Selected slides from

Dynamics and control of space structures

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Control system design – introduction to state-space methods (part 2)

2020/2021

Outline

Control system design – introduction to state-space methods

Part 1

- Introduction and motivation
- State space fundamentals
- Controllability and observability

Part 2

- Pole placement
- Linear quadratic regulator (LQR)
- Steady-state tracking

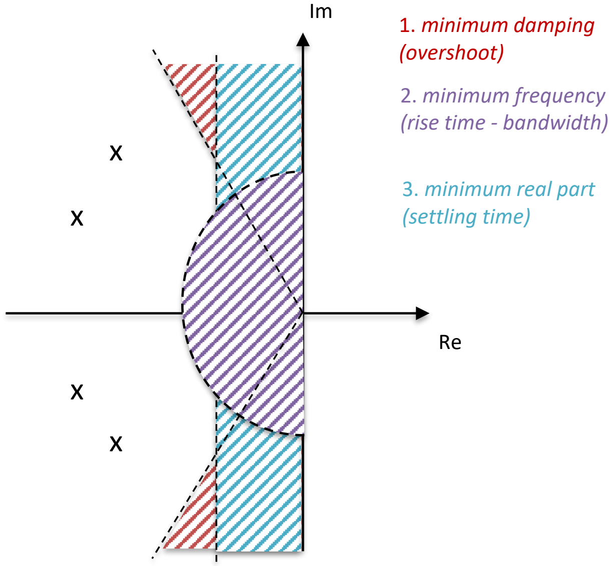
Part 3

- Linear observer
- Guidelines for selecting weighting matrices in LQR
- Finite-horizon optimal control

POLE PLACEMENT

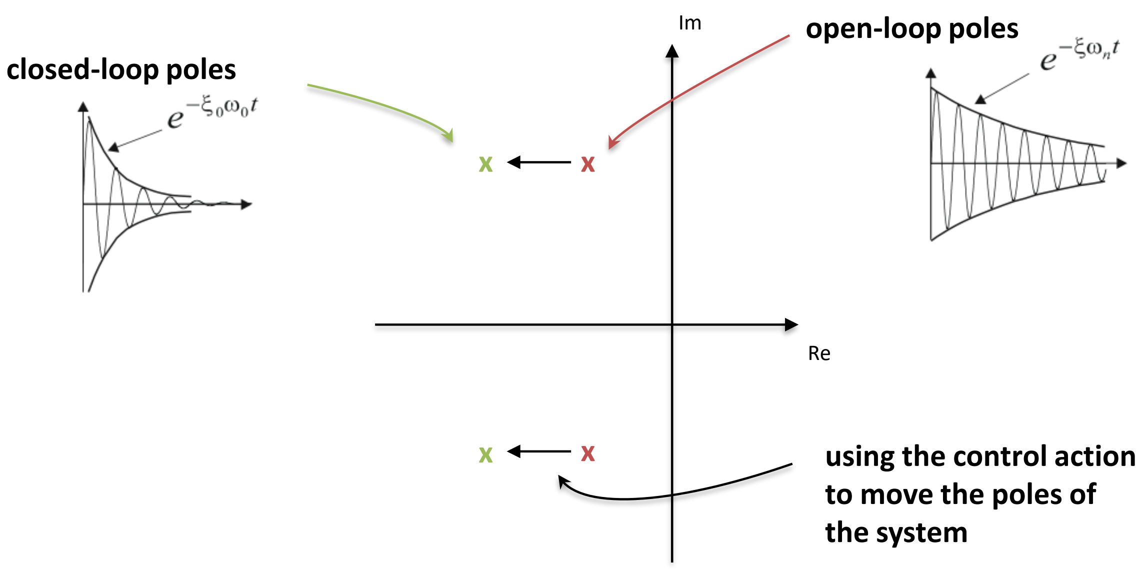
A simple method of designing an active control for a LTI system in which all the state variables are accessible (i.e., they can be measured in some way) is the so-called *pole-placement technique*. It is based on enforcing a desired closed-loop response by specifying the closed-loop poles of the system. *If the system is fully controllable*, the closed-loop poles can be located *anywhere we wish* in the complex plane.

Relationship between characteristics of the step response (rise time, settling time, overshoot) and the location of closed-loop poles.



Pole placement

A simple method of designing an active control for a LTI system in which all the state variables are accessible (i.e., they can be measured in some way) is the so-called *pole-placement technique*. It is based on enforcing a desired closed-loop response by specifying the closed-loop poles of the system. *If the system is fully controllable*, the closed-loop poles can be located *anywhere we wish* in the complex plane.



Control system design – state-space methods
Pole placement

Let consider a structural system with a *single control input* $u(t)$. The structure is assumed to be excited by a generic disturbance input $d(t)$. Therefore, the dynamics of the system is represented by the following state equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}_u u(t) + \mathbf{b}_d d(t) \tag{17.1}$$

where $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{b}_u \in \mathbb{R}^{n\times 1}$, $\mathbf{b}_d \in \mathbb{R}^{n\times 1}$, and n is the number of states. The pole-placement approach is implemented by a full state feedback law as follows

$$\boxed{u(t) = -\mathbf{g}\,\mathbf{x}(t)} \tag{17.2}$$

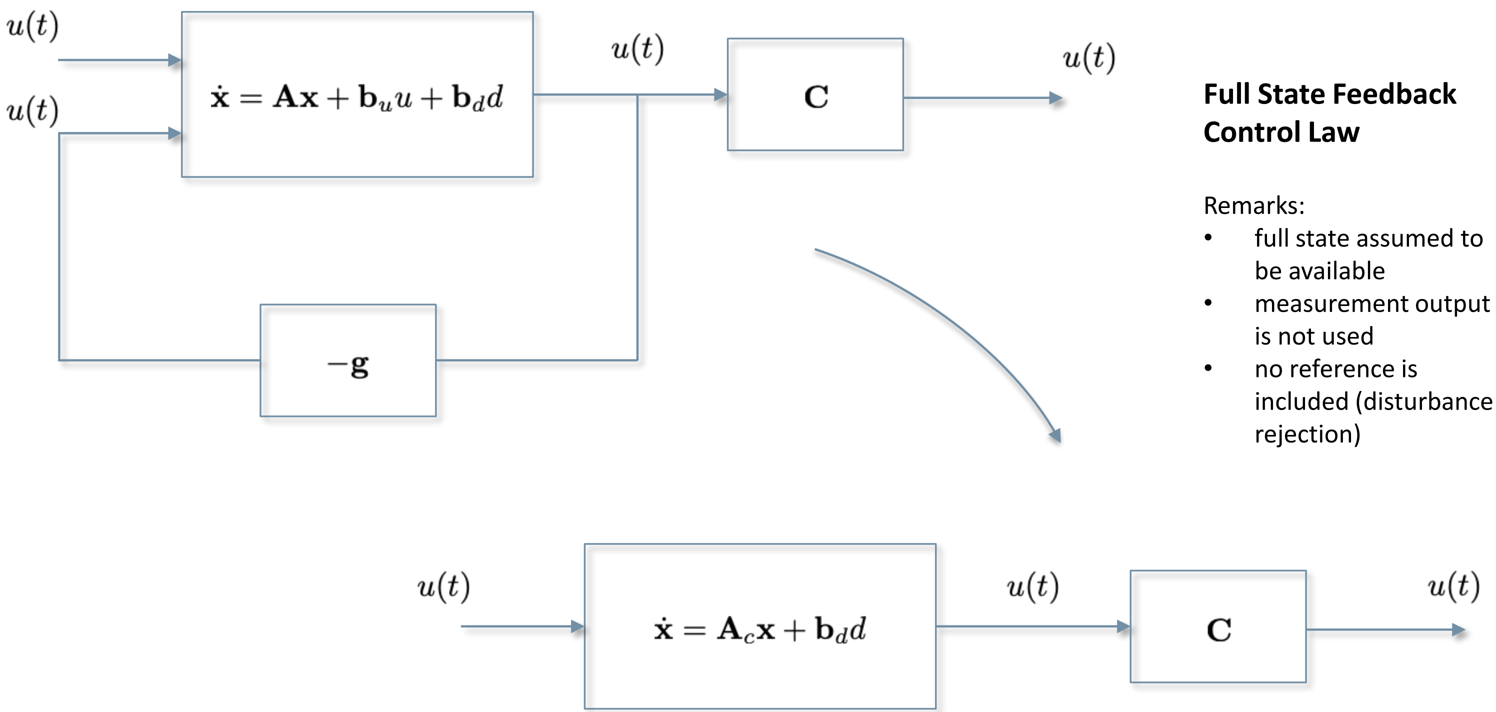
where $\mathbf{g} \in \mathbb{R}^{1\times n}$ is the row vector of control gains, which should be selected to achieve desirable properties of the closed-loop dynamics. For a LTI system with n states, there are n feedback gains g_i that can be adjusted independently. Substituting Eq. (17.2) into Eq. (17.1) yields

