ORBITAL MECHANICS

CASE 2: given θ_1 where apse line rotation take place find apse line rotation η (or $\Delta\omega$) and e_2.

$\Delta v \rightarrow$ will in general change v_r and v_θ

From $h = r v_\theta$

$$\boxed{h_2 = \underbrace{r v_\theta}_{h_1} + r\Delta v_\theta = h_1 + r\Delta v_\theta} \qquad (3.27)$$

Radial component of v

$$v_{r_1} = \frac{\mu}{h_1} e_1 \sin\theta_1 \qquad (3.28)$$

$$v_{r_2} = v_{r_1} + \Delta v_r = \frac{\mu}{h_2} e_2 \sin\theta_2 \qquad (3.29)$$

And $\theta_2 = \theta_1 - \eta$ see eq (3.23)

Known variables v_{r_1}, Δv_r, h_2 (from eq (3.27)) $\rightarrow$? e_2, θ_2

we can invert (3.29) $\rightarrow \theta_2$

$$\sin\theta_2 = \frac{1}{e_2} \frac{h_2}{\mu} \left(v_{r_1} + \Delta v_r \right) \qquad (3.30)$$

we could use the condition of the intersection point

$$\begin{cases} r_I = \dfrac{h_1^2}{\mu} \dfrac{1}{1 + e_1\cos\theta_1} \\ r_I = \dfrac{h_2^2}{\mu} \dfrac{1}{1 + e_2\cos\theta_2} \end{cases} \qquad (3.31)$$

$$\frac{h_1^2}{\mu}\,\frac{1}{1+e_1\cos\theta_1} = \frac{h_2^2}{\mu}\,\frac{1}{1+e_2\cos\theta_2}$$

$$h_1^2\left(1+e_2\cos\theta_2\right) = h_2^2\left(1+e_1\cos\theta_1\right)$$

$$\cos\theta_2 = \frac{1}{e_2 h_1^2}\left(h_2^2\left(1+e_1\cos\theta_1\right) - h_1^2\right) \qquad (3.32)$$

where $h_2 = h_1 + \Delta v_\theta r$

$$\rightarrow \ tg\,\theta_2 = \frac{\sin\theta_2 \ \leftarrow 3.30}{\cos\theta_2 \ \leftarrow 3.32}$$

we have eliminated e_2

$$tg\,\theta_2 = \frac{h_2}{\mu}\left(v_{r_1} + \Delta v_{r_1}\right) h_1^2 \,\frac{1}{h_2^2\left(1+e_1\cos\theta_1\right) - h_1^2}$$

but $h_1 = r v_{\theta_1}$

$$h_2 = \underbrace{r v_\theta}_{h_1} + r\Delta v_\theta$$

$$\Rightarrow tg\,\theta_2 = \frac{h_1 + r\Delta v_\theta}{\mu}\left(v_{r_1} + \Delta v_{r_1}\right)\frac{h_1^2}{\left(h_1^2 + r^2\Delta v_\theta^2 + 2h_1 r\Delta v_\theta\right)\left(1+e_1\cos\theta_1\right) - h_1^2}$$

$$= \frac{h_1 + r\Delta v_\theta}{\mu}\left(v_{r_1} + \Delta v_{r_1}\right)\frac{h_1^2}{r^2\Delta v_\theta^2 + 2h_1 r\Delta v_\theta + e_1\cos\theta_1\left(h_1^2 + r^2\Delta v_\theta^2 + 2h_1 r\Delta v_\theta\right)}$$

$$= \frac{r v_{\theta_1} + r\Delta v_\theta}{\mu}\left(v_{r_1} + \Delta v_{r_1}\right)\frac{r^2\Delta v_{\theta_1}^2}{r^2\Delta v_\theta^2 + 2r^2 v_{\theta_1}\Delta v_\theta + e_1\cos\theta_1\left(h_1^2 + r^2\Delta v_\theta^2 + 2r^2 v_{\theta_1}\Delta v_\theta\right)}$$

$$= \frac{r^3\left(v_{\theta_1} + \Delta v_\theta\right)\left(v_{r_1} + \Delta v_r\right) v_{\theta_1}^2}{\mu r^2\left(\Delta v_\theta^2 + 2v_{\theta_1}\Delta v_\theta + \left(v_{\theta_1} + \Delta v_\theta\right)^2 e_1\cos\theta_1\right)}$$

$$\boxed{\operatorname{tg}\theta_2 = \frac{v_{\theta_1}^2 r}{\mu}\,\frac{(v_{\theta_1}+\Delta v_\theta)(v_{r_1}+\Delta v_r)}{(v_{\theta_1}+\Delta v_\theta)^2 e_1\cos\theta_1 + \Delta v_\theta(\Delta v_\theta + 2v_{\theta_1})}} \quad (3.33)$$

