ORBITAL MECHANICS

Equation (1.47) to define eccentricity of transfer orbit.

$$\boxed{e_T = \frac{r_a - r_p}{r_a + r_p}} \qquad (3.7)$$

From eq. (1.40)

$$\boxed{a_T = \frac{r_a + r_p}{2}} \qquad (3.8)$$

For (T) from A To B $\quad r_a = r_2 \quad r_p = r_1$

From energy equation (1-56] we get

$$\boxed{v = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a}\right)}} \qquad (3.9)$$

v_A^-, v_A^+, v_B^-, v_B^+ → calculated by inserting in 3.9 a of the correct orbit and the position of point A and B

⇓

Δv_A, Δv_B

An Alternative method uses the definition of h

Recall from eq (1.38)

$$r_p = \frac{h^2}{\mu}\frac{1}{1+e}$$

↳ eq 3.7 for e_T

$$r_p = \frac{h^2}{\mu}\frac{1}{1+\frac{r_a-r_p}{r_a+r_p}} = \frac{h^2}{\mu}\frac{r_a+r_p}{r_a+\cancel{r_p}+r_a-\cancel{r_p}} = \frac{h^2}{\mu}\frac{r_a+r_p}{2r_a}$$

let's isolate h

$$\boxed{h = \sqrt{\frac{2\mu\, r_p r_a}{r_a + r_p}}} \qquad (3.10)$$

From eq (1.14) $\quad \boxed{h = r\, v_\theta} \qquad (3.11)$

at A and B $v_A^-, v_A^+, v_B^-, v_B^+$ are all transversal ($\perp$ to r)

Considering that when an circular orbit (1, 2)

$$r_a = r_p = r \qquad h_c = \sqrt{\frac{\cancel{2}\mu\, r^{\cancel{2}}}{\cancel{2r}}} \longrightarrow \boxed{h_{circ} = \sqrt{\mu r}} \qquad (3.12)$$

Therefore from A to B

$$h_1 = \sqrt{\mu r_1} \longrightarrow v_A^- = v_{A1} = \frac{h_1}{r_1}$$

$$h_2 = \sqrt{\mu r_2} \longrightarrow v_B^+ = v_{B2} = \frac{h_2}{r_2}$$

$$h_T = \sqrt{2\mu \frac{r_a r_p}{r_a + r_p}} \longrightarrow v_A^- = v_{AT} = \frac{h_T}{r_1}$$

$$\longrightarrow v_B^- = v_{BT} = \frac{h_T}{r_2}$$

$$\left.\begin{aligned} \Delta v_A &= v_A^+ - v_A^- \\ \Delta v_B &= v_B^+ - v_B^- \end{aligned}\right\} \longrightarrow \Delta v_{TOT} = |\Delta v_A| + |\Delta v_B|$$

From B to A

$$v_B^+ = v_{BT} = \frac{h_T}{r_2}$$

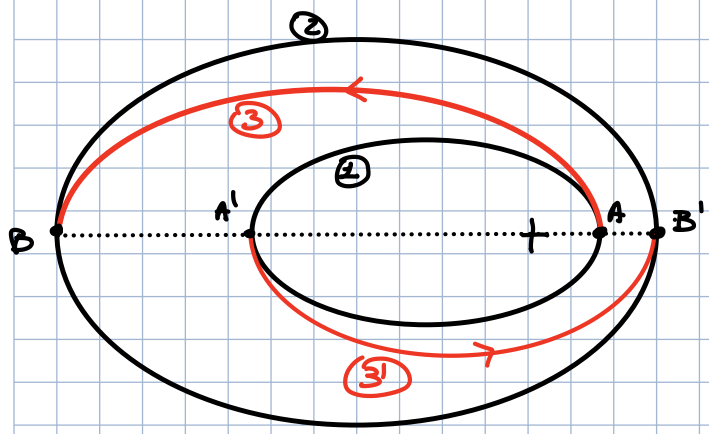
$$v_B^- = v_{B2} = h_2/r_2$$

$$v_A^- = v_{AT} = \frac{h_T}{r_1}$$

$$v_A^+ = v_{A1} = \frac{h_1}{r_1}$$

$$\begin{aligned} \Delta v_B &= v_B^+ - v_B^- \\ \Delta v_A &= v_A^+ - v_A^- \end{aligned} \longrightarrow \Delta v_{TOT} = |\Delta v_A| + |\Delta v_B|$$

