POLITECNICO MILANO 1863

Spacecraft Attitude Dynamics

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Euler Equations

Fundamental Properties

$$\underline{h} = I\underline{\omega}$$

$$T = \frac{1}{2}\underline{\omega} \cdot I\underline{\omega}$$

If the body fixed frame is chosen to coincide with the principle axes then:

$$\underline{h} = [I_x\omega_x \quad I_y\omega_y \quad I_z\omega_z]^T$$

$$T = \frac{1}{2}(I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2)$$

Euler Equations

$$\underline{h} = I\underline{\omega}$$

Recall the Transport Theorem:

$$\frac{{}^{N}d}{dt}\underline{x} = \frac{{}^{B}d}{dt}\underline{x} + \omega \times \underline{x}$$

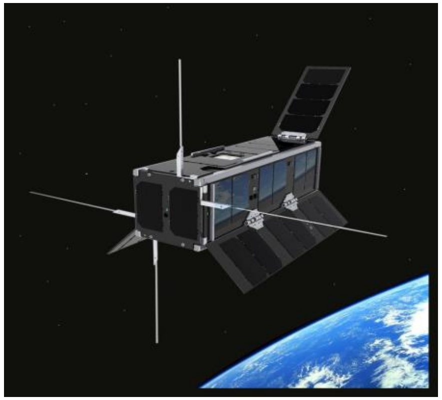
Assuming we are in the principal axis then:

Euler Equations for a rigid-body:

$$I\frac{d\underline{\omega}}{dt} = I\underline{\omega} \times \underline{\omega} + \underline{M}$$

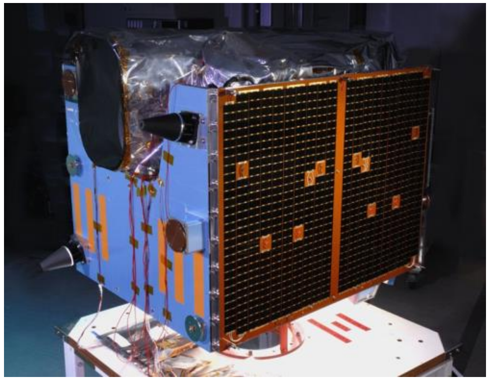
$$\dot{\omega}_x = \frac{I_y - I_z}{I_x}\omega_y\omega_z + \frac{M_x}{I_x}$$
$$\dot{\omega}_y = \frac{I_z - I_x}{I_y}\omega_x\omega_z + \frac{M_y}{I_y}$$
$$\dot{\omega}_z = \frac{I_x - I_y}{I_z}\omega_y\omega_x + \frac{M_z}{I_z}$$

Principal moments of inertia (kg m²)



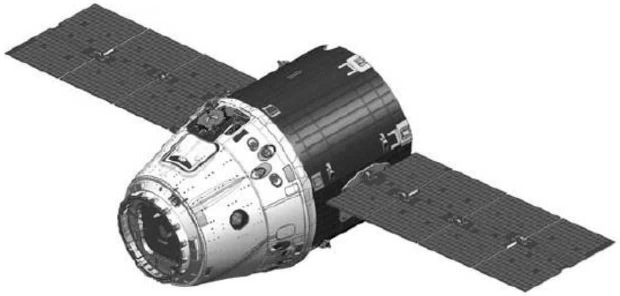
UKube-1 – 3U CubeSat

$I_1 = 0.0109kgm^2, I_2 = 0.0504kgm^2, I_3 = 0.055kgm^2$



Rapid Eye - Micro-spacecraft (150 kg)

