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Spacecraft Attitude Dynamics

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Attitude determination

The way we can transform the informations given by sensor to determinate the Attitude of the s/c $\Rightarrow$ Angular rates

Orientation wrt a given frame

Reference Attitude Sensors

→ Not all sensor provide the same accuracy with the measurement. All of these fact must be taken into account for the attitude determination.

With 3 sensors an algebraic method can be used

Reference Object	Potential accuracy
Sun	1 arc minute
Earth (horizon)	6 arc minutes
Magnetometer	30 arc minutes

1 arcminute is 1/60th of a degree

1 arcsecond is 1/360th of a degree

Algebraic method

s_i -> unit vectors measured by the sensors, in body-fixed frame
v_i -> unit vectors of the same celestial bodies but referred to a given reference frame

$$s_i = A_{B/N} v_i$$

for a real case the measurement s_i will have some error

$s_i = A_{B/N} v_i + \varepsilon_i$

↳ we understand that the algebraic solution is not the best one because we are going to get an error in the evaluation of the attitude of the s/c

The simplest case refers to 3 measurements:

$$[s_1 \quad s_2 \quad s_3] = A_{B/N}[v_1 \quad v_2 \quad v_3]$$

$$S = A_{B/N} V$$
$$A_{B/N} = S V^{-1}$$

If the number of measurements available is greater than 3

$$[s_1 \quad s_2 \quad s_3 \quad s_4] = A_{B/N}[v_1 \quad v_2 \quad v_3 \quad v_4]$$

$$SV^T = AVV^T$$
$$SV^T (VV^T)^{-1} = A_{B/N}$$
$$V^* = V^T (VV^T)^{-1}$$
$$A_{B/N} = SV^*$$

$S = AV$

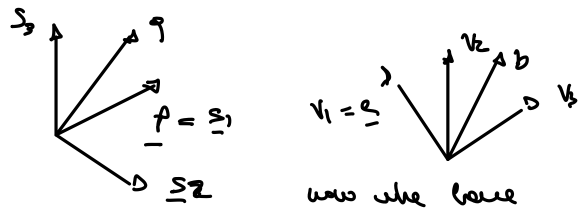
$V^T S = AVV^T$

$S\underbrace{V^T (VV^T)^{-1}}_{V^*} = A$

$A = SV^*$

Algebraic method

why two measurement are enougth



Case with only two measurements available

- call the 2 measurements p and q
- call a and b their corresponding directions in the reference system (can be inertial)
- build 2 orthogonal frames (s_1–s_2–s_3 and v_1–v_2–v_3)

$$s_1 = p \qquad\qquad v_1 = a$$

$$s_2 = \frac{p \wedge q}{|p \wedge q|} \qquad\qquad v_2 = \frac{a \wedge b}{|a \wedge b|}$$

$$s_3 = p \wedge s_2 \qquad\qquad v_3 = a \wedge v_2$$

$s_2 \perp s_1 \equiv p$

$s_3 = s_1 \wedge s_2$

$s_2 \perp q$, $q \in$ plane s_3 and s_1

$v_2 \perp v_1 = a$

↳ the three unit vectors are orthogonal

the 3 unit vectors s_1, s_2, s_3 and v_1, v_2, v_3 are orthogonal

$$V^{-1} = V^T$$

$$A_{B/N} = SV^{-1} = SV^T$$

Problem highly dependent on the choice of s_1
it is important that s1 should be aligned to the highest precision measurement.

To minimize errors, vector p should be measured with the maximum possible precision and q should be as orthogonal to p as possible

Statistical methods

→ The measurement is not perfect but it is affected by errors so we are not looking for the exact association of s_i from v_i but we are trying to minimize the error ε → Best fit.
$\varepsilon_i = s_i - A v_i$

need at least 2 vector measurements and knowledge of relative precision of the sensors

minimize the weighted error function $$J(A) = \frac{1}{2}\sum_{i=1}^{N} \alpha_i \left|s_{Bi} - A_{B/N} v_{Ni}\right|^2 \quad N \geq 2$$

α_i weight because we should rely more on the more accurate measurement.