$$\dot{\mathbf{x}}(t) = \mathbf{A}_c\mathbf{x}(t) + \mathbf{b}_d d(t) \tag{17.3}$$

where

$$\boxed{\mathbf{A}_c = \mathbf{A} - \mathbf{b}_u\mathbf{g}} \tag{17.4}$$

is the closed-loop system matrix. The eigenvalues of $\mathbf{A}_c$ are the closed-loop poles of the system. The objective of the design is to find the vector of control gains $\mathbf{g}$ which places the poles of $\mathbf{A}_c$ at desired locations. Therefore, the first step is to select the desired locations of the closed-loop poles. Then, a technique is sought to compute $\mathbf{g}$ such that $\mathbf{A}_c$ will have the prescribed eigenvalues.



Full State Feedback Control Law

Remarks:

- full state assumed to be available
- measurement output is not used
- no reference is included (disturbance rejection)

Feedback gain formula for Controller Canonical Form
(Bass-Gura formula)

Let's consider a state space representation in the following controller canonical form

$$\mathbf{A}_{\mathrm{CF}} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & \dots & & \\ 0 & 0 & 0 & \dots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \dots & -a_1 \end{bmatrix} \qquad \mathbf{b}_{\mathrm{CF}} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

The related characteristic polynomial is given by

$\det(s\mathbf{I} - \mathbf{A}_{\mathrm{CF}})$

$$s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_{n-1} s + a_n$$

where the coefficients correspond to the elements of the last row of the state matrix.

Feedback gain formula for Controller Canonical Form
(Bass-Gura formula)

Full state feedback control law:

$$u(t) = -\begin{bmatrix} g_1 & g_2 & \dots & g_{n-1} & g_n \end{bmatrix} \mathbf{x}(t)$$

the closed-loop system matrix is given in this case by

$$\mathbf{A}_c = \mathbf{A}_{\mathrm{CF}} - \mathbf{b}_{\mathrm{CF}}\mathbf{g} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & \dots & & \\ 0 & 0 & 0 & \dots & 1 \\ -(a_n + g_1) & -(a_{n-1} + g_2) & -(a_{n-2} + g_3) & \dots & -(a_1 + g_n) \end{bmatrix}$$

The characteristic equation is then expressed as

$$s^n + (a_1 + g_n)s^{n-1} + (a_2 + g_{n-1})s^{n-2} + \dots + (a_{n-1} + g_2)s + (a_n + g_1) = 0$$

Feedback gain formula for Controller Canonical Form
(Bass-Gura formula)

Open-loop poles: $$s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n = 0$$

Closed-loop poles: $$s^n + \underline{(a_1 + g_n)} s^{n-1} + \underline{(a_2 + g_{n-1})} s^{n-2} + \cdots + \underline{(a_{n-1} + g_2)} s + \underline{(a_n + g_1)} = 0$$

Desired poles: $$\prod_{i=1}^{n} \left(s - s_i^{\mathrm{C}}\right) = s^n + \underline{\alpha_1} s^{n-1} + \underline{\alpha_2} s^{n-2} + \cdots + \underline{\alpha_{n-1}} s + \underline{\alpha_n} = 0$$

In order to have closed-loop poles at the same location of desired poles, the coefficients of the two characteristic polynomials must be the same

$$\Rightarrow \begin{cases} g_1 = \alpha_n - a_n \\ g_2 = \alpha_{n-1} - a_{n-1} \\ \vdots \\ g_{n-1} = \alpha_2 - a_2 \\ g_n = \alpha_1 - a_1 \end{cases}$$

Feedback gain formula for Controller Canonical Form
(Bass-Gura formula)

The previous design procedure requires that the state equation is expressed through the first companion form. However, in practical applications, this condition is rarely met. What can be done is to transform the original state-space formulation of the problem under investigation to the first companion form.

$$\mathbf{x}_{\mathrm{CF}}(t) = \mathbf{T}\mathbf{x}(t) \quad \Rightarrow \quad \mathbf{A}_{\mathrm{CF}} = \mathbf{T}\mathbf{A}\mathbf{T}^{-1} \qquad \mathbf{b}_{\mathrm{CF}} = \mathbf{T}\mathbf{b}_u$$

$$u(t) = -\mathbf{g}_{\mathrm{CF}}\mathbf{x}_{\mathrm{CF}}(t)$$

T is a similarity transformation, i.e.
A and $\mathbf{A}_{CF}$ have the same eigenvalues
(we can assign the poles of the system in the first companion form since they are the same poles of the original system)

Feedback gain formula for Controller Canonical Form
(Bass-Gura formula)

$$\mathbf{x}_{\mathrm{CF}}(t) = \mathbf{T}\mathbf{x}(t)$$

$$u(t) = -\mathbf{g}_{\mathrm{CF}}\mathbf{x}_{\mathrm{CF}}(t)$$

$$u(t) = -\mathbf{g}_{\mathrm{CF}}\mathbf{T}\mathbf{x}(t) = -\mathbf{g}\,\mathbf{x}(t)$$

$$\boxed{\mathbf{g} = \mathbf{g}_{\mathrm{CF}}\mathbf{T}}$$

The transformation matrix $\mathbf{T}$ can be determined by using the relation between the controllability matrix $\mathcal{C}$ of the original system and the controllability matrix of the transformed system, which is given by $\mathcal{C}_{\mathrm{CF}} = \mathbf{T}\mathcal{C}$

$$\mathcal{C}_{\mathrm{CF}} = \begin{bmatrix} a_{n-1} & a_{n-2} & \dots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \dots & 1 & 0 \\ \vdots & & & & \\ a_1 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix}^{-1}$$

$$\mathbf{T} = \mathcal{C}_{\mathrm{CF}}\mathcal{C}^{-1}$$

$$\mathbf{g} = \mathbf{g}_{\mathrm{CF}}\mathcal{C}_{\mathrm{CF}}\mathcal{C}^{-1}$$

Feedback gain formula for Controller Canonical Form
(Bass-Gura formula)

– Summary and implementation

State space model of the system

$(\mathbf{A}, \mathbf{b}_u)$

$a_1, a_2, \ldots, a_n$

coefficients of the characteristic polynomial of A

Desired set of closed-loop poles

$(s_1^{\mathrm{C}}, s_2^{\mathrm{C}}, \ldots, s_n^{\mathrm{C}})$

$\alpha_1, \alpha_2, \ldots, \alpha_n$

coefficients of the desired characteristic polynomial

$$\mathbf{g}_{\mathrm{CF}} = [\alpha_n - a_n \quad \ldots \quad \alpha_1 - a_1]$$

This is how we will get the control law

control gains in canonical form

Feedback gain formula for Controller Canonical Form
(Bass-Gura formula)

– Summary and implementation

Controllability matrix

$$\mathcal{C} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \dots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}$$

