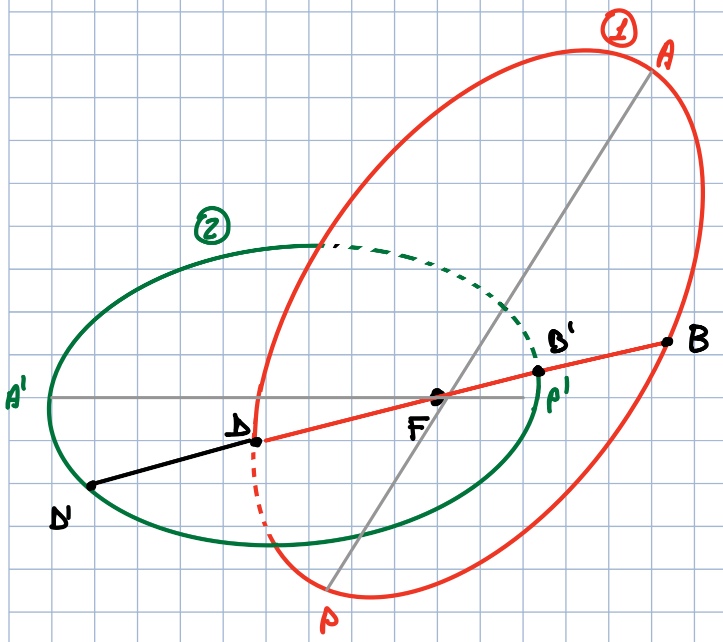
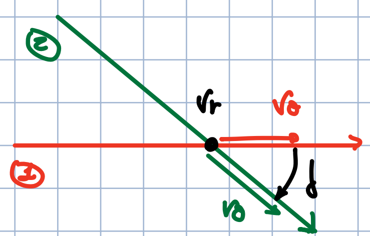
## OUT OF PLANE MANOEUVRE




DB intersection line

Cofocal orbit do not lie on the same plane, $F \in \overline{DB}$

To change plane the manoeuvre has to be done on the intersection line by a single $\Delta v$ manoeuvre




$v_r$ belongs to the intersection line

To change the plane of orbit 1 we need to ROTATE $v_\theta$ around the intersection line through diedral angle $\delta$

- if $v_r$ remains unchanged in magnitude $\Rightarrow$ rigid rotation of orbit 1 only orientation is changes where shape and size remain unchanged.

- if $v_r$ changes both direction and magnitude $\Rightarrow$ also modification of orbit 2 in plane.

Let's look at the most generic case :

velocity after manoeuvre $\underline{v_2} = v_{r2}\,\hat{r} + v_{\theta 2}\,\hat{\theta}_2$

velocity before manoeuvre $\underline{v_1} = v_{r1}\,\hat{r} + v_{\theta 1}\,\hat{\theta}_1$

$\hat{r}$ unit vector along intersection line of 2 planes, does not change during manoeuvre.

$\hat{\theta} \perp$ to $\hat{r}$ lie on the orbit plane $\hat{\theta}_1 \in$ orbit plane 1

$\hat{\theta}_2 \in$ orbit plane 2

$\hat{\theta}$ rotates through the diedral angle $\delta$

$$\underline{\Delta v} = \underline{v_2} - \underline{v_1} = (v_{r2} - v_{r1})\,\hat{r} + v_{\theta 2}\,\hat{\theta}_2 - v_{\theta 1}\,\hat{\theta}_1$$

$$\Delta v^2 = \underline{\Delta v} \cdot \underline{\Delta v} = (v_{r2} - v_{r1})^2 + v_{\theta 2}^2 + v_{\theta 1}^2 - 2 v_{\theta 1} v_{\theta 2} (\hat{\theta}_1 \cdot \hat{\theta}_2)$$

as $\hat{r} \cdot \hat{r} = \hat{\theta}_1 \cdot \hat{\theta}_1 = \hat{\theta}_2 \cdot \hat{\theta}_2 = 1$

$\hat{r} \cdot \hat{\theta}_1 = \hat{r} \cdot \hat{\theta}_2 = \emptyset$ because $\hat{r} \perp \hat{\theta}_1$ and $\hat{r} \perp \hat{\theta}_2$

but $\hat{\theta}_1 \cdot \hat{\theta}_2 = 1 \cdot 1 \cdot \cos\delta = \cos\delta$

$$\boxed{\Delta v = \sqrt{(v_{r2} - v_{r1})^2 + v_{\theta 2}^2 + v_{\theta 1}^2 - 2 v_{\theta 1} v_{\theta 2} \cos\delta}} \qquad (3.65)$$

From the definition of the flight path angle

$$\left.\begin{array}{ll} v_{r1} = v_1 \sin\gamma_1 & v_{\theta 1} = v_1 \cos\gamma_1 \\ v_{r2} = v_2 \sin\gamma_2 & v_{\theta 2} = v_2 \cos\gamma_2 \end{array}\right\} \qquad (3.66)$$