↳ stands for of an error ⇒ when we want to minimized a function we want to set the gradient to zero.

known as Wahba's problem

In general, the vector of weights α is normalized to 1 $$\sum_{i=1}^{N} \alpha_i = 1$$

Relative weighting is the best option → $\Sigma \alpha_i = 1$ and $\alpha_i \neq \alpha_j$ generally.

$$J(\underline{\underline{A}}) = \frac{1}{2}\sum_{i=1}^{N} \alpha_i\left(\underline{s}_i^T \underline{s}_i + \underline{v}_i^T \underline{\underline{A}}^T \underline{\underline{A}} \underline{v}_i - 2\underline{s}_i^T \underline{\underline{A}} \underline{v}_i\right) = 1 - \sum_{i=1}^{N} \alpha_i\left(\underline{s}_i^T \underline{\underline{A}} \underline{v}_i\right)$$

$\underline{s}_i$ is normalized as $\underline{v}_i$

we know that $\underline{\underline{A}}^T\underline{\underline{A}} = \underline{\underline{I}}$

$$\cancel{\frac{1}{2}} \sum_{i=1}^{N} \underbrace{\alpha_i (1+1)}_{\cancel{2}\alpha_i} - \cancel{2}\alpha_i \underline{s}_i^T \underline{\underline{A}} \underline{v}_i = \sum_{i=1}^{N} \alpha_i - \sum_{i=1}^{N} \alpha_i \underline{s}_i^T \underline{\underline{A}} \underline{v}_i$$
$$= 1 - \sum_{i=1}^{N} \alpha_i \underline{s}_i^T \underline{\underline{A}} \underline{v}_i$$

if every measurement is perfect $\underline{s}_i = \underline{\underline{A}}\,\underline{v}_i$ so $\sum_{i=1}^{N} \alpha_i \underline{s}_i^T \underline{\underline{A}} \underline{v}_i = 1$ ⇒ $J(\underline{\underline{A}}) = 0$ perfect

if $\underline{s}_i \neq \underline{\underline{A}}\,\underline{v}_i$ but $\underbrace{\underline{s}_i^T \underline{\underline{A}} \underline{v}_i}_{\text{scalar product of}}$ $\tilde{\underline{s}}_i^T \underline{s}_i < 1$ scalar product

Example

first step is to normalized the weight → we will use the accuracy of each sensor to weight the coefficient. → if a measure is precise then we should rely on it, so the weight will be inversely proportional to the accuracy.

Reference Object	Potential accuracy
Sun	1 arc minute
Earth (horizon)	6 arc minutes
Magnetometer	30 arc minutes

$\hat{\alpha}_1 = \frac{1}{1}$ $\hat{\alpha}_2 = \frac{1}{6}$ $\hat{\alpha}_3 = \frac{1}{30}$

we need to normalized them so that their sum is equal to 1.

$\alpha_i = \frac{\alpha_i}{\hat{\alpha}_1 + \hat{\alpha}_2 + \hat{\alpha}_3}$

$$J(A) = \frac{1}{2}\sum_{i=1}^{N} \alpha_i \left|s_{Bi} - A_{B/N} v_{Ni}\right|^2 \quad N = 3$$

$$J(A_{B/N}) = \alpha_1 \left|\underline{s}_1 - A_{B/N}\underline{v}_1\right|^2 + \alpha_2 \left|\underline{s}_2 - A_{B/N}\underline{v}_2\right|^2 + \alpha_3 \left|\underline{s}_3 - A_{B/N}\underline{v}_3\right|^2$$

$\hat{\alpha}_1 = 1, \hat{\alpha}_2 = \frac{1}{6}, \hat{\alpha}_3 = \frac{1}{30}$ Normalize the weights $\alpha_i = \frac{\hat{\alpha}_i}{\hat{\alpha}_1 + \hat{\alpha}_2 + \hat{\alpha}_3}$

to compute Euler angle we should use this one but we do not have any guarantee on the amount of time that it will take and this is unacceptable on a/c flight computers.

Unconstrained minimization

Then we need to minimize the error function $J(\underline{A})$

we usually calculate the gradient of that function and then we set the gradient equal to zero.

$$J(A) = \alpha_1 \left| \underline{s}_1 - A_{B/N} \underline{v}_1 \right|^2 + \alpha_2 \left| \underline{s}_2 - A_{B/N} \underline{v}_2 \right|^2 + \alpha_3 \left| \underline{s}_3 - A_{B/N} \underline{v}_3 \right|^2$$

$$A_{B/N} = \begin{bmatrix} \cos\psi\cos\vartheta & \cos\psi\sin\vartheta\sin\phi + \sin\psi\cos\phi & -\cos\psi\sin\vartheta\cos\phi + \sin\psi\sin\phi \\ -\sin\psi\cos\vartheta & -\sin\psi\sin\vartheta\sin\phi + \cos\psi\cos\phi & \sin\psi\sin\vartheta\cos\phi + \cos\psi\sin\phi \\ \sin\vartheta & -\cos\vartheta\sin\phi & \cos\vartheta\cos\phi \end{bmatrix}$$