Controllability matrix (canonical form)

$$\mathcal{C}_{\mathrm{CF}} = \begin{bmatrix} a_{n-1} & a_{n-2} & \dots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \dots & 1 & 0 \\ \vdots & & & & \\ a_1 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix}^{-1}$$

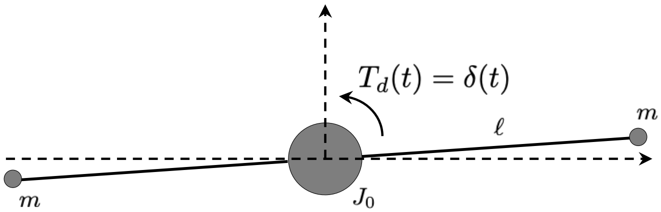
$$\mathbf{T} = \mathcal{C}_{\mathrm{CF}}\mathcal{C}^{-1}$$

$$\mathbf{g} = \mathbf{g}_{\mathrm{CF}}\mathbf{T}$$

control gains

Example

Rigid spacecraft (J = 100 kg m^2) subject to an **impulse disturbance** of amplitude 1 N m



__Open-loop behavior__

Equation of motion: $$J\ddot{\theta}(t) = \delta(t)$$

State-space representation: $$\mathbf{x}(t) = \begin{Bmatrix} \theta(t) \\ \dot{\theta}(t) \end{Bmatrix}$$

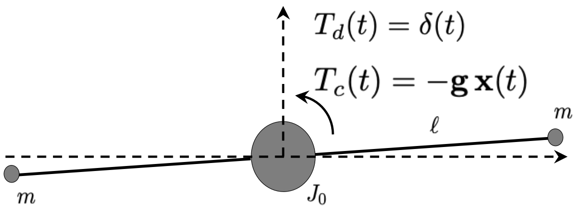
$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1/J \end{bmatrix} \delta(t)$$

$$\mathbf{b}_d = \begin{bmatrix} 0 \\ 1/J \end{bmatrix} \qquad \mathbf{y}(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(t)$$

→ system controllable in the canonical form

$$\mathbf{c}_y = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Example – rigid spacecraft subject to an impulse disturbance



Closed-loop behavior

Equation of motion:

$$J\ddot{\theta}(t) = T_c(t) + \delta(t)$$

State feedback control law:

$$T_c(t) = -\mathbf{g}\,\mathbf{x}(t)$$

→ state feedback control law

→ gain on the angular rate

gain on the angle

$$= -\begin{bmatrix} g_1 & g_2 \end{bmatrix} \begin{Bmatrix} \theta(t) \\ \dot{\theta}(t) \end{Bmatrix}$$

→ looks similar to a PD controller.

$$= -g_1\theta(t) - g_2\dot{\theta}(t)$$

(PD-like control)

Closed-loop equation of motion:

$$J\ddot{\theta}(t) + g_2\dot{\theta}(t) + g_1\theta(t) = \delta(t)$$

Example – rigid spacecraft subject to an impulse disturbance

Closed-loop behavior

State-space representation:

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1/J \end{bmatrix} T_c(t) + \begin{bmatrix} 0 \\ 1/J \end{bmatrix} \delta(t)$$

$$T_c(t) = -\mathbf{g}\,\mathbf{x}(t)$$

$$\mathbf{A}_c = \begin{bmatrix} 0 & 1 \\ -g_1/J & -g_2/J \end{bmatrix}$$

$$\mathbf{b}_d = \begin{bmatrix} 0 \\ 1/J \end{bmatrix}$$

$$\mathbf{c}_y = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ -g_1/J & -g_2/J \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1/J \end{bmatrix} \delta(t)$$

$$\mathbf{y}(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(t)$$

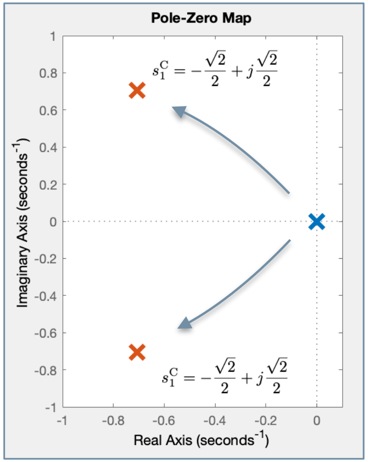
Example – rigid spacecraft subject to an impulse disturbance

Closed-loop behavior

Pole placement design – desired closed-loop poles

$$s_1^{\mathrm{C}} = -\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}$$

$$s_2^{\mathrm{C}} = -\frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2}$$



desired closed-loop characteristic polynomial

$$(s - s_1^{\mathrm{C}})(s - s_2^{\mathrm{C}}) =$$
$$s^2 - (s_1^{\mathrm{C}} + s_2^{\mathrm{C}})s + s_1^{\mathrm{C}} s_2^{\mathrm{C}} =$$
$$s^2 + \sqrt{2}s + 1$$

