicated by NASA in its report NASA SP-8018, summarized in the following two tables.

Table I.—*Criteria for Magnetic Properties Control*

	Class I	Class II	Class III
Design	Formal specification on magnetic properties control; approved materials and parts lists; cancellation of moments by preferred mounting arrangements and control of current loops.	Advisory specifications and guidelines for material and parts selection. Avoidance of "soft" magnetic materials or current loops and awareness of good design practices.	Nominal control over current loops; guidelines for avoidance of "soft" magnetic materials.
Quality control	Complete magnetic inspection of parts and testing of subassemblies.	Inspection or test of suspect parts.	Test of subassemblies that are potentially major sources of dipole moment.
Test and compensation	Deperming either at subassembly or spacecraft level; test of final spacecraft assembly and compensation if required.	Deperming and compensation frequently used.	Test and compensation optional.

Note.—Class I—Magnetic torques dominant when compared with other torques.
II—Magnetic torques comparable to other torques.
III—Magnetic torques insignificant when compared with other torques.

Table IV.—*Factors for Estimating Spacecraft Dipole Moment (M).*

Category of magnetic properties control (see table I)	Estimate of dipole moment per unit mass for nonspinning spacecraft		Estimate of dipole moment per unit mass for spinning spacecraft	
	A-m²/kg	(pole-cm/lb)	A-m²/kg	(pole-cm/lb)
Class I	1×10^{-3}	(0.45)	0.4×10^{-3}	(0.18)
Class II	3.5×10^{-3}	(1.6)	1.4×10^{-3}	(0.63)
Class III	10×10^{-3} and higher	(4.5)	4×10^{-3} and higher	(1.8)

Torque due to atmospheric drag

The interaction between satellite and upper atmosphere generates aerodynamic forces, that in turn can generate a torque about the center of mass. Aerodynamic torques represent the major disturbance for satellites orbiting at altitudes lower than 400 Km.
To have an approximated mathematical model of the forces, we assume that air particles hit the external surface of the satellite and their kinetic energy is totally transferred to the satellite. The aerodynamic force acting on the elementary area dA defined by its perpendicular direction $\hat{N}$ is:

$$d\bar{F}_i = \frac{1}{2} C_D \cdot \rho \cdot v^2 (\hat{N} \bullet \bar{v}) \bar{u}_v \cdot dA$$

where ρ is the air density, $\bar{v}$ is the relative velocity and C_D is the drag coefficient that is evaluated experimentally. It is a constant parameter that depends on the type of surface and on the local angle of attack.

The aerodynamic torque is then:

$$\bar{M}_{aero} = \int_S \bar{r}_i \Lambda d\bar{F}_i.$$

The integral is evaluated for all the surfaces of the satellites for which $\widehat{N} \bullet \bar{v} > 0$: surfaces defined by a negative scalar product are not exposed to aerodynamic flow. The relative velocity of the elementary area dA is:

$$\bar{v} = \bar{v}_0 + \omega \Lambda \bar{r}$$

where $\bar{v}_0$ is the relative velocity of the center of mass and ω is the angular velocity of the satellite. After substitution we obtain:

$$\bar{M}_{aero} = \frac{1}{2} C_D \cdot \rho \left(v^2 \int_S (\widehat{N} \bullet \hat{v}_0)(\hat{v}_0 \Lambda \bar{r}_i) dA + \bar{v}_0 \int_S \{\widehat{N}(\omega \Lambda \bar{r}_i)(\hat{v}_0 \Lambda \bar{r}_i) + (\widehat{N} \bullet \hat{v}_0)[(\omega \Lambda \bar{r}_i) \Lambda \bar{r}_i]\} dA \right)$$

where the first term represents the torque due to the misalignment between satellite center of mass and center of pressure, while the second term represents the torque produced by the rotation of the satellite.
It is possible to decompose the outer surface of the satellite into a larger series of simple geometric shape, in order to evaluate the aerodynamic forces and torque as a sum of elementary terms. The force due to atmospheric drag is given by

$$\bar{F}_{a_i} = -\frac{1}{2} C_D \rho(h,t) {v_r}^2 \frac{\underset{\sim}{v}_r}{\|\underset{\sim}{v}_r\|} A_{cross}$$

