Selected slides from

Dynamics and control of space structures

Lorenzo Dozio

Control system design – introduction to state-space methods (part 3)

2020/2021

Outline

Control system design – introduction to state-space methods

Part 1

- Introduction and motivation
- State space fundamentals
- Controllability and observability

Part 2

- Pole placement
- Linear quadratic regulator (LQR)
- Steady-state tracking

Part 3

- Linear observer
- Guidelines for selecting weighting matrices in LQR
- Finite-horizon optimal control

LINEAR OBSERVER

Linear observer

The pole-placement/LQR techniques are based on the feedback of the *full* state vector. Unfortunately, **in many practical situations, the state is not fully directly accessible to measurement**.

In some cases, it is possible to have access to a subset of state variables and to estimate those state variables that are not accessible to measurement using the measurement data from those state variables that are accessible.

However, **in the general case, there is the need of estimating in some way the entire state vector**.

This job is undertaken by the so-called **linear state observer** or **state estimator**.

when we are not able to access directly the state variable we need a way to estimate them with what we have available

A linear observer is a dynamic system whose state variables are the estimates of the state variables of another system. Thus, the aim of the observer is to reconstruct the state vector from a model of the system. The approach we will show in the following is based on the reconstruction of ***x****(t)* from the input vector ***u****(t)* and the output measurement ***y****(t)*.

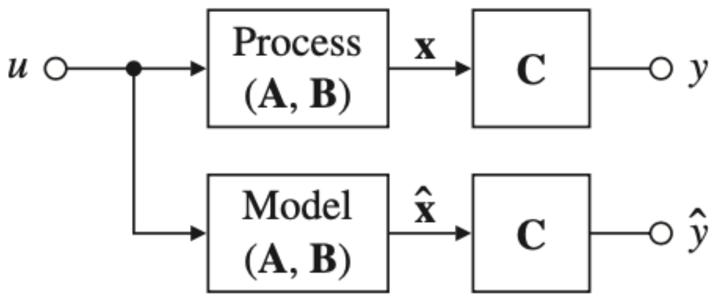
Control system design – state-space methods
Linear observer

Let's consider a method of estimating the state by replicating the model of the plant dynamics (**open-loop observer**):

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}_u\mathbf{u}(t)$$

$$\hat{\mathbf{y}}(t) = \mathbf{C}_y\hat{\mathbf{x}}(t)$$

$$\hat{\mathbf{x}}(0) = \mathbf{x}(0)$$



$\mathbf{x}(t)$	true state
$\mathbf{x}(t)$	estimated state

Open-loop observer: $$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}_u\mathbf{u}(t) \qquad \hat{\mathbf{y}}(t) = \mathbf{C}_y\hat{\mathbf{x}}(t)$$

$$\hat{\mathbf{x}}(0) = \mathbf{x}(0)$$

A perfect estimate of ***x(0)*** would cause the estimated state to track the true state exactly.

But **what about a poor estimate of *x(0)*?**

observation error

Let's define the estimation (reconstruction/observation) error ***e(t)*** as the difference between the true state and the estimated state

$$\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$$

The dynamics of the error is given by $$\dot{\mathbf{e}}(t) = \dot{\mathbf{x}}(t) - \dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\mathbf{e}(t)$$

We have no ability to influence the above dynamics (the error could continually grow or could go to zero too slowly to be of use).

How to overcome this limitation

Control system design – state-space methods
Linear observer

Let's change the previous approach (**closed-loop observer**):

$$\dot{\hat{\mathbf{x}}}(t) = \hat{\mathbf{A}}\hat{\mathbf{x}}(t) + \hat{\mathbf{B}}_u\mathbf{u}(t) + \mathbf{L}\mathbf{y}(t)$$

feed back of the true output of the systems
rectangular matrix
L N x y
number of states ; number of input

Note: **L** *is a rectangular matrix having as many rows as the number of states and as many columns as the number of outputs*

where $\hat{\mathbf{A}}$, $\hat{\mathbf{B}}_u$ and $\mathbf{L}$ are undetermined matrices (at this stage).

Now we can write parallel to this model

The output of the above dynamic system is the estimate of the state vector at any time instant *t*. Note that the state estimation is achieved by using as inputs the control input vector ***u****(t)* and the output vector ***y****(t)* arising from measurement.

Let's compute the dynamics of the estimated error according to this method:

$$\begin{aligned}
\dot{\mathbf{e}}(t) &= \dot{\mathbf{x}}(t) - \dot{\hat{\mathbf{x}}}(t) \\
&= \mathbf{A}\mathbf{x}(t) + \mathbf{B}_u\mathbf{u}(t) - \hat{\mathbf{A}}\hat{\mathbf{x}}(t) - \hat{\mathbf{B}}_u\mathbf{u}(t) - \mathbf{L}\mathbf{y}(t) \\
&= \mathbf{A}\mathbf{x}(t) + \mathbf{B}_u\mathbf{u}(t) - \hat{\mathbf{A}}\left[\mathbf{x}(t) - \mathbf{e}(t)\right] - \hat{\mathbf{B}}_u\mathbf{u}(t) - \mathbf{L}\mathbf{C}_y\mathbf{x}(t) \\
&= \hat{\mathbf{A}}\mathbf{e}(t) + \left(\mathbf{A} - \hat{\mathbf{A}} - \mathbf{L}\mathbf{C}_y\right)\mathbf{x}(t) + \left(\mathbf{B}_u - \hat{\mathbf{B}}_u\right)\mathbf{u}(t)
\end{aligned}$$

Linear observer

We impose that the error goes to zero asymptotically, independent to ***x***(t) and ***u***(t). Then, the coefficients of ***x*** and ***u*** in the equation representing the dynamics of the error must be equal to zero

$$\mathbf{A} - \hat{\mathbf{A}} - \mathbf{L}\mathbf{C}_y = \mathbf{0} \qquad \Rightarrow \qquad \hat{\mathbf{A}} = \mathbf{A} - \mathbf{L}\mathbf{C}_y$$

$$\mathbf{B}_u - \hat{\mathbf{B}}_u = \mathbf{0} \qquad \Rightarrow \qquad \hat{\mathbf{B}}_u = \mathbf{B}_u$$

Accordingly, the dynamics of the estimated error is represented as

$$\boxed{\dot{\mathbf{e}}(t) = (\mathbf{A} - \mathbf{L}\mathbf{C}_y)\,\mathbf{e}(t)}$$

The equation governing the dynamics of the observer becomes

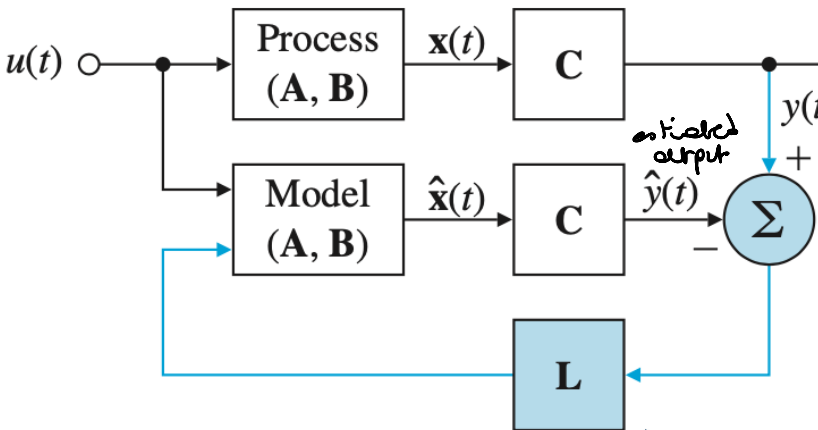
still unknown → tuning the dynamics of the observer

$$\boxed{\dot{\hat{\mathbf{x}}}(t) = (\mathbf{A} - \mathbf{L}\mathbf{C}_y)\,\hat{\mathbf{x}}(t) + \mathbf{B}_u\mathbf{u}(t) + \mathbf{L}\mathbf{y}(t)}$$

Control system design – state-space methods
Linear observer

The equation governing the dynamics of the observer can be also written as

$$\begin{aligned}\dot{\hat{\mathbf{x}}}(t) &= \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}_u\mathbf{u}(t) + \mathbf{L}\left(\mathbf{y}(t) - \mathbf{C}_y\hat{\mathbf{x}}(t)\right) \\ &= \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}_u\mathbf{u}(t) + \mathbf{L}\left(\mathbf{y}(t) - \hat{\mathbf{y}}(t)\right)\end{aligned}$$



→ only variable we can measure from our system

feedback matrix: observer gain matrix

Control system design – state-space methods
Linear observer

we would like to have the characteristic time of the linear observer equals to a fraction of the characteristic time of the system

The gain matrix **L** is chosen to achieve satisfactory error characteristics

$$\dot{\mathbf{e}}(t) = (\mathbf{A} - \mathbf{L}\mathbf{C}_y)\,\mathbf{e}(t)$$

→ Helpful because the observer will give an exact result after some time because error goes to zero faster than the primary system change.

observation poles

The characteristic equation of the error is given by

$$\det\left[s\mathbf{I} - (\mathbf{A} - \mathbf{L}\mathbf{C}_y)\right] = \mathbf{0}$$

If we can choose **L** such that **A** – **L** $\mathbf{C}_y$ has stable and reasonably fast eigenvalues, ***e***(*t*) will decay to zero and remain there – independent of the known forcing function ***u****(t)* and its effect on the state.

Furthermore, we can choose the dynamics of the error to be stable as well as much faster than the open-loop dynamics determined by **A**.

Remarks:

1. We have assumed that **A**, $\mathbf{B}_u$, and $\mathbf{C}_y$ are identical in the physical plant and in the estimator. If we do not have an accurate model of the plant (A, B, C), the dynamics of the error are no longer governed by the previous equation. However, we can typically choose **L** so that the error system is still at least stable, and the error remains acceptably small, even with (small) modeling errors and disturbing inputs.
2. The nature of the plant and the estimator are quite different – the plant is a physical system, whereas **the estimator is usually a digital processor**.

Linear observer – design by duality

The control and the estimation problems are mathematically equivalent. This property is called **duality**.

The control problem is to select the matrix **G** for satisfactory placement of the poles of the system matrix **A** − **BG**.

The estimator problem is to select the matrix **L** for satisfactory placement of the poles of **A** − **LC**.

Since the poles of A−LC are equal to those of $(A-LC)^T$,
the estimator problem is to select the matrix L for satisfactory placement of the poles of $A^T - C^T L^T$.

In this form, **the algebra of the design for L^T is identical to that for G**.

Thus duality allows us to use the same design tools for estimator problems as for control problems with proper substitutions.

Duality

Control	Estimation
A	A^T
B	C^T
C	B^T

Pole-placement

```
G = place(A,B,pc)

LT = place(A',C',pe)
L = LT'
```

Observer design by pole-placement.

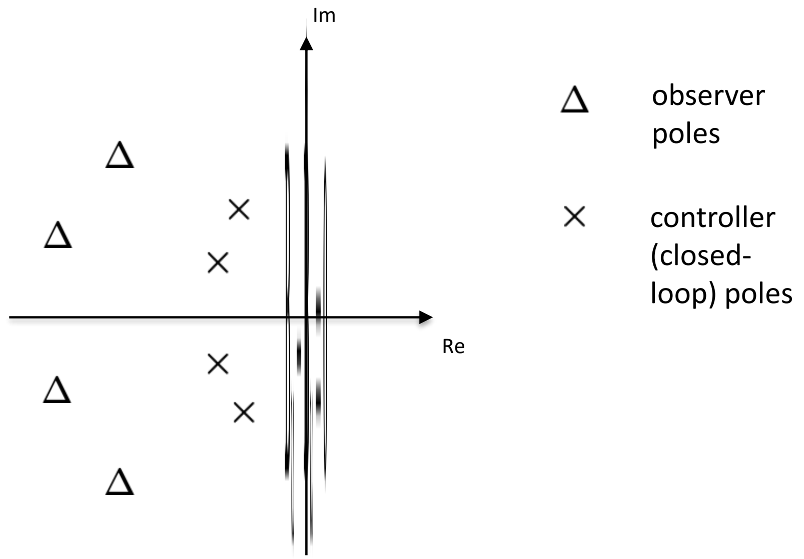
If the system is *fully observable*, it is possible to locate everywhere in the complex plane the observer poles.

```
LT = place(A',C',pe)
L = LT'
```

As a rule of thumb**, the estimator poles can be chosen to be faster than the controller poles by a factor of 2 to 6**.