open-loop characteristic polynomial

$$s^2$$

Example – rigid spacecraft subject to an impulse disturbance

Closed-loop behavior

$$\mathbf{A}_c = \begin{bmatrix} 0 & 1 \\ -g_1/J & -g_2/J \end{bmatrix}$$

$$\mathbf{b}_d = \begin{bmatrix} 0 \\ 1/J \end{bmatrix}$$

$$\mathbf{c}_y = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$\mathbf{g} = 100 \begin{bmatrix} 1 & \sqrt{2} \end{bmatrix}$$

```
impulse(sysc)
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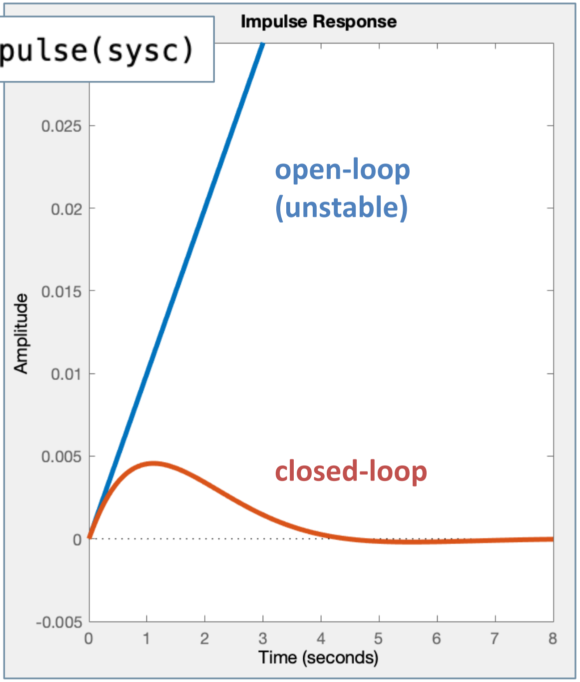
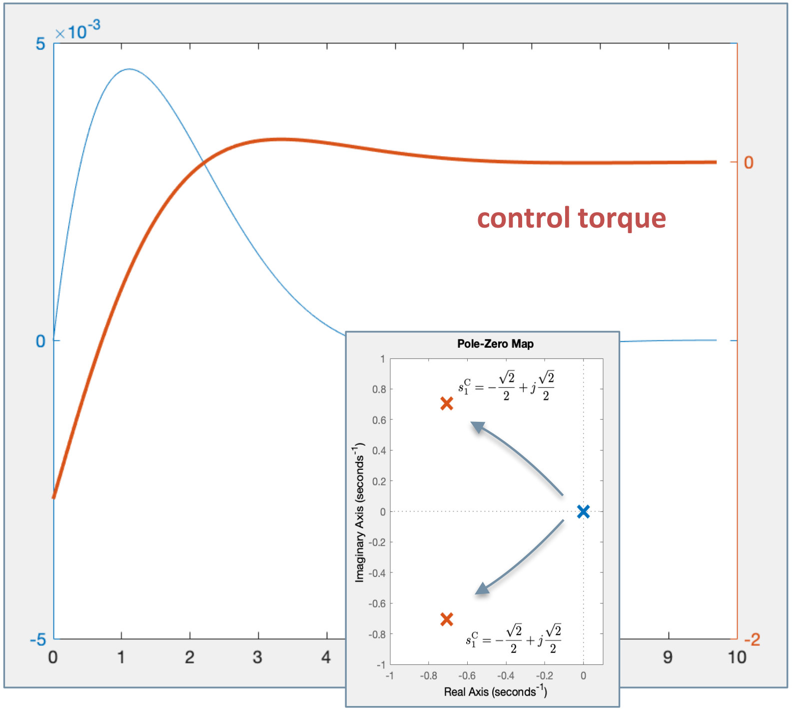


Example – rigid spacecraft subject to an impulse disturbance

Closed-loop behavior

$$\mathbf{g} = 100\,[1 \quad \sqrt{2}]$$

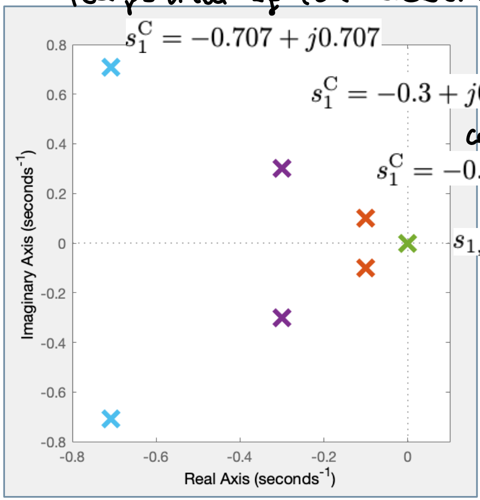
$$T_c(t) = -\mathbf{g}\,\mathbf{x}(t)$$



Example – rigid spacecraft subject to an impulse disturbance

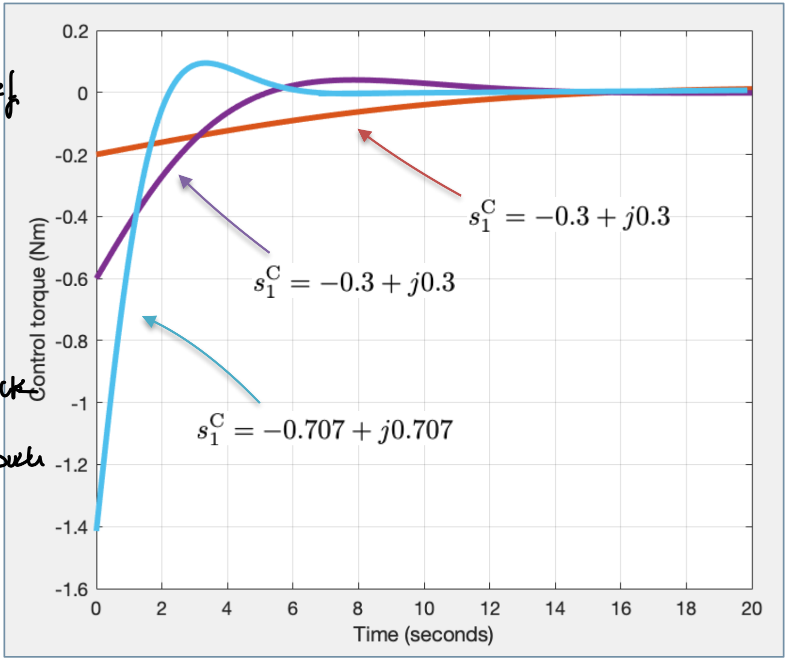
Closed-loop behavior

We need to understand what is the effect of the position of the modes on the response of the system and also to the value of the control torque



We need to check if our system is able to provide such kind of control action.

Further implications to where we need to position the poles.



The more distance will be the poles and higher will be the control effort so we need to keep a cost on that.

Until now, we have considered the design of single-input systems. If the system under consideration has more than one control input, it means that the state feedback control law is expressed as

$$\mathbf{u}(t) = -\mathbf{G}\mathbf{x}(t) \tag{17.37}$$

where $\mathbf{G}$ is now a matrix with n_u rows (as the number of inputs) and n columns. The closed-loop dynamics is expressed as

$$\dot{\mathbf{x}}(t) = \mathbf{A}_c\mathbf{x}(t) + \mathbf{b}_d d(t) \tag{17.38}$$

where

$$\mathbf{A}_c = \mathbf{A} - \mathbf{B}_u\mathbf{G} \tag{17.39}$$

is the closed-loop system matrix. Therefore, the design process involves the computation of the control matrix $\mathbf{G}$. Since each row of $\mathbf{G}$ furnishes n gains that can be adjusted, it is clear that in a controllable system there will be more gains available than are needed to place all of the closed-loop poles. This can be considered as an increased flexibility for the designer since he/she can place the poles and, at the same time, satisfy some other requirements. For example, the control system structure can be simplified by setting some of the gains to zero. Another possibility is to use the extra degrees of freedom to find a solution which minimizes the sensitivity of the closed-loop poles to perturbations in the $\mathbf{A}$ and $\mathbf{B}$ matrices. This method is adopted by the MATLAB function

↳ Different logic to do it

