POLITECNICO
MILANO 1863

ITECNICO
ILANO 1863

Spacecraft Attitude Dynamics

Prof. Franco Bernelli

Fundamental Properties

Transport Theorem

Let N and B be two coordinate frames with a relative angular velocity $\underline{\omega}_{B/N}$

$\underline{x}$ is a generic vector then

Rule of the derivation on a vector moving in a rotating frame.

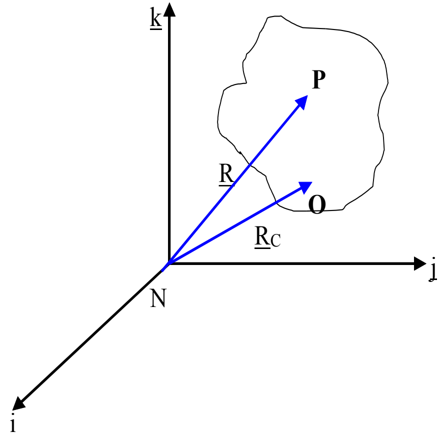
angular velocity of frame B seen from N.

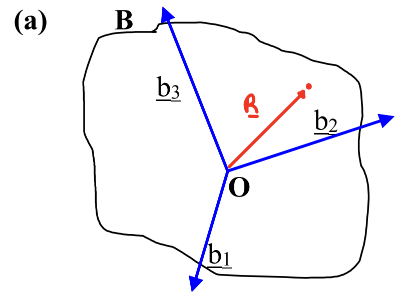
$$\frac{{}^{N}d}{dt}\underline{x} = \frac{{}^{B}d}{dt}\underline{x} + \underline{\omega}_{B/N} \times \underline{x}$$

Time derivative took in the moving frame

$$\dot{\underline{x}} = \frac{{}^{N}d}{dt}\underline{x}$$

Angular Momentum of a Rigid Body → Abstract model nothing is really infinitely rigid.





The angular momentum of a rigid body B is defined with respect to an origin fixed on the rigid body. For an infinitesimal point mass we have:

$$\boxed{d\underline{h}_o = \underline{R} \times \underline{\dot{R}}dm}$$

→ contribution of the point mass to the total angular momentum.

The total angular momentum is then:

$$\underline{h}_o = \underbrace{\underline{R}_C \times \underline{\dot{R}}_C M}_{\text{angular momentum of the mass centre about the origin}} + \underbrace{\int_B (\underline{r} \times \underline{\dot{r}})dm}_{\text{angular momentum of rigid body about its centre of mass } \underline{h}}$$

angular momentum of the total mass concentrated in the center of mass wrt origin.

angular momentum of the rigid body wrt the center of mass

if $O \equiv C$ → $\underline{h}_o = \int_B (\underline{r} \wedge \underline{\dot{r}})dm$

Rotational Angular Momentum of a Rigid Body (Exercise)

$$\underline{h} = \int_B \underline{r} \times (\underline{\omega} \times \underline{r})\mathrm{d}m$$

velocity in the body frame.

Show that the angular momentum about the centre of mass evaluated in the body frame is:

$$\underline{h} = I\underline{\omega}$$

!! Important

$\underline{\underline{I}} = \underline{\underline{I}}^T$ simmetric.

where

Inertia of the rigid body around a specified axis

Inertial product → they can be equally positive or negative terms. off-diagonal terms are always simmetrical

$$I = \begin{bmatrix} I_{xx} = \int_B (y^2 + z^2)dm & I_{xy} = \int_B -xydm & I_{xz} = \int_B -xzdm \\ I_{yx} = \int_B -yxdm & I_{yy} = \int_B (x^2 + z^2)dm & I_{yz} = \int_B -yzdm \\ I_{zx} = \int_B -zxdm & I_{zy} = \int_B -zydm & I_{zz} = \int_B (x^2 + y^2)dm \end{bmatrix}$$

along the diagonal there are only positive values

Properties of the inertia matrix

$$I_{xx}+I_{yy} = \int_B (x^2+y^2+2z^2)\,dm \geq I_{zz} = \int_B (x^2+y^2)\,dm$$

Equals only if the satellite has no mass on the z axis.