Hohmann transfer can be generalized between two coplanar cofocal, coaxial (same apse line).



$$h_1 = \sqrt{2\mu \frac{r_A r_{A'}}{r_A + r_{A'}}}$$

$$h_2 = \sqrt{2\mu \frac{r_B r_{B'}}{r_B + r_{B'}}}$$

$$h_3 = \sqrt{2\mu \frac{r_A r_B}{r_A + r_B}}$$

$$h_{3'} = \sqrt{2\mu \frac{r_{A'} r_{B'}}{r_{A'} + r_{B'}}}$$

@ A $\quad v_{A1} = \frac{h_1}{r_A} \qquad v_{A3} = \frac{h_3}{r_A}$

@ B $\quad v_{B2} = \frac{h_2}{r_B} \qquad v_{B3} = \frac{h_3}{r_B}$

@ A' $\quad v_{A'1} = \frac{h_1}{r_{A'}} \qquad v_{A'3'} = \frac{h_{3'}}{r_{A'}}$

@ B' $\quad v_{B'2} = \frac{h_2}{r_{B'}} \qquad v_{B'3'} = \frac{h_{3'}}{r_{B'}}$

$\Delta v_A = v_{A3} - v_{A1} \qquad \Delta v_B = v_{B2} - v_{B3}$

$\Delta v_{A'} = v_{A'3'} - v_{A'1} \qquad \Delta v_{B'} = v_{B'2} - v_{B'3'}$

$\Delta v_{TOT3} = |\Delta v_A| + |\Delta v_B|$

$\Delta v_{TOT3'} = |\Delta v_{A'}| + |\Delta v_{B'}|$

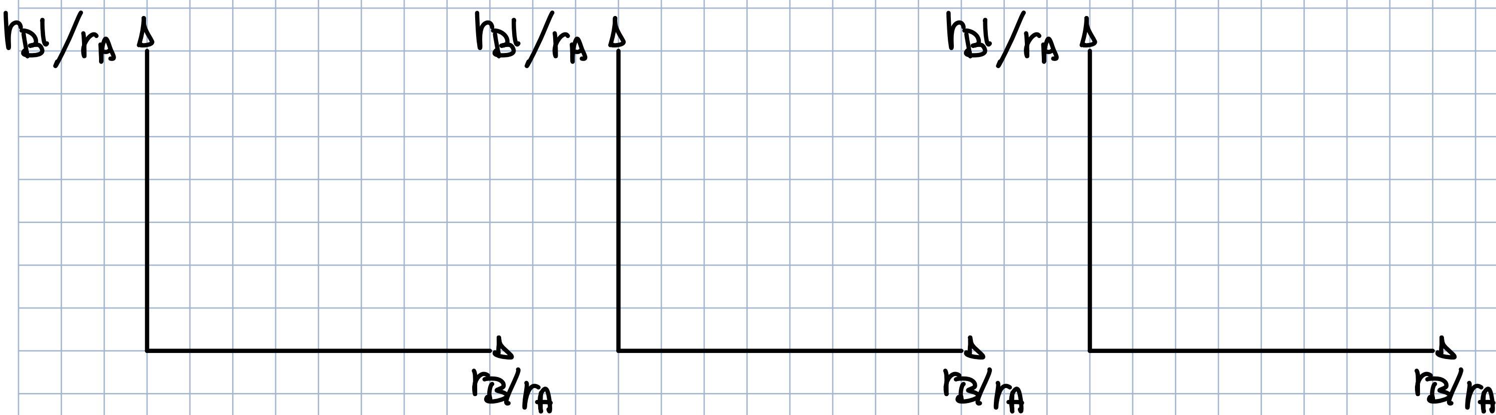
if $\Delta v_{TOT3'} / \Delta v_{TOT3} > 1 \Rightarrow$ orbit 3 is more fuel efficient than orbit 3'

if $\Delta v_{TOT3'} / \Delta v_{TOT3} < 1 \Rightarrow$ orbit 3' is more fuel efficient than orbit 3