NOTE if $\Delta v_r = -v_{r_1}$ $\Rightarrow$ manoeuvre is given at apse line of the orbit (orbit 2 will have $v_{r_2}=0$ at manoeuvre point) $\Rightarrow$ point θ_2 is on the apse line.

$$\sin\theta_2 = \frac{1}{e_2}\,\frac{h_2}{\mu}(v_{r_1}+\Delta v_r) \rightarrow \boxed{e_2 = \frac{1}{\sin\theta_2}\,\frac{h_2}{\mu}(v_{r_1}+\Delta v_r) \quad (3.34)}$$

Case 2 $\Rightarrow$ useful for flight dynamics because we know everything of orbit 1 and the Δv.

PARTICULAR CASE OF APSE LINE ROTATION

Shape of orbit 1 and 2 are maintained

1: e_1, h_1, ω_1

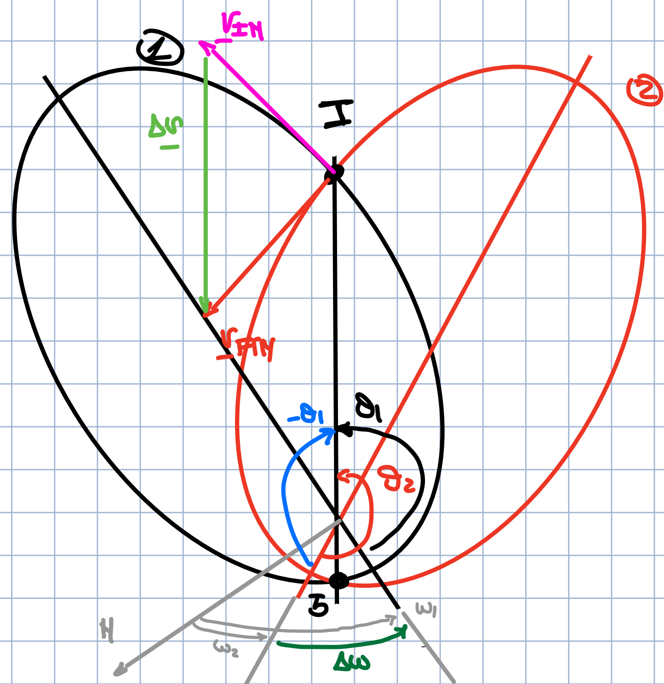
2: $e_2 = e_1$, $h_1 = h_2$, ω_2

$\theta_{1I} = \theta_1$

$\theta_{2I} = \theta_2 = -\theta_1$

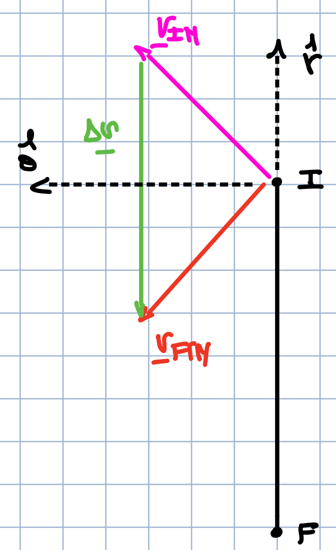
$\Delta\omega = \Delta\theta = \theta_2 - \theta_1 = -2\theta$

$$\boxed{\theta = -\frac{\Delta\omega}{2}}$$



$\hat{r}$ $\quad v_{r_{IN}} = v_r \qquad v_{r_{FIN}} = -v_r$

$$v_{r_{IN}} = -v_{r_{FIN}}$$



$\hat{\theta}$ $\quad v_{\theta_{IN}} = v_\theta \qquad v_{\theta_{FIN}} = v_\theta$

$$v_{\theta_{IN}} = v_{\theta_{FIN}}$$

$$\Delta \underline{v} = \| \underline{v}_{FIN} - \underline{v}_{IN} \|$$

$$\Delta v = 2 v_r = 2\sqrt{\frac{\mu}{p}}\, e \sin\theta$$

$$\boxed{\Delta v = 2 v_r = 2\sqrt{\frac{\mu}{p}}\, e \sin\frac{\Delta\omega}{2}} \qquad (3.35)$$

