ORBITAL MECHANICS

$$\ddot{\underline{R}}_{s/c} = -\frac{G m_s}{R_{s/c}^3}\underline{R}_{s/c} - \frac{G m_p}{r^3}\underline{r} \qquad (4.11)$$

It can be written also as:

$$\ddot{\underline{R}}_{s/c} = \underline{A}_s + \underline{P}_p$$

$\underline{P}_p$ ↳ perturbing effect acceleration of the planet

$\underline{A}_s$ ↳ main effect acceleration due to the sun

$$\underline{A}_s = -\frac{G m_s}{R_{s/c}^3}\underline{R}_{s/c}$$

$$\underline{P}_p = -\frac{G m_p}{r^3}\underline{r}$$

$$\frac{P_p}{A_s} = \frac{\cancel{G} m_p}{r^2}\frac{R_{s/c}^2}{\cancel{G} m_s} = \frac{m_p}{m_s}\left(\frac{R_{s/c}}{r}\right)^2 \simeq \frac{m_p}{m_s}\left(\frac{R}{r}\right)^2 \qquad (4.12)$$

$$\underline{R}_{s/c} = \underline{R} + \underline{r} \simeq \underline{R}$$

2) EQUATION OF MOTION OF THE PLANET RELATIVE TO INERTIAL REF. FRAME

$$m_p \ddot{\underline{R}} = \underline{F}_{sc}^{P} + \underline{F}_{s}^{P} = G\frac{m_p m_{s/c}}{r^3}\underline{r} - G\frac{m_p m_s}{R^3}\underline{R}$$

$$\boxed{\ddot{\underline{R}} = G\frac{m_{sc}}{r^3}\underline{r} - G\frac{m_s}{R^3}\underline{R}} \qquad (4.13)$$

subtract (4.13) from (4.11)

$$\ddot{\underline{R}}_{s/c} - \ddot{\underline{R}} = -\frac{G m_s}{R_{s/c}^3}\underline{R}_{s/c} - \frac{G m_p}{r^3}\underline{r} - G\frac{m_{s/c}}{r^3}\underline{r} + \frac{G m_s}{R^3}\underline{R}$$

Recall $\underline{R}_{s/c} = \underline{R} + \underline{r}$

$$\boxed{\ddot{\underline{r}} = -\frac{G m_p}{r^3}\underline{r}\left(1 + \frac{m_{s/c}}{m_p}\right) - \frac{G m_s}{R_{s/c}^3}\left\{\underline{r} + \underline{R}\left[1 - \left(\frac{R_{s/c}}{R}\right)^3\right]\right\}} \qquad (4.14)$$

Eqs of motion of s/c wrt the planet

Recall (4.10) $R_{s/c} \simeq R$ and considering $m_{s/c} \ll m_p$ we can approximate to

$$\boxed{\ddot{\underline{r}} = -\frac{G m_p}{r^3}\underline{r} - \frac{G m_s}{R_{s/c}^3}\underline{r}} \qquad (4.15)$$

$\frac{m_{s/c}}{m_p} \simeq 0 \qquad \frac{R_{sc}}{R} \simeq 1 \qquad \simeq R^3$

→ equation that we usually use where we are in the proximity of planet 1 (SOI of planet 1) and we want to consider the effect of the third body with some approx.

$$\ddot{\underline{r}} = \underline{a_p} + \underline{\rho_s}$$

$\underline{a_p}$ → primary acceleration

$\underline{\rho_s}$ → Perturbating due to Sun

$$\underline{a_p} = -\frac{G m_p}{r^3}\underline{r}$$

$$\underline{\rho_s} = -\frac{G m_s}{R^3}\underline{r}$$

$$\frac{\rho_s}{a_p} = \frac{\cancel{G} m_s \cancel{r}}{R^3}\,\frac{r^3}{\cancel{G} m_p \cancel{r}} = \frac{m_s}{m_p}\,\frac{r^3}{R^3} \qquad (4.16)$$