This ensures a faster decay of the estimator errors compared with the desired dynamics, thus causing the controller poles to dominate the total response.

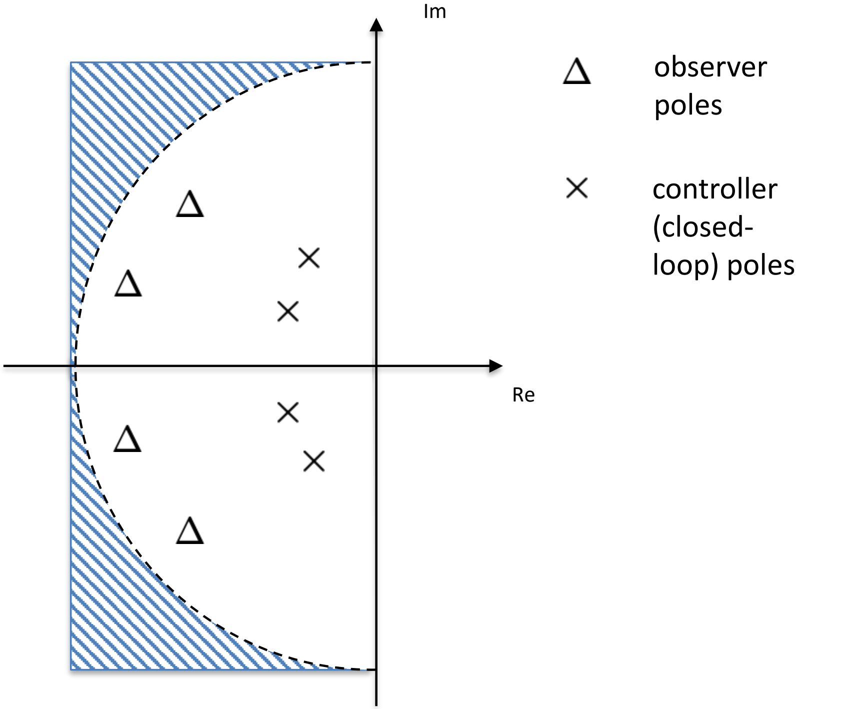
Why not arbitrarily fast?



Why the observer poles are not taken as arbitrarily fast?

The important consequence of increasing the speed of response of an estimator is that the bandwidth of the estimator becomes higher, thus causing **more sensor noise to pass on to the control actuator**.

The best estimator design is a balance between good transient response and low-enough bandwidth that sensor noise does not significantly impair actuator activity.



The separation principle

According to the procedures outlined before, the full-state feedback control has been designed by assuming that the full state vector is available, and the linear observer has been designed without considering the feedback controller.

$$\dot{\mathbf{x}}(t) = (\mathbf{A} - \mathbf{B}_u\mathbf{G})\mathbf{x}(t) \qquad \dot{\mathbf{e}}(t) = (\mathbf{A} - \mathbf{L}\mathbf{C}_y)\,\mathbf{e}(t)$$

We have implicitly assumed that the controller design and the observer design can be carried out separately or independently, in such a way that one design does not affect the other.

Is that true?

The separation principle

The reconstruction error is given by $\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$

and it is governed by the following dynamics $\dot{\mathbf{e}}(t) = (\mathbf{A} - \mathbf{L}\mathbf{C}_y)\,\mathbf{e}(t)$ (1)

The estimated state can be expressed as $\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$

Accordingly, the feedback law is given by

$$\mathbf{u}(t) = -\mathbf{G}\hat{\mathbf{x}}(t) = -\mathbf{G}\mathbf{x}(t) + \mathbf{G}\mathbf{e}(t)$$

→ feedback of the observed state = feedback of the true state + contribution of the obs error.

The closed-loop dynamics is described by

$$\dot{\mathbf{x}}(t) = (\mathbf{A} - \mathbf{B}_u\mathbf{G})\,\mathbf{x}(t) + \overbrace{\mathbf{B}_u\mathbf{G}\,\mathbf{e}(t)}^{\text{Non zero in the initial phase of the system response but in the steady state goes to zero}} \quad (2)$$

By coupling (1) and (2)

$$\begin{Bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{e}}(t) \end{Bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{B}_u\mathbf{G} & \mathbf{B}_u\mathbf{G} \\ \mathbf{0} & \mathbf{A} - \mathbf{L}\mathbf{C}_y \end{bmatrix} \begin{Bmatrix} \mathbf{x}(t) \\ \mathbf{e}(t) \end{Bmatrix}$$

Lucky ⇒ upper Triangular square matrix
The eigenvalues of this matrix are the combination of the eigenvalues of $A - B_uG$ and $A - LC_y$
→ Dynamic of the joined system
⇒ Dynamics of the system obtained + the dynamics of the observer

The separation principle

The characteristic equation is

$$\det\begin{bmatrix} s\mathbf{I}-\mathbf{A}+\mathbf{B}_u\mathbf{G} & -\mathbf{B}_u\mathbf{G} \\ \mathbf{0} & s\mathbf{I}-\mathbf{A}+\mathbf{L}\mathbf{C}_y \end{bmatrix}=0$$

Since the matrix is block triangular, it follows that

$$\det(s\mathbf{I}-\mathbf{A}+\mathbf{B}_u\mathbf{G})\det(s\mathbf{I}-\mathbf{A}+\mathbf{L}\mathbf{C}_y)=0$$

the eigenvalues of the closed-loop system are the eigenvalues of the diagonal blocks

$$(\mathbf{A}-\mathbf{B}_u\mathbf{G}) \quad (\mathbf{A}-\mathbf{L}\mathbf{C}_y)$$

As a result, the poles of the controller are not affected by the poles of the observer (also the opposite is true) when the two subsystems are put together and the controller design and the observer design can be carried out independently.

This is known as **separation principle**, which assures that the poles of the closed-loop dynamic system will be the poles of the full-state feedback system and those selected for the state estimation.

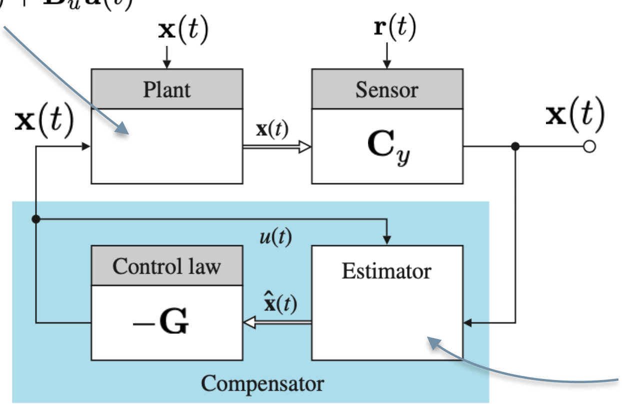
Control system design – state-space methods
Compensator

The combination of the state feedback controller and the observer gives rise to the so-called **compensator**.

Feedback control law $\mathbf{u}(t) = -\mathbf{G}\hat{\mathbf{x}}(t)$

Substituting this into the equation of the observer $\dot{\hat{\mathbf{x}}}(t) = (\mathbf{A} - \mathbf{L}\mathbf{C}_y)\,\hat{\mathbf{x}}(t) + \mathbf{B}_u\mathbf{u}(t) + \mathbf{L}\mathbf{y}(t)$

$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}_u\mathbf{u}(t)$



$$\dot{\hat{\mathbf{x}}}(t) = (\mathbf{A} - \mathbf{L}\mathbf{C}_y)\,\hat{\mathbf{x}}(t) + \mathbf{B}_u\mathbf{u}(t) + \mathbf{L}\mathbf{y}(t)$$

yields

$$\dot{\hat{\mathbf{x}}}(t) = (\mathbf{A} - \mathbf{L}\mathbf{C}_y - \mathbf{B}_u\mathbf{G})\,\hat{\mathbf{x}}(t) + \mathbf{L}\mathbf{y}(t)$$
$$\mathbf{u}(t) = -\mathbf{G}\hat{\mathbf{x}}(t)$$

Compensator

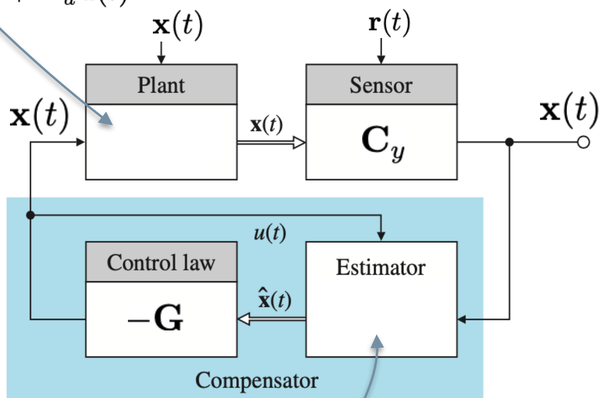
Compensator (time-domain representation):

We can guarantee the stability of this part

$$\dot{\hat{\mathbf{x}}}(t) = (\underbrace{\mathbf{A} - \mathbf{LC}_y}_{\text{We have no guarantee about the stability of the compensator}} - \mathbf{B}_u\mathbf{G})\,\hat{\mathbf{x}}(t) + \mathbf{Ly}(t)$$

$$\mathbf{u}(t) = -\mathbf{G}\hat{\mathbf{x}}(t)$$

$$\dot{\mathbf{x}}(t) = \mathbf{Ax}(t) + \mathbf{B}_u\mathbf{u}(t)$$



$$\dot{\hat{\mathbf{x}}}(t) = (\mathbf{A} - \mathbf{LC}_y)\,\hat{\mathbf{x}}(t) + \mathbf{B}_u\mathbf{u}(t) + \mathbf{Ly}(t)$$

Remark

The characteristic equation of the compensator is given by

$$\det\left(s\mathbf{I} - \mathbf{A} + \mathbf{LC}_y + \mathbf{B}_u\mathbf{G}\right) = 0 \qquad (1)$$

Note that we never specified the roots of Eq. (1) nor used them in our discussion of the state-space design technique.

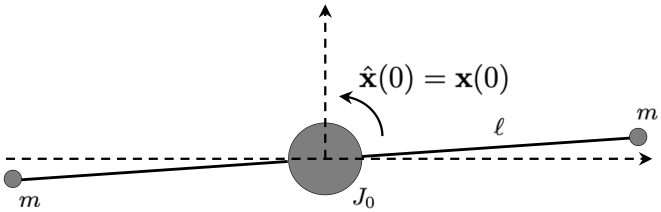
Note also that the compensator is not guaranteed to be stable; the roots of Eq. (1) can be in the right-half plane.

But the unstable compensator is made stable by y(t)

Either one could be unstable but they are made stable by appropriate input.

Example

Design a compensator for a rigid spacecraft (J = 100 kg m^2) subjected to an **impulse disturbance** of amplitude 1 N m.