TRANSFER BETWEEN TWO SPECIFIED POINT: LAMBERT'S PROBLEM

Transfer from any point in space to any point in space

Application:

- initial orbit determination

  know $\underline{r}_1, \underline{r}_2$ of P_1 and $P_2 \in$ path of S/C

  Δt between $\underline{r}_1$ and $\underline{r}_2$ between two consecutive observatio

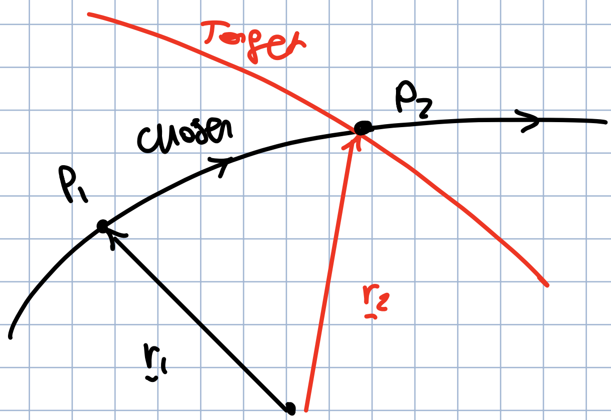
  find $\underline{v}_1, \underline{v}_2$ $\equiv$ orbital path of unclassified object (e.g. space debries)

- spacecraft targeting

  find target: planet or S/C moving on known orbit we want to

  - intersect $\quad$ match only position (e.g. fly by)
  - rendez-vous $\triangleq$ match position and velocity

(e.g. intersection in orbit ISS, inspection, Land on a planet)



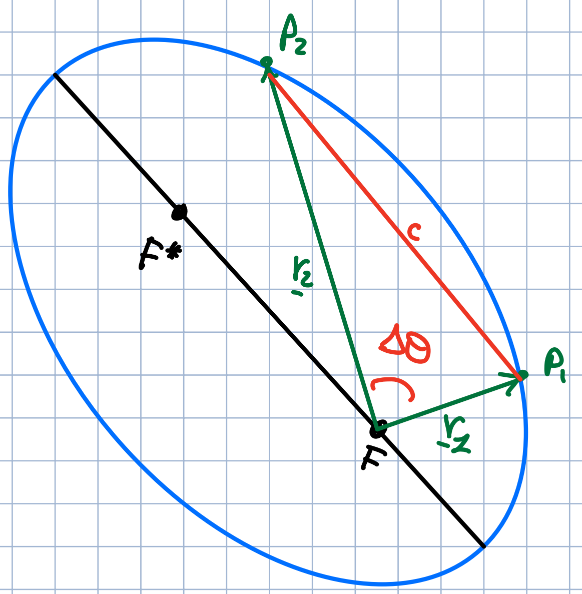
Need to determinate the two-body-problem orbit that connects $\underline{r}_1$ and $\underline{r}_2$ (initial and final position) in a given time $\Delta T \Rightarrow$ Time of flight between $\underline{r}_1$ and $\underline{r}_2$

$\underline{r}_1, \underline{r}_2, \Delta t \longrightarrow$ orbit between them LAMBERT'S PROBLEM (1761)

$\underline{v}_1, \underline{v}_2$

we do not know the velocity so the Lambert's problem is also known as BOUNDARY VALUE PROBLEM

- Lambert geometrical formulation: equation to define the minimum energy orbit
- Gauss' formulation: geometrical insight
- Battin's formulation: universal variables $\Rightarrow$ LambertMR.m