- ρ_s / a_p Eq (4.16) deviation from a 2BP (Kepler 2 Body Problem) for s/c wrt planet
- P_p / A_s Eq (4.12) deviation from a 2BP for s/c wrt Sun

$$\text{if} \quad \frac{\rho_s}{a_p} < \frac{P_p}{A_s} \qquad (4.17)$$

Then the perturbing effect of the Sun on s/c orbit around the planet is less than the perturbing effect of the planet on s/c orbit around the Sun

Substitute Eqs (4.12) and (4.16) into condition (4.17)

$$\frac{m_S}{m_p}\left(\frac{r}{R}\right)^3 < \frac{m_p}{m_S}\left(\frac{R}{r}\right)^2$$

$$\left(\frac{r}{R}\right)^5 < \left(\frac{m_p}{m_S}\right)^2 \longrightarrow \frac{r}{R} < \left(\frac{m_p}{m_S}\right)^{2/5}$$

$$\boxed{r_{SOI} = R\left(\frac{m_p}{m_S}\right)^{2/5}} \qquad (4.18)$$

SPHERE OF INFLUENCE

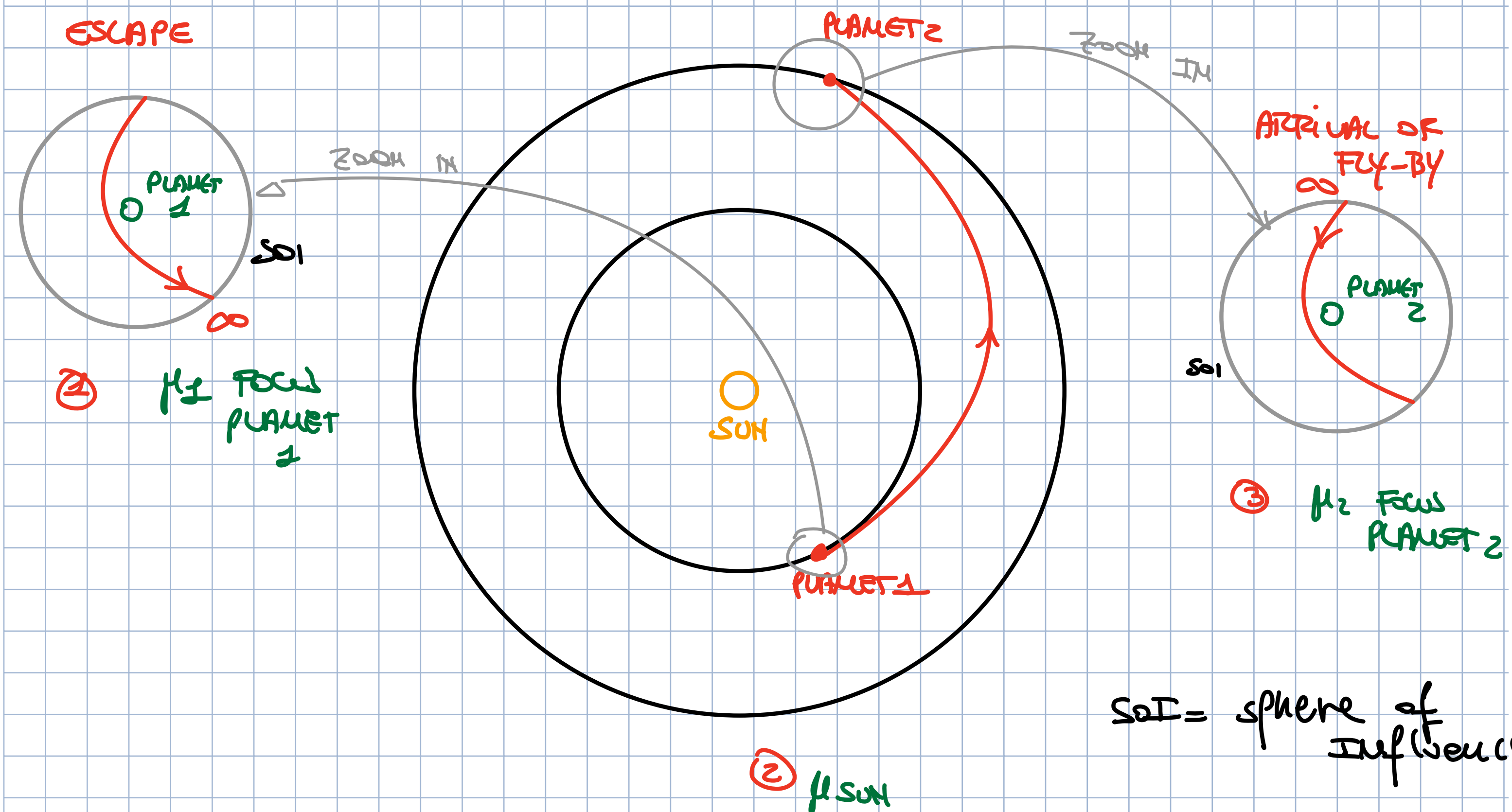
Definition of the sphere of influence if $r < r_{SOI}$ the planet has a greater effect than the sun on the satellite.