$I_1 = 19.5kgm^2, I_2 = 19kgm^2, I_3 = 12.6kgm^2$



Space X ‘s unmanned 10 tonne spacecraft

$I_1 = 20{,}000kgm^2, I_2 = 20{,}000kgm^2, I_3 = 25{,}000kgm^2$

Conservation shown in coordinate form

$$\underline{h} = [I_x\omega_x \quad I_y\omega_y \quad I_z\omega_z]^T$$

$$T = \frac{1}{2}(I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2)$$

Using the chain rule:

$$\frac{d(\square)}{dt} = \frac{d(\square)}{d\omega_x}\frac{d\omega_x}{dt} + \frac{d(\square)}{d\omega_y}\frac{d\omega_y}{dt} + \frac{d(\square)}{d\omega_z}\frac{d\omega_z}{dt}$$

$$\dot{\omega}_x = \frac{I_y - I_z}{I_x}\omega_y\omega_z + \frac{M_x}{I_x}$$
$$\dot{\omega}_y = \frac{I_z - I_x}{I_y}\omega_x\omega_z + \frac{M_y}{I_y}$$
$$\dot{\omega}_z = \frac{I_x - I_y}{I_z}\omega_y\omega_x + \frac{M_z}{I_z}$$

$$\frac{dT}{dt} = 0$$

$$\frac{d(\underline{h}\cdot\underline{h})}{dt} = 0$$

Exact Solution for a symmetric spacecraft

$I_x = I_y = I$

$$\dot{\omega}_x = \frac{I_y - I_z}{I_x}\omega_y\omega_z$$
$$\dot{\omega}_y = \frac{I_z - I_x}{I_y}\omega_x\omega_z$$
$$\dot{\omega}_z = \frac{I_x - I_y}{I_z}\omega_y\omega_x$$

$$T = \frac{1}{2}(I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2)$$

$$\underline{h} = [I_x\omega_x \quad I_y\omega_y \quad I_z\omega_z]^T$$

Exact Solution is:

$$\omega_x = \omega_{x0}\cos(\lambda t) - \omega_{y0}\sin(\lambda t)$$
$$\omega_y = \omega_{x0}\sin(\lambda t) + \omega_{y0}\cos(\lambda t)$$
$$\omega_z = \omega_{z0}$$

$$\lambda = \frac{(I_z - I)\omega_{z0}}{I}$$

Equilibrium configurations

$$\dot{\omega}_x = \frac{I_y - I_z}{I_x}\omega_y\omega_z$$
$$\dot{\omega}_y = \frac{I_z - I_x}{I_y}\omega_x\omega_z$$
$$\dot{\omega}_z = \frac{I_x - I_y}{I_z}\omega_y\omega_x$$

Equilibrium Points:

$$\omega_x = 0, \omega_y = 0, \omega_z = 0$$
$$\omega_x = \omega_x(0), \omega_y = 0, \omega_z = 0$$
$$\omega_x = 0, \omega_y = \omega_y(0), \omega_z = 0$$
$$\omega_x = 0, \omega_y = 0, \omega_z = \omega_z(0)$$

Stability definitions of equilibrium points.

Stability definitions

Consider an autonomous nonlinear dynamical system

$$\dot{\underline{x}} = f(\underline{x}), \underline{x}(0) = \underline{x}_0$$

defined on an open set containing the origin, and *f* is continuous on this open set. Then an equilibrium point x_e is said to be:

1. **Lyapunov stable**, if, for every $\varepsilon > 0$, there exists a $\partial > 0$ such that, if $\|x(0) - x_e\| < \partial$, then for every $t > 0$ we have $\|x(t) - x_e\| < \varepsilon$.
2. The equilibrium of the above system is said to be **asymptotically stable** if it is Lyapunov stable and if $\|x(t) - x_e\| \to 0$ as $t \to \infty$

Equilibrium configurations

$$\dot{\omega}_x = \frac{I_y - I_z}{I_x}\omega_y\omega_z$$
$$\dot{\omega}_y = \frac{I_z - I_x}{I_y}\omega_x\omega_z$$
$$\dot{\omega}_z = \frac{I_x - I_y}{I_z}\omega_y\omega_x$$

Equilibrium Points:

$$\omega_x = 0, \omega_y = 0, \omega_z = 0$$
$$\omega_x = \omega_x(0), \omega_y = 0, \omega_z = 0$$
$$\omega_x = 0, \omega_y = \omega_y(0), \omega_z = 0$$
$$\omega_x = 0, \omega_y = 0, \omega_z = \omega_z(0)$$