Contour plot e.g. three fixed values of r_A'/r_A

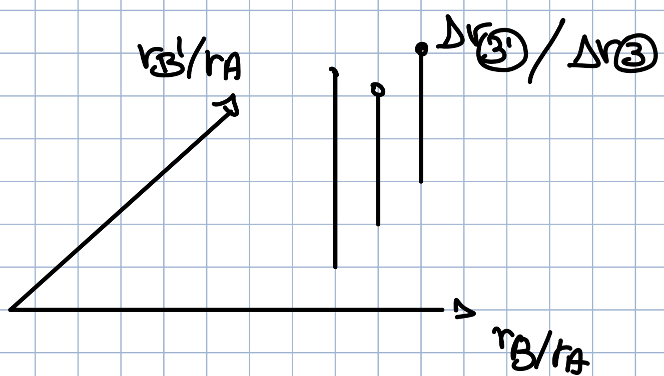
Let's pick 3 values of r_A'/r_A

- $\frac{r_A'}{r_A} = 3$ (r_A' = apoapsis r_A = periapsis)
- $\frac{r_A'}{r_A} = 1$ (orbit 1 is a circular orbit)
- $\frac{r_A'}{r_A} = \frac{1}{3}$ (r_A' = periapsis r_A = apoapsis)



(a) $r_A'/r_A = 3$ (b) $r_A'/r_A = 1$ (c) $r_A'/r_A = 1/3$

How to compute a contour plot $\Rightarrow$ we are defining a grid and for each value of the x, y apsis we calculate on the z axis the value of what we want to plot.



use matlab contour

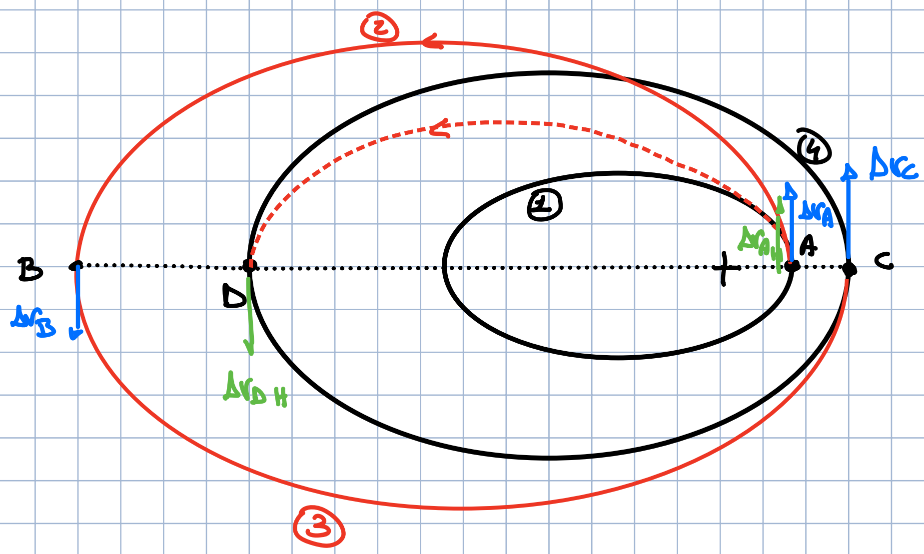
$\rightarrow$ help contour

and use x, y grid (n, m)

makes easy to spot error.

Bi-Elliptic Transfer

A further extension of the Hohmann transfer orbit.