Open-loop behavior

Equation of motion: $$J\ddot{\theta}(t) = \delta(t)$$

State-space representation: $$\mathbf{x}(t) = \begin{Bmatrix} \theta(t) \\ \dot{\theta}(t) \end{Bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{b}_d = \begin{bmatrix} 0 \\ 1/J \end{bmatrix} \qquad \mathbf{b}_u = \begin{bmatrix} 0 \\ 1/J \end{bmatrix}$$

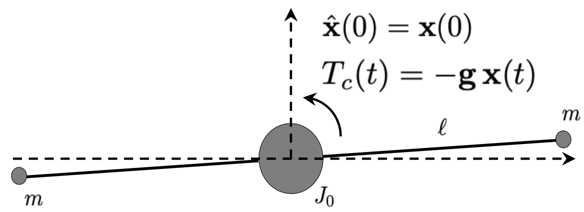
$$\mathbf{c}_y = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1/J \end{bmatrix} \delta(t)$$

$$\mathbf{y}(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(t)$$

Compensator - example

Example – rigid spacecraft subjected to an impulse disturbance



Controller design

Equation of motion: $$J\ddot{\theta}(t) = T_c(t) + \delta(t)$$

State feedback control law: $$\mathbf{u}(t) = -\mathbf{G}\hat{\mathbf{x}}(t)$$

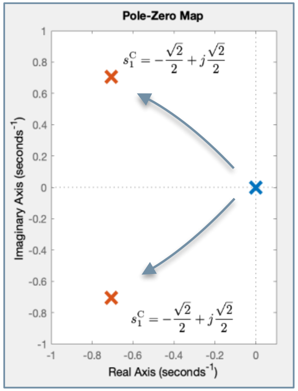
Desired closed-loop poles

$$\omega = 1\,\mathrm{rad/s}$$

$$\xi = 0.7$$

$$s_1^{\mathrm{C}} = -\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}$$

$$s_2^{\mathrm{C}} = -\frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2}$$



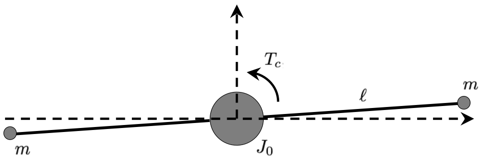
Control gain vector (through pole-placement) $$\mathbf{g} = 100\,[1 \quad \sqrt{2}]$$

```
% pole placement
g = place(A,b_u,sC);
```

Alredy seen in another lesson.

Control system design – state-space methods
Compensator - example

Example – rigid spacecraft subjected to an impulse disturbance



Observer design

Desired observer poles

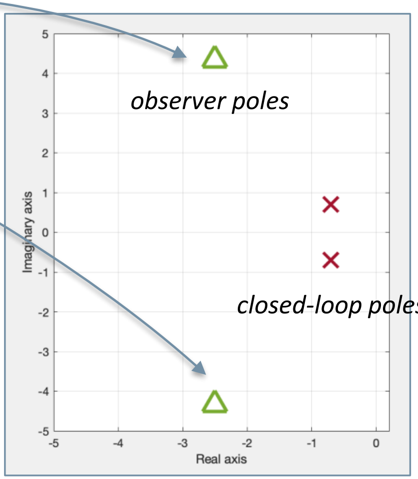
as a first guess

$$\omega = 1\,\mathrm{rad/s}$$

$$\xi = 0.5$$

$$s_1^O = -\frac{5}{2} + j5\sqrt{\frac{3}{4}}$$

$$s_2^O = -\frac{5}{2} - j5\sqrt{\frac{3}{4}}$$



Observer gain vector
(through pole-placement)

$$\mathbf{l} = \begin{bmatrix} 5 \\ 25 \end{bmatrix}$$

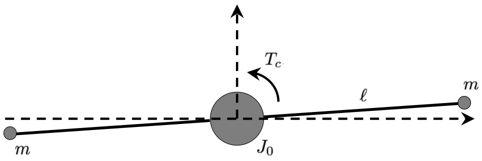
MATLAB code:

```
ome_0 = 5;  % rad/s
xi_0 = 0.5;
Re_s0 = -xi_0*ome_0;
Im_s0 = ome_0*sqrt(1-xi_0^2);
s0_1 = Re_s0 + 1i*Im_s0;
s0_2 = Re_s0 - 1i*Im_s0;
s0 = [s0_1 s0_2];

% pole placement
lT = place(A',c_y',s0);
l = lT';
```

Compensator - example

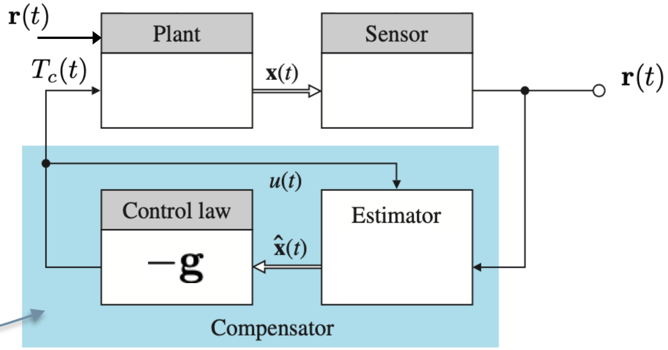
Example – rigid spacecraft subjected to an impulse disturbance



System:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}_u T_c(t) + \mathbf{b}_d \delta(t)$$

$$y(t) = \mathbf{c}_y \mathbf{x}(t)$$

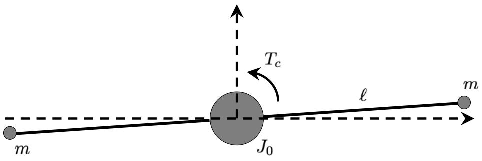


Compensator (time-domain):

$$\dot{\hat{\mathbf{x}}}(t) = (\mathbf{A} - \mathbf{l}\mathbf{c}_y - \mathbf{b}_u \mathbf{g})\,\hat{\mathbf{x}}(t) + \mathbf{l}y(t)$$

$$T_c(t) = -\mathbf{g}\,\hat{\mathbf{x}}(t)$$

Example – rigid spacecraft subjected to an impulse disturbance



$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}_u T_c(t) + \mathbf{b}_d \delta(t)$$
$$y(t) = \mathbf{c}_y \mathbf{x}(t)$$

$$\dot{\hat{\mathbf{x}}}(t) = (\mathbf{A} - \mathbf{l}\mathbf{c}_y - \mathbf{b}_u \mathbf{g})\,\hat{\mathbf{x}}(t) + \mathbf{l}y(t)$$
$$T_c(t) = -\mathbf{g}\,\hat{\mathbf{x}}(t)$$

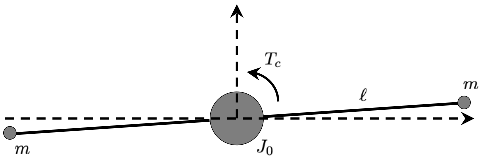
$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) - \mathbf{b}_u \mathbf{g}\,\hat{\mathbf{x}}(t) + \mathbf{b}_d \delta(t)$$
$$\dot{\hat{\mathbf{x}}}(t) = (\mathbf{A} - \mathbf{l}\mathbf{c}_y - \mathbf{b}_u \mathbf{g})\,\hat{\mathbf{x}}(t) + \mathbf{l}\mathbf{c}_y \mathbf{x}(t)$$
$$y(t) = \mathbf{c}_y \mathbf{x}(t)$$

Let's define:

$$\mathbf{x}_{\mathrm{aug}}(t) = \begin{Bmatrix} \mathbf{x}(t) \\ \hat{\mathbf{x}}(t) \end{Bmatrix}$$

$$\dot{\mathbf{x}}_{\mathrm{aug}}(t) = \mathbf{A}_{\mathrm{aug}} \mathbf{x}_{\mathrm{aug}}(t) + \mathbf{b}_{\mathrm{aug}} \delta(t)$$
$$y(t) = \mathbf{c}_{\mathrm{aug}} \mathbf{x}_{\mathrm{aug}}(t)$$

Example – rigid spacecraft subjected to an impulse disturbance

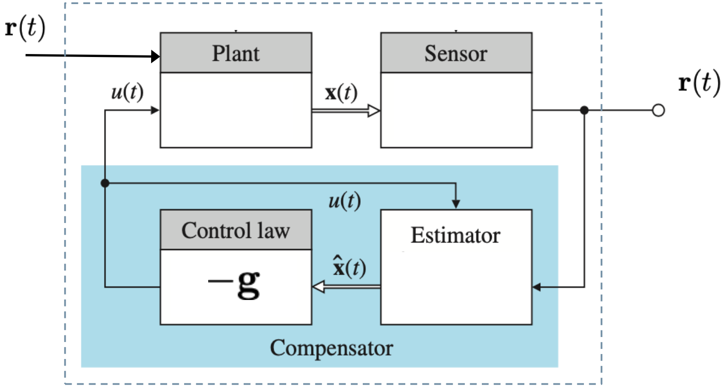


$$\mathbf{x}_{\text{aug}}(t) = \begin{Bmatrix} \mathbf{x}(t) \\ \hat{\mathbf{x}}(t) \end{Bmatrix}$$

$$\dot{\mathbf{x}}_{\text{aug}}(t) = \mathbf{A}_{\text{aug}}\mathbf{x}_{\text{aug}}(t) + \mathbf{b}_{\text{aug}}\delta(t)$$
$$y(t) = \mathbf{c}_{\text{aug}}\mathbf{x}_{\text{aug}}(t)$$

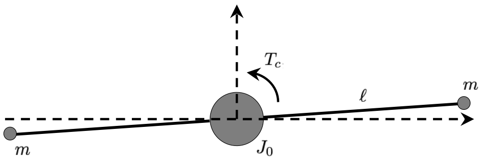
$$\mathbf{A}_{\text{aug}} = \begin{bmatrix} \mathbf{A} & -\mathbf{b}_u\mathbf{g} \\ \mathbf{l}\mathbf{c}_y & \mathbf{A} - \mathbf{l}\mathbf{c}_y - \mathbf{b}_u\mathbf{g} \end{bmatrix}$$

$$\mathbf{b}_{\text{aug}} = \begin{bmatrix} \mathbf{b}_d \\ \mathbf{0} \end{bmatrix} \qquad \mathbf{c}_{\text{aug}} = \begin{bmatrix} \mathbf{c}_y & \mathbf{0} \end{bmatrix}$$



Control system design – state-space methods
Compensator - example

Example – rigid spacecraft subjected to an impulse disturbance

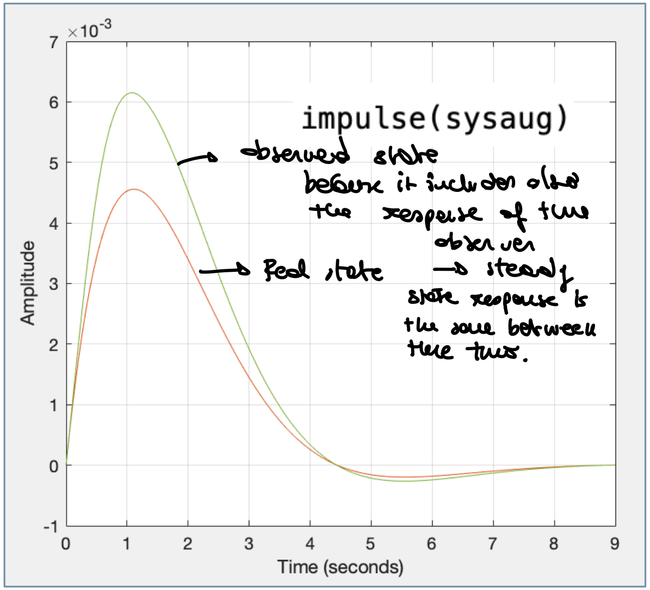


$$\dot{\mathbf{x}}_{\text{aug}}(t) = \mathbf{A}_{\text{aug}}\mathbf{x}_{\text{aug}}(t) + \mathbf{b}_{\text{aug}}\delta(t)$$

$$y(t) = \mathbf{c}_{\text{aug}}\mathbf{x}_{\text{aug}}(t)$$

$$\mathbf{A}_{\text{aug}} = \begin{bmatrix} \mathbf{A} & -\mathbf{b}_u\mathbf{g} \\ \mathbf{l}\mathbf{c}_y & \mathbf{A} - \mathbf{l}\mathbf{c}_y - \mathbf{b}_u\mathbf{g} \end{bmatrix}$$