Stability Analysis of equilibrium configurations

$$\omega_x = C, \omega_y = 0, \omega_z = 0$$

Look at the perturbed solution

$$\omega_x = C + \partial\omega_x, \omega_y = 0 + \partial\omega_y, \omega_z = 0 + \partial\omega_z$$

$$\dot{\omega}_x = \frac{I_y - I_z}{I_x}\omega_y\omega_z$$
$$\dot{\omega}_y = \frac{I_z - I_x}{I_y}\omega_x\omega_z$$
$$\dot{\omega}_z = \frac{I_x - I_y}{I_z}\omega_y\omega_x$$

Conditions for stability

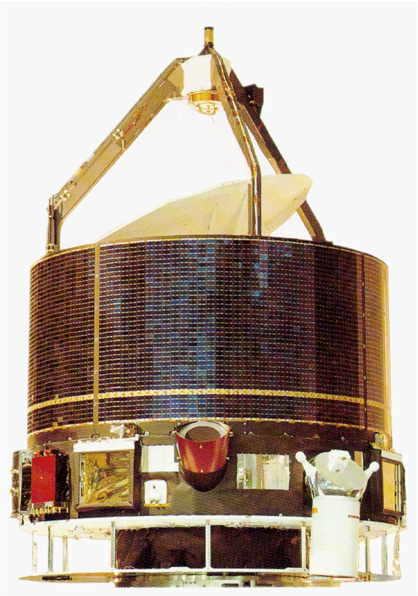
$$\partial\ddot{\omega}_y + \frac{(I_x - I_z)(I_x - I_y)}{I_y I_z} C_1^2 \partial\omega_y = 0$$

Stable if:

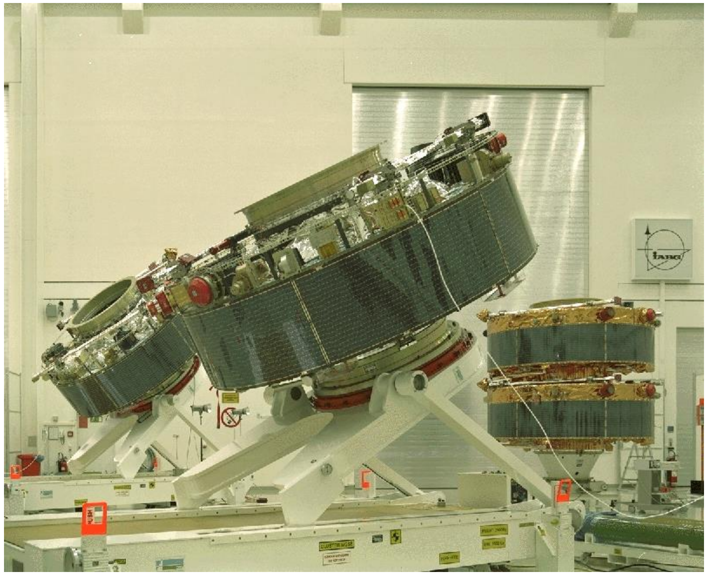
$$\frac{(I_x - I_z)(I_x - I_y)}{I_y I_z} C_1^2 > 0$$

Spin stabilisation

- Simple and low cost method of attitude stabilisation (largely passive)
- Generally not suitable for imaging payloads (but can use a scan platform)
- Poor power efficiency since entire spacecraft body covered with solar cells