$r_A = r_{①}$

$r_B = r_{④}$

r_B = radius of the auxiliary orbit

NOTE for Δv_C we are not considering the value $\Rightarrow$ we will obtain a negative value.

if we put B far $\Rightarrow$ $\Delta v_B \downarrow$

$r_B \to \infty \quad \Delta v_B \to 0$ but $\Delta v_A, \Delta v_C \uparrow$

Is the Δv_{TOT} of the bielliptical transfer is smaller than the Δv_{TOT} of the Hohmann transfer orbit?

$$\Delta v_{TOT}\big|_{Bi} < \Delta v_{TOT}\big|_{HOH}$$

Let's define $\quad v_0 = \sqrt{\frac{\mu}{r_A}}$ $\to$ velocity on the initial circular orbit.

$$\alpha = \frac{r_C}{r_A}, \quad \beta = \frac{r_B}{r_A}$$

$$\Delta v_{TOT}\big|_{Bi-Ell} = |\Delta v_A| + |\Delta v_B| + |\Delta v_C| \qquad (3.13)$$

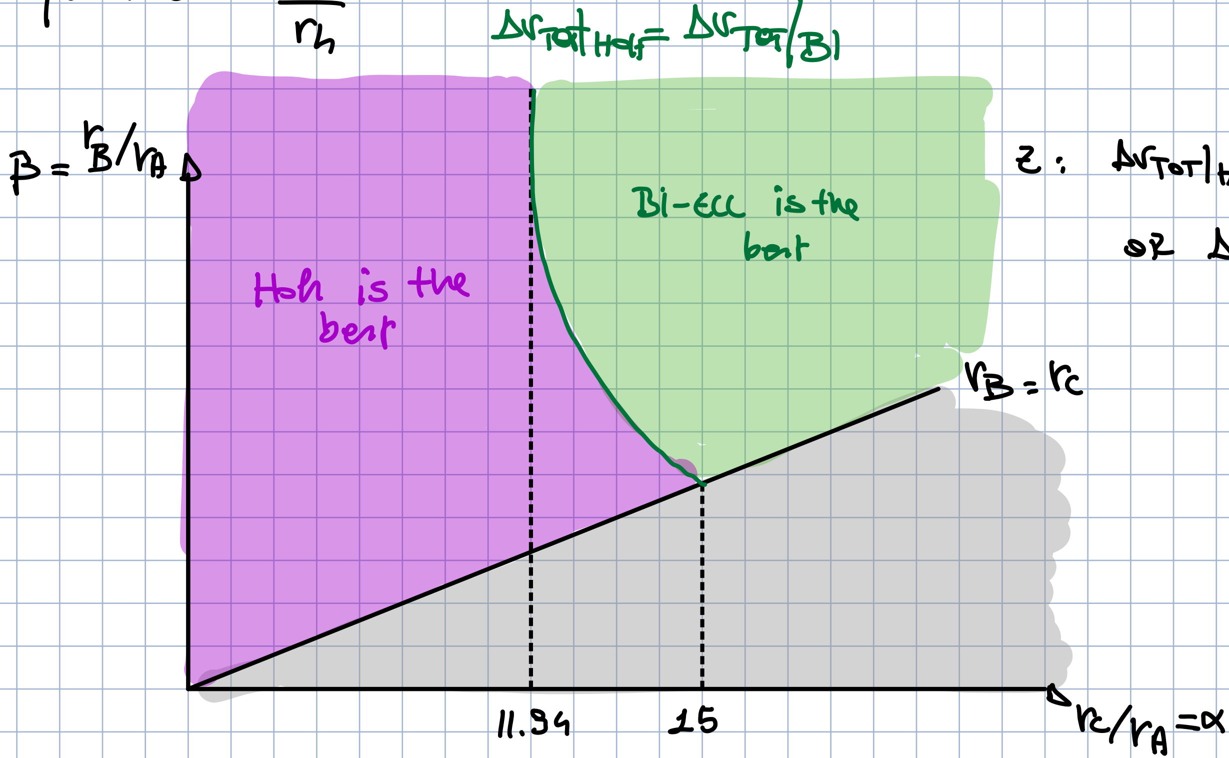
$$\Delta v_{TOT}\big|_{HOH} = |\Delta v_{AH}| + |\Delta v_{DH}|$$

$$\frac{\Delta v_{TOT}|_{HOH}}{v_0} = \frac{1}{\sqrt{\alpha}} - \frac{\sqrt{2}\,(1-\alpha)}{\sqrt{\alpha(1+\alpha)}} - 1$$

$$\frac{\Delta v_{TOT}|_{BI-ECC}}{v_0} = \sqrt{\frac{2(\alpha+\beta)}{\alpha\beta}} - \frac{1+\sqrt{\alpha}}{\sqrt{\alpha}} - \sqrt{\frac{2}{\beta(1+\beta)}}\,(1-\beta)$$