We could use the Newton method to find the zero of the gradient function.

$$\underline{x}_{n+1} = \underline{x}_n - \left[\frac{\partial \nabla J}{\partial \underline{x}}(\underline{x}_n) \right]^{-1} \nabla J(\underline{x}_n)$$

$$\underline{x}_n = [\psi_n \quad \vartheta_n \quad \phi_n]^T$$

$$\nabla J = \begin{bmatrix} \dfrac{\partial J}{\partial \psi_n} \\ \dfrac{\partial J}{\partial \vartheta_n} \\ \dfrac{\partial J}{\partial \phi_n} \end{bmatrix}$$

The solution might not be unique because if we add 2π to one of the 3 angle obtained we will obtain an equivalent solution.

Constrained minimization → Not as easy !! fairly difficult to implement

Minimize

$$J(A) = \alpha_1 \left|\underline{s}_1 - A_{B/N}\underline{v}_1\right|^2 + \alpha_2 \left|\underline{s}_2 - A_{B/N}\underline{v}_2\right|^2 + \alpha_3 \left|\underline{s}_3 - A_{B/N}\underline{v}_3\right|^2$$

Subject to the constraints

$$A_{B/N}A_{B/N}^T = I, \det A_{B/N} = 1$$

Many constrained optimization methods can be used - Simulated annealing, random search, genetic algorithms etc.

One way to do not have many solution.

Analytic solution to Wabha's problem

It can be shown that minimizing

$$J(A) = \frac{1}{2}\sum_{i=1}^{N} \alpha_i \left|s_{Bi} - A_{B/N} v_{Ni}\right|^2$$

Is equivalent to maximizing

look above for proof.

$$\tilde{J}(A) = \sum_{i=1}^{N} \alpha_i \, s_{Bi}^T A_{B/N} v_{Ni} = tr(A_{B/N} B^T)$$

$$B = \sum_{i=1}^{N} \alpha_i \, s_{Bi} v_{Ni}^T$$

$$A_{B/N} = (B^T)^{-1}(B^T B)^{1/2} = B(B^T B)^{-1/2}$$

?

$\tilde{J}(A=B) = tr(B(B^TB)^{-1}B^T)$

$A_{B/N} = (B^T)^{-1}(B^TB)^{1/2} = B(B^TB)^{-1/2}B^T$

How to compute this.

Analytic solution to Wabha's problem

matrix B can be decomposed using singular value decomposition

$$B = U\Sigma^T V^T = \mathrm{U\ diag}[\Sigma_{11} \quad \Sigma_{22} \quad \Sigma_{33}]^T V^T$$

U and V are unitary orthogonal matrices
U represents the eigenvectors of matrix BB^T
V U represents the eigenvectors of matrix B^TB
$UU^T=I$
$VV^T=I$
singular values Σ are the square roots of the eigenvalues of matrix B^TB

$$A_{B/N} = \left(B^T\right)^{-1}\left(B^TB\right)^{1/2} = B\left(B^TB\right)^{-1/2}$$

Can be expressed in the simple form:

$$A_{B/N} = UMV^T$$
$$M = \mathrm{diag}\,[1 \quad 1 \quad (\det U)(\det V)]$$

q-method

Minimizing
$$J(A) = \frac{1}{2}\sum_{i=1}^{N} \alpha_i \left|s_{Bi} - A_{B/N} v_{Ni}\right|^2 = \sum_{i=1}^{N} \alpha_i \left(1 - s_{Bi}^T A_{B/N} v_{Ni}\right)$$

Is equivalent to maximising
$$\tilde{J}(A) = \sum_{i=1}^{N} \alpha_i \, s_{Bi}^T A_{B/N} v_{Ni} = tr\left(A_{B/N} B^T\right)$$

This becomes much simpler in quaternion form

$$\mathrm{A}_{\mathrm{B/N}} = \begin{bmatrix} \mathrm{q}_1^2 - \mathrm{q}_2^2 - \mathrm{q}_3^2 + \mathrm{q}_4^2 & 2(\mathrm{q}_1\mathrm{q}_2 + \mathrm{q}_3\mathrm{q}_4) & 2(\mathrm{q}_1\mathrm{q}_3 - \mathrm{q}_2\mathrm{q}_4) \\ 2(\mathrm{q}_1\mathrm{q}_2 - \mathrm{q}_3\mathrm{q}_4) & -\mathrm{q}_1^2 + \mathrm{q}_2^2 - \mathrm{q}_3^2 + \mathrm{q}_4^2 & 2(\mathrm{q}_2\mathrm{q}_3 + \mathrm{q}_1\mathrm{q}_4) \\ 2(\mathrm{q}_1\mathrm{q}_3 + \mathrm{q}_2\mathrm{q}_4) & 2(\mathrm{q}_2\mathrm{q}_3 - \mathrm{q}_1\mathrm{q}_4) & -\mathrm{q}_1^2 - \mathrm{q}_2^2 + \mathrm{q}_3^2 + \mathrm{q}_4^2 \end{bmatrix}$$