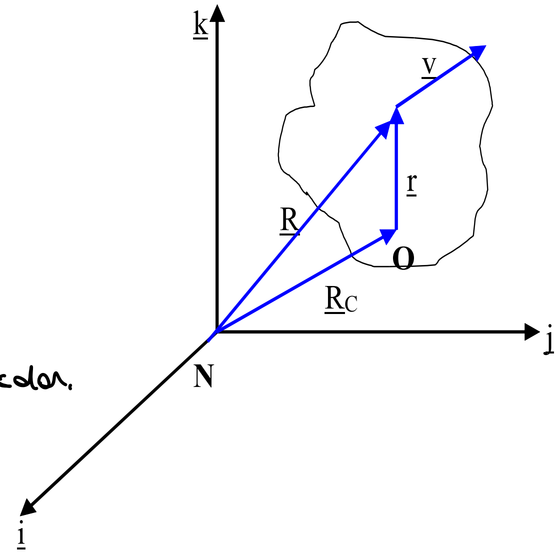
$$I_{xx}-I_{yy} = \int_B (y^2-x^2)\,dm \leq I_{zz} = \int_B (x^2+y^2)\,dm$$

$$\left.\begin{aligned} &I_{xx}+I_{yy}\geq I_{zz} \\ &I_{xx}-I_{yy}\leq I_{zz} \\ &I_{xx}\geq 2I_{zy} \end{aligned}\right\} \text{Triangular inequalities}$$

$$I_{xx} = \int_B (z^2+y^2)\,dm \geq 2I_{zy} = 2\int_B -zy\,dm$$

Rotational Kinetic Energy of a Rigid Body

$$2T = \int_B \underline{v} \cdot \underline{v} dm$$



Which evaluated in body coordinates is:

$$T = \frac{1}{2}\underline{\omega} \cdot \underline{H} = \frac{1}{2}\underline{\omega} \cdot I\underline{\omega}$$ → It is a scalar.

$$I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$ → If I is a diagonal matrix → Using a principal inertial frame

$$I = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \rightarrow T = \frac{1}{2}\left(I_{xx}\omega_x^2 + I_{yy}\omega_y^2 + I_{zz}\omega_z^2\right)$$

Fundamental properties

→ semplification are valid only if
I is diagonal (using a principal inertial frame)
axis 1,2,3 are called principal axis.

$$I = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

Some text book use xx, yy, zz instead of 1,2,3 for the principal axis.

↳ It is a valid approximation usually the bulk of the mass is in an uniform "cubical" shape while appendices are small or light.

Angular Momentum Vector

$$\underline{h} = [I_1\omega_1 \quad I_2\omega_2 \quad I_3\omega_3]^T$$

$$\underline{h} = [H_1 \quad H_2 \quad H_3]^T$$

$$I_i\omega_i = H_i$$

Rotational Kinetic Energy function

$$T = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2)$$

$$T = \frac{1}{2}\left(\frac{H_1^2}{I_1} + \frac{H_2^2}{I_2} + \frac{H_3^2}{I_3}\right)$$

Geometric Interpretation

$2\mathrm{T} = I_\eta \omega^2 = I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2$ → overall the sum must be constant where every single component could change.

$$\frac{\omega_x^2}{2T/I_x} + \frac{\omega_y^2}{2T/I_y} + \frac{\omega_z^2}{2T/I_z} = 1$$ → kinetic energy ellipsoid

