ORBITAL MECHANICS

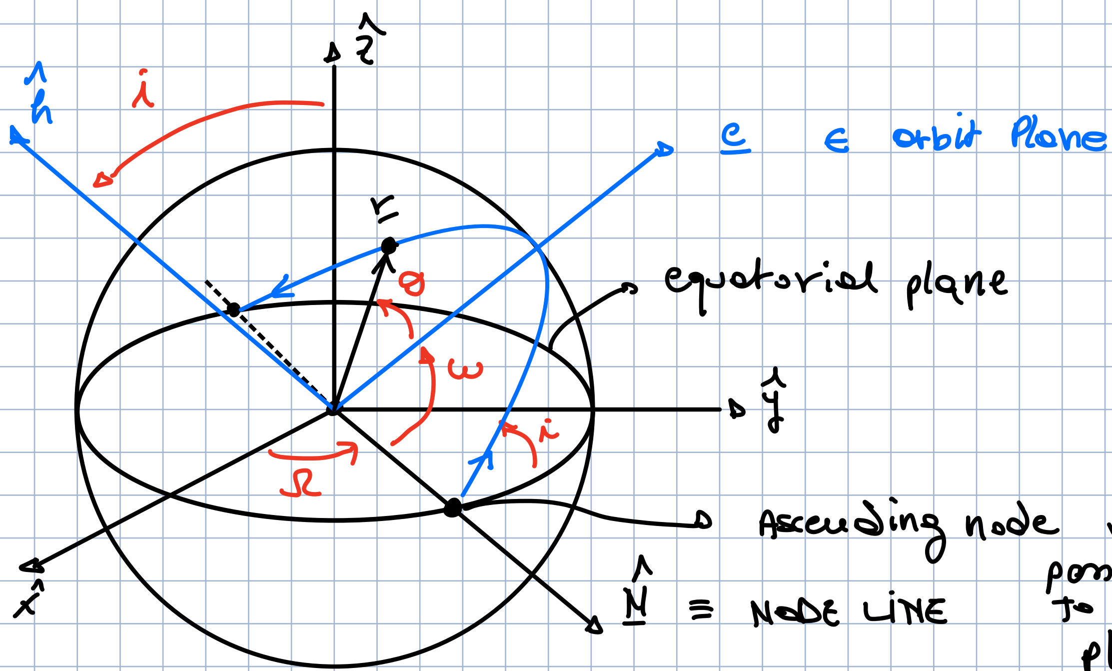
ORBIT ELEMENTS AND START VECTOR (3D ORBIT)

3D orbits requires 3 additional parameters → Those parameters are essentially the euler angles

ECEI (Earth centered Earth inertial) $\hat{x} \equiv \gamma$

$\hat{z}$ = Earth rotation axis

$\hat{y}$ completes reference frame



① Ω = RAAN $\quad 0 \leq \Omega < 2\pi \quad$ RIGHT ASCENSION ASCENDING NODE

② i = INCLINATION $\quad 0 \leq i \leq \pi$

dihedral angle between the orbit plane and the equatorial plane or it can be calculated as the angle between the $\hat{z}$ axis and the $\hat{h}$ axis.

$0 \leq i < \frac{\pi}{2}$ PROGRADE ORBIT ($\hat{h}$ northern emisphere)

$\frac{\pi}{2} \leq i < \pi$ RETROGRADE ORBIT ($\hat{h}$ southern emisphere)

③ ω = ARGUMENT of PERIGEE. $0 \le \omega < 2\pi$

a = SEMI-MAJOR AXIS — it is a mesure of energy or dimension of the orbit.

ENERGY, DIMENSION

e = ECCENTRICITY — SHAPE

i = INCLINATION

Ω = RIGHT ASCENSION OF ASCENDING NODE

ω = ARGUMENT OF PERIGEE

θ = TRUE ANOMALY.

NOTE The ecliptic plane does not have anything to do to with we have done to-day (at least so far).