It tell us where to switch our reference system.

Now it is possible to go back to:

PATCHED CONIC METHOD

Each phase of the mission is modelled on a keplerian orbit



SOI = sphere of influence

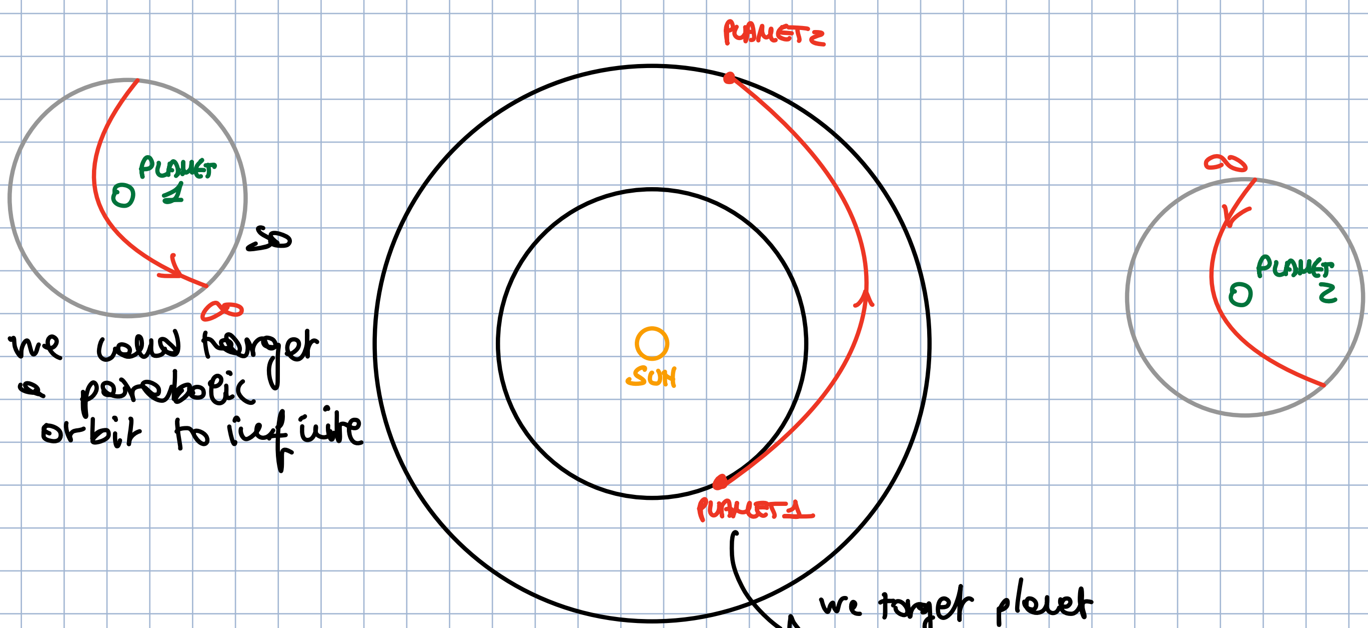
– give us wore approx.