<u>11:00 ritiro da Moriccas</u> 20-30€

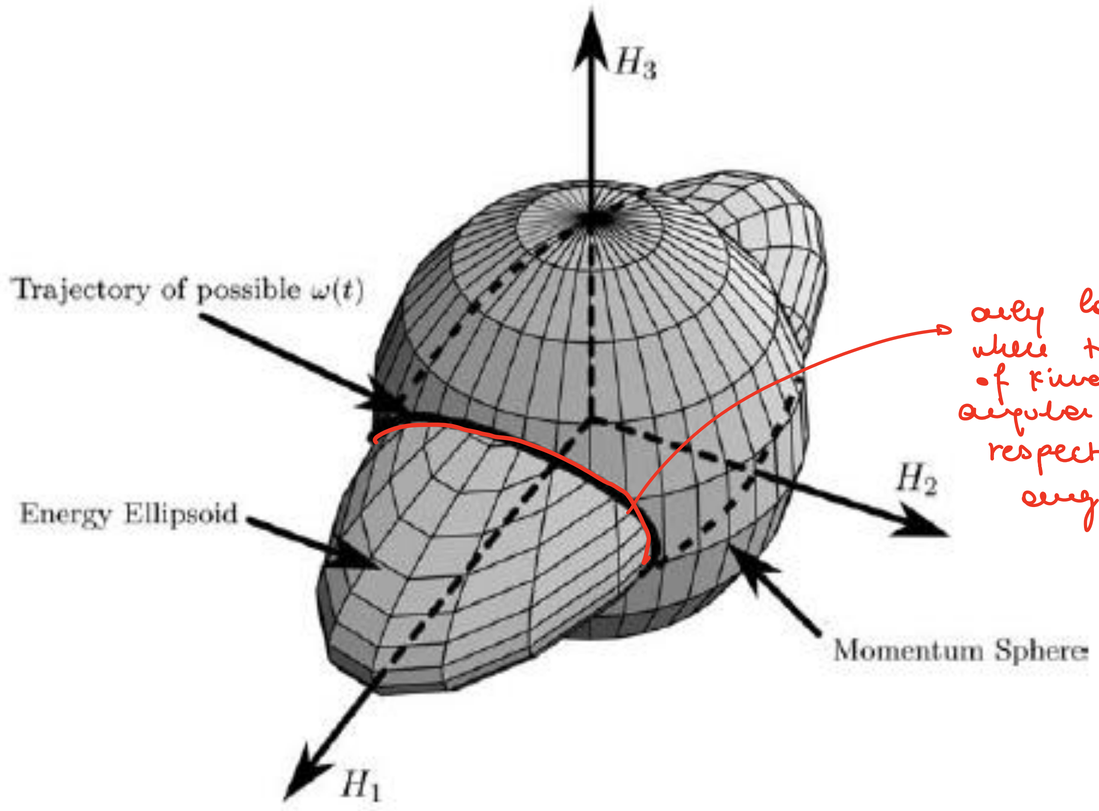
$$h^2 = \underline{h} \cdot \underline{h} = I_x^2\omega_x^2 + I_y^2\omega_y^2 + I_z^2\omega_z^2$$

$$\frac{\omega_x^2}{h^2/I_x^2} + \frac{\omega_y^2}{h^2/I_y^2} + \frac{\omega_z^2}{h^2/I_z^2} = 1$$ → angular momentum ellipsoid

ellipsoids represents all possible angular velocities compatible either with the given kinetic energy or with the given angular momentum