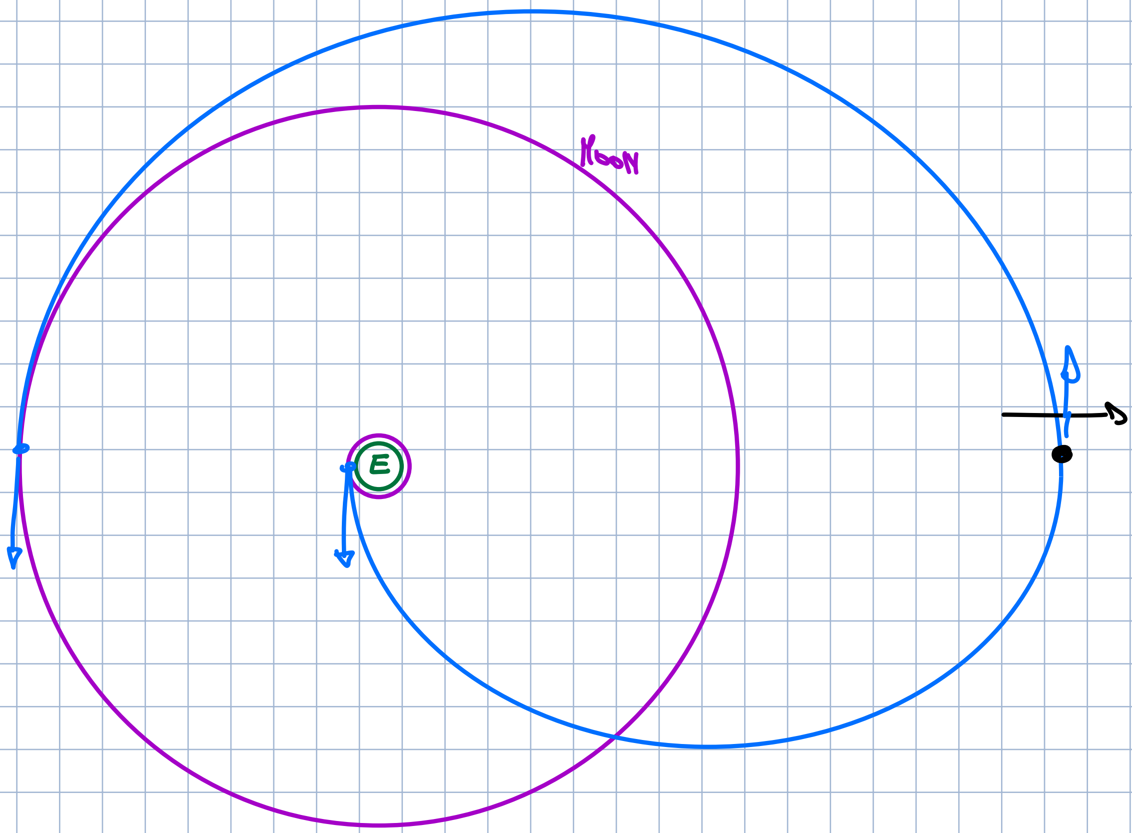
NOTE

Hohmann transfer is not always the cheapest transfer → some time it could be the bielliptical transfer depending on the fraction $\frac{r_1}{r_h}$.



z: $\Delta v_{TOT}|_{HOH}$, $\Delta v_{TOT}|_{BI\text{-}ECC}$ or $\Delta v_{TOT}|_{HOH} / \Delta v_{TOT}|_{BI\text{-}ECC}$

NOTE The Hohmann transfer is the best two burn strategy to consume less fuel as possible. But if we are allowed to use more than two burn we can use less fuel in certain case → As shown in the graph above.



SUN +
EARTH +
MOON + S/C
4 BODY
PROBLEM

WEAK
STABILITY
boundary transfer

change plane at
low cost