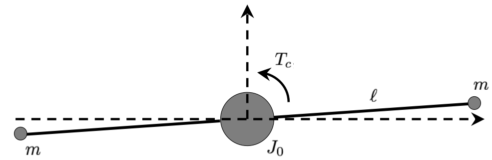
$$\mathbf{b}_{\text{aug}} = \begin{bmatrix} \mathbf{b}_d \\ \mathbf{0} \end{bmatrix} \qquad \mathbf{c}_{\text{aug}} = \begin{bmatrix} \mathbf{c}_y & \mathbf{0} \end{bmatrix}$$



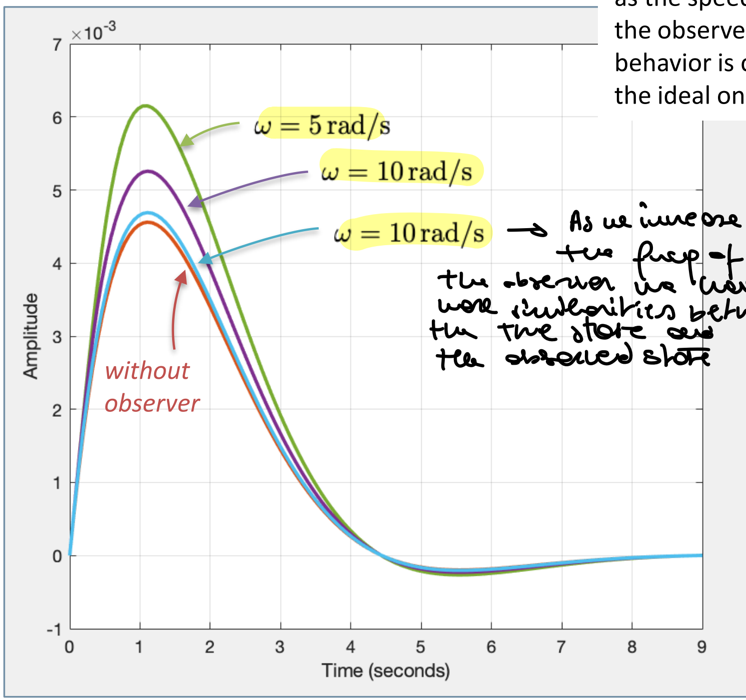
MATLAB code:

```
% augmented system (plant + state observer)
sysaug = ss([A -b_u*g; l*c_y A-l*c_y-b_u*g], ...
            [b_d ; zeros(2,1)], ...
            [c_y zeros(1,2)], 0);
```

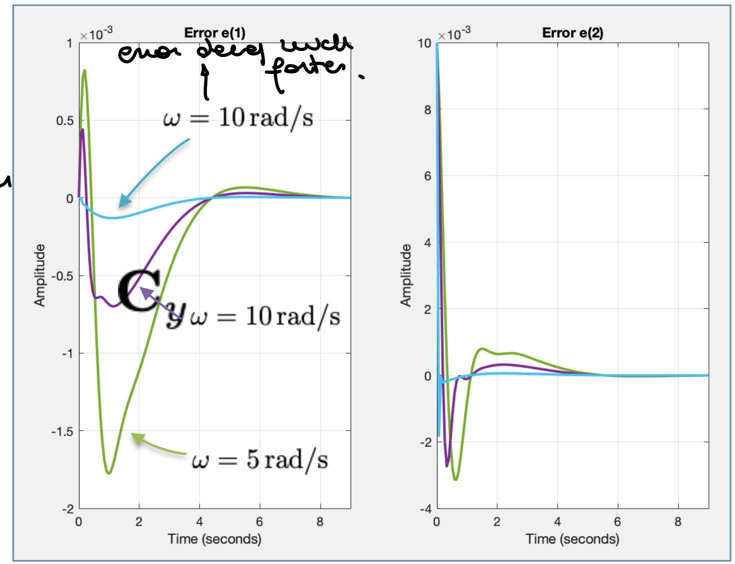
Example – rigid spacecraft subjected to an impulse disturbance



as the speed of the response of the observer increases, the behavior is closer and closer to the ideal one…

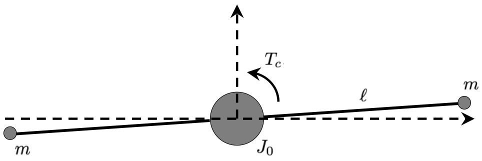


As we increase the freq of the observer we have more similarities between the true state and the observed state



error decay much faster.

Example – rigid spacecraft subjected to an impulse disturbance



as the speed of the response of the observer increases, the behavior is closer and closer to the ideal one…

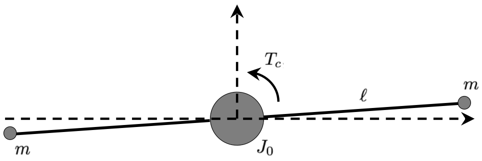
…however, take care about <u>measurement noise</u>

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}_u T_c(t) + \mathbf{b}_d \delta(t)$$
$$y(t) = \mathbf{c}_y \mathbf{x}(t)$$

$$\dot{\hat{\mathbf{x}}}(t) = (\mathbf{A} - \mathbf{l}\mathbf{c}_y - \mathbf{b}_u \mathbf{g})\,\hat{\mathbf{x}}(t) + \mathbf{l}y(t)$$
$$T_c(t) = -\mathbf{g}\,\hat{\mathbf{x}}(t)$$

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) - \mathbf{b}_u \mathbf{g}\,\hat{\mathbf{x}}(t) + \mathbf{b}_d \delta(t)$$
$$\dot{\hat{\mathbf{x}}}(t) = (\mathbf{A} - \mathbf{l}\mathbf{c}_y - \mathbf{b}_u \mathbf{g})\,\hat{\mathbf{x}}(t) + \mathbf{l}\mathbf{c}_y \mathbf{x}(t) + \mathbf{l}\,n(t)$$

Example – rigid spacecraft subjected to an impulse disturbance



as the speed of the response of the observer increases, the behavior is closer and closer to the ideal one…

…however, take care about <u>measurement noise</u>

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) - \mathbf{b}_u\mathbf{g}\,\hat{\mathbf{x}}(t) + \mathbf{b}_d\delta(t)$$

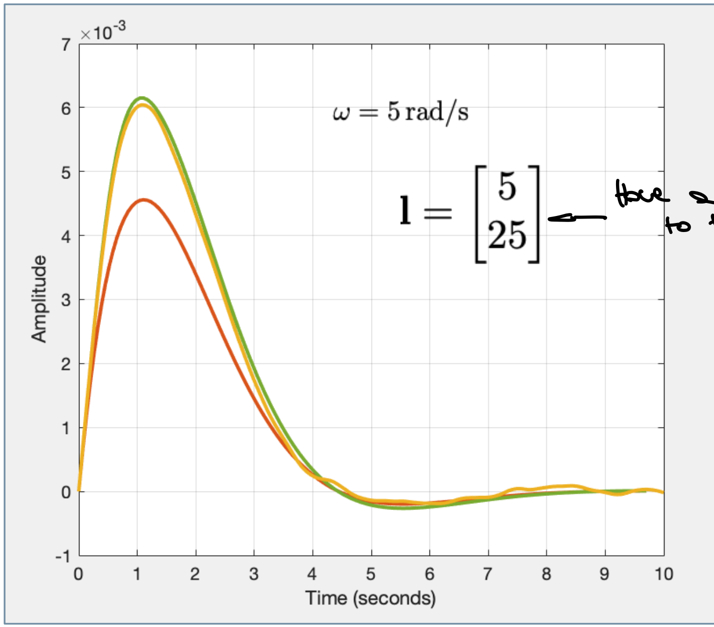
$$\dot{\hat{\mathbf{x}}}(t) = (\mathbf{A} - \mathbf{l}\mathbf{c}_y - \mathbf{b}_u\mathbf{g})\,\hat{\mathbf{x}}(t) + \mathbf{l}\mathbf{c}_y\mathbf{x}(t) + \mathbf{l}\,n(t)$$

$$\begin{Bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\hat{\mathbf{x}}}(t) \end{Bmatrix} = \begin{bmatrix} \mathbf{A} & -\mathbf{b}_u\mathbf{g} \\ \mathbf{l}\mathbf{c}_y & \mathbf{A} - \mathbf{l}\mathbf{c}_y - \mathbf{b}_u\mathbf{g} \end{bmatrix} \begin{Bmatrix} \mathbf{x}(t) \\ \hat{\mathbf{x}}(t) \end{Bmatrix} + \begin{bmatrix} \mathbf{b}_d \\ \mathbf{0} \end{bmatrix} \delta(t) + \begin{bmatrix} \mathbf{0} \\ \mathbf{l} \end{bmatrix} n(t)$$

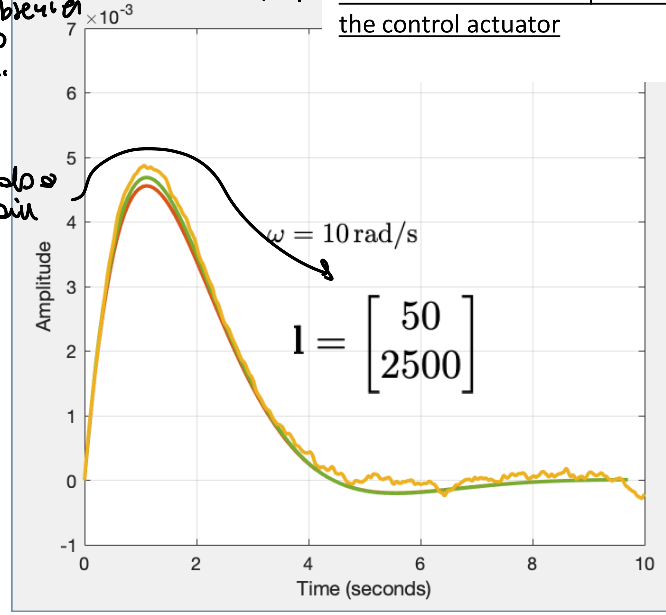
Example – rigid spacecraft subjected to an impulse disturbance

Not good practice to increase too much the frequency of the observer due to noise.

as the speed of the response of the observer increases, <u>more measurement noise is passed to the control actuator</u>



Here a look also to the gain



Spillover effects

→ Effect of the unmodelled dynamics on the system when we have already modelled the control system. What we are not modelling has a freq. content above what we are modelling → system dynamics in reality will be pretty close so control system will have good performance.
If we have unmodelled contributions with the same frequency content close to the modelled system we will have a big impact and the control system will have non satisfactory performances

We learned that the closed-loop poles of a system using a compensator designed by the separation principle has its poles at the poles of the observer and at the poles of the full-state feedback control system.

→ This is the worst case scenario considering the unmodelled behaviours are completely unknown.

Since both sets of poles have been selected by the designer,
we can assume that their location is favourable to overall system operation.

We made use of the fact that the observer design includes an exact dynamic model of the plant.

There are several reasons why it is impractical to assume that the dynamic model of the plant is exact.