INTERPLANETARY LEG → already modelled with an Hohmann transfer

- Outside the SOI (sphere of influence) of the planet the s/c follows a 2BP wrt Sun μ_{SUN}
- for the heliocentric trajectory we can NEGLECT the size of the SOI of planet, from the sun point of view.
  The SOI of the planets is so small that we could not consider it → we can target directly the center of the planet (it's position given by Ephemerides)

PLANETOCENTRIC LEG

- from planet point of view the SOI is so large that we can consider it to be INFINITE.



we could target a parabolic orbit to infinite

we target planet directly from ephemerides

Design

1- Heliocentric leg to intersect the two planet
   helio leg take SOI of pl 1 to SOI of pl 2 (target planet center)
2 - Compute planet-centred leg at SOI of the planet → calculate $\underline{v}_\infty$ ($\underline{r}_\infty \to \infty$) (modulus and direction) to

patch the 3 arcs of conics.

Approx good for interplanetary trajectory, not good for lunar trajectory.

Earth-moon distance 384000 km ← R_{EM}

moon SOI

$$r_{SOI_{MOON}} = R_{EM}\left(\frac{m_{moon}}{m_{EARTH}}\right)^{2/5} = 66200\ km$$

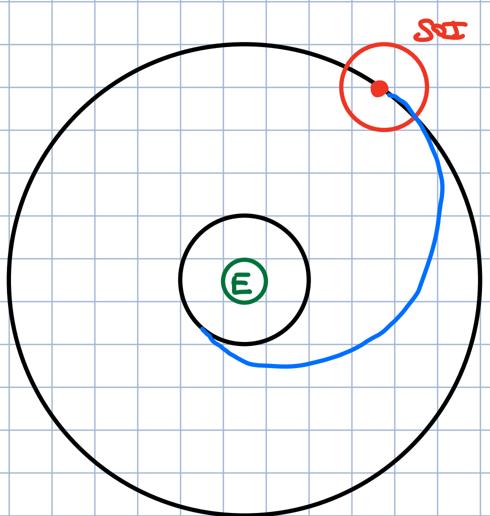
$r_{SOI_{MOON}} > \frac{1}{6} R_{EM}$ so not a good approx.

all depends on the ratio of masses.

NOTE we are not asked to model the escape and the arrival on the planet for the lab assignment.

Difficult is to optimize the departure fly by and the arrival.

↳ we should put our effort in the optimization (vector optimization)



Moon's orbit
Need at any time to consider Earth + Moon

↳ 3 BODY PROBLEM

PLANET DEPARTURE

$(r \to \infty)$

In order to travel to the heliocentric leg s/c must reach the ∞ within SOI of planet with a relative velocity v_∞ greater than zero

v_∞ hyperbolic excess velocity.

We have two options:

- Parabolic orbit at $r \to \infty$, $v_\infty = 0$
  the s/c will not inject in a sun centred leg but for the sun point of view it will be at the same velocity of the planet.

  v_∞ velocity at end of SOI.

  $$v_\infty = \Delta V_D = 0 \Rightarrow \underbrace{\Delta V_D = V_{D s/c} - V_1}_{\text{wrt sun}} \Rightarrow V_{D s/c} = V_1$$

  ↳ relative velocity wrt planet at ∞

  I am stuck to the planet.

  the s/c orbit around the Sun at same speed of planet

  (NICER INTERPRETATION IN THE 3 BODY PROBLEM)

Note CAPITAL wrt SUN
small wrt PLANET

- Hyperbolic orbit at $r \to \infty$, $v_\infty \neq 0$

  example — At SOI of planet $V_{D s/c}$ is parallel both to the asymptote of departure hyperbola AND to the planet heliocentric velocity V_1