```
G = place(A,B,p)
```

↳ The most important task is where to position the pole p.

LINEAR QUADRATIC REGULATOR (LQR)

(infinite-horizon/steady state LQR)

Introduction – shortcomings of pole placement technique

Through the assignment of the closed-loop poles, the pole-placement technique can specify the speed (bandwidth) and damping of the dynamic response of LTI systems.

However, it suffers from some shortcomings.

1. The first shortcoming is related to the **multiple input case**. We have seen that, in this case, there are more gains than those needed. The resulting flexibility provides an undetermined control solution, since there are infinitely many ways by which the same closed-loop poles can be attained. The lack of a definitive algorithm gives rise to a natural question: **which way of assigning poles is best**?

2. Another shortcoming of the pole-placement technique is that the designer has **no direct information on the control effort associated with a particular solution**. Choosing pole locations far from the open-loop poles (and from the origin) will typically require large control signals, which can exceed the available power source. However, it is difficult to predict the amount of control effort and to identify the poles that dominate the response.

3. Finally, it is well known that the **transient response of LTI systems can be also strongly affected by the zeros**. The pole-placement technique is focused only on the pole locations.

Problem definition

The above limitations can be overcome by using another state-space design technique, which is known as steady-state optimal linear quadratic control or **linear quadratic regulator** (**LQR**).

The LQR problem is formulated as follows.

As done before with the pole-placement approach, again we seek a stabilizing linear state feedback with constant gain matrix

$$\mathbf{u}(t) = -\mathbf{G}\mathbf{x}(t)$$

Here, however, the gain matrix **G** is selected such that the following quadratic cost function (or performance index) is minimized

looking forward in time

quadratic function of the control input

$$\boxed{J = \frac{1}{2}\int_0^{\infty} \left(\mathbf{z}^T\mathbf{W}_{zz}\mathbf{z} + \mathbf{u}^T\mathbf{W}_{uu}\mathbf{u}\right) dt}$$

control input

performance function

can be either the state or contain other variable over the state,

Problem definition

Cost function: $$J = \frac{1}{2}\int_0^\infty \left(\mathbf{z}^T \mathbf{W}_{zz}\mathbf{z} + \mathbf{u}^T \mathbf{W}_{uu}\mathbf{u}\right) dt$$

$\mathbf{z}(t)$ is a selected performance vector

$\mathbf{W}_{zz}$ is a nonnegative symmetric weighting matrix ($\mathbf{W}_{zz} \geq 0$) associated with the performance

$\mathbf{W}_{uu}$ is a symmetric weighting matrix related to the control effort

Remarks:

- minimization of *J* also minimizes *aJ*, where *a* is any positive constant – so the problem is not altered if we multiply the cost function by any positive value.
- minimization of *J* will depend on the selection of the weighting matrices $\mathbf{W}_{zz}$ and $\mathbf{W}_{uu}$ – the corresponding solution will then be strongly affected by the weighting matrices, which therefore play a fundamental role in the design process.
- x the cost function contains two contributions – the first represents the penalty on the deviation of the performance vector **z** from the origin, the second represents the cost of control and is included in order to limit the magnitude of the control variables (avoid saturation).

Weighting matrices

$$J = \frac{1}{2}\int_0^{\infty} \left(\mathbf{z}^T \mathbf{W}_{zz}\mathbf{z} + \mathbf{u}^T \mathbf{W}_{uu}\mathbf{u}\right) dt$$

The weighting matrices can be used to specify the relative importance of the various components of the performance vector and the control input vector. For example, for a system with two components of the performance vector z_1 and z_2, the selection of the following weighting matrix

$$\mathbf{W}_{zz} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{17.42}$$

will introduce a penalty for the first performance variable z_1 without imposing any restriction for z_2. Since the previous choice might lead to values of z_2 larger than desired, a limitation can be imposed by selecting the following diagonal weighting matrix

$$\mathbf{W}_{zz} = \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix} \tag{17.43}$$

where the weight c can be regulated to achieve a desired behavior of z_2 compared to the behavior of z_1.

Weighting matrices

$$J = \frac{1}{2}\int_0^\infty \left(\mathbf{z}^T\mathbf{W}_{zz}\mathbf{z} + \mathbf{u}^T\mathbf{W}_{uu}\mathbf{u}\right) dt$$

The same procedure can be applied to the control inputs.

Referring to a LTI system with three control inputs u_1, u_2 and u_3, a diagonal control weighting matrix $\mathbf{W}_{uu}$ as follows

$$\mathbf{W}_{uu} = \begin{bmatrix} \rho_1 & 0 & 0 \\ 0 & \rho_2 & 0 \\ 0 & 0 & \rho_3 \end{bmatrix}$$

can be used to regulate the relative control effort of the three components by tuning the ratios ρ_2/ρ_1 and ρ_3/ρ_1.

Weighting matrices

$$J = \frac{1}{2}\int_0^{\infty} \left(\mathbf{z}^T \mathbf{W}_{zz}\mathbf{z} + \mathbf{u}^T \mathbf{W}_{uu}\mathbf{u}\right) dt$$

Remarks:

1. It is very difficult and, in most cases, impractical to accurately predict the effect of the selection of a given pair of weighting matrices on the closed-loop dynamic response of the system. This is one of the most restrictive aspect of the LQR design. It means that **there is a gap between what the LQR controller achieves and the desired control system performance**. The optimization problem expressed by the minimization of the cost function *J* may have very little to do with more meaningful control system specifications like levels of disturbance rejection, overshoot in tracking, stability margins, and so on. This aspect should always be kept in mind when using the LQR technique.