Substitute them in Eq (3.64)

$$\Delta v^2 = v_1^2 \sin\gamma_1^2 + v_2^2 \sin\gamma_2^2 - 2 v_1 v_2 \sin\gamma_1 \sin\gamma_2 + v_2^2 \cos\gamma_2^2 + v_1^2 \cos\gamma_1^2 +$$
$$- 2 v_1 v_2 \cos\gamma_1 \cos\gamma_2 \cos\delta$$

but
$$\sin\gamma_1^2 + \cos\gamma_1^2 = 1$$
$$\sin\gamma_2^2 + \cos\gamma_2^2 = 1$$

$$\Delta v^2 = v_1^2 + v_2^2 - 2 v_1 v_2 \left( \sin\gamma_1 \sin\gamma_2 + \cos\gamma_1 \cos\gamma_2 \cos\delta \right)$$

<u>Recall</u> $\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$

$$\Delta v^2 = v_1^2 + v_2^2 - 2 v_1 v_2 ( \sin\gamma_1 \sin\gamma_2 + \cos\gamma_1 \cos\gamma_2 - \cos\gamma_1 \cos\gamma_2$$
$$+ \cos\gamma_1 \cos\gamma_2 \cos\delta )$$

$$\Delta v^2 = v_1^2 + v_2^2 - 2 v_1 v_2 \left[ \cos\Delta\gamma - \cos\gamma_1 \cos\gamma_2 (1 - \cos\delta) \right]$$

$$\boxed{\Delta v = \sqrt{v_1^2 + v_2^2 - 2 v_1 v_2 \left[ \cos\Delta\gamma - \cos\gamma_1 \cos\gamma_2 (1 - \cos\delta) \right]}} \quad (3.67)$$

↳ very general manoeuvre

with $\Delta\gamma = \gamma_2 - \gamma_1$

(A) If no plane change $\delta = 0 \Rightarrow \cos\delta = 1$ Eq (3.67) reduces to

$$\boxed{\Delta v = \sqrt{v_1^2 + v_2^2 - 2 v_1 v_2 \cos\Delta\gamma}} \quad (3.68)$$

NO PLANE CHANGE MANOEUVRE (single manoeuvre in plane)

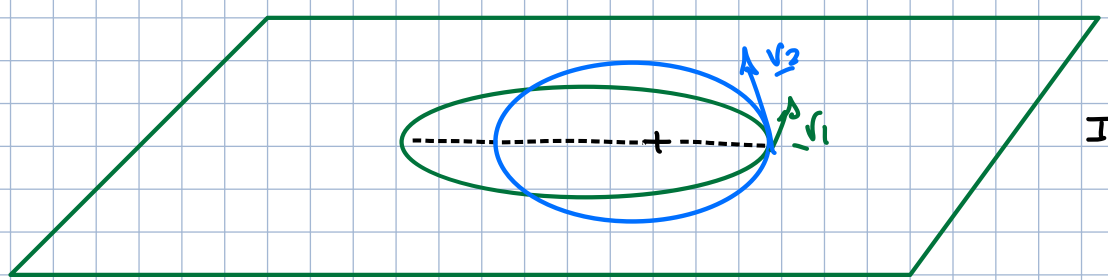
this is the cosine law for in-plane manoeuvre

Eq 3.68 is a special case of Eq (3.67)

(B) To keep $\Delta v$ at minimum to change plane the radial component should remain unchanged during the plane change manoeuvre (see eq (3.65))

(C) To keep $\Delta v$ at minimum → the manoeuvre needs to take place when $v_\theta$ is minimum (i.e. apoapsis). This is possible only if apoapsis belongs to the intersection line.

example




$v_{r_1} = v_{r_2} = 0$ @ apoapsis

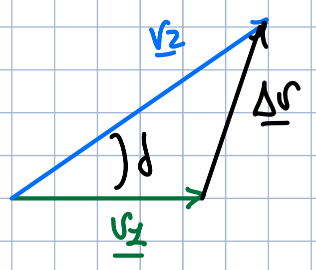
$v_{\theta_1} = v_1$
$v_{\theta_2} = v_2$ } ∈ to ≠ plane

Eq (3.65) reduces to
$$\boxed{\Delta v = \sqrt{v_1^2 + v_2^2 - 2 v_1 v_2 \cos\delta}} \quad (3.69)$$

Rotation of an orbit around a common apse line

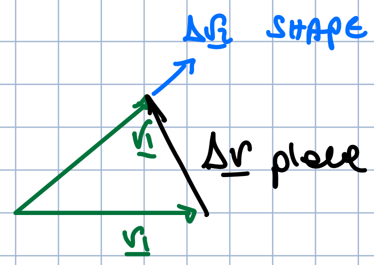
Eq (3.69) correspond a speed change (@ apse line) accompanied by a plane change.