  particular case

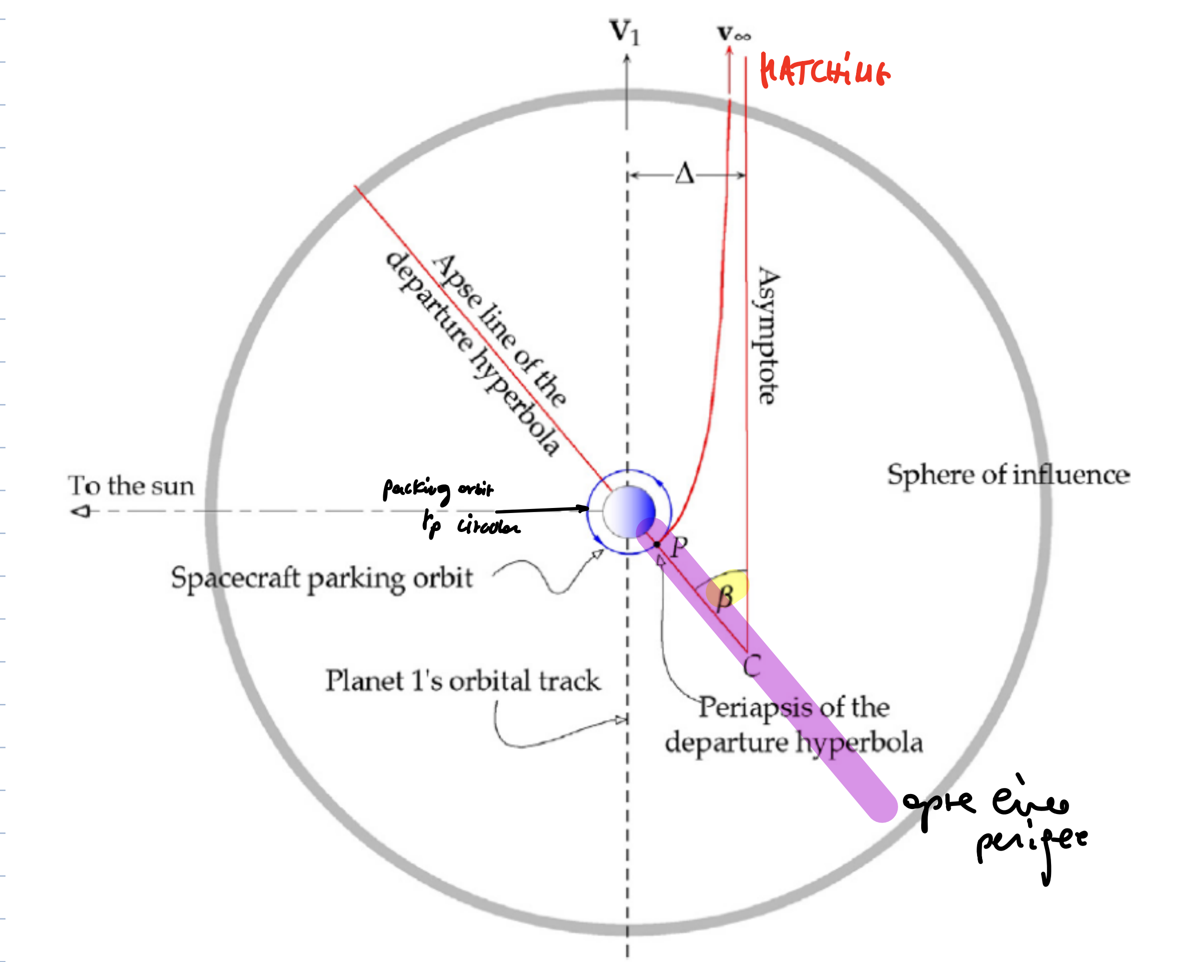
  particular case

- $V_{D s/c}$, V_1 are parallel and in the same direction. if we want to go from

• In the project we do not do these kind of transfer inner → outer planet

$\Delta V_D > 0$

ΔV_D is the hyperbolic excess velocity



$V_1 \parallel V_\infty$

$v_\infty = \Delta V_D$ is hyperbolic excess speed

$$V_{s/c}^{D} = V_1 + \underbrace{\Delta V_D}_{V_\infty}$$

MATCHING VELOCITIES

PATCHED CONIC METHOD

$$v_\infty = \underbrace{\sqrt{\frac{\mu_{SUN}}{R_1}}\left(\sqrt{\frac{2R_2}{R_1+R_2}}-1\right)}_{\text{SUN centred } \Delta V_D} \qquad [\text{Eq } (4.1)]$$

planet centred

$V_{s/c}^{D} \parallel V_1$

$V_{s/c}^{D} > V_1$

The s/c is injected into the escape hyperbola from the parking circular orbit with radius r_p.