2. An LQR design which is optimal does not imply that it meets the performance goals, since performance requirements are not given in terms of minimizing quadratic costs. It is the job of the designer to use the LQR tool wisely. According to what discussed, the design process is typically carried out iteratively, by starting from a guess pair of $\mathbf{W}_{zz}$ and $\mathbf{W}_{uu}$, then simulating the closed-loop response, and changing the weighting matrices so that the corresponding gain matrix **G** will produce the response closest to the design objectives.

Control system design – state-space methods
LQR

Optimal solution

Open-loop behavior:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}_u\mathbf{u}(t)$$
$$\mathbf{z}(t) = \mathbf{C}_z\mathbf{x}(t) + \mathbf{D}_{zu}\mathbf{u}(t)$$
$$\mathbf{x}(0) = \mathbf{x}_0$$

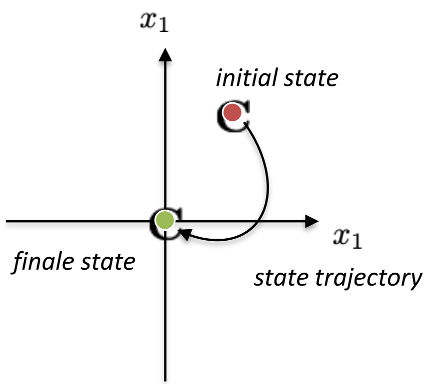
optimal gain matrix

$$\mathbf{u}(t) = -\mathbf{G}\mathbf{x}(t)$$

Closed-loop behavior:

$$\dot{\mathbf{x}}(t) = \mathbf{A}_c\mathbf{x}(t)$$
$$\mathbf{z}(t) = \mathbf{C}_{zc}\mathbf{x}(t)$$

$$\mathbf{A}_c = \mathbf{A} - \mathbf{B}_u\mathbf{G}$$
$$\mathbf{C}_{zc} = \mathbf{C}_z - \mathbf{D}_{zu}\mathbf{G}$$



$$\mathbf{x}(t) = e^{\mathbf{A}_c t}\mathbf{x}_0$$

$$\mathbf{x}(t) \to 0$$

it can be shown that the closed-loop system is asymptotically stable

Optimal solution

$$\mathbf{z}(t) = \mathbf{C}_z\mathbf{x}(t) + \mathbf{D}_{zu}\mathbf{u}(t)$$

$$J = \frac{1}{2}\int_0^\infty \left(\mathbf{z}^T\mathbf{W}_{zz}\mathbf{z} + \mathbf{u}^T\mathbf{W}_{uu}\mathbf{u}\right) dt$$

the cost function can be written as

$$J = \frac{1}{2}\int_0^\infty \left(\mathbf{x}^T\mathbf{Q}\mathbf{x} + 2\mathbf{x}^T\mathbf{S}\mathbf{u} + \mathbf{u}^T\mathbf{R}\mathbf{u}\right) dt$$
$$= \frac{1}{2}\int_0^\infty \begin{Bmatrix}\mathbf{x}\\\mathbf{u}\end{Bmatrix}^T \begin{bmatrix}\mathbf{Q} & \mathbf{S}\\\mathbf{S}^T & \mathbf{R}\end{bmatrix} \begin{Bmatrix}\mathbf{x}\\\mathbf{u}\end{Bmatrix} dt$$

where

- $\mathbf{Q} = \mathbf{C}_z^T\mathbf{W}_{zz}\mathbf{C}_z$ is the nonnegative symmetric weighting matrix related to the states
- $\mathbf{R} = \mathbf{W}_{uu} + \mathbf{D}_{zu}^T\mathbf{W}_{zz}\mathbf{D}_{zu}$ is the positive definite symmetric weighting matrix related to the control effort
- $\mathbf{S} = \mathbf{C}_z^T\mathbf{W}_{zz}\mathbf{D}_{zu}$ is the coupled weighting matrix

Control system design – state-space methods
LQR

Optimal solution

weight on the control input

solution of the equation that we will see

It can be shown that
the gain matrix corresponding to
the minimum of *J* is given by

$$\boxed{\mathbf{G} = \mathbf{R}^{-1}\left(\mathbf{B}_u^T\mathbf{P} + \mathbf{S}^T\right)}$$

where **P** is the symmetric positive matrix solution of the following equation

$$\mathbf{P}\left(\mathbf{A} - \mathbf{B}_u\mathbf{R}^{-1}\mathbf{S}^T\right) + \left(\mathbf{A}^T - \mathbf{S}\mathbf{R}^{-1}\mathbf{B}_u^T\right)\mathbf{P} - \mathbf{P}\mathbf{B}_u\mathbf{R}^{-1}\mathbf{B}_u^T\mathbf{P} + \mathbf{Q} - \mathbf{S}\mathbf{R}^{-1}\mathbf{S}^T = \mathbf{0}$$

(algebraic Riccati equation)

It can be shown that:

1. The existence and uniqueness of the solution is guaranteed if (**A**,**B**) is a controllable pair (stabilizable is in fact enough) and (**A**,$\mathbf{Q}^{1/2}$) is observable.
2. The LQR (with full state feedback) has an infinite gain margin and a phase margin of at least 60°.

LQR design

NOTE if we want to have the accelerations performance as they are never included in the state we need to choose accordingly a combination of the state variables

Steps:

1. Select the performance ***z*** and write the corresponding performance equation.

2. Select the weighting matrices $\mathbf{W}_{zz}$ and $\mathbf{W}_{uu}$ associated with the penalty of the performance vector and the cost of control, respectively.

— from that point I can do on matlab

3. Compute the corresponding weighting matrices **Q**, **R** and **S**.

4. Solve the (algebraic) Riccati equation to compute the matrix **P**.

5. Compute the gain matrix $\mathbf{G} = \mathbf{R}^{-1}\left(\mathbf{B}_u^T\mathbf{P} + \mathbf{S}^T\right)$

with $\mathbf{S} = \mathbf{0}$

the solution is given by

$$\mathbf{G} = \mathbf{R}^{-1}\mathbf{B}_u^{\mathrm{T}}\mathbf{P}$$

$$\mathbf{PA} + \mathbf{A}^T\mathbf{P} - \mathbf{PB}_u\mathbf{R}^{-1}\mathbf{B}_u^T\mathbf{P} + \mathbf{Q} = \mathbf{0}$$

Example – rigid spacecraft subject to an impulse disturbance

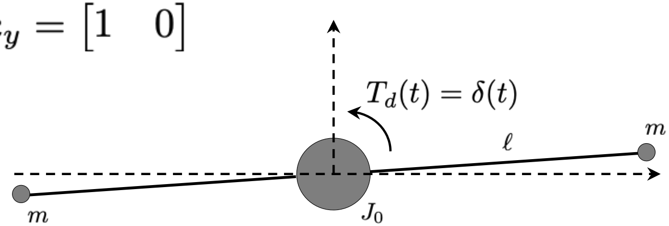
Open-loop behavior

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}_d\delta(t)$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{b}_d = \begin{bmatrix} 0 \\ 1/J \end{bmatrix}$$

$$\mathbf{c}_y = \begin{bmatrix} 1 & 0 \end{bmatrix}$$



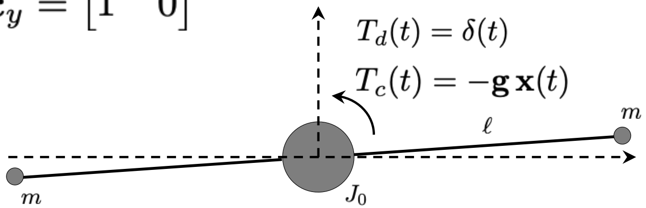
Closed-loop behavior

$$\dot{\mathbf{x}}(t) = \mathbf{A}_c\mathbf{x}(t) + \mathbf{b}_d\delta(t)$$

$$\mathbf{A}_c = \begin{bmatrix} 0 & 1 \\ -g_1/J & -g_2/J \end{bmatrix}$$

$$\mathbf{b}_d = \begin{bmatrix} 0 \\ 1/J \end{bmatrix}$$

$$\mathbf{c}_y = \begin{bmatrix} 1 & 0 \end{bmatrix}$$



Control system design – state-space methods
LQR - example

Example – rigid spacecraft subject to an impulse disturbance

Closed-loop behavior

LQR design

1. performance $z(t) = \theta(t)$ $\Rightarrow$ $z(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(t) = \mathbf{c}_z \mathbf{x}(t)$

2. weighting matrices $W_{zz} = 1 \quad W_{uu} = \rho$

3. weighting matrices $\mathbf{Q} = \mathbf{c}_z^{\mathrm{T}} W_{zz} \mathbf{c}_z = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$$\mathbf{R} = W_{uu} + \mathbf{d}_{zu}^{\mathrm{T}} W_{zz} \mathbf{d}_{zu} = \rho$$

to be varied for tuning the closed-loop behavior

↳ This is corresponding to that when the input reach the maximum the contribution will be 1.

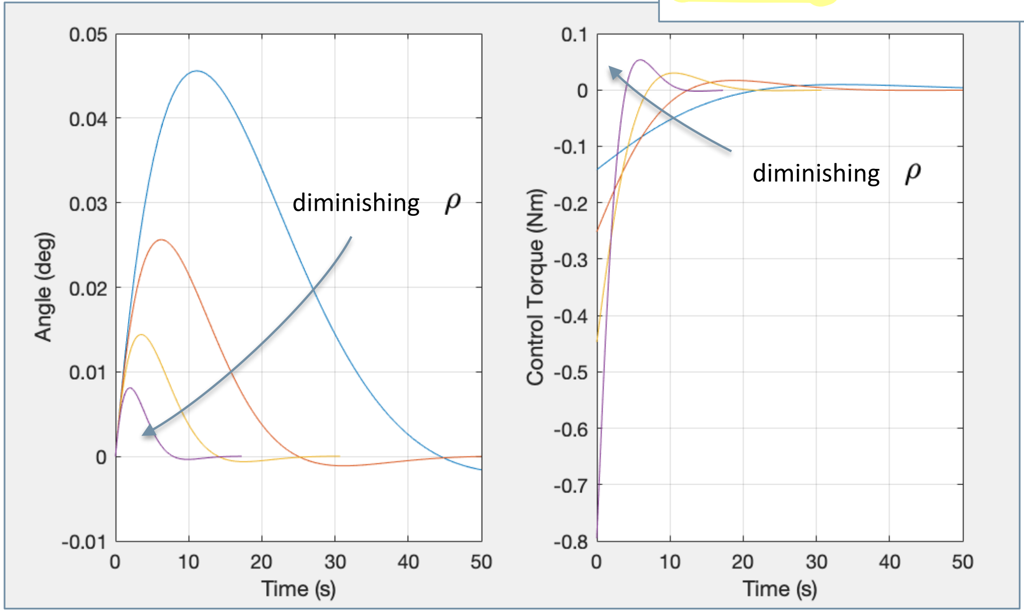
$$\mathbf{S} = \mathbf{c}_z^{\mathrm{T}} W_{zz} \mathbf{d}_{zu} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

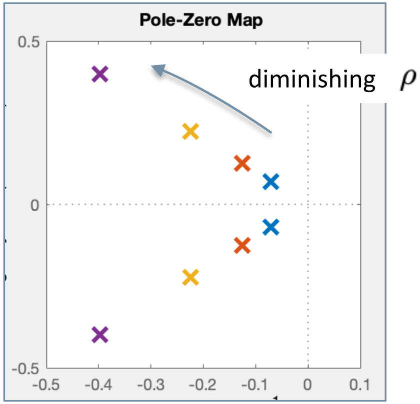
Control system design – state-space methods
LQR - example

Example – rigid spacecraft subject to an impulse disturbance

Closed-loop behavior