Know F, $P_1(\underline{r}_1)$, $P_2(\underline{r}_2)$, Δt

chord (shortest segment between P_1 and P_2) = c

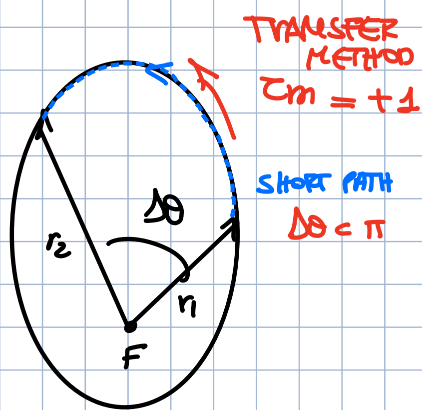
$\Delta\theta$ = transfer angle

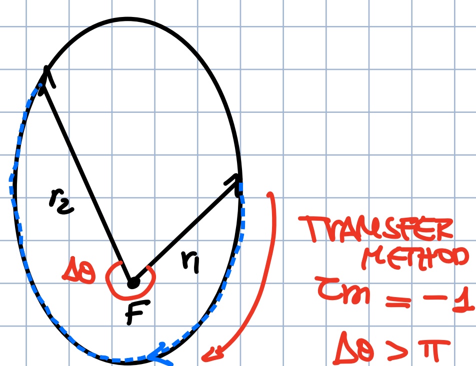
NOTE we do not know F^* so we do not know θ_1 and θ_2

Find transfer orbit between P_1 and P_2 (F^*)

NOTE: $\underline{r}_1$, $\underline{r}_2$ identify a plane $\rightarrow$ $\underline{r}_1 \wedge \underline{r}_2 = \hat{h}\,\bar{d}$ of the transfer orbit

if the transfer method is identified $\Rightarrow$ the problem has only 1 solution





NOTE we are considering less than one full revolution $\rightarrow$ this will help to simplify the solution

$$\boxed{\cos\Delta\theta = \frac{\underline{r}_1 \cdot \underline{r}_2}{r_1\, r_2}} \qquad (3.36)$$

$$\boxed{\sin\Delta\theta = \tau_m \sqrt{1-\cos^2\Delta\theta}} \qquad (3.37)$$

Property of ellipses

unknown

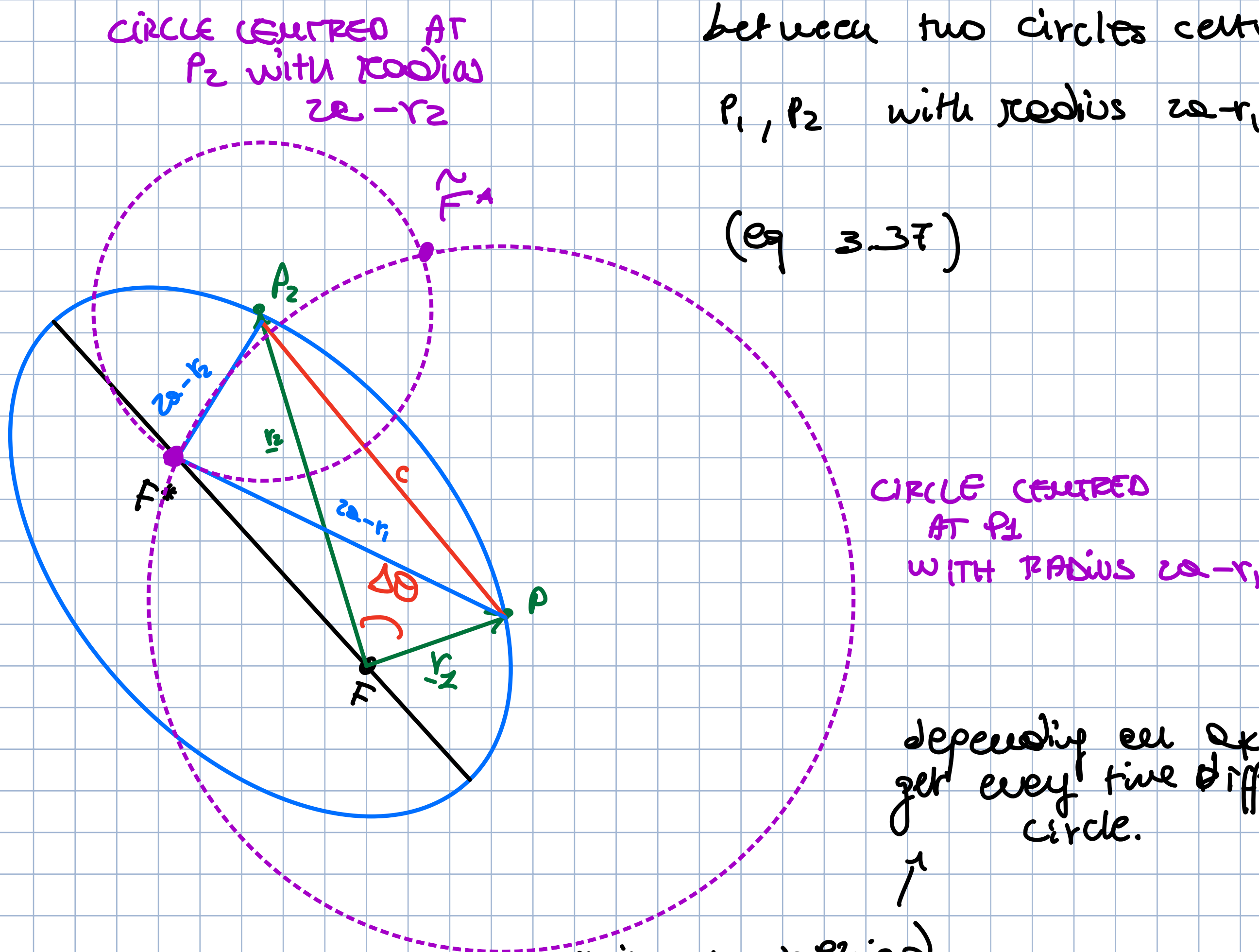
$$P_1F + P_1F^* = 2a \rightarrow P_1F^* = 2a - r_1$$