From eq (1.35)

$$r_p = \frac{h^2}{\mu_1} \frac{1}{1+e} \qquad (4.18)$$

μ_1 = planetary constant
h = specific angular momentum of the hyperbola
e = eccentricity of the hyperbola

let's remind some expression that we are going to use:
From Eq (1.33) and (1.34)

$$v_r = \frac{\mu}{h} e \sin\theta \qquad (1.33)$$

$$v_\theta = \frac{\mu}{h} (1+e\cos\theta) \qquad (1.34)$$

At ∞ we can define θ_∞ (after p $(\theta = \frac{\pi}{2})$ before $\theta = \pi$)
Recall

$$r = \frac{h^2}{\mu} \frac{1}{1+e\cos\theta} \qquad r \to \infty \quad \text{as} \quad 1+e\cos\theta \to 0$$

$$\Rightarrow \cos\theta_\infty = -\frac{1}{e} \qquad \text{Eq } (1.75)$$

$$\sin\theta_\infty = \frac{\sqrt{e^2-1}}{e} \qquad \text{Eq } (1.76)$$

substitute in v_r and v_θ

$$v_{r\infty} = \frac{\mu}{h} e \frac{\sqrt{e^2-1}}{e} = \frac{\mu}{h}\sqrt{e^2-1}$$ radial component of the velocity at ∞

$$v_{\theta\infty} = \frac{\mu}{h}\left(1+e\left(-\frac{1}{e}\right)\right) = 0$$

$$\Rightarrow \quad \boxed{v_\infty = v_{r\infty} = \frac{\mu}{h}\sqrt{e^2-1}} \quad (4.20)$$

we have buly the component that leads the s/c away.

NOTE at ∞ we move away r in radial direction away from planet.

Let's rearrange (4.19) to link conditions at perigee to conditions at ∞

$$\text{eq } (4.20) \rightarrow \quad h = \frac{\mu_1}{v_\infty}\sqrt{e^2-1} \quad \rightarrow \text{eq } (4.19)$$

$$r_p = \frac{\mu_1^2}{v_\infty^2}(e^2-1)\frac{1}{\mu_1}\frac{1}{1+e}$$

$$r_p = \frac{\mu_1(e-1)}{v_\infty^2} \rightarrow e \qquad \boxed{e_{Hyp} = \frac{r_p v_\infty^2}{\mu_1} + 1} \quad (4.21)$$

condition at ∞ on Hyp

perigee radius = parking orbit radius

Back in Eq (4.20)

$$h = \frac{\mu_1\sqrt{e^2-1}}{v_\infty} \qquad e = \frac{r_p v_\infty^2}{\mu_1} + 1$$

$$h = \frac{\mu_1}{v_\infty}\sqrt{1 + \frac{2 r_p v_\infty^2}{\mu_1} + \left(\frac{r_p v_\infty^2}{\mu_1}\right)^2 - 1}$$

$$= r_p\sqrt{\frac{v_\infty^2}{\mu_1^2}\left(\frac{v_\infty^4}{\mu_1^2} + \frac{2 v_\infty^2}{r_p \mu_1}\right)} = r_p\sqrt{v_\infty^2 + \frac{2\mu_1}{r_p}}$$

$$\boxed{h_{Hyp} = r_p\sqrt{v_\infty^2 + \frac{2\mu_1}{r_p}}} \qquad (4.22)$$

|| hyperbola e, h as function of r_p, v_∞
|| chosen r_p and $v_\infty \Rightarrow$ I can fully characterized the helio-leg e_{Hyp}, h_{Hyp}

$h \longrightarrow$ velocity at perigee

$$\boxed{v_p = \frac{h}{r_p} = \sqrt{v_\infty^2 + \frac{2\mu}{r_p}}}$$ (4.23) velocity at the periapsis

↳ could have been computed from the energy conservation

$$-\frac{\mu}{2a} = -\frac{\mu}{r} + \frac{v^2}{2}$$

at perigee = at ∞

Now we can calculate the Impulse.

S/c is injected from parking orbit

$$v_c = \sqrt{\frac{\mu}{r_p}}$$

$$\Delta v = v_p - v_c = \sqrt{\frac{\mu}{r_p}}\left(\sqrt{\frac{v_\infty^2}{v_c^2}+2} - 1\right)$$

Δv required to enter the hyperbola.