```
[y,t,x] = impulse(sysc);
u = -g*x';
```





$\rho = 1$

$\rho = 0.1$

$\rho = 0.01$

$\rho = 0.001$

STEADY-STATE TRACKING

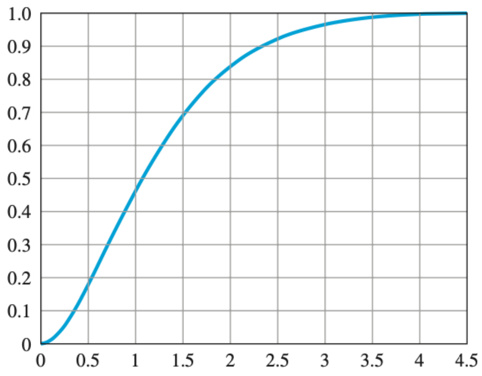
Control system design – state-space methods
Steady-state tracking

Thus far, the control has been given by $\mathbf{u}(t) = -\mathbf{G}\mathbf{x}(t)$

In order to study the response of the pole-placement/LQR designs to input commands, it is necessary to **introduce the reference input into the system**.

We'll discuss two common methods for adding **steady-state tracking** capabilities to full state feedback laws.

1. Feedforward input
2. Integral action

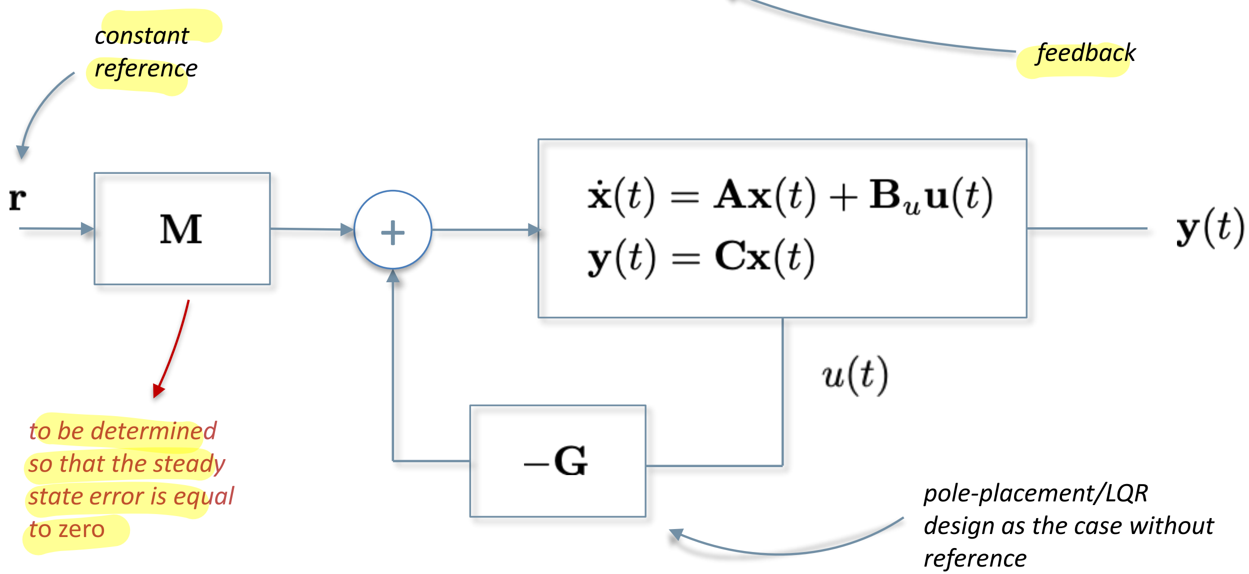


This approach is based on writing the feedback control law as

$$\mathbf{u}(t) = -\mathbf{G}\mathbf{x}(t) + \mathbf{M}\mathbf{r}$$

feedforward input (proportional to the reference to track)

feedback



Control system design – state-space methods
Steady-state tracking – feedforward input

feed forward term

$$\mathbf{u}(t) = -\mathbf{G}\mathbf{x}(t) + \mathbf{M}\mathbf{r}$$

close loop state control

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}_u\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

$$\dot{\mathbf{x}}(t) = (\mathbf{A} - \mathbf{B}_u\mathbf{G})\,\mathbf{x}(t) + \mathbf{B}_u\mathbf{M}\mathbf{r}$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

Steady-state condition:

$$\mathbf{0} = (\mathbf{A} - \mathbf{B}_u\mathbf{G})\,\mathbf{x}_{\mathrm{ss}} + \mathbf{B}_u\mathbf{M}\mathbf{r}$$
$$\mathbf{y}_{\mathrm{ss}} = \mathbf{C}\mathbf{x}_{\mathrm{ss}}$$

From the first equation:

$$\mathbf{x}_{\mathrm{ss}} = -(\mathbf{A} - \mathbf{B}_u\mathbf{G})^{-1}\mathbf{B}_u\mathbf{M}\mathbf{r}$$

Putting into the second equation:

$$\mathbf{y}_{\mathrm{ss}} = \mathbf{C}\mathbf{x}_{\mathrm{ss}} = -\mathbf{C}(\mathbf{A} - \mathbf{B}_u\mathbf{G})^{-1}\mathbf{B}_u\mathbf{M}\mathbf{r}$$

Control system design – state-space methods
Steady-state tracking – feedforward input

Steady state output: $\mathbf{y}_{ss} = \mathbf{C}\mathbf{x}_{ss} = -\mathbf{C}\left(\mathbf{A} - \mathbf{B}_u\mathbf{G}\right)^{-1}\mathbf{B}_u\mathbf{M}\mathbf{r}$

Imposing $\mathbf{y}_{ss} = \mathbf{r}$

yields the following condition $-\mathbf{C}\left(\mathbf{A} - \mathbf{B}_u\mathbf{G}\right)^{-1}\mathbf{B}_u\mathbf{M} = \mathbf{I}$

$$\boxed{\mathbf{M} = -\left[\mathbf{C}\left(\mathbf{A} - \mathbf{B}_u\mathbf{G}\right)^{-1}\mathbf{B}_u\right]^{-1}}$$

feed forward matrix depend directly on the gain just set.

Since

$$\mathbf{A}_c = \mathbf{A} - \mathbf{B}_u\mathbf{G} \quad \Rightarrow \quad \mathbf{M} = -\left[\mathbf{C}\mathbf{A}_c^{-1}\mathbf{B}_u\right]^{-1}$$

(inverse of the static gain)

Remarks:

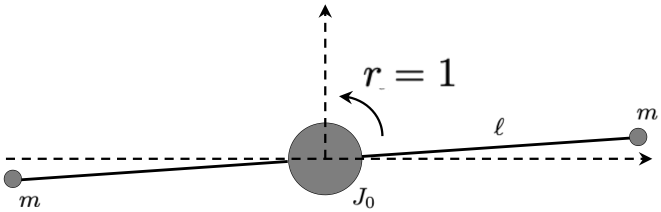
1. number of inputs equal to the number of outputs

2. the matrix $C(A-B_uG)^{-1}B_u$ must be invertible

3. the method requires accurate knowledge of the open-loop state equations coefficient matrices (it is not robust wrt any change in the plant parameters)

Example

Rigid spacecraft (J = 100 kg m^2) aimed at **tracking a unit step reference**



Open-loop behavior

Equation of motion: $J\ddot{\theta}(t) = T_c(t)$

State-space representation: $\mathbf{x}(t) = \begin{Bmatrix} \theta(t) \\ \dot{\theta}(t) \end{Bmatrix}$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1/J \end{bmatrix} T_c(t)$$

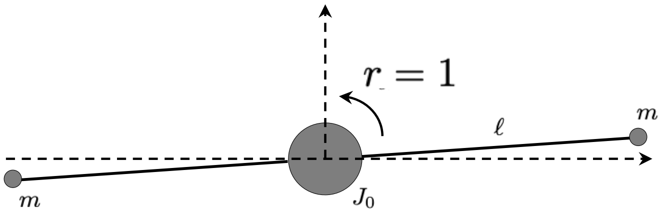
$$\mathbf{b}_u = \begin{bmatrix} 0 \\ 1/J \end{bmatrix}$$

$$\mathbf{y}(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(t)$$

$$\mathbf{c}_y = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Example

Rigid spacecraft (J = 100 kg m^2) aimed at **tracking a unit step reference**



Closed-loop behavior

State feedback control law:

$$T_c(t) = -\mathbf{g}\,\mathbf{x}(t) + Mr$$
