POLITECNICO
MILANO 1863

Spacecraft Attitude Dynamics

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Fundamental Properties

Transport Theorem

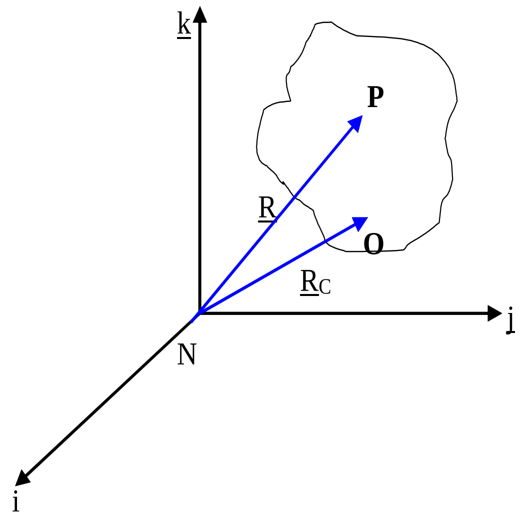
Let N and B be two coordinate frames with a relative angular velocity $\underline{\omega}_{B/N}$

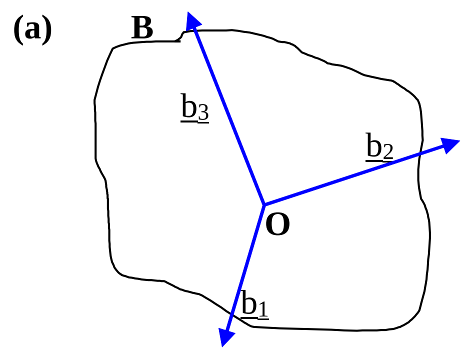
$\underline{x}$ is a generic vector then

$$\frac{{}^{N}d}{dt}\underline{x} = \frac{{}^{B}d}{dt}\underline{x} + \underline{\omega}_{B/N} \times \underline{x}$$

$$\dot{\underline{x}} = \frac{{}^{N}d}{dt}\underline{x}$$

Angular Momentum of a Rigid Body





The angular momentum of a rigid body B is defined with respect to an origin fixed on the rigid body. For an infinitesimal point mass we have:

$$d\underline{h}_o = \underline{R} \times \dot{\underline{R}}\, dm$$

The total angular momentum is then:

$$\underline{h}_o = \underbrace{\underline{R}_C \times \dot{\underline{R}}_C M}_{\text{angular momentum of the mass centre about the origin}} + \underbrace{\int_B (\underline{r} \times \dot{\underline{r}})\, dm}_{\text{angular momentum of rigid body about its centre of mass } \underline{h}}$$

Rotational Angular Momentum of a Rigid Body (Exercise)

$$\underline{h} = \int_B \underline{r} \times (\underline{\omega} \times \underline{r}) \mathrm{d}m$$

Show that the angular momentum about the centre of mass evaluated in the body frame is:

$$\underline{h} = I\underline{\omega}$$

where

$$I = \begin{bmatrix} I_{xx} = \int_B (y^2 + z^2) dm & I_{xy} = \int_B -xydm & I_{xz} = \int_B -xzdm \\ I_{yx} = \int_B -yxdm & I_{yy} = \int_B (x^2 + z^2) dm & I_{yz} = \int_B -yzdm \\ I_{zx} = \int_B -zxdm & I_{zy} = \int_B -zydm & I_{zz} = \int_B (x^2 + y^2) dm \end{bmatrix}$$