After entered the hyperbola we do not need to give more energy because ΔV_p is only a switch of ref frame.

parking orbit $\xrightarrow[\text{at perigee}]{\Delta v}$ hyperbola $v_\infty = \Delta V_p$ $\xrightarrow[\text{TRANSFER}]{\text{NO}}$ Heliocentric leg

$r_p \longrightarrow \infty$

ε constant

$\frac{v^2}{2}$ is exchanged with $-\frac{\mu}{r}$

HOHMANN TRANSFER

without approx

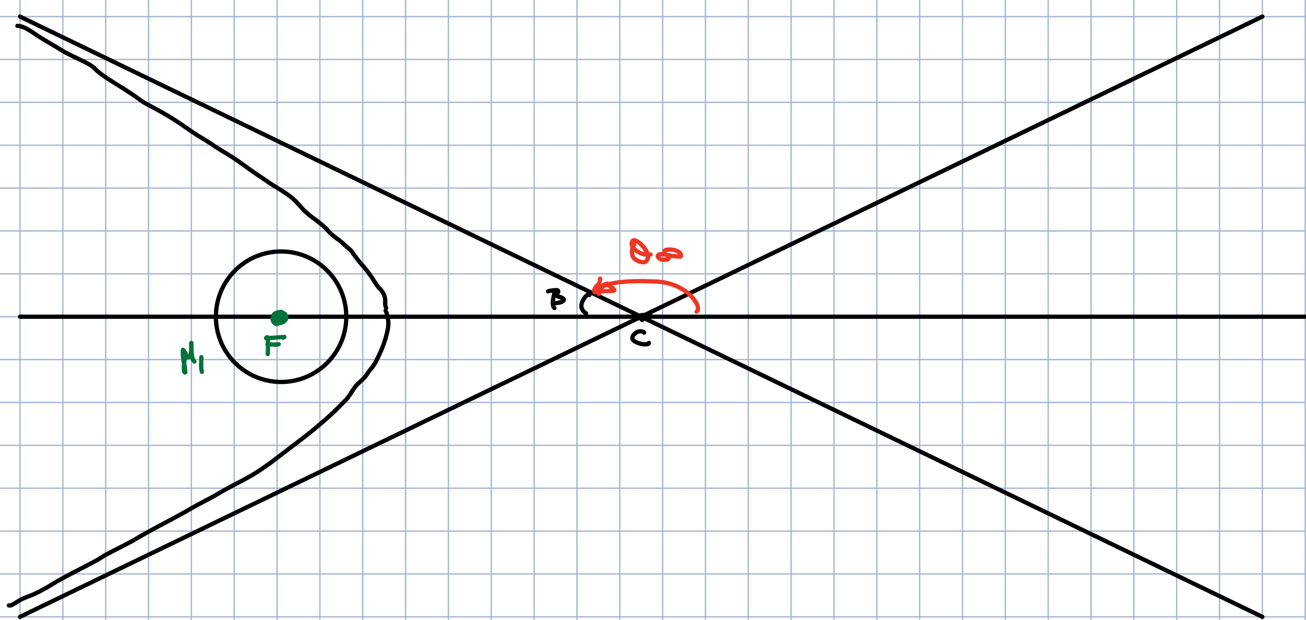
as in the lab

In actual mission the helio leg is generic (Lambert) but the procedure is the same. It will just be a vectorial summ.

$$\underline{V_{\Delta s/c}} = \underline{v_1} + \underline{v_\infty}$$ see exercises.

To have the two $v_\infty \parallel v_1$ it is necessary that the apse line of the hyperbola has to be oriented so that every thing work. I need to give the Δv in a proper way, I could not give it at any time.