The physics of the plant may be understood only approximately
(modelling errors)

or

The exact dynamics may be known but too complicated to include in the control system design
(discarded dynamics)

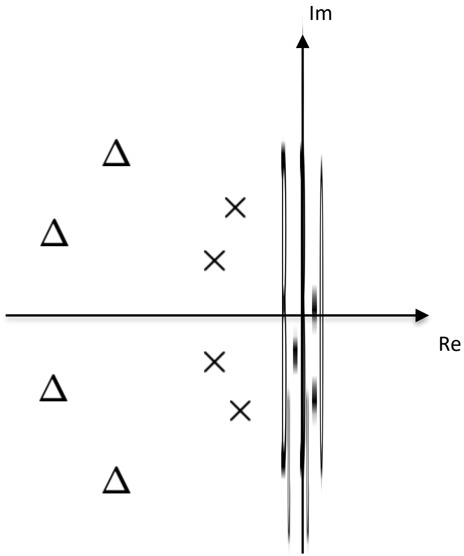
Spillover effects

The best that the designer can do is to design a compensator on the basis of a nominal plant model, i.e., a plant model defined for purpose of design.

Since the true plant will (almost) never be the same as the nominal plant, the closed-loop poles will (almost) never be located in the exact locations intended.

If their actual locations are not far from their intended locations when the actual plant does not differ greatly from the nominal plant, the nominal design will probably be satisfactory.

If, on the other hand, a small change in the plant causes a large change in the closed-loop pole locations-perhaps going so far as to move them into the right half-plane-then the nominal design will surely be unsatisfactory.



Effect of modelling errors

Observer designed on the nominal plant:

$$\dot{\hat{\mathbf{x}}}(t) = (\mathbf{A} - \mathbf{L}\mathbf{C}_y)\,\hat{\mathbf{x}}(t) + \mathbf{B}_u\mathbf{u}(t) + \mathbf{L}\mathbf{y}(t)$$

Variations in the state and output matrix:

$$\mathbf{A} \quad \rightarrow \quad \mathbf{A} + \delta\mathbf{A}$$
$$\mathbf{C}_y \quad \rightarrow \quad \mathbf{C}_y + \delta\mathbf{C}$$

True system dynamics:

$$\dot{\mathbf{x}}(t) = (\mathbf{A} + \delta\mathbf{A})\,\mathbf{x}(t) + \mathbf{B}_u\mathbf{u}(t)$$
$$\mathbf{y}(t) = (\mathbf{C}_y + \delta\mathbf{C})\,\mathbf{x}(t)$$

True estimation error:

$$\begin{aligned}\dot{\mathbf{e}}(t) &= \dot{\mathbf{x}}(t) - \dot{\hat{\mathbf{x}}}(t)\\ &= (\mathbf{A} + \delta\mathbf{A})\,\mathbf{x}(t) + \mathbf{B}_u\mathbf{u}(t) - (\mathbf{A} - \mathbf{L}\mathbf{C}_y)\,\hat{\mathbf{x}}(t) - \mathbf{B}_u\mathbf{u}(t) - \mathbf{L}\mathbf{y}(t)\\ &= (\mathbf{A} + \delta\mathbf{A})\,\mathbf{x}(t) + (\mathbf{A} - \mathbf{L}\mathbf{C}_y)\,(\mathbf{e}(t) - \mathbf{x}(t)) - \mathbf{L}\,(\mathbf{C}_y + \delta\mathbf{C})\,\mathbf{x}(t)\end{aligned}$$

Spillover effects

Effect of modelling errors

$$\mathbf{A} \quad \rightarrow \quad \mathbf{A} + \delta\mathbf{A}$$
$$\mathbf{C}_y \quad \rightarrow \quad \mathbf{C}_y + \delta\mathbf{C}$$

True estimation error:

$$\dot{\mathbf{e}}(t) = (\mathbf{A} - \mathbf{L}\mathbf{C}_y)\,\mathbf{e}(t) + (\delta\mathbf{A} - \mathbf{L}\delta\mathbf{C})\,\mathbf{x}(t) \qquad (1)$$

The feedback law is given by

$$\mathbf{u}(t) = -\mathbf{G}\hat{\mathbf{x}}(t) = -\mathbf{G}\mathbf{x}(t) + \mathbf{G}\mathbf{e}(t)$$

The closed-loop dynamics is described by

$$\dot{\mathbf{x}}(t) = (\mathbf{A} + \delta\mathbf{A} - \mathbf{B}_u\mathbf{G})\,\mathbf{x}(t) + \mathbf{B}_u\mathbf{G}\,\mathbf{e}(t) \qquad (2)$$

By coupling (1) and (2)

The design of the control system can be done indipendently from the state estimator → No longer true because of this terms

$$\begin{Bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{e}}(t) \end{Bmatrix} = \begin{bmatrix} \mathbf{A} + \delta\mathbf{A} - \mathbf{B}_u\mathbf{G} & \mathbf{B}_u\mathbf{G} \\ \boxed{\delta\mathbf{A} - \mathbf{L}\delta\mathbf{C}} & \mathbf{A} - \mathbf{L}\mathbf{C}_y \end{bmatrix} \begin{Bmatrix} \mathbf{x}(t) \\ \mathbf{e}(t) \end{Bmatrix}$$

Has an impact on the close loop poles of the system.

Same logic used before

Generally δA or δC are unknown

because of the presence of this term, the closed-loop poles are no longer at the design positions but at other locations that depend on $\delta\mathbf{A} - \mathbf{L}\delta\mathbf{C}$

Effect of modelling errors $\mathbf{B}_u \quad \rightarrow \quad \mathbf{B}_u + \delta\mathbf{B}$

Homework: show that the separation principle is not valid.
(hint: follow similar steps as before)

Effect of residual dynamics → We have similar effect when we look at the residual dynamics → We know they exist but we do not include them in the design phase.

Nominal dynamics
(included into the design model)

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}_u\mathbf{u}(t)$$

Residual dynamics
(not included into the design model)

$$\dot{\mathbf{x}}_r(t) = \mathbf{A}_r\mathbf{x}_r(t) + \mathbf{B}_{ur}\mathbf{u}(t)$$

Output equation:

$$\mathbf{y}(t) = \mathbf{C}_y\mathbf{x}(t) + \mathbf{C}_{yr}\mathbf{x}_r(t)$$

Observer designed on the nominal plant:

$$\dot{\hat{\mathbf{x}}}(t) = (\mathbf{A} - \mathbf{L}\mathbf{C}_y)\,\hat{\mathbf{x}}(t) + \mathbf{B}_u\mathbf{u}(t) + \mathbf{L}\mathbf{y}(t)$$

Effect of residual dynamics

True estimation error:

$$
\begin{aligned}
\dot{\mathbf{e}}(t) &= \dot{\mathbf{x}}(t) - \dot{\hat{\mathbf{x}}}(t) \\
&= \mathbf{A}\mathbf{x}(t) + \mathbf{B}_u\mathbf{u}(t) - (\mathbf{A} - \mathbf{L}\mathbf{C}_y)(\mathbf{x}(t) - \mathbf{e}(t)) \\
&\quad - \mathbf{B}_u\mathbf{u}(t) - \mathbf{L}(\mathbf{C}_y\mathbf{x}(t) + \mathbf{C}_{yr}\mathbf{x}_r(t)) \\
&= (\mathbf{A} - \mathbf{L}\mathbf{C}_y)\,\mathbf{e}(t) + \mathbf{L}\mathbf{C}_{yr}\mathbf{x}_r(t) \qquad (1)
\end{aligned}
$$

The feedback law is given by

$$\mathbf{u}(t) = -\mathbf{G}\hat{\mathbf{x}}(t) = -\mathbf{G}\mathbf{x}(t) + \mathbf{G}\mathbf{e}(t)$$

Closed-loop nominal dynamics

$$\dot{\mathbf{x}}(t) = (\mathbf{A} - \mathbf{B}_u\mathbf{G})\,\mathbf{x}(t) + \mathbf{B}_u\mathbf{G}\,\mathbf{e}(t) \qquad (2)$$

Closed-loop residual dynamics
(the control acts also on this)

$$\dot{\mathbf{x}}_r(t) = \mathbf{A}_r\mathbf{x}_r(t) - \mathbf{B}_{ur}\mathbf{G}\,\mathbf{x}(t) + \mathbf{B}_{ur}\mathbf{G}\,\mathbf{e}(t) \qquad (3)$$

↳ residual dynamics affected by the control
→ make sure it is not disturbed too much due to the nominal control

Effect of residual dynamics

By coupling (1), (2) and (3)

Meta system

↳ No more block diagonal due to this contributions to have an effect on the overall system dynamics.

observation spillover

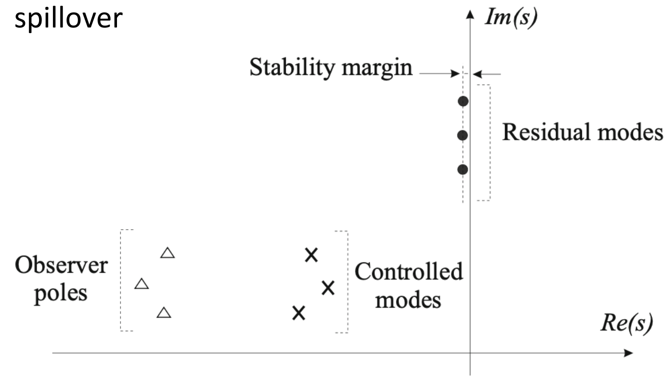
$$\begin{Bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{e}}(t) \\ \dot{\mathbf{x}}_r(t) \end{Bmatrix} = \begin{bmatrix} \mathbf{A}-\mathbf{B}_u\mathbf{G} & \mathbf{B}_u\mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}-\mathbf{L}\mathbf{C}_y & -\mathbf{L}\mathbf{C}_{yr} \\ -\mathbf{B}_{ur}\mathbf{G} & \mathbf{B}_{ur}\mathbf{G} & \mathbf{A}_r \end{bmatrix} \begin{Bmatrix} \mathbf{x}(t) \\ \mathbf{e}(t) \\ \mathbf{x}_r(t) \end{Bmatrix}$$

control spillover

It is clearly observed that the above system matrix is neither block diagonal nor block triangular.

This implies that the poles of the closed-loop system cannot be considered as the union of the poles of the controller, the poles of the observer and the poles associated with the residual modes.

The poles of the controller and the observer are affected by the residual dynamics.



GUIDELINES FOR SELECTING WEIGHTING MATRICES IN LQR

Selection of weighting matrices in LQR

LQR is based on the full state feedback control

$$\mathbf{u}(t) = -\mathbf{G}\mathbf{x}(t)$$

where the gain matrix **G** is selected such that the following quadratic cost functional (or performance index) is minimized

$$J = \frac{1}{2}\int_0^{\infty} \left(\mathbf{z}^T \mathbf{W}_{zz}\mathbf{z} + \mathbf{u}^T \mathbf{W}_{uu}\mathbf{u}\right) dt$$

Minimization of J will depend on the selection of the weighting matrices $\mathbf{W}_{zz}$ and $\mathbf{W}_{uu}$ – the corresponding solution will then be strongly affected by the weighting matrices, which therefore play a fundamental role in the design process.

Different techniques for the choice of $\mathbf{W}_{zz}$ and $\mathbf{W}_{uu}$

1. maximum values
2. prescribed degree of stability
3. implicit model following
4. spillover reduction
5. sensitivity-weighted LQR
6. frequency-shaped LQR

1. Maximum values

Cost function: $$J = \frac{1}{2}\int_0^{\infty} \left(\mathbf{z}^T \mathbf{W}_{zz}\mathbf{z} + \mathbf{u}^T \mathbf{W}_{uu}\mathbf{u}\right) dt$$

→ Cost function becomes non dimensional with this logic.