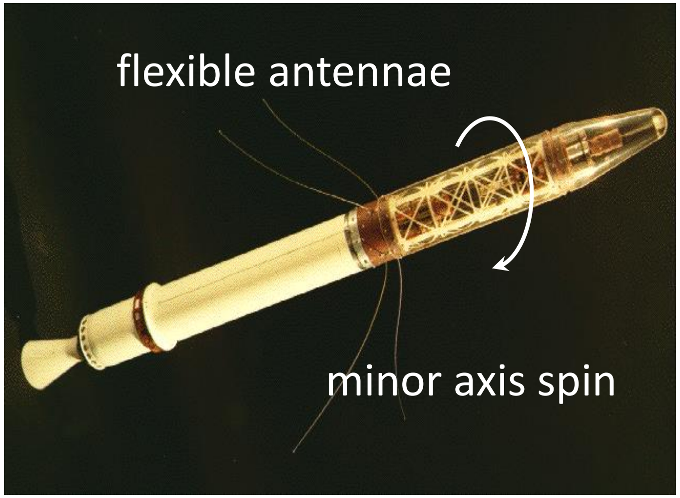
ESA Giotto spacecraft

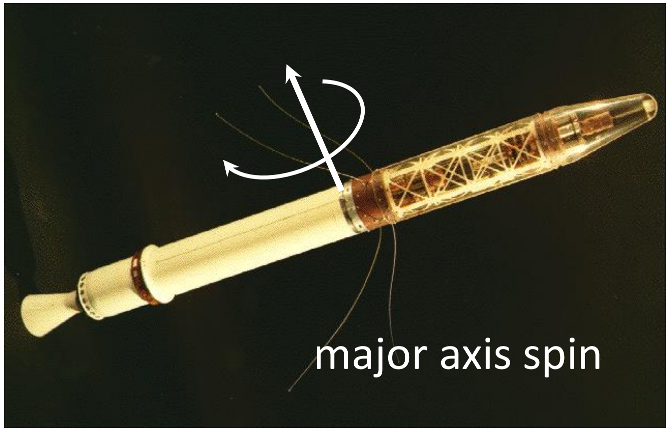


ESA Cluster spacecraft

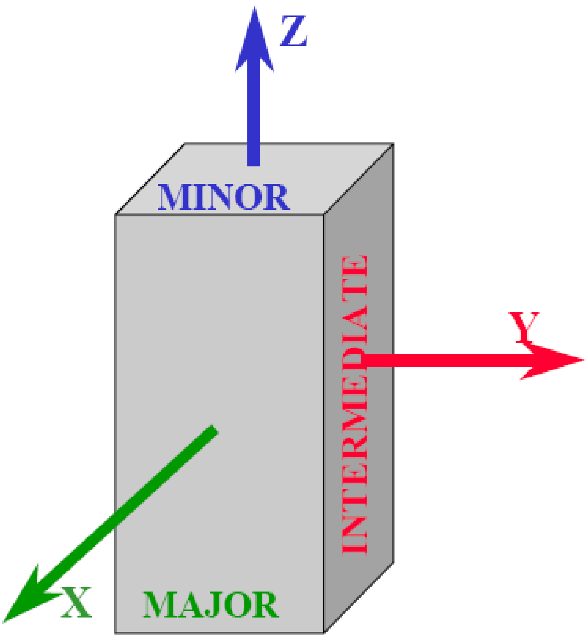
Explorer 1

Explorer 1 in the Figure below (first US satellite, 1958) was designed as a minor axis spinner





Major axis spin rule



- $I_{xx} > I_{yy} > I_{zz}$
- Major axis spin is stable
- Minor axis spin is stable
- Intermediate axis spin is unstable
- Energy dissipation changes these results
  $\rightarrow$ Minor axis spin becomes unstable
- This is called the Major-Axis Rule

Major axis spin rule

energy dissipation, no torque → $\dot{T} < 0$, $|h| = const$

$$\dot{T} < 0$$
$$|h| = const$$

Spin around major or minor axis →

$$2T_{I\,zmax} = I_{z\,max}\omega_{z\,max}^2$$
$$2T_{I\,zmin} = I_{z\,min}\omega_{z\,min}^2$$
$$|h| = I_{z\,max}\omega_{z\,max} = I_{z\,min}\omega_{z\,min}$$

conservation of angular momentum

$\omega_{zmax} < \omega_{zmin}$

$$T_{I\,zmin} > T_{I\,zmax}$$

Solutions in the phase space

Assume the external torques are zero → $\begin{cases} I_x\dot{\omega}_x + (I_z - I_y)\omega_z\omega_y = 0 \\ I_y\dot{\omega}_y + (I_x - I_z)\omega_x\omega_z = 0 \\ I_z\dot{\omega}_z + (I_y - I_x)\omega_y\omega_x = 0 \end{cases}$ → $\begin{matrix} 2\mathrm{T} = \mathrm{const} \\ h^2 = \mathrm{const} \end{matrix}$