Geometric Interpretation

$I_3 > I_2 > I_1 \longleftarrow$ Assumption

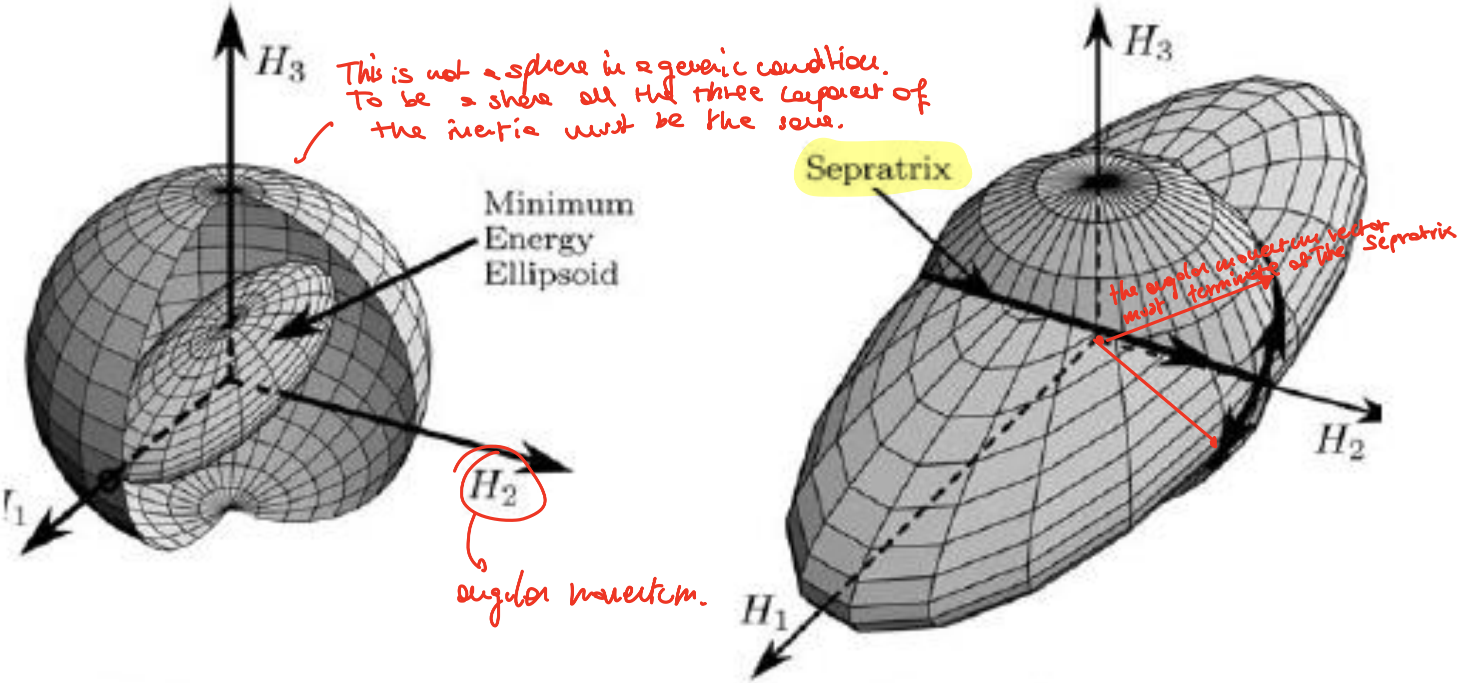


only locus of points where the conservation of kinetic energy and angular momentum is respected by the angular velocity.

Geometric Interpretation

$I_3 > I_2 > I_1$

This is not a sphere in a generic condition. To be a shere all the three component of the inertia must be the same.



angular momentum.

the angular momentum vector must terminate at the Sepratrix

(a) Minimum energy case

(b) Intermediate energy case

NOTE The only case where the intersection is not a line but a surface is with a perfecty spherical body.

Geometric Interpretation

the intersection of the two ellipsoids is evaluated as

$$\frac{\omega_x^2}{2T/I_x}+\frac{\omega_y^2}{2T/I_y}+\frac{\omega_z^2}{2T/I_z}=1$$

$$\frac{\omega_x^2}{h^2/I_x^2}+\frac{\omega_y^2}{h^2/I_y^2}+\frac{\omega_z^2}{h^2/I_z^2}=1$$

shere only if all three semi apsis are equal and if we look at the equation it happens only when the inertia moment are equals.

The initial condition assign the angular momentum and the kinetic energy + I_i can't change.
→ Equation in function of ω_i.

$$\boxed{\omega_x^2\overbrace{\left[I_x\left(\frac{I_x}{h^2}-\frac{1}{2T}\right)\right]}^{>0}+\omega_y^2\overbrace{\left[I_y\left(\frac{I_y}{h^2}-\frac{1}{2T}\right)\right]}^{\text{not sure}}+\omega_z^2\overbrace{\left[I_z\left(\frac{I_z}{h^2}-\frac{1}{2T}\right)\right]}^{<0}=0}$$

→ Are all number

In order to obtain a real solution, the three terms $I_{x,y,z}-\frac{h^2}{2T}$ must show differences in sign

It is possible to find the solution at least one different sign.

Then must be a change in sign in one of three → in order to have a real solution to the problem.

Assuming $I_x > I_y > I_z$ $\quad$ $I_x > \frac{h^2}{2T} > I_z$

If this is the case there will be a change in the sign in one of $I_i - \frac{h^2}{2T}$

Geometric Interpretation

we still assume that $I_x > I_y > I_z$

analyze its projections onto the three coordinate planes

This conic section is representing an ellipse

$$\omega_x^2\left(\frac{I_x(I_x-I_z)}{\underset{>0}{h^2-2TI_z}}\right)+\omega_y^2\left(\frac{I_y(I_y-I_z)}{\underset{>0}{h^2-2TI_z}}\right)=1$$

projection onto the (x-y) plane

this conic section is representing an ellipse

$$\omega_y^2\left(\frac{I_y(\overset{<0}{I_y-I_x})}{\underset{<0}{h^2-2TI_x}}\right)+\omega_z^2\left(\frac{I_z(\overset{<0}{I_z-I_x})}{\underset{<0}{h^2-2TI_x}}\right)=1$$

projection onto the (y-z) plane

This conic section is representing an hyperbola

$$\omega_x^2\left(\frac{I_x(\overset{>0}{I_x-I_y})}{h^2-2TI_y}\right)+\omega_z^2\left(\frac{I_z(\overset{<0}{I_z-I_y})}{h^2-2TI_y}\right)=1$$

projection onto the (x-z) plane

we can not say a lot about the denominator → but for sure we will have different sign of the coefficient