PROCEDURE TO OBTAIN ORBITAL ELEMENTS FROM STATE VECTOR

$\underline{r}, \underline{v}$ ECEI reference frame

① $\underline{r} = x\hat{i} + y\hat{j} + z\hat{k}$ $\quad r = \sqrt{\underline{r}\cdot\underline{r}} = \sqrt{x^2+y^2+z^2}$ (2.34)

distance

② $\underline{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$ $\quad v = \sqrt{\underline{v}\cdot\underline{v}} = \sqrt{v_x^2+v_y^2+v_z^2}$ (2.35)

↳ module of velocity

$$v_r = \frac{\underline{v}\cdot\underline{r}}{r} = \frac{(x v_x + y v_y + z v_z)}{r}$$

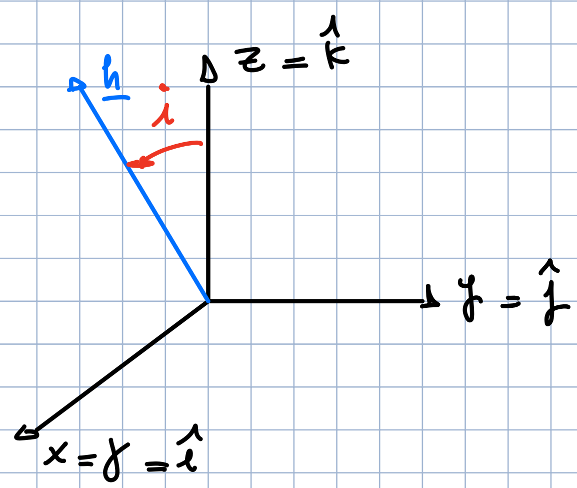
Radial components of the velocity

③ $\underline{h} = \underline{r} \wedge \underline{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ v_x & v_y & v_z \end{vmatrix}$ (2.36)

specific angular momentum

the next step is calculate the magnitude of $\underline{h}$

(4) $$h = \sqrt{\underline{h} \cdot \underline{h}} \qquad (2.37)$$



(5) INCLINATION

$$\cos i = \frac{\underline{h} \cdot \hat{k}}{|\underline{h}|} \qquad \boxed{i = \cos^{-1}\left(\frac{h_z}{h}\right)} \qquad (2.38)$$

The inclination define a cone around the z axis

$0 \le i < \pi \Rightarrow$ so there is no quadrant ambiguity using the 2.38 equation.

$h_z > 0 \qquad 0 \le i < \frac{\pi}{2}$ PROGRADE ORBIT

$h_z < 0 \qquad \frac{\pi}{2} \le i < \pi$ RETROGRADE ORBIT

(6) IDENTIFY $\hat{N}$

$$\left.\begin{array}{l} \hat{N} \in xy \text{ plane} \Rightarrow \hat{N} \perp \hat{z} \\ \hat{N} \in \text{orbit plane} \Rightarrow \hat{N} \perp \hat{h} \end{array}\right\} \Rightarrow \hat{N} \perp \text{ to the plane that contains } \hat{z} \text{ and } \hat{h}$$

$$\hat{N} = \frac{\hat{k} \wedge \underline{h}}{|\hat{k} \wedge \underline{h}|} \quad \text{or} \quad \underline{N} = \hat{k} \wedge \underline{h} \qquad \hat{N} = \frac{\underline{N}}{|\underline{N}|}$$

$$\underline{N} = \hat{K} \wedge \underline{h} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 1 \\ h_x & h_y & h_z \end{vmatrix} = \begin{Bmatrix} -h_y \, \hat{i} \\ h_x \, \hat{j} \\ 0 \, \hat{k} \end{Bmatrix} \qquad (2.39)$$

$\hat{N}$ node line

⑦ Magnitude of N

$$N = \sqrt{\underline{N} \cdot \underline{N}} \qquad (2.40) \qquad \hat{N} = \frac{\underline{N}}{N}$$

⑧ Calculate Ω