$$\mathrm{A}_{\mathrm{B/N}} = \left(q_4^2 - \underline{q}^T \underline{q}\right) I + 2\underline{q}\underline{q}^T - 2\mathrm{q}_4[q \wedge]$$

QUEST-method

Before conversion to quaternions we needed to minimize a function

$$J(A) = 1 - \sum_{i=1}^{N} \alpha_i \left(s_{Bi}^T A_{B/N} v_{Ni} \right) = \sum_{i=1}^{N} \alpha_i - q^T K q$$

which means to maximize $\tilde{J}(q) = q^T K q$ with the constraint $q^T q = 1$

$$K = \begin{bmatrix} S - \sigma I & Z \\ Z^T & \sigma \end{bmatrix}$$

$$B = \sum_{i=1}^{N} \alpha_i \, s_{Bi} v_{Ni}^T$$

$$S = B + B^T$$

$$Z = [B_{23} - B_{32}, B_{31} - B_{13}, B_{12} - B_{21}]^T$$

$$\sigma = tr[B]$$

QUEST-method

Maximising $\tilde{J}$ with $q^T q = 1$

Leads to maximising $\tilde{G}(q) = q^T K q - \lambda(q^T q - 1)$

Evaluate the gradient of $\tilde{G}$ and equate to zero

$$Kq = \lambda q$$

back substituting into the expression of $\tilde{J}$

$$\tilde{J}(q) = q^T K q = q^T \lambda q = \lambda$$

The maximum value of $\tilde{J}$ is then the maximum eigenvalue of K

I can compute the optimal attitude with procedures using eigenvalue and eigenvectors analysis

the optimal attitude, from a statistical point of view, is the eigenvector associated to the maximum eigenvalue of K

QUEST-method

If expected errors in attitude determination are small, the cost function J is close to 0

Then rewrite the equation $Kq = \lambda q$ in partitioned form

$$(S - \sigma \mathrm{I})\underline{q} + Zq_4 = \lambda_{\mathrm{MAX}}\underline{q}$$
$$Z^T\underline{q} + \sigma q_4 = \lambda_{\mathrm{MAX}} q_4$$

Converting to the Gibb's vector and taking the approximation λ_{MAX} =1 we get

$$[S - (\sigma + 1)I]\bar{g} = -Z$$

$$\bar{g} = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix}$$ approximate Gibbs vector that provides the satellite attitude

Angular velocity estimation

ω can be expressed as a function of quaternions

$$\dot{q} = \frac{1}{2}\Omega q$$

reformulated as

$$\Omega q = \begin{bmatrix} q_4 & -q_3 & q_2 \\ q_3 & q_4 & -q_1 \\ -q_2 & q_1 & q_4 \\ -q_1 & -q_2 & -q_3 \end{bmatrix} \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} = Q\omega$$

makes the derivative smoother
and less sensible to errors.

$$\omega = 2Q^*\dot{q}$$

Q^* is the pseudo inverse of Q

It is easy to verify that $Q^T Q = I$

Gyro noise

$$\underline{\omega}_i^M = \underline{\omega}_i + \underline{n} + \underline{b}$$

Where the white-noise is characterised by zero mean

$$\underline{n} = \sigma_n \underline{\zeta}_n$$ → Average random walk

$$\dot{\underline{b}} = \sigma_b \underline{\zeta}_b$$ → Bias random walk

} function of a white noise → random they are not related to the construction of the sensor but is related to the electronics components inside the sensor.

Usually manufacturers state the ARW and RRW or bias stability as in the Sensonor STIM300
https://www.sensonor.com/products/inertial-measurement-units/stim300/

$$\text{ARW} = \sigma_{ARW} = 0.15\,\text{deg}/\sqrt{hr} = \sigma_n \sqrt{T_s}$$

$$RRW = \sigma_{RRW} = 0.0003\,\text{deg}/\,hr^{3/2} = \sigma_b \sqrt{T_s}$$

RRW=Bias Random Walk=Bias instability x$\sqrt{T_s}$

(38.6mm x 44.8mm x 21.5mm)

$f = 262Hz$ $\qquad T_s = 1/f$