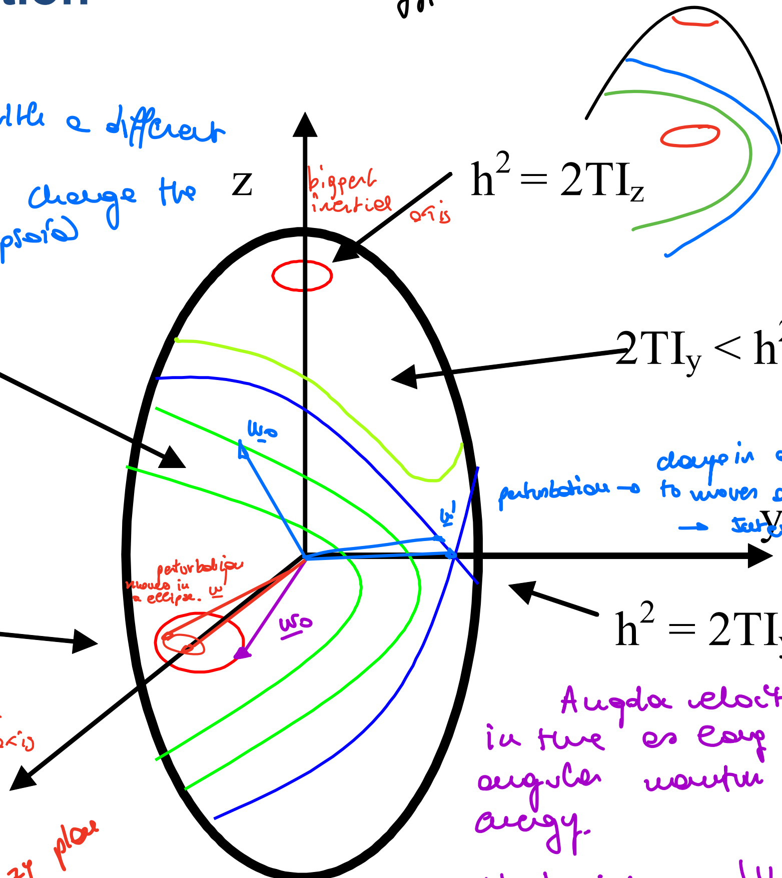
Analyzing the planar conic curves and the signs of all coefficients, we have that

$$h^2-2TI_z>0$$
$$h^2-2TI_x<0$$
$$h^2-2TI_y \quad ?$$

Geometric Interpretation

All the time keeps always the same amount of kinetic energy.

$\underline{w_0}$ same level of energy with a different angular momentum.
Different angular momentum change the intersection on the ellipsoid



$h^2 = 2TI_z$

$2TI_x < h^2 < 2TI_y$

$2TI_y < h^2 < 2TI_z$

$h^2 = 2TI_x$

$h^2 = 2TI_y$

perturbation → change in angular momentum leads to moves along the sepratrix
→ intermediate axis of inertia is not stable

Angular velocity could change in time as long as it conserve angular momentum and kinetic energy.

Given this initial condition $\underline{w}$ could change only along the red line.

eye = looking from infinite projection on the zy plane

Riascoltare fine lezione Bonelli

We have given a notion of stability looking only with a geometric prospective. In future we will look at the same problem with a different notation of stability. (mathematical one).

TRANSPORT THEOREM DEMOSTRATION

STATEMENT $\Rightarrow$ $$\frac{{}^{N}d\underline{x}}{dt} = \frac{{}^{B}d\underline{x}}{dt} + \underline{\omega}_{B/N} \wedge \underline{x}$$

NOTE Valid for $O = O'$ i think

$$\underline{x}\big|_{N} = \underline{x}\big|_{B}$$ same vector rappresented in two different reference frame