**STRATEGY 1**




speed change with a plane change combined

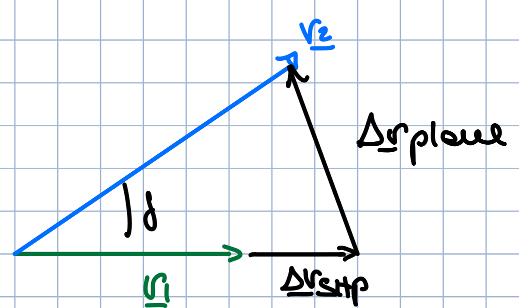
**STRATEGY 2**




plane change $\Delta v_p$ followed by a shape change

This is also a demonstration that doing a speed change and a change of plane at the same time is better than doing two different manoeuvres one to change shape and one to change plane.

**STRATEGY 3**




change of velocity followed by a change plane

As we can see the optimal manoeuvre is the first → we save a smaller $\Delta v$ because we are just doing the transfer along one side of the speed triangle.

**STRATEGY 1 (COMBINED)**

$$\Delta v_1 = \sqrt{v_1^2 + v_2^2 - 2 v_1 v_2 \cos\delta}$$

if no speed change, only plane $\quad v_1 = v_2$

Eq (3.69) becomes

$$\Delta v_1 = \sqrt{v_1^2 + v_2^2 - 2 v_1 v_2 \left(1 - 2\sin^2\frac{\delta}{2}\right)} \qquad \text{recall } \cos\delta = 1 - 2\sin^2\frac{\delta}{2}$$

$$\Delta v_1 = \sqrt{(v_1 - v_2)^2 + 4 v_1 v_2 \sin^2\frac{\delta}{2}} \qquad \text{if } v_1 = v_2 = v \text{ then}$$

$$\Delta v_{\text{PLANE CHANGE ONLY}} = \Delta v_p = \sqrt{4 v^2 \sin^2\frac{\delta}{2}} \qquad \boxed{\Delta v = 2 \overset{v_0?}{v} \sin\frac{\delta}{2}} \quad (3.70)$$

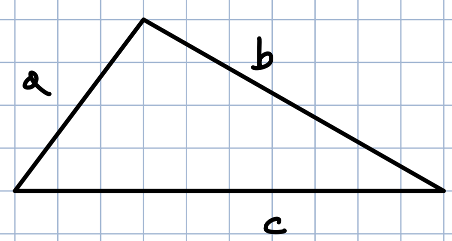
**STRATEGY 2** — **PURE ROTATION OF THE VELOCITY VECTOR**

$$\Delta v_2 = 2v_1 \sin\frac{\delta}{2} + |v_2 - v_1|$$

**STRATEGY 3**

$$\Delta v_3 = |v_2 - v_1| + 2v_2 \sin\frac{\delta}{2}$$

Recall




$a + b > c$

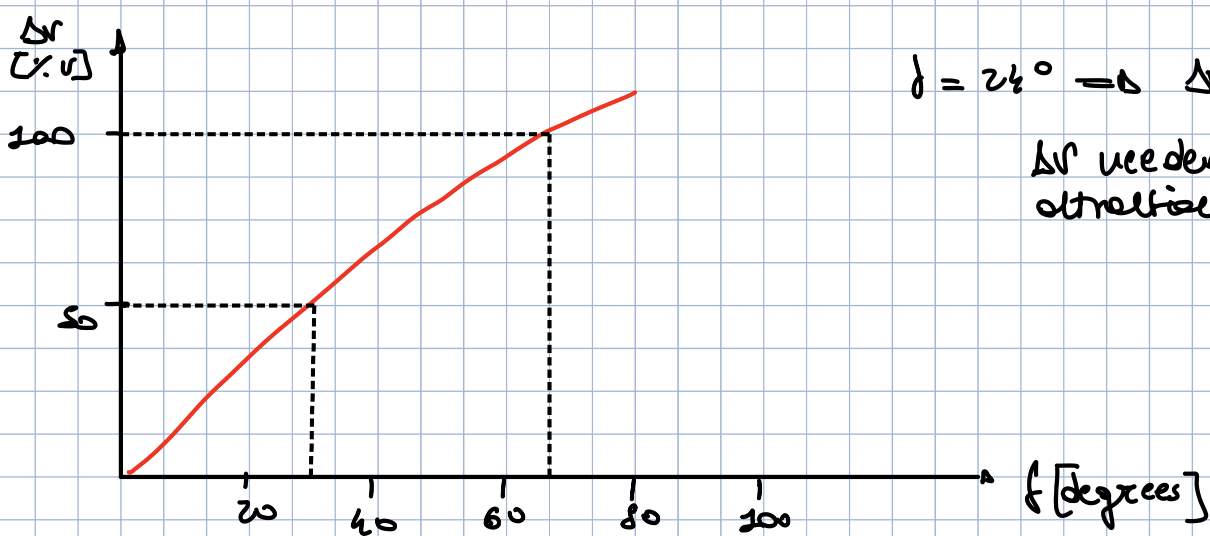
$a + c > b$

$b + c > a$

$\Rightarrow$ evident that $\Delta v_2 > \Delta v_1$, $\Delta v_3 > \Delta v_1$ — better combined

**Note**

The plane change is a very expensive manoeuvre




$\delta = 24° \Rightarrow \Delta v = 41.4\%$

$\Delta v$ needed for escape the attraction of the earth.