Where Δv must be given?



$$\beta = \pi - \theta_\infty$$

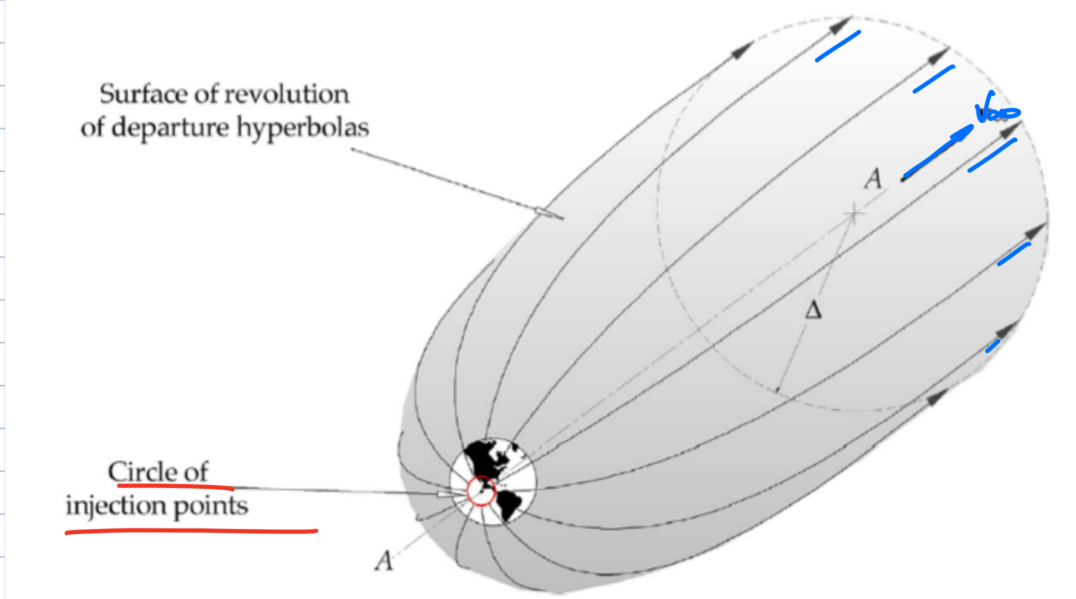
$$\cos\beta = \cos(\pi - \theta_\infty) \longrightarrow \cos\beta = +\frac{1}{e}$$

$$\boxed{\beta = \cos^{-1}\left(\frac{1}{1 + \frac{r_p v_\infty^2}{\mu}}\right)} \qquad (4.24)$$

constraints

- hyperbola plane must contain the focus (PLANET)
- exit asymptote (c to ∞) must be // to v_1 and $v_D^{S/C}$

↳ only these two constrain must be satisfied

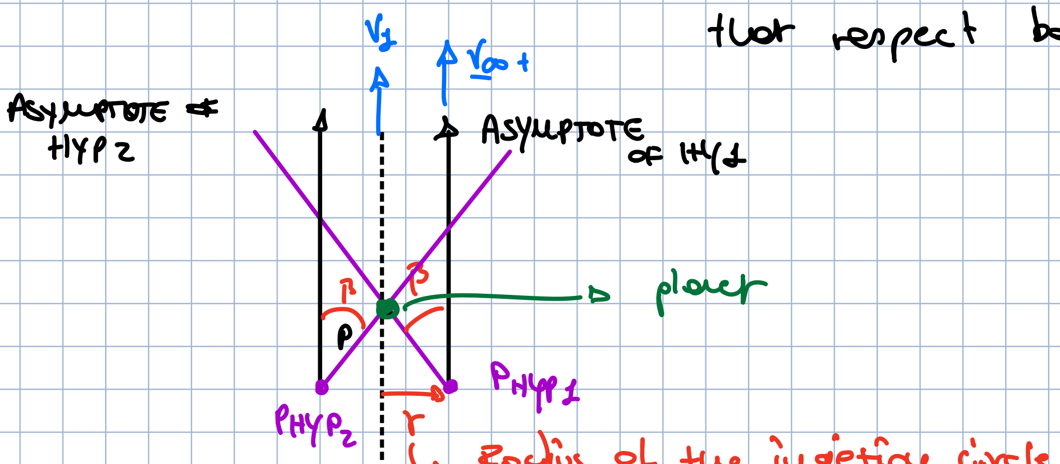


We can define the shape and size of the hyperbola but we have not defined its plane.

The injection point can be any point on this circle O.

↳ set of perigee point possible

I can draw another hyperbola that respect both constraints



The perigee sweeps a radius of ↳ $r_p \sin\beta$

The choice of the hyperbola plane does not depend on heliocentric plane (we have same v_∞, $\Delta V_{\Delta SC}$) but on the injection capabilities.