$\{\underline{e_1}, \underline{e_2}, \underline{e_3}\}$ Base for reference frame B $\quad \underline{x} = x_1 \underline{e_1} + x_2 \underline{e_2} + x_3 \underline{e_3}$ in the B referance frame

$$\frac{{}^{A}d\underline{x}}{dt} = \frac{dx_1}{dt}\underline{e_1} + \frac{dx_2}{dt}\underline{e_2} + \frac{dx_3}{dt}\underline{e_3} + x_1 \frac{{}^{A}d\underline{e_1}}{dt} + x_2 \frac{{}^{A}de_2}{dt} + x_3 \frac{{}^{A}de_3}{dt}$$

$$= \frac{{}^{B}d\underline{x}}{dt} + x_1 \frac{{}^{A}d\underline{e_1}}{dt} + x_2 \frac{{}^{A}de_2}{dt} + x_3 \frac{{}^{A}de_3}{dt} \quad (3)$$

$$e_i \circ e_j = 1 \quad \text{if } i = j$$

$$e_i \circ e_j = \emptyset \quad \text{if } i \neq j$$

recall that $$\frac{{}^{A}d(\underline{e_i} \circ \underline{e_i})}{dt} = \emptyset$$

$$\frac{{}^{A}d(\underline{e_i} \circ \underline{e_i})}{dt} = \underline{e_i} \circ \frac{{}^{A}de_i}{dt} + \frac{{}^{A}de_i}{dt} \circ \underline{e_i} = \underline{e_i} \circ \frac{{}^{A}de_i}{dt} = \emptyset$$

$$\Rightarrow e_i \perp \frac{{}^{A}de_i}{dt} \quad (1)$$

$$\frac{{}^{A}d(\underline{e_i} \circ \underline{e_j})}{dt} = \emptyset \rightarrow \underline{e_i} \circ \frac{{}^{A}de_j}{dt} = - \underline{e_j} \circ \frac{{}^{A}de_i}{dt} \quad (2)$$

Next we will write each derivative of the unit vector as a funtion of three angular velocity component.

$$\frac{{}^{A}de_1}{dt} = w_{11}\underline{e_1} + w_{12}\underline{e_2} + w_{13}\underline{e_3}$$

$$\frac{{}^{A}de_2}{dt} = w_{21}\underline{e_1} + w_{22}\underline{e_2} + w_{33}\underline{e_3} \qquad \text{due to } (1) \quad w_{11} = w_{22} = w_{33} = \emptyset$$

$$\frac{{}^{A}de_3}{dt} = w_{31}\underline{e_1} + w_{32}\underline{e_2} + w_{33}\underline{e_3}$$

Due to ② $\underline{e_2} \frac{{}^A d\underline{e_1}}{dt} = - \underline{e_1} \cdot \frac{{}^A d\underline{e_2}}{dt}$

$$\begin{cases} \omega_{12} = -\omega_{21} \\ \omega_{13} = -\omega_{31} \\ \omega_{23} = -\omega_{32} \end{cases}$$

$$\frac{{}^A d\underline{e_1}}{dt} = \omega_{12}\underline{e_2} + \omega_{13}\underline{e_3}$$

$$\frac{{}^A d\underline{e_2}}{dt} = -\omega_{21}\underline{e_1} + \omega_{23}\underline{e_3} \quad ④ \qquad \text{now}$$

$$\frac{{}^A d\underline{e_3}}{dt} = -\omega_{31}\underline{e_1} - \omega_{32}\underline{e_2}$$

$$\omega_1 = \omega_{32}$$

$$\omega_2 = -\omega_{31}$$

$$\omega_3 = \omega_{21}$$

$A = N$

$$\underline{\omega}_{B/N} = \omega_1\underline{e_1} + \omega_2\underline{e_2} + \omega_3\underline{e_3} \quad ③$$

③ → ④ we are $\frac{{}^N d\underline{e_i}}{dt} = \underline{\omega}_{B/A} \wedge \underline{e_i}$

Substituting this into ① we will get

$$\frac{{}^N dx}{dt} = \frac{{}^B dx}{dt} + x_1\,\underline{\omega_{B/N}} \wedge \underline{e_1} + x_2\,\underline{\omega_{B/N}} \wedge \underline{e_2} + x_3\,\underline{\omega_{B/N}} \wedge \underline{e_3}$$

$$\frac{{}^N dx}{dt} = \frac{{}^B dx}{dt} + \underline{\omega_{B/N}} \wedge \underbrace{(x_1\underline{e_1} + x_2\underline{e_2} + x_3\underline{e_3})}_{\underline{x}}$$

$$\boxed{\frac{{}^N dx}{dt} = \frac{{}^B dx}{dt} + \underline{\omega_{B/N}} \wedge \underline{x}}$$