POLITECNICO MILANO 1863

Spacecraft Attitude Dynamics

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Attitude kinematics

Direction cosines - kinematics

How attitude parameters change with time, and how they are related to the angular velocity expressed in body frame $\rightarrow$ $\frac{dA}{dt} = f(\omega_u, \omega_v, \omega_w)$

Rule for sequence of rotations $\rightarrow$ $A(t + \Delta t) = A' A(t)$

$$A' = I \cos\phi + (1 - \cos\phi) \underline{e}\,\underline{e}^T - \sin\phi [\underline{e} \wedge]$$

short time intervals, φ small $\rightarrow$ $A' = I - \phi [\underline{e} \wedge]$

$$\phi = \omega \Delta t \qquad \underline{\omega} = \omega \underline{e}$$

$$\phi[\underline{e} \wedge] = \omega \Delta t \begin{bmatrix} 0 & -e_w & e_v \\ e_w & 0 & -e_u \\ -e_v & e_u & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\omega_w & \omega_v \\ \omega_w & 0 & -\omega_u \\ -\omega_v & \omega_u & 0 \end{bmatrix} \Delta t = [\underline{\omega} \wedge] \Delta t$$

Direction cosines - kinematics

$$A' = I - [\underline{\omega} \wedge]\Delta t$$
$$A(t+\Delta t) = A(t) - \Delta t[\underline{\omega} \wedge]A(t)$$

$$\frac{dA}{dt} = \lim_{\Delta t \to 0} \frac{A(t+\Delta t) - A(t)}{\Delta t} = \lim_{\Delta t \to 0} - \frac{\Delta t[\underline{\omega} \wedge]A(t)}{\Delta t} = -[\underline{\omega} \wedge]A(t)$$

Need to preserve the structure -> Use a standard numerical integration scheme and orthonormalize the matrix at every time step using:

$$A_{k+1}(t) = A_k(t)*3/2 - A_k(t)*A_k^T(t)*A_k(t)/2$$

If performing this at every time step a single iteration is normally enough:

$$A(t) = A_0(t)*3/2 - A_0(t)*A_0^T(t)*A_0(t)/2$$

Euler axis / angle - kinematics

No rule for sequence of rotations -> no rule for kinematics

Quaternion - kinematics

$$q(t+\Delta\mathrm{t}) = \begin{bmatrix} q_4' & q_3' & -q_2' & q_1' \\ -q_3' & q_4' & q_1' & q_2' \\ q_2' & -q_1' & q_4' & q_3' \\ -q_1' & -q_2' & -q_3' & q_4' \end{bmatrix} q(t) \qquad \text{with} \begin{cases} q_1' = e_u \sin\dfrac{\phi}{2} \\ q_2' = e_v \sin\dfrac{\phi}{2} \\ q_3' = e_w \sin\dfrac{\phi}{2} \\ q_4' = \cos\dfrac{\phi}{2} \end{cases}$$

$$q(t+\Delta t) = \left\{ I\,cos\frac{\phi}{2} + \begin{bmatrix} 0 & e_w & -e_v & e_u \\ -e_w & 0 & e_u & e_v \\ e_v & -e_u & 0 & e_w \\ -e_u & -e_v & -e_w & 0 \end{bmatrix} sin\frac{\phi}{2} \right\} q(t)$$

Quaternion - kinematics

short intervals Δt → $\phi = \omega \Delta t \qquad cos\frac{\phi}{2} = 1 \qquad sin\frac{\phi}{2} = \frac{\phi}{2} = \frac{\omega \Delta t}{2}$

evaluate e_u, e_v, e_w, as a function of $\underline{\omega} = \omega\ \underline{e}$

$$q(t + \Delta t) = \left[I + \frac{1}{2}\Omega \Delta t\right] q(t)$$

$$\Omega = \begin{bmatrix} 0 & \omega_w & -\omega_v & \omega_u \\ -\omega_w & 0 & \omega_u & \omega_v \\ \omega_v & -\omega_u & 0 & \omega_w \\ -\omega_u & -\omega_v & -\omega_w & 0 \end{bmatrix}$$

limit for $t \rightarrow 0$

$$\frac{dq}{dt} = \lim_{\Delta t \rightarrow 0} \frac{q(t + \Delta t) - q(t)}{\Delta t} = \frac{1}{2}\Omega q(t)$$

Gibbs vector - kinematics

$$\underline{g}(t+\Delta t)=\frac{\underline{g}(t)+\underline{g}'-\underline{g}'\wedge\underline{g}(t)}{1-\underline{g}(t)\cdot\underline{g}'}$$

$$\underline{g}'=\underline{e}\,tan\frac{\phi}{2}=\xrightarrow{small\ \Delta t}=\frac{1}{2}\underline{\omega}\Delta t$$