Properties of the inertia matrix

$$I_{xx} + I_{yy} \geq I_{zz}$$

$$I_{xx} - I_{yy} \leq I_{zz}$$

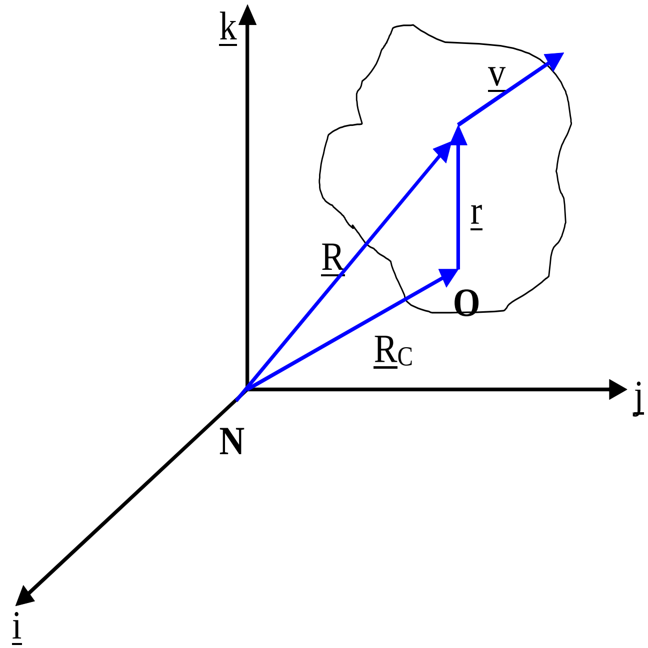
$$I_{xx} \geq 2I_{zy}$$

Typical order of magnitude of inertia moments:

Cubesat (1U to 12U): 0.01 $\rightarrow$ 1 kg·m^2
Envisat (10 x 2.8 x 2.6 m main body, mass 7800 kg): 17000 $\rightarrow$ 129000 kg·m^2

Rotational Kinetic Energy of a Rigid Body

$$2T = \int_B \underline{v} \cdot \underline{v} dm$$



Which evaluated in body coordinates is:

$$T = \frac{1}{2} \underline{\omega} \cdot H = \frac{1}{2} \underline{\omega} \cdot I \underline{\omega}$$

$$I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

Fundamental properties

Angular Momentum Vector

$$\underline{h} = [I_1\omega_1 \quad I_2\omega_2 \quad I_3\omega_3]^T$$

$$\underline{h} = [H_1 \quad H_2 \quad H_3]^T$$

$$I_i\omega_i = H_i$$

Rotational Kinetic Energy function

$$T = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2)$$

$$T = \frac{1}{2}(\frac{H_1^2}{I_1} + \frac{H_2^2}{I_2} + \frac{H_3^2}{I_3})$$

Geometric Interpretation

$$2\mathrm{T} = I_\eta \omega^2 = I_x \omega_x^2 + I_y \omega_y^2 + I_z \omega_z^2$$

$$\frac{\omega_x^2}{2T/I_x} + \frac{\omega_y^2}{2T/I_y} + \frac{\omega_z^2}{2T/I_z} = 1 \quad \Rightarrow \quad \text{kinetic energy ellipsoid}$$