$$= -\begin{bmatrix} g_1 & g_2 \end{bmatrix} \begin{Bmatrix} \theta(t) \\ \dot{\theta}(t) \end{Bmatrix} + Mr$$
$$= -g_1\theta(t) - g_2\dot{\theta}(t) + Mr$$

Closed-loop equation of motion:

$$J\ddot{\theta}(t) + g_2\dot{\theta}(t) + g_1\theta(t) = Mr$$

Example

Rigid spacecraft ($J = 100\ \mathrm{kg\ m^2}$) aimed at **tracking a unit step reference**

Closed-loop behavior

State-space representation:

$$T_c(t) = -\mathbf{g}\,\mathbf{x}(t) + Mr$$

$$\mathbf{A}_c = \begin{bmatrix} 0 & 1 \\ -g_1/J & -g_2/J \end{bmatrix}$$

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ -g_1/J & -g_2/J \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1/J \end{bmatrix} \delta(t)$$

$$\mathbf{c}_y = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

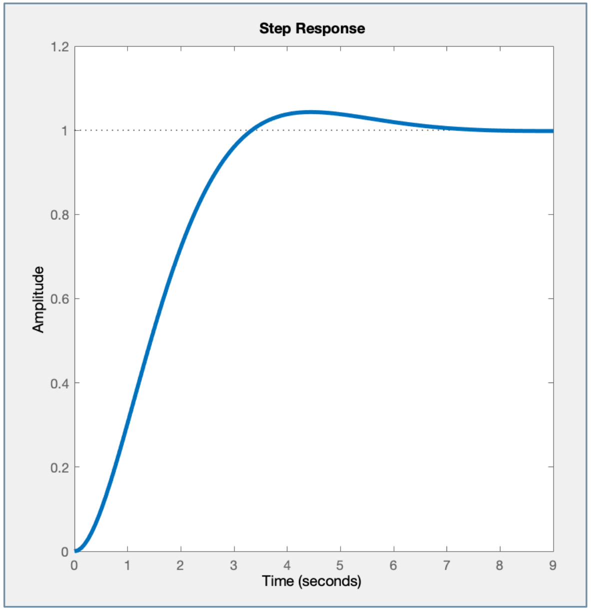
$$\mathbf{y}(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(t)$$

$$M = -\left(\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -g_1/J & -g_2/J \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1/J \end{bmatrix} \right)^{-1}$$

Example

Rigid spacecraft ($J = 100$ kg m^2) aimed at **tracking a unit step reference**

<u>Closed-loop behavior</u>



Example

Rigid spacecraft (J = 100 kg m^2) aimed at **tracking a unit step reference**

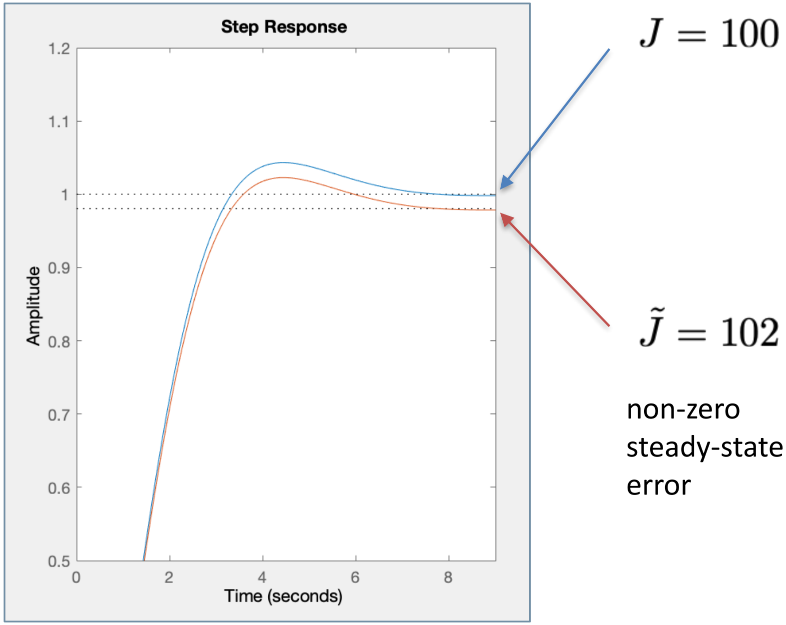
<u>Closed-loop behavior</u>

Remarks:

1. number of inputs equal to the number of outputs

2. the matrix $C(A-B_uG)^{-1}B_u$ must be invertible

3. **the method requires accurate knowledge of the open-loop state equations coefficient matrices (it is not robust wrt any change in the plant parameters)**



non-zero steady-state error

Another common approach of forcing to zero the steady-state error is **adding the integral of the system error to the dynamic equations**.

Let's assume the control law given by

Tracking error

$$\mathbf{u}(t) = -\mathbf{G}\mathbf{x}(t) - \mathbf{G}_I \int \mathbf{e}(t)dt$$

expected output

actual output

$$\mathbf{e}(t) = \mathbf{r} - \mathbf{y}(t)$$

(tracking error)

Calling $\mathbf{x}_I(t) = \int \mathbf{e}(t)dt$ → Extended state → additional state including the integral of the tracking error.

the new state representing the integral of the tracking error, the control law can be expressed as

$$\mathbf{u}(t) = -\begin{bmatrix} \mathbf{G} & \mathbf{G}_I \end{bmatrix} \begin{Bmatrix} \mathbf{x}(t) \\ \mathbf{x}_I(t) \end{Bmatrix}$$

augmented state

gain matrix to be determined (pole-placement/LQR)

Dynamics of the augmented state:

$$\dot{\mathbf{x}}(t) = \mathbf{A}_c\mathbf{x}(t) + \mathbf{b}_d\delta(t)$$

$$\dot{\mathbf{x}}_I(t) = \mathbf{e}(t) = \mathbf{r} - \mathbf{y}(t) = \mathbf{r} - \mathbf{C}\mathbf{x}(t)$$

↳ function only of the state.

$$\mathbf{x}_{\text{aug}}(t) = \begin{Bmatrix} \mathbf{x}(t) \\ \mathbf{x}_I(t) \end{Bmatrix}$$

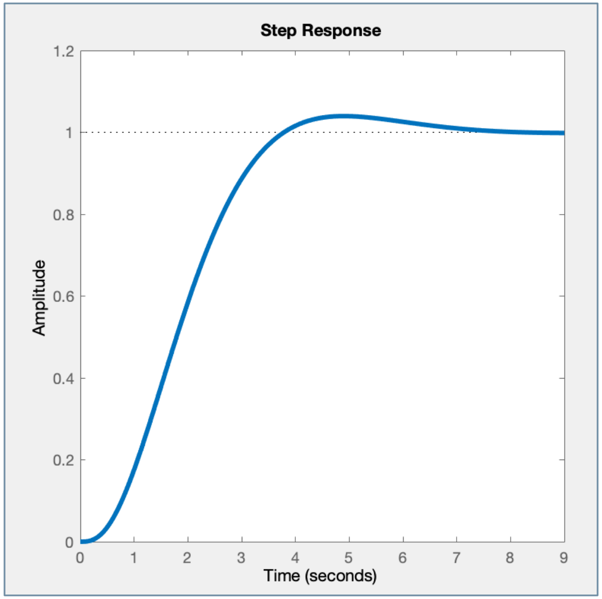
augmented dynamics

$$\dot{\mathbf{x}}_{\text{aug}}(t) = \mathbf{A}_{\text{aug}}\mathbf{x}_{\text{aug}}(t) + \mathbf{B}_{u\text{aug}}\mathbf{u}(t) + \mathbf{B}_{r\text{aug}}\mathbf{r}$$

$$\mathbf{A}_{\text{aug}} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & \mathbf{0} \end{bmatrix} \qquad \mathbf{x}(t) = \begin{Bmatrix} \theta(t) \\ \dot{\theta}(t) \end{Bmatrix} \qquad \mathbf{B}_{r\text{aug}} = \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix}$$

Example

Rigid spacecraft ($J = 100$ kg m^2)
aimed at **tracking a unit step reference**



Example

Rigid spacecraft (J = 100 kg m^2)
aimed at **tracking a unit step reference**

Because it is not required the exact knowledge of the system → steady state that computed using the feedforward matrix computed only by the error.

Remark:
the approach of including the integral action is more robust with respect to any change in the plant parameters