$$P_2F + P_2F^* = 2a \rightarrow P_2F^* = 2a - r_2 \qquad (3.37)$$

Let's assume $r_2 > r_1$ which does not imply any loss of generality.

For a given a the vacant focus will be located at the intersection between two circles centred at P_1, P_2 with radius $2a - r_1$, $2a - r_2$.

(eq 3.37)



depending on a_K we get every time different circle.

for a given a_K (because it is not defined) the two circles intersect at 2 points F^* and $\tilde{F}^*$ → They are equidistant from chord c

As there are 2 apparent focii ⇒ there exist 2 possible transfer between P_1 and P_2.

These two different transfer have different e, different transfer time but they have the same energy (a_K)

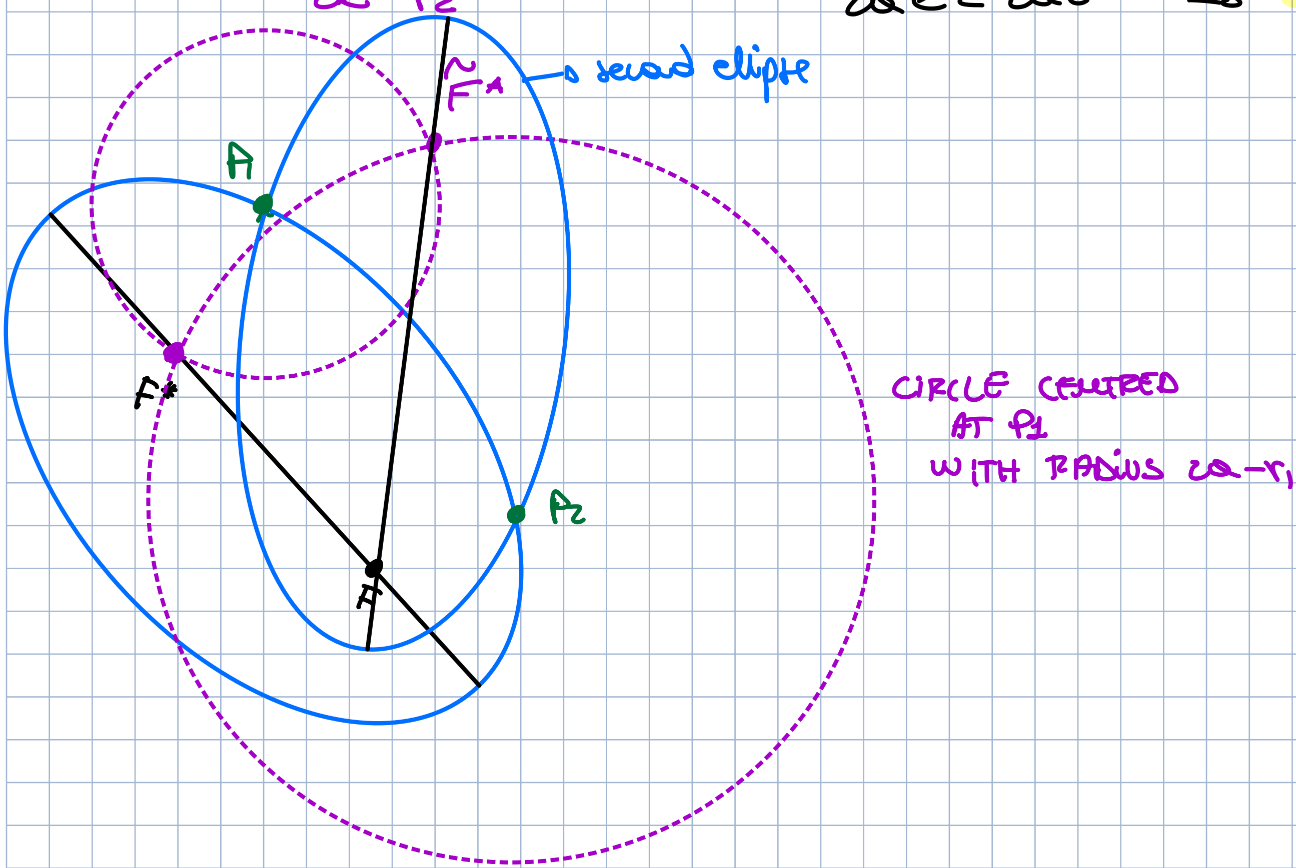
in the figure below $FF^* < F\tilde{F}^*$ $(FF^* = 2ae)$

$$FF^* = 2ae$$
$$F\tilde{F}^* = 2a\tilde{e}$$

because $FF^* < F\tilde{F}^*$

$2ae < 2a\hat{e} \Rightarrow \boxed{e < \tilde{e}}$

CIRCLE CENTRED AT P_2 WITH RADIUS $2a - r_2$



when the two circle are tangent to one another we will get only one focus and in this case we will get the minimum semi-major axis so we will get the minimum energy to do the transfer. All of that is done by decreasing the value of a_t.