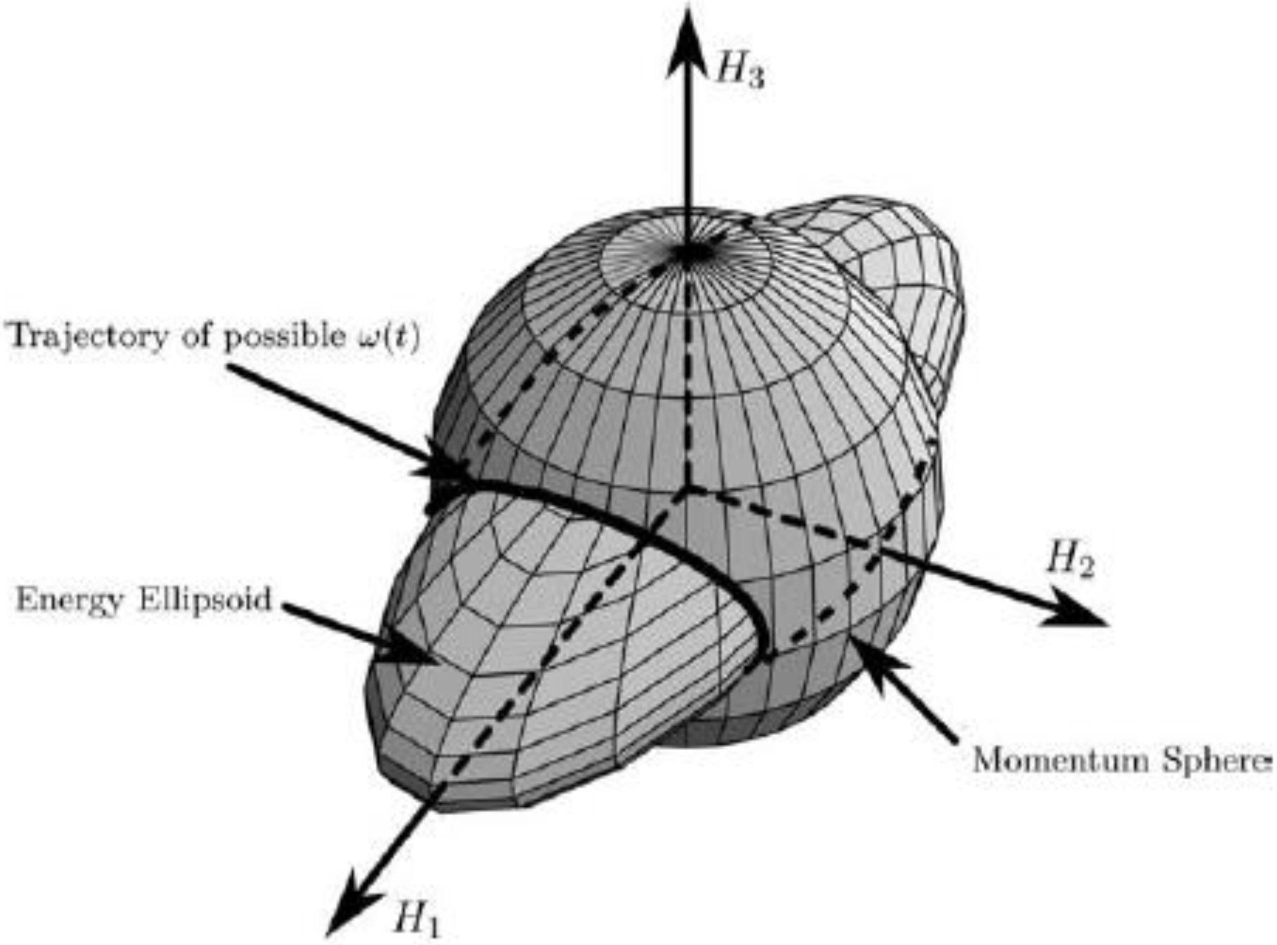
$$h^2 = \underline{h} \cdot \underline{h} = I_x^2 \omega_x^2 + I_y^2 \omega_y^2 + I_z^2 \omega_z^2$$

$$\frac{\omega_x^2}{h^2/I_x^2} + \frac{\omega_y^2}{h^2/I_y^2} + \frac{\omega_z^2}{h^2/I_z^2} = 1 \quad \Rightarrow \quad \text{angular momentum ellipsoid}$$

ellipsoids represents all possible angular velocities compatible either with the given kinetic energy or with the given angular momentum