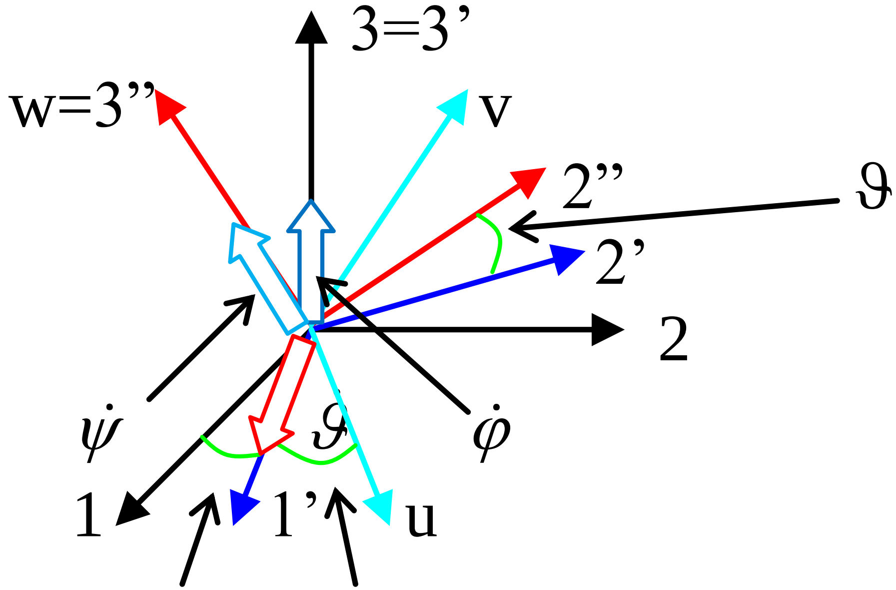
$$\frac{d\underline{g}}{dt}=\frac{1}{2}\left[\underline{\omega}-\underline{\omega}\wedge\underline{g}(t)+\left(\underline{g}(t)\cdot\underline{\omega}\right)\underline{g}(t)\right]$$

Euler angles - kinematics

no convenient rule for combining two consecutive rotations

consider the sequence 313 (φ,ϑ,ψ)

$\underline{\omega} = \dot{\phi}\underline{3} + \dot{\vartheta}\underline{1}' + \dot{\psi}\underline{w}$



Euler angles - kinematics

$$\underline{\omega} = \dot{\phi}\underline{3} + \dot{\vartheta}\underline{1}' + \dot{\psi}\underline{w} \longrightarrow \begin{cases} \omega_u = \underline{\omega}\cdot\underline{u} = \dot{\phi}\underline{3}\cdot\underline{u} + \dot{\vartheta}\underline{1}'\cdot\underline{u} + \dot{\psi}\underline{w}\cdot\underline{u} = \dot{\phi}\underline{3}\cdot\underline{u} + \dot{\vartheta}\underline{1}'\cdot\underline{u} \\ \omega_v = \underline{\omega}\cdot\underline{v} = \dot{\phi}\underline{3}\cdot\underline{v} + \dot{\vartheta}\underline{1}'\cdot\underline{v} + \dot{\psi}\underline{w}\cdot\underline{v} = \dot{\phi}\underline{3}\cdot\underline{v} + \dot{\vartheta}\underline{1}'\cdot\underline{v} \\ \omega_w = \underline{\omega}\cdot\underline{w} = \dot{\phi}\underline{3}\cdot\underline{w} + \dot{\vartheta}\underline{1}'\cdot\underline{w} + \dot{\psi}\underline{w}\cdot\underline{w} \end{cases}$$

$\underline{3}\,\underline{u}$, $\underline{3}\,\underline{v}$, $\underline{3}\,\underline{w}$ $\longrightarrow$ third column of matrix A_{313}

$\underline{1}'\underline{u}$, $\underline{1}'\underline{v}$, $\underline{1}'\underline{w}$ $\longrightarrow$ first column of matrix A_{313} if $\varphi = 0$

$$\begin{cases} \omega_u = \dot{\phi}\sin\vartheta\sin\psi + \dot{\vartheta}\cos\psi \\ \omega_v = \dot{\phi}\sin\vartheta\cos\psi - \dot{\vartheta}\sin\psi \\ \omega_w = \dot{\phi}\cos\vartheta + \dot{\psi} \end{cases} \longrightarrow \begin{cases} \dot{\phi} = \dfrac{(\omega_u\sin\psi + \omega_v\cos\psi)}{\sin\vartheta} \\ \dot{\vartheta} = \omega_u\cos\psi - \omega_v\sin\psi \\ \dot{\psi} = \omega_w - (\omega_u\sin\psi + \omega_v\cos\psi)\dfrac{\cos\vartheta}{\sin\vartheta} \end{cases}$$

Euler angles - kinematics

sequence 313 $\begin{cases} \dot{\phi} = \dfrac{(\omega_u sin\psi + \omega_v cos\psi)}{sin\vartheta} \\ \dot{\vartheta} = \omega_u cos\psi - \omega_v sin\psi \\ \dot{\psi} = \omega_w - (\omega_u sin\psi + \omega_v cos\psi)\dfrac{cos\vartheta}{sin\vartheta} \end{cases}$

sequence 312 $\begin{cases} \dot{\phi} = \dfrac{(\omega_w cos\psi - \omega_u sin\psi)}{cos\vartheta} \\ \dot{\vartheta} = \omega_u cos\psi + \omega_w sin\psi \\ \dot{\psi} = \omega_v - (\omega_w cos\psi - \omega_u sin\psi)\dfrac{sin\vartheta}{cos\vartheta} \end{cases}$

Direction Cosine Matrices (DCM)

Advantages

- Singularity free.
- Uniquely defines every possible rotation.
- Intuitive.

Disadvantages

- 9 components to evaluate.
- Requires orthonormalization during integration.

Euler axis / angle ?

Quaternion ?

Gibbs vector ?

Euler angles ?