Weighting matrices: $$\mathbf{W}_{zz} = \mathrm{Diag}\left\{\frac{1}{z_{i\max}^2}\right\} \qquad \mathbf{W}_{uu} = \mathrm{Diag}\left\{\frac{1}{u_{i\max}^2}\right\}$$

↳ Ideal maximum acceptable

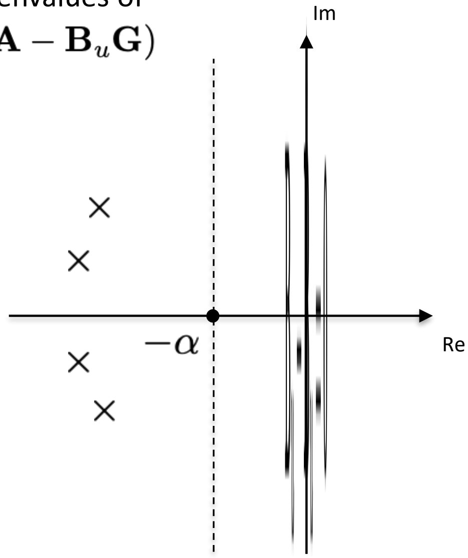
2. Prescribed degree of stability

Let's refer to the cost function J expressed as follows (we know that we can always transform the problem to reach this form)

$$J = \frac{1}{2}\int_0^{\infty} \left(\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u}\right) dt$$

This procedure allows to design an optimal controller with a prescribed degree of stability – i.e., **the closed-loop poles are guaranteed of lying on the left of a vertical line identified by the parameter** α.

eigenvalues of $(\mathbf{A} - \mathbf{B}_u\mathbf{G})$



2. Prescribed degree of stability

The idea is to modify the cost function as follows

exponential terms

$$J = \frac{1}{2}\int_0^\infty e^{2\alpha t}\left(\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u}\right) dt \quad \text{if } (\alpha > 0) \rightarrow$$

J over time is exponentially growing

if I am capable of obtaining J = minimum

→ The state of the system is decaying faster than the expont α.

$$J = \frac{1}{2}\int_0^\infty \left(e^{\alpha t}\mathbf{x}^T\mathbf{Q}\mathbf{x}e^{\alpha t} + e^{\alpha t}\mathbf{u}^T\mathbf{R}\mathbf{u}e^{\alpha t}\right) dt$$

Define

→ modified state

$$\hat{\mathbf{x}} = \mathbf{x}e^{\alpha t}$$

$$\hat{\mathbf{u}} = \mathbf{u}e^{\alpha t}$$

so that

$$J = \frac{1}{2}\int_0^\infty \left(\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u}\right) dt$$

2. Prescribed degree of stability

Note that the modified cost function

$$J = \frac{1}{2}\int_0^{\infty} \left(\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u}\right) dt$$

affected negatively by the term: $\alpha \longrightarrow$ I am trying to make a system that is destabilized by α stable again.

is associated with the following dynamics

$$\dot{\hat{\mathbf{x}}}(t) = (\mathbf{A} + \alpha\mathbf{I})\,\hat{\mathbf{x}}(t) + \mathbf{B}_u\hat{\mathbf{u}}(t)$$

Proof:

$$\hat{\mathbf{x}} = \mathbf{x}e^{\alpha t}$$

$$\hat{\mathbf{u}} = \mathbf{u}e^{\alpha t}$$

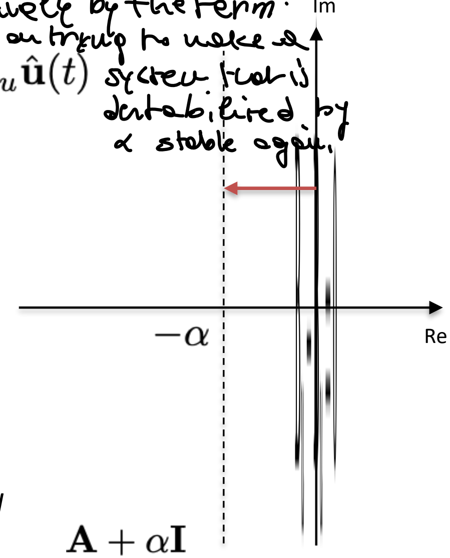
$$\begin{aligned}
\dot{\hat{\mathbf{x}}}(t) &= \dot{\mathbf{x}}(t)e^{\alpha t} + \mathbf{x}(t)\alpha e^{\alpha t} \\
&= (\mathbf{A}\mathbf{x} + \mathbf{B}_u\mathbf{u})\,e^{\alpha t} + \mathbf{x}(t)\alpha e^{\alpha t} \\
&= (\mathbf{A} + \alpha\mathbf{I})\,\mathbf{x}e^{\alpha t} + \mathbf{B}_u\mathbf{u}e^{\alpha t} \\
&= (\mathbf{A} + \alpha\mathbf{I})\,\hat{\mathbf{x}}(t) + \mathbf{B}_u\hat{\mathbf{u}}(t)
\end{aligned}$$

The optimal solution is given by:

$$\mathbf{G} = \mathbf{R}^{-1}\mathbf{B}_u^T\mathbf{P}$$

$$\mathbf{P}\left(\mathbf{A} + \alpha\mathbf{I}\right) + \left(\mathbf{A}^T + \alpha\mathbf{I}\right)\mathbf{P}$$

$$+\,\mathbf{Q} - \mathbf{P}\mathbf{B}_u\mathbf{R}^{-1}\mathbf{B}_u^T\mathbf{P} = \mathbf{0}$$



the prescribed degree of stability is achieved by solving a classical LQR problem for the modified system with state matrix $\mathbf{A} + \alpha\mathbf{I}$

2. Prescribed degree of stability - example

Let’s consider an LTI system having the following dynamics $\ddot{\theta}(t) = u(t)$

Compute the gain matrix (vector) of an LQR controller to achieve a prescribed degree of stability of α when $\mathbf{Q} = \mathbf{0}_{2x2}$ and R = 1.

State-space model:

$$\mathbf{x}(t) = \begin{Bmatrix} \theta(t) \\ \dot{\theta}(t) \end{Bmatrix} \qquad \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}_u u(t)$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \mathbf{b}_u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Weighting matrices of the LQR controller:

$$\mathbf{b}_u = \begin{bmatrix} 0 \\ 1/J \end{bmatrix} \qquad R = 1$$

2. Prescribed degree of stability - example

Shifted-**A** matrix: $$\mathbf{A}+\alpha\mathbf{I}=\begin{bmatrix}\alpha & 1\\ 0 & \alpha\end{bmatrix}$$

The Riccati equation:

$$\mathbf{P}\left(\mathbf{A}+\alpha\mathbf{I}\right)+\left(\mathbf{A}^T+\alpha\mathbf{I}\right)\mathbf{P}-\mathbf{P}\mathbf{b}_u\mathbf{b}_u^T\mathbf{P}=\mathbf{0} \quad \Rightarrow \quad \mathbf{P}=\begin{bmatrix}8\alpha^3 & 4\alpha^2\\ 4\alpha^2 & 4\alpha\end{bmatrix}$$

Gain matrix: $$\mathbf{g}=\mathbf{b}_u^T\mathbf{P}=\begin{bmatrix}4\alpha^2 & 4\alpha\end{bmatrix}$$

Closed-loop eigenvalues: $$\det\left(s\mathbf{I}-\mathbf{A}+\mathbf{b}_u\mathbf{g}\right)=\det\begin{bmatrix}s & -1\\ 4\alpha^2 & s+4\alpha\end{bmatrix}=0$$

$$\Downarrow$$

$$p_{1,2}^{\mathrm{C}}=-2\alpha$$

3. Implicit model following

Let's assume that **we want to have a closed-loop dynamics as similar as possible to a desired dynamics** (i.e., follow a prescribed model)

open-loop dynamics:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}_u\mathbf{u}(t)$$

desired closed-loop dynamics:

desired closed loop dynamics → What I can do is to define the close loop dynamic that I want.

$$\hat{\mathbf{y}}(t) = \mathbf{C}_y\hat{\mathbf{x}}(t)$$

The performance is defined as the difference between the two dynamics

$$\mathbf{z}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}_u\mathbf{u}(t) - \mathbf{A}_d\mathbf{x}(t)$$
$$= (\mathbf{A} - \mathbf{A}_d)\,\mathbf{x}(t) + \mathbf{B}_u\mathbf{u}(t)$$

The cost function is taken as

$$J = \frac{1}{2}\int_0^\infty \mathbf{z}^T\mathbf{z}\,dt$$

→ minimize the cost function.

3. Implicit model following

Substituting the performance into the cost function yields

$$J = \frac{1}{2}\int_0^{\infty} \left(\mathbf{x}^T\mathbf{Q}\mathbf{x} + 2\mathbf{x}^T\mathbf{S}\mathbf{u} + \mathbf{u}^T\mathbf{R}\mathbf{u}\right) dt$$

$$\mathbf{Q} = \left(\mathbf{A} - \mathbf{A}_d\right)^T \left(\mathbf{A} - \mathbf{A}_d\right)$$
$$\mathbf{S} = \left(\mathbf{A} - \mathbf{A}_d\right)^T \mathbf{B}_u$$
$$\mathbf{R} = \mathbf{B}_u^T\mathbf{B}_u$$

the implicit model following is achieved by solving a classical LQR problem with the above weighting matrices

3. Implicit model following - example

Let's consider an LTI system having the following dynamics $\ddot{\theta}(t) = u(t)$

Compute the weighting matrices to be used in the design of an LQR controller with implicit model following capabilities when the desired dynamics is expressed as

$$\ddot{\theta}(t) + 2\xi_0\omega_0\dot{\theta}(t) + \omega_0^2\theta(t) = 0$$

State-space model: $\mathbf{x}(t) = \begin{Bmatrix} \theta(t) \\ \dot{\theta}(t) \end{Bmatrix}$ $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}_u u(t)$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \mathbf{b}_u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

3. Implicit model following - example

Desired dynamics:

$$\ddot{\theta}(t) = -\omega_0^2\theta(t) - 2\xi_0\omega_0\dot{\theta}(t)$$

$$\mathbf{x}(t) = \begin{Bmatrix} \theta(t) \\ \dot{\theta}(t) \end{Bmatrix} \qquad \hat{\mathbf{y}}(t) = \mathbf{C}_y\hat{\mathbf{x}}(t)$$

$$\mathbf{A}_d = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2\xi_0\omega_0 \end{bmatrix}$$

Weighting matrices:

$$\mathbf{Q} = (\mathbf{A} - \mathbf{A}_d)^T (\mathbf{A} - \mathbf{A}_d) = \begin{bmatrix} \omega_0^4 & 2\xi_0\omega_0^3 \\ 2\xi_0\omega_0^3 & 4\xi_0^2\omega_0^2 \end{bmatrix}$$

$$\mathbf{S} = (\mathbf{A} - \mathbf{A}_d)^T \mathbf{b}_u = \begin{bmatrix} \omega_0^2 \\ 2\xi_0\omega_0 \end{bmatrix} \qquad \mathbf{R} = \mathbf{b}_u^T\mathbf{b}_u = 1$$

4. Spillover reduction

We have seen that a spillover phenomenon arises from the excitation of the residual dynamics by the control (control spillover -$\mathbf{B}_{ur}\mathbf{G}$).