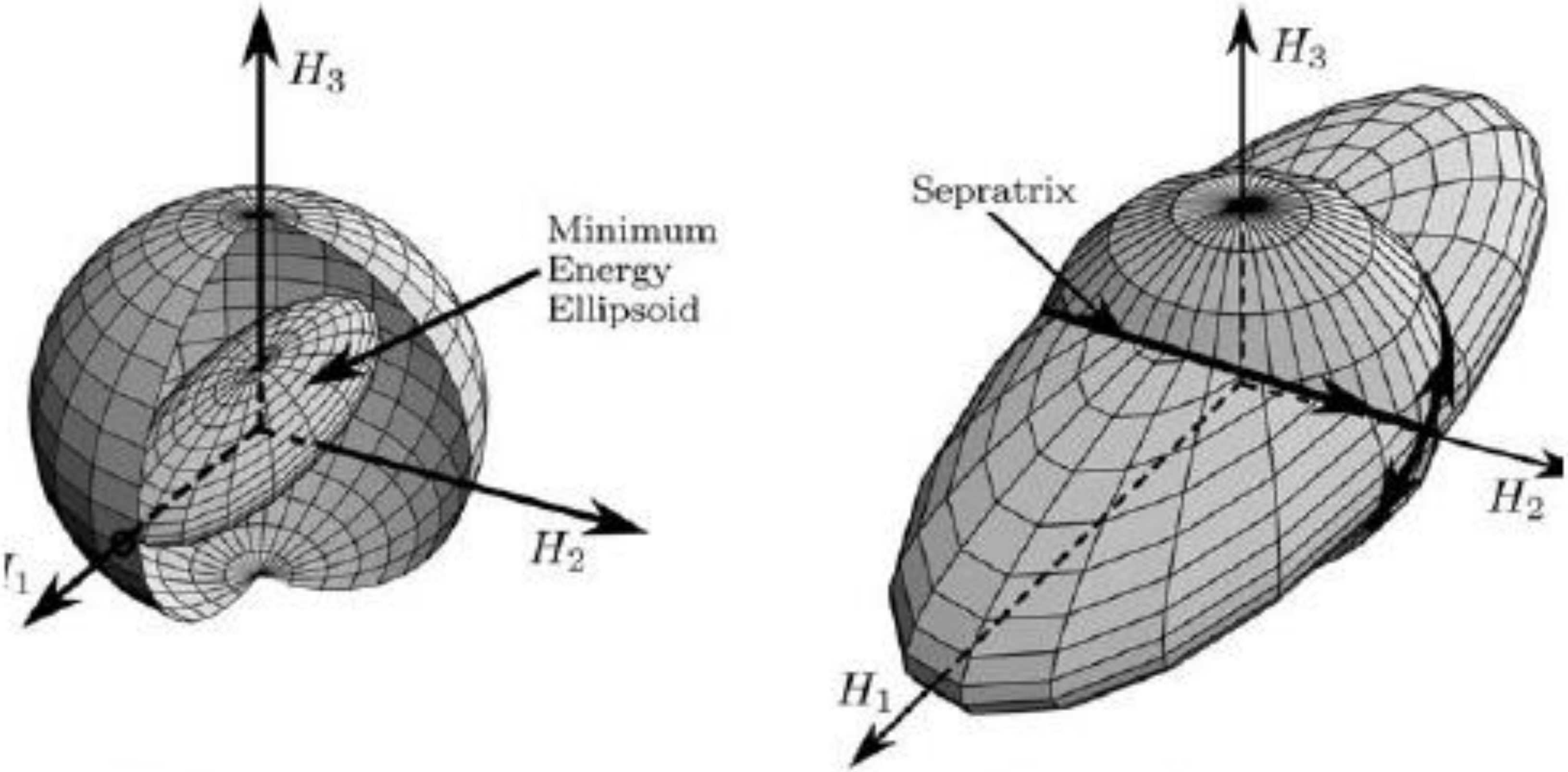
Geometric Interpretation

$I_3 > I_2 > I_1$



Geometric Interpretation

$$I_3 > I_2 > I_1$$



(a) Minimum energy case

(b) Intermediate energy case

Geometric Interpretation

the intersection of the two ellipsoids is evaluated as

$$\frac{\omega_x^2}{2T/I_x} + \frac{\omega_y^2}{2T/I_y} + \frac{\omega_z^2}{2T/I_z} = 1$$

$$\frac{\omega_x^2}{h^2/I_x^2} + \frac{\omega_y^2}{h^2/I_y^2} + \frac{\omega_z^2}{h^2/I_z^2} = 1$$

$$\omega_x^2 \left[I_x \left(\frac{I_x}{h^2} - \frac{1}{2T} \right) \right] + \omega_y^2 \left[I_y \left(\frac{I_y}{h^2} - \frac{1}{2T} \right) \right] + \omega_z^2 \left[I_z \left(\frac{I_z}{h^2} - \frac{1}{2T} \right) \right] = 0$$

In order to obtain a real solution, the three terms $I_{x,y,z} - \frac{h^2}{2T}$ must show differences in sign

Assuming $I_x > I_y > I_z$ $\quad I_x > \frac{h^2}{2T} > I_z$

Geometric Interpretation

analyze its projections onto the three coordinate planes

$$\omega_x^2\left(\frac{I_x(I_x-I_z)}{h^2-2TI_z}\right)+\omega_y^2\left(\frac{I_y(I_y-I_z)}{h^2-2TI_z}\right)=1$$

projection onto the (x-y) plane

$$\omega_y^2\left(\frac{I_y(I_y-I_x)}{h^2-2TI_x}\right)+\omega_z^2\left(\frac{I_z(I_z-I_x)}{h^2-2TI_x}\right)=1$$

projection onto the (y-z) plane

$$\omega_x^2\left(\frac{I_x(I_x-I_y)}{h^2-2TI_y}\right)+\omega_z^2\left(\frac{I_z(I_z-I_y)}{h^2-2TI_y}\right)=1$$

projection onto the (x-z) plane

Analyzing the planar conic curves and the signs of all coefficients, we have that

$$h^2-2TI_z>0$$
$$h^2-2TI_x<0$$
$$h^2-2TI_y \quad ?$$

Geometric Interpretation