where $\rho(h,t)$ is the air density which depends on the altitude of the orbit, $\underset{\sim}{v}_r$ is the relative velocity between the satellite and the atmosphere, C_D is the drag coefficient that is evaluated experimentally ($C_D \approx 2.2, 1.5 < C_D < 2.6$ assuming a flat plate) and A_{cross} is the cross sectional surface area perpendicular to $\frac{\underset{\sim}{v}_r}{\|\underset{\sim}{v}_r\|}$ which can be expressed as $A_{cross} = (\underset{\sim}{n}_s \cdot \frac{\underset{\sim}{v}_r}{\|\underset{\sim}{v}_r\|})A$ where $\underset{\sim}{n}_s$ is the unit vector normal to the surface and A is the area of the surface. Then it is possible to write for a given surface:

$$\bar{F}_{a_i} = -\frac{1}{2} C_D \rho(h,t) {v_r}^2 \frac{\underset{\sim}{v}_r}{\|\underset{\sim}{v}_r\|} (\bar{n}_s \cdot \frac{\underset{\sim}{v}_r}{\|\underset{\sim}{v}_r\|}) A$$

Each surface evaluated for $\bar{n}_{si} \cdot \frac{\underset{\sim}{v}_r}{\|\underset{\sim}{v}_r\|} > 0$
Then the total torque is given by

$$M = \sum \bar{r}_i \wedge \bar{F}_{a_i}$$

where $\bar{r}_i$ is the position vector from the centre of mass of the spacecraft to the centre of pressure of the surface A_i. Recall, to model the torque in the body frame the above equations must be evaluated in body coordinates. For example, evaluating the relative velcoity $\underset{\sim}{v}_r$ with respect to the inertial frame we have:

$$\underset{\sim}{v}_r = \underset{\sim}{v}_o - \underset{\sim}{\omega}_\oplus \times \underset{\sim}{R}$$

Where $\underset{\sim}{v}_o$is the velocity of the spacecraft in the inertial frame, $\underset{\sim}{R}$is the position of the spacecraft and $\underset{\sim}{\omega}_\oplus$ is the angular rotation of the Earth (assuming the atmosphere rotates with the Earth). In Cartesian coordinates and assuming an equatorial orbit we could write:

$$\underline{v}_r = \begin{bmatrix} \dot{x} + \omega_{\oplus} y \\ \dot{y} - \omega_{\oplus} x \\ \dot{z} \end{bmatrix}$$

where $\omega_{\oplus} = 0.000072921 rad/sec$. However, this must be expressed in the body frame such that

$$\underline{F}_i^b = -\frac{1}{2} C_D \rho(h,t) v_r{}^2 \frac{\underline{v}_r^b}{\left\|\underline{v}_r^b\right\|} (\hat{\underline{n}}_{si}^b \cdot \frac{\underline{v}_r^b}{\left\|\underline{v}_r^b\right\|}) A_i$$

where $\underline{v}_r^b = A_{B/N} \underline{v}_r$. Furthermore, the normal to the surfaces $\hat{\underline{n}}_{si}^b$ in the body frame. However, these are fixed with respect to the body frame and so are constant values. For example, a simple cuboid spacecraft, with the principal axis aligned with the normal to the 6 surfaces would be:

$$\hat{\underline{n}}_{s1}^b = [1 \quad 0 \quad 0]^T, \hat{\underline{n}}_{s2}^b = [0 \quad 1 \quad 0]^T, \hat{\underline{n}}_{s3}^b = [0 \quad 0 \quad 1]^T$$
$$\hat{\underline{n}}_{s4}^b = [-1 \quad 0 \quad 0]^T, \hat{\underline{n}}_{s5}^b = [0 \quad -1 \quad 0]^T, \hat{\underline{n}}_{s6}^b = [0 \quad 0 \quad -1]^T$$

To perform all the calculations it is necessary to evaluate correctly the aerodynamic coefficient and the air density as a function of the distance from Earth. To make the simplified expression compatible with the experimental data, the air density can be made dependent upon the altitude as well as other parameters, such as temperature, solar activity (F10.7 constant), surface and material properties, and for ease of use the drag coefficient is taken as the flat plate drag coefficient. One example of tabular data of atmospheric density as a function of altitude is reported here below.