As we vary the value of a_t F^* and $\tilde{F}^*$ (vacant foci) describe a locus of points that is formed by the intersection of the different circle of vary radii that are the two circle centred at P_1 and P_2.
We can demonstrate that the locus of point of the vacant focii is an hyperbola.

LOCUS OF POINT OF VACANT FOCII :

for any point on it the difference in the distance to P_1 and P_2 is equal to $r_1 - r_2$ and is constant.

let's demonstrate it :

$$F^* P_1 = 2a_k - r_1$$

$$F^* P_2 = 2a_k - r_2$$

$$F^* P_1 - F^* P_2 = 2a_k - r_1 - 2a_k + r_2$$

$$F^* P_1 - F^* P_2 = r_2 - r_1 \qquad (3.38)$$

$\Rightarrow$ The locus itself is an hyperbole with focii at P_1 and P_2 !!

if a_k is decreased the 2 vacant focii approach the focus F^*_m on the chord $\overline{P_1 P_2}$

if $a < a_m \Rightarrow$ no intersection between the circles $\Rightarrow$ no transfer exist with $a < a_m \Rightarrow$

a_m caracteristic minimum energy transfer

$$F_m^* P_1 = 2a_m - r_1$$

$$F_m^* P_2 = 2a_m - r_2$$

$$P_1 P_2 = c$$

$$F_m^* P_1 + F_m^* P_2 = c$$

$$2a_m - r_1 + 2a_m - r_2 = c \Rightarrow \boxed{a_m = \frac{r_1 + r_2 + c}{4}} \quad (3.39)$$

But the semi-perimeter of the space triangle $(\underline{r}_1, \underline{r}_2, c)$ is

$$s = \frac{r_1 + r_2 + c}{2} \Rightarrow \boxed{a_m = \frac{s}{2}} \qquad (3.40)$$