Control spillover can be alleviated by minimizing the amount of energy fed into the residual dynamics. This can be achieved by supplementing the cost functional used in the regulator design by a quadratic term in the control spillover:

$$J = \frac{1}{2}\int_0^\infty \left(\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u}\right) dt + \frac{1}{2}\int_0^\infty \mathbf{u}^T\mathbf{B}_{ur}^T\mathbf{W}_{rr}\mathbf{B}_{ur}\mathbf{u}\, dt$$

where the weighting matrix $\mathbf{W}_{rr}$ allows us to penalize some specific dynamics.
This amounts to using the modified control weighting matrix $\mathbf{R} + \mathbf{B}_{ur}^T\mathbf{W}_{rr}\mathbf{B}_{ur}$

This control weighting matrix penalizes the excitation ***u***(t) whose shape is favorable to the residual dynamics; **this tends to produce a control which is orthogonal to the residual dynamics**.
This is achieved when there are many actuators.

5. Sensitivity-weighted LQR (SWLQR)

It is a variation of the standard LQR problem that **can be used to increase the stability robustness of LQR controllers to parametric modeling errors**.

When parametric errors exist, the model will be able to capture the system's nominal dynamics, but the model cannot capture the exact behavior.

Parametric errors occur often because it is difficult to know the exact properties of the physical devices that make up the system and because the physical properties may vary with time or the environment in which they are placed.

Designing feedback controllers that are robust with respect to parametric uncertainty is a very difficult problem. SWLQR is one of the methods available.

The philosophy behind SWLQR is that it aims to desensitize the LQR controller to parametric uncertainty. The advantage is that **it produces a technique which is nothing more than a special way to choose the weighting matrices**.

5. Sensitivity-weighted LQR (SWLQR)

Let's assume that the uncertain parameter to which you wish to desensitize a standard LQR controller is denoted by α .

A quadratic term as an **additional penalty on the sensitivity of the state to the uncertain variable** is introduced in the cost function as follows

$$J = \frac{1}{2}\int_0^{\infty}\left[\mathbf{x}^T\hat{\mathbf{Q}}\mathbf{x} + \mathbf{u}^T\hat{\mathbf{R}}\mathbf{u} + \left(\frac{\partial \mathbf{x}}{\partial \alpha}\right)^T \mathbf{W}_{\alpha\alpha}\left(\frac{\partial \mathbf{x}}{\partial \alpha}\right)\right] dt$$

where the derivative is to be evaluated at the nominal value of the parameter.

The larger the value of $\mathbf{W}_{\alpha\alpha}$, the more we tell the cost that we are uncertain about the influence of parameter α in our model on the trajectory of the state.

5. Sensitivity-weighted LQR (SWLQR)

By taking the derivative of the state equation with respect to the uncertain parameter yields

$$\frac{\partial \dot{\mathbf{x}}}{\partial \alpha} = \frac{\partial}{\partial \alpha}\left(\mathbf{A}\mathbf{x} + \mathbf{B}_u\mathbf{u}\right) = \frac{\partial \mathbf{A}}{\partial \alpha}\mathbf{x} + \mathbf{A}\frac{\partial \mathbf{x}}{\partial \alpha} + \frac{\partial \mathbf{B}_u}{\partial \alpha}\mathbf{u}$$

Since we are seeking for an infinite horizon solution (steady-state LQR controller), we set $\frac{\partial \dot{\mathbf{x}}}{\partial \alpha} = \mathbf{0}$ (steady-state solution)

such that $$\frac{\partial \mathbf{x}}{\partial \alpha} = -\mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial \alpha}\mathbf{x} - \mathbf{A}^{-1}\frac{\partial \mathbf{B}_u}{\partial \alpha}\mathbf{u}$$

The cost function becomes

$$J = \frac{1}{2}\int_0^{\infty}\left(\mathbf{x}^T\mathbf{Q}\mathbf{x} + 2\mathbf{x}^T\mathbf{S}\mathbf{u} + \mathbf{u}^T\mathbf{R}\mathbf{u}\right)dt$$

$$\mathbf{Q} = \hat{\mathbf{Q}} + \frac{\partial \mathbf{A}^T}{\partial \alpha}\mathbf{A}^{-T}\mathbf{W}_{\alpha\alpha}\mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial \alpha}$$

$$\mathbf{S} = \frac{\partial \mathbf{A}^T}{\partial \alpha}\mathbf{A}^{-T}\mathbf{W}_{\alpha\alpha}\mathbf{A}^{-1}\frac{\partial \mathbf{B}_u}{\partial \alpha}$$

$$\mathbf{R} = \hat{\mathbf{R}} + \frac{\partial \mathbf{B}_u^T}{\partial \alpha}\mathbf{A}^{-T}\mathbf{W}_{\alpha\alpha}\mathbf{A}^{-1}\frac{\partial \mathbf{B}_u}{\partial \alpha}$$

5. Sensitivity-weighted LQR (SWLQR) - example

Homework: see DCSS-homework-3

6. Frequency-shaped LQR

It is a variation of the standard LQR problem that **can be used to increase the stability robustness of LQR controllers to unmodeled dynamics** (un-modeled non-linear behavior and/or neglected high-frequency dynamics which lie beyond the control bandwidth of interest). It is based on the following

Parseval's Theorem:

$$\int_{-\infty}^{+\infty} \mathbf{z}^T(t)\mathbf{z}(t)dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathbf{z}^*(j\omega)\mathbf{z}(j\omega)d\omega$$

Proof:

$$\begin{aligned}\int_{-\infty}^{+\infty} \mathbf{z}^T(t)\mathbf{z}(t)dt &= \int_{-\infty}^{+\infty} \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathbf{z}^T(j\omega)e^{j\omega t}d\omega\,\mathbf{z}(t)dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathbf{z}^T(j\omega)\int_{-\infty}^{+\infty} \mathbf{z}(t)e^{-(-j\omega t)}dt\,d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathbf{z}^T(j\omega)\mathbf{z}(-j\omega)d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathbf{z}^T(-j\omega)\mathbf{z}(j\omega)d\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathbf{z}^*(j\omega)\mathbf{z}(j\omega)d\omega\end{aligned}$$

$\mathbf{z}(j\omega)$ is the Fourier transform of $\mathbf{r}(t)$

$$\mathbf{z}^*(j\omega) = \mathbf{z}^T(-j\omega)$$

6. Frequency-shaped LQR

Let's consider the cost function expressed as

$$J = \int_0^{\infty} \left(\mathbf{z}^T(t)\mathbf{z}(t) + \mathbf{u}^T(t)\mathbf{u}(t) \right) dt$$

Applying the Parseval's Theorem
it is equivalent to the following

$$J = \frac{1}{2\pi} \int_0^{\infty} \left(\mathbf{z}^*(j\omega)\mathbf{z}(j\omega) + \mathbf{u}^*(j\omega)\mathbf{u}(j\omega) \right) d\omega$$

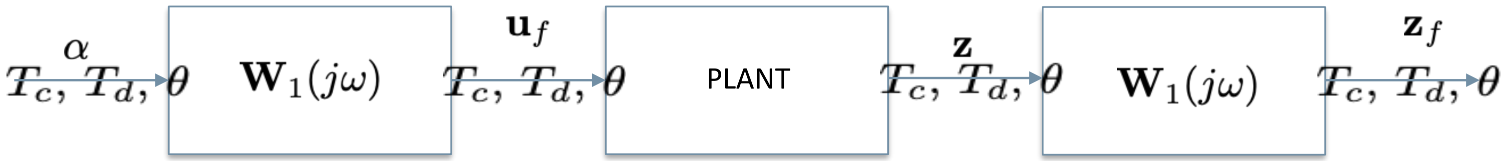
The frequency-shaped LQR is based on minimizing a filtered version of ***z*** and ***u***

$$J = \frac{1}{2\pi} \int_0^{\infty} \left(\mathbf{z}_f^*(j\omega)\mathbf{z}_f(j\omega) + \mathbf{u}_f^*(j\omega)\mathbf{u}_f(j\omega) \right) d\omega$$

where $\mathbf{z}_f(j\omega) = \mathbf{W}_1(j\omega)\mathbf{z}(j\omega)$ $\qquad \mathbf{u}_f(j\omega) = \mathbf{W}_2(j\omega)\mathbf{u}(j\omega)$

6. Frequency-shaped LQR

$$\mathbf{z}_f(j\omega) = \mathbf{W}_1(j\omega)\mathbf{z}(j\omega) \qquad\qquad \mathbf{u}_f(j\omega) = \mathbf{W}_2(j\omega)\mathbf{u}(j\omega)$$



$$J = \frac{1}{2\pi}\int_0^{\infty} \left(\mathbf{z}_f^*(j\omega)\mathbf{z}_f(j\omega) + \mathbf{u}_f^*(j\omega)\mathbf{u}_f(j\omega)\right) d\omega$$

$$J = \frac{1}{2\pi}\int_0^{\infty} \left(\mathbf{z}^*(j\omega)\mathbf{W}_1^*(j\omega)\mathbf{W}_1(j\omega)\mathbf{z}(j\omega) + \mathbf{u}^*(j\omega)\mathbf{W}_2^*(j\omega)\mathbf{W}_2(j\omega)\mathbf{u}(j\omega)\right) d\omega$$

6. Frequency-shaped LQR

$$J = \frac{1}{2\pi}\int_0^\infty \left(\mathbf{z}^*(j\omega)\mathbf{W}_1^*(j\omega)\mathbf{W}_1(j\omega)\mathbf{z}(j\omega) + \mathbf{u}^*(j\omega)\mathbf{W}_2^*(j\omega)\mathbf{W}_2(j\omega)\mathbf{u}(j\omega)\right) d\omega$$

$$J = \frac{1}{2\pi}\int_0^\infty \left(\mathbf{z}^*(j\omega)\mathbf{W}_{zz}(j\omega)\mathbf{z}(j\omega) + \mathbf{u}^*(j\omega)\mathbf{W}_{uu}(j\omega)\mathbf{u}(j\omega)\right) d\omega$$

frequency-shaped weighting matrices:

$$\mathbf{W}_{zz}(j\omega) = \mathbf{W}_1^*(j\omega)\mathbf{W}_1(j\omega)$$

$$\mathbf{W}_{uu}(j\omega) = \mathbf{W}_2^*(j\omega)\mathbf{W}_2(j\omega)$$

6. Frequency-shaped LQR

Actual implementation.

$$\mathbf{z}_f(j\omega) = \mathbf{W}_1(j\omega)\mathbf{z}(j\omega) \qquad \mathbf{z}_f(s) = \mathbf{W}_1(s)\mathbf{z}(s)$$

$$\mathbf{u}_f(j\omega) = \mathbf{W}_2(j\omega)\mathbf{u}(j\omega) \qquad \mathbf{u}_f(s) = \mathbf{W}_2(s)\mathbf{u}(s)$$

State-space realization of $W_1(s)$

$$\dot{\mathbf{x}}_1(t) = \mathbf{A}_1\mathbf{x}_1(t) + \mathbf{B}_1\mathbf{z}(t)$$
$$\mathbf{z}_f(t) = \mathbf{C}_1\mathbf{x}_1(t) + \mathbf{D}_1\mathbf{z}(t)$$

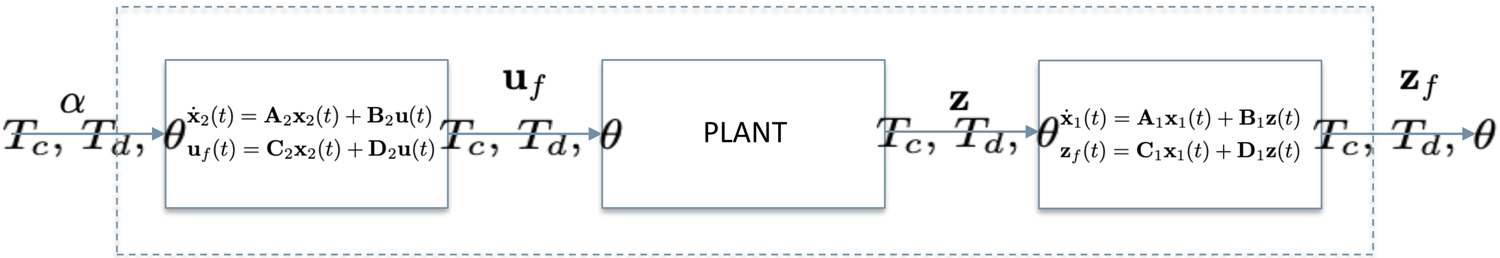
$$\mathbf{W}_1(s) = \mathbf{C}_1\left(s\mathbf{I} - \mathbf{A}_1\right)^{-1}\mathbf{B}_1 + \mathbf{D}_1$$