altit. (km)	10	20	30	40	50	60	70	80	90	100
density (g/cm^3)	4,02e-04	8,34e-05	1,57e-05	3,18e-06	8,37e-07	2,33e-07	5,86e-08	1,40e-08	2,99e-09	5,17e-10
altit. (km)	110	120	130	140	150	160	170	180	190	200
density (g/cm^3)	8,42e-11	1,84e-11	7,36e-12	3,78e-12	2,19e-12	1,37e-12	9,00e-13	6,15e-13	4,32e-13	3,10e-13
altit. (km)	210	220	230	240	250	260	270	280	290	300
density (g/cm^3)	2,27e-13	1,68e-13	1,26e-13	9,58e-14	7,35e-14	5,68e-14	4,43e-14	3,48e-14	2,75e-14	2,18e-14
altit. (km)	310	320	330	340	350	360	370	380	390	400
density (g/cm^3)	1,74e-14	1,40e-14	1,13e-14	9,10e-15	7,39e-15	6,02e-15	4,92e-15	4,03e-15	3,31e-15	2,72e-15
altit. (km)	410	420	430	440	450	460	470	480	490	500
density (g/cm^3)	2,25e-15	1,86e-15	1,54e-15	1,28e-15	1,07e-15	8,89e-16	7,43e-16	6,22e-16	5,22e-16	4,39e-16
altit. (km)	510	520	530	540	550	560	570	580	590	600
density (g/cm^3)	3,71e-16	3,13e-16	2,66e-16	2,26e-16	1,93e-16	1,65e-16	1,41e-16	1,22e-16	1,05e-16	9,14e-17
altit. (km)	610	620	630	640	650	660	670	680	690	700
density (g/cm^3)	7,96e-17	6,97e-17	6,12e-17	5,41e-17	4,79e-17	4,27e-17	3,82e-17	3,44e-17	3,10e-17	2,82e-17
altit. (km)	710	720	730	740	750	760	770	780	790	800
density (g/cm^3)	2,57e-17	2,35e-17	2,16e-17	1,99e-17	1,84e-17	1,71e-17	1,59e-17	1,49e-17	1,39e-17	1,31e-17
altit. (km)	810	820	830	840	850	860	870	880	890	900
density (g/cm^3)	1,23e-17	1,16e-17	1,10e-17	1,04e-17	9,86e-18	9,37e-18	8,91e-18	8,48e-18	8,09e-18	7,72e-18
altit. (km)	910	920	930	940	950	960	970	980	990	1000
density (g/cm^3)	7,37e-18	7,04e-18	6,74e-18	6,45e-18	6,18e-18	5,92e-18	5,68e-18	5,44e-18	5,22e-18	5,02e-18

Aerodynamic torques are never used to control the satellite rotation, and their effect is totally negligible above 700 km altitude. Aerodynamic torques represent the major disturbance for satellites orbiting at altitudes lower than 400 Km.

Solar radiation torque

Solar radiation illuminating the outer surface of a satellite generates a pressure, that in turn generates a force and a torque around the center of mass of the satellite.
The main sources of electromagnetic radiation are the direct solar radiation, the solar radiation reflected by the Earth (or by any other planet) and the radiation directly emitted by the Earth. Solar radiation intensity varies with the inverse square distance from the source, so that direct solar radiation is almost constant for geocentric orbits, independent from the orbit radius since the distance to the Sun does not change significantly. On the contrary, the reflected radiation and the Earth radiation intensity are strongly dependent on the orbit radius.

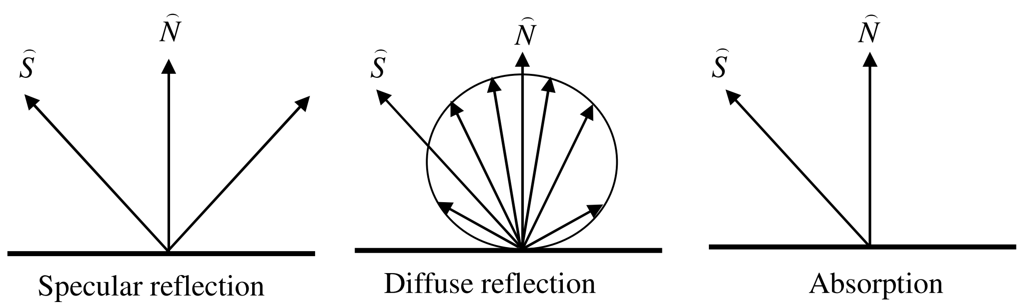
Altitude (Km)	**Direct solar radiation** (W/m^2)	**Radiation reflected by the Earth** (W/m^2)	**Earth radiation** (W/m^2)
500	1358	600	150
1000	1358	500	117
2000	1358	300	89
4000	1358	180	62
8000	1358	75	38
15000	1358	30	14
30000	1358	12	3
60000	1358	7	2

The average pressure due to radiation can be evaluated as

$$P = \frac{F_e}{c}$$

where c is the speed of light and F_e is the power per unit surface.