Time derivative of x equation → $I_x\ddot{\omega}_x + (I_z - I_y)\dot{\omega}_z\omega_y + (I_z - I_y)\omega_z\dot{\omega}_y = 0$

Compute $\dot{\omega}_y$ and $\dot{\omega}_z$ from other 2 equations

$$I_x\ddot{\omega}_x + (I_z - I_y)\left(\frac{I_x - I_y}{I_z}\right)\omega_y^2\omega_x + (I_z - I_y)\left(\frac{I_z - I_x}{I_y}\right)\omega_z^2\omega_x = 0$$

Solutions in the phase space

From definitions

$$h^2 - 2\mathrm{TI}_z = I_x^2\omega_x^2 + I_y^2\omega_y^2 - I_x I_z \omega_x^2 - I_y I_z \omega_y^2$$

$$\omega_y^2\left(I_y - I_z\right)I_y = h^2 - 2\mathrm{TI}_z - \omega_x^2(I_x - I_z)I_x$$

$$\omega_y^2\left(I_y - I_z\right) = \frac{h^2 - 2\mathrm{TI}_z - \omega_x^2(I_x - I_z)I_x}{I_y}$$

$$h^2 - 2\mathrm{TI}_y = I_z^2\omega_z^2 + I_x^2\omega_x^2 - I_y I_z \omega_z^2 - I_x I_y \omega_x^2$$

$$\omega_z^2\left(I_z - I_y\right)I_z = h^2 - 2\mathrm{TI}_y + \omega_x^2\left(I_y - I_x\right)I_x$$

$$\omega_z^2\left(I_z - I_y\right) = \frac{h^2 - 2\mathrm{TI}_y + \omega_x^2\left(I_y - I_x\right)I_x}{I_z}$$

Substituting in the previous equation

$$\ddot{\omega}_x + \left[\frac{\left(I_y - I_x\right)(h^2 - 2TI_z) + (I_z - I_x)\left(h^2 - 2TI_y\right)}{I_x I_y I_z}\right]\omega_x + \left[\frac{2(I_z - I_x)\left(I_y - I_x\right)I_x}{I_x I_y I_z}\right]\omega_x^3 = 0$$

Solutions in the phase space

Similarly the other 2 equations

$$\ddot{\omega}_y + \left[\frac{\left(I_z - I_y\right)(h^2 - 2\mathrm{TI}_x) + \left(I_x - I_y\right)(h^2 - 2\mathrm{TI}_z)}{I_x I_y I_z}\right]\omega_y + \left[\frac{2\left(I_z - I_y\right)\left(I_x - I_y\right)I_y}{I_x I_y I_z}\right]\omega_y^3 = 0$$

$$\ddot{\omega}_z + \left[\frac{(I_x - I_z)\left(h^2 - 2\mathrm{TI}_y\right) + \left(I_y - I_z\right)(h^2 - 2\mathrm{TI}_x)}{I_x I_y I_z}\right]\omega_z + \left[\frac{2(I_x - I_z)\left(I_y - I_z\right)I_z}{I_x I_y I_z}\right]\omega_z^3 = 0$$

The three equations have the structure $\ddot{\omega} + P\omega + Q\omega^3 = 0$

Integrating the x equation $\dot{\omega}_x^2 + \omega_x^2\left(P_x + \frac{1}{2}Q_x\omega_x^2\right) = K_x$

↓

conic section in the phase plane $(\dot{\omega},\omega)$

Solutions in the phase space

Assume $I_z > I_y > I_x$ $\Rightarrow$ $I_x < \frac{h^2}{2T} < I_z$

$\Downarrow$

P_x ? $\quad$ $P_y > 0$ $\quad$ P_z ?

$Q_x > 0$ $\quad$ $Q_y < 0$ $\quad$ $Q_z > 0$

x and z phase planes

- conic sections are ellipses for large velocities
- for small velocities the type of conic section is undefined.

y phase plane

- solution must be such that the angular velocity is small enough to prevent the trace from being a hyperbola

Solutions in the phase space