State-space realization of $W_2(s)$

$$\dot{\mathbf{x}}_2(t) = \mathbf{A}_2\mathbf{x}_2(t) + \mathbf{B}_2\mathbf{u}(t)$$
$$\mathbf{u}_f(t) = \mathbf{C}_2\mathbf{x}_2(t) + \mathbf{D}_2\mathbf{u}(t)$$

$$\mathbf{W}_1(s) = \mathbf{C}_1\left(s\mathbf{I} - \mathbf{A}_1\right)^{-1}\mathbf{B}_1 + \mathbf{D}_1$$

6. <u>Frequency-shaped LQR</u>



Augmented system:

$$\mathbf{x}_{\text{aug}}(t) = \begin{Bmatrix} \mathbf{x}(t) \\ \mathbf{x}_1(t) \\ \mathbf{x}_2(t) \end{Bmatrix}$$

$$\dot{\mathbf{x}}_{\text{aug}}(t) = \mathbf{A}_{\text{aug}}\mathbf{x}_{\text{aug}}(t) + \mathbf{B}_{u\text{aug}}\mathbf{u}(t)$$

$$\dot{\mathbf{x}}_{\text{aug}}(t) = \mathbf{A}_{\text{aug}}\mathbf{x}_{\text{aug}}(t) + \mathbf{B}_{u\text{aug}}\mathbf{u}(t)$$

the frequency-shaped LQR is achieved by solving a classical LQR problem on the dynamic system augmented with the corresponding filters on z and u

6. Frequency-shaped LQR - example

Homework: see DCSS-homework-4

FINITE HORIZON OPTIMAL CONTROL

The *finite-horizon* optimal control problem of a LTI system is defined as the solution of the control input vector $\mathbf{u}(t)$ which is capable of minimizing the following cost functional

$$J = \frac{1}{2}\int_{t_0}^{t_f} \left(\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u}\right) dt + \frac{1}{2}\mathbf{x}_f^T\mathbf{Z}\mathbf{x}_f \tag{17.201}$$

@ tf the final state could be different from zero so we need to take into account x_f.
To do so we will introduce in the cost function the final state.

subjected to initial state vector

$$\mathbf{x}(t_0) = \mathbf{x}_0$$

and to the constraint imposed by the state equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}_u\mathbf{u}(t)$$

where t_f is the specified final time and $\mathbf{x}_f$ is a free (not prescribed) final state vector. It is assumed that

- $\mathbf{Q}$ is a symmetric nonnegative matrix
- $\mathbf{R}$ is a symmetric positive definite matrix
- $\mathbf{Z}$ is a symmetric nonnegative matrix

Note that the above cost function differ from what presented for the LQR design in two aspects. First, it is observed that the optimal tradeoff between regulation performance and control effort is sought in a finite time interval $[t_0, t_f]$. Second, a terminal cost is added to introduce a penalty on the final system state. The optimization problem at hand is to determine an optimal input signal with the goal of maintaining the state trajectory *close* to the equilibrium at the origin while expending *moderate* control effort.

The solution of the optimization problem can be obtained by a variational approach. The constrained minimization is related to the following cost functional

$$J = \frac{1}{2}\int_{t_0}^{t_f} \left[\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u} + 2\boldsymbol{\lambda}^T\left(\mathbf{A}\mathbf{x} + \mathbf{B}_u\mathbf{u} - \dot{\mathbf{x}}\right)\right] dt + \frac{1}{2}\mathbf{x}_f^T\mathbf{Z}\mathbf{x}_f \tag{17.202}$$

where $\boldsymbol{\lambda}$ is the vector of Lagrange multipliers. The necessary condition for the optimum is that the variation of J vanishes, i.e.,

$$\delta J = 0 \tag{17.203}$$

Control system design – state-space methods
Finite-horizon optimal control

It can be shown that the optimum is achieved if

$$
\begin{aligned}
\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}_u\mathbf{u}(t) \\
\dot{\boldsymbol{\lambda}}(t) &= -\mathbf{A}^T\boldsymbol{\lambda}(t) - \mathbf{Q}\mathbf{x}(t) \\
\mathbf{R}\mathbf{u}(t) &+ \mathbf{B}_u^T\boldsymbol{\lambda}(t) = \mathbf{0} \\
\mathbf{Z}\mathbf{x}_f &= \boldsymbol{\lambda}_f
\end{aligned}
\tag{17.206}
$$

From the third equation we obtain the optimal control as

$$\mathbf{u}(t) = -\mathbf{R}^{-1}\mathbf{B}_u^T\boldsymbol{\lambda}(t) \tag{17.207}$$

where we notice that $\mathbf{R}$ must be positive definite so that its inverse exists. By inserting such optimal solution into the first equation of the set (17.206) we can write the following differential equations

$$
\begin{cases}
\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) - \mathbf{B}_u\mathbf{R}^{-1}\mathbf{B}_u^T\boldsymbol{\lambda}(t) \\
\dot{\boldsymbol{\lambda}}(t) = -\mathbf{Q}\mathbf{x}(t) - \mathbf{A}^T\boldsymbol{\lambda}(t)
\end{cases}
\tag{17.208}
$$

→ Lagrange multipliers dynamics

This give the solution for an optimal control

which is subjected to the following conditions

$$
\begin{aligned}
\mathbf{x}(t_0) &= \mathbf{x}_0 \\
\boldsymbol{\lambda}(t_f) &= \mathbf{Z}\mathbf{x}_f
\end{aligned}
\tag{17.209}
$$

The differential problem, known as Hamiltonian system, can be put in matrix form as

$$\begin{Bmatrix} \dot{\mathbf{x}} \\ \dot{\boldsymbol{\lambda}} \end{Bmatrix} = \begin{bmatrix} \mathbf{A} & -\mathbf{E} \\ -\mathbf{Q} & -\mathbf{A}^T \end{bmatrix} \begin{Bmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{Bmatrix} \tag{17.210}$$

where $\mathbf{E} = \mathbf{B}_u \mathbf{R}^{-1} \mathbf{B}_u^T$. The vector $\boldsymbol{\lambda}$ is also called *costate*. The final condition on the costate $\boldsymbol{\lambda}_f$ together with the initial condition on the state $\mathbf{x}_0$ and the Hamiltonian system of equations (17.210) form a two-point boundary value problem.

The above problem can be formulated as a closed-loop optimal control solution. The goal is to write the optimal control law, which is written in Eq. (17.207) as a function of the costate $\boldsymbol{\lambda}(t)$, in terms of the state vector $\mathbf{x}(t)$. This can be done by examining the final condition on $\boldsymbol{\lambda}_f$, which relates the final costate in terms of the final state. Similarly, we may like to connect the costate with the state not only at the final time but for the complete time horizon $[t_0, t_f]$. Thus, let us assume

$$\boldsymbol{\lambda}(t) = \mathbf{P}(t)\mathbf{x}(t) \tag{17.211}$$

where the time-variant $\mathbf{P}(t)$ matrix is yet to be determined. Accordingly, the optimal control input is expressed as

$$\mathbf{u}(t) = -\mathbf{R}^{-1}\mathbf{B}_u^T\mathbf{P}(t)\mathbf{x}(t) \tag{17.212}$$

which is now a full state feedback control. Note that it can be also written as

$$\mathbf{u}(t) = -\mathbf{G}(t)\mathbf{x}(t) \tag{17.213}$$

where $\mathbf{G} = \mathbf{R}^{-1}\mathbf{B}_u^T\mathbf{P}$ is the time-variant gain matrix. Differentiating Eq. (17.211) with respect to time yields

$$\dot{\boldsymbol{\lambda}} = \dot{\mathbf{P}}\mathbf{x} + \mathbf{P}\dot{\mathbf{x}} \tag{17.214}$$

Using the equations of the Hamiltonian system, we get

$$-\mathbf{Q}\mathbf{x}(t) - \mathbf{A}^T\mathbf{P}(t)\mathbf{x}(t) = \dot{\mathbf{P}}(t)\mathbf{x}(t) + \mathbf{P}(t)\left[\mathbf{A}\mathbf{x}(t) - \mathbf{E}\mathbf{P}(t)\mathbf{x}(t)\right] \tag{17.215}$$

which must hold for any value of $\mathbf{x}(t)$. This clearly means that the matrix $\mathbf{P}(t)$ satisfies the matrix differential equation

$$\dot{\mathbf{P}}(t) + \mathbf{P}(t)\mathbf{A} + \mathbf{A}^T\mathbf{P}(t) - \mathbf{P}(t)\mathbf{B}_u\mathbf{R}^{-1}\mathbf{B}_u^T\mathbf{P}(t) + \mathbf{Q} = \mathbf{0} \tag{17.216}$$

which is known as *differential Riccati equation*. Comparing the final condition on the costate with the transformation (17.211), we have the final condition on $\mathbf{P}(t)$ as

$$\mathbf{P}(t_f) = \mathbf{Z} \tag{17.217}$$

Thus, the Riccati equation (17.216) is to be solved *backward* in time from the final condition (17.217) to obtain the solution $\mathbf{P}(t)$ for the entire interval $[t_0, t_f]$.

The function of the optimal control law presented thus far is to hold the state vector near zero, that is to guarantee closed-loop stability. Another fundamental optimal design problem is to control a system so that a specified output $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$ follows a given nonzero reference trajectory $\mathbf{r}(t)$. An example is controlling a spacecraft to follow a desired step input command (e.g., change in attitude). This is called *optimal tracking*. For this purpose, the performance index is modified as follows

$$J = \frac{1}{2}\int_{t_0}^{t_f} \left(\mathbf{e}^T\mathbf{Q}\mathbf{e} + \mathbf{u}^T\mathbf{R}\mathbf{u}\right) dt + \frac{1}{2}\mathbf{e}_f^T\mathbf{Z}\mathbf{e}_f \quad (17.223)$$

where

$$\mathbf{e}(t) = \mathbf{y}(t) - \mathbf{r}(t) \quad (17.224)$$

It can be shown that

$$\mathbf{u}(t) = -\mathbf{R}^{-1}\mathbf{B}_u^T\mathbf{P}(t)\mathbf{x}(t) - \mathbf{R}^{-1}\mathbf{B}_u^T\mathbf{s}(t)$$

$$\dot{\mathbf{P}}(t) + \mathbf{P}(t)\mathbf{A} + \mathbf{A}^T\mathbf{P}(t) - \mathbf{P}(t)\mathbf{B}_u\mathbf{R}^{-1}\mathbf{B}_u^T\mathbf{P}(t) + \mathbf{C}^T\mathbf{Q}\mathbf{C} = \mathbf{0}$$

$$\mathbf{P}(t_f) = \mathbf{C}^T\mathbf{Z}\mathbf{C}$$

$$\dot{\mathbf{s}}(t) + \left(\mathbf{A}^T - \mathbf{P}\mathbf{B}_u\mathbf{R}^{-1}\mathbf{B}_u^T\right)\mathbf{s}(t) - \mathbf{C}^T\mathbf{Q}\mathbf{r}(t) = \mathbf{0}$$

$$\mathbf{s}(t_f) = \mathbf{C}^T\mathbf{Z}\mathbf{r}(t_f)$$