Radiation forces can be modeled assuming that part of the incident radiation is absorbed, part is reflected specularly and part reflected with diffusion.



Therefore the three components of the resulting force acting on a surface element dA having orthogonal direction $\hat{N}$ will be:

$$\begin{cases} d\bar{F}_{assorbita} = -Pc_a \cos\vartheta \cdot \hat{S} dA & (0 \le \vartheta \le 90°) \\ d\bar{F}_{diffusa} = Pc_d \left(-\frac{2}{3}\cos\vartheta \cdot \hat{N} - \cos\vartheta \cdot \hat{S}\right) dA & (0 \le \vartheta \le 90°) \\ d\bar{F}_{speculare} = -2Pc_s \cos^2\vartheta \cdot \hat{N} dA & (0 \le \vartheta \le 90°) \end{cases}$$

with $\hat{S}$ denoting the unit direction of the satellite-Sun vector and ϑ is the angle between the surface orthogonal direction and direction $\hat{S}$.
When ϑ is greater than 90°, so that $\cos\vartheta$ is negative, the surface is not illuminated and not subject to radiation forces.
Coefficients c_a, c_d, c_s are respectively the coefficient of absorption, diffuse reflection and specular reflection, and are constrained by the following equality:

$$c_a + c_d + c_s = 1$$

Merging the three components due to the three possible effects, the total force due to radiation is evaluated as:

$$d\bar{F}_i = -P \int_S \left[(1 - c_s)\hat{S} + 2\left(c_s \cos\vartheta + \frac{1}{3}c_d\right)\hat{N}\right] \cos\vartheta \cdot dA$$

The torque due to radiation will then be:

$$\bar{M}_{sol} = \int_S \bar{r}_i \Lambda d\bar{F}_i$$

where the integral has to be evaluated for all surfaces illuminated by radiation. In many cases it is advisable to decompose the external surface into several simple planar surfaces, so that the resulting torque will be the sum of the contributions of each surface element:

$$\bar{M}_{sol} = \sum_{i=1}^{N} \bar{r}_i \Lambda \bar{F}_i.$$

Taking the case of a flat panel of area *A*, the resulting force is

$$\bar{F}_i = -A \cdot P \left[(1 - c_s)\hat{S} + 2\left(c_s \cos\vartheta + \tfrac{1}{3}c_d\right)\hat{N}\right] \cos\vartheta \qquad \vartheta = \cos^{-1}(\hat{S} \cdot \hat{N}).$$

In the case of a cylindrical surface, the effects can be divided into forces acting on the base of the cylinder and forces acting on the lateral surface.

$$F_{lateral} = PA \sin\vartheta \left\{\left[\left(1 + \frac{1}{3}c_s\right) + \frac{\pi}{6}c_d\right] \cdot \hat{S} + \left[-\frac{4}{3}c_s - \frac{\pi}{6}c_d\right] \cdot \cos\vartheta \cdot \hat{N}\right\}$$

$$F_{base} = PA \cos\vartheta \left[(1 - c_s) \cdot \hat{S} + 2\left(c_s \cos\vartheta + \frac{1}{3}c_d\right) \cdot \hat{N}\right]$$

The relative position of the Sun with respect to the spacecraft measured in the inertial frame is given as $\hat{S}_N$ and in the body frame is given as:

$$\hat{S}_b = A_{B/N}\hat{S}_N$$

Then the torque is given as

$$\underline{T}_{SRP} = \begin{cases} \sum_{i=1}^{n} \underline{r}_i \Lambda \, \underline{F}_i & \underline{\hat{S}}_b \cdot \underline{\hat{n}}_s^b > 0 \\ 0 & \underline{\hat{S}}_b \cdot \underline{\hat{n}}_s^b < 0 \end{cases}$$

where $\underline{r}_i$is the position of the centre of pressure of the flat surface with respect to the centre of mass. As before a simple cuboid geometry with the principal axis aligned with the normal to the surfaces can be evaluated using:

$$\underline{\hat{n}}_{s1}^b = [1 \quad 0 \quad 0]^T, \underline{\hat{n}}_{s2}^b = [0 \quad 1 \quad 0]^T, \underline{\hat{n}}_{s3}^b = [0 \quad 0 \quad 1]^T$$
$$\underline{\hat{n}}_{s4}^b = [-1 \quad 0 \quad 0]^T, \underline{\hat{n}}_{s5}^b = [0 \quad -1 \quad 0]^T, \underline{\hat{n}}_{s6}^b = [0 \quad 0 \quad -1]^T$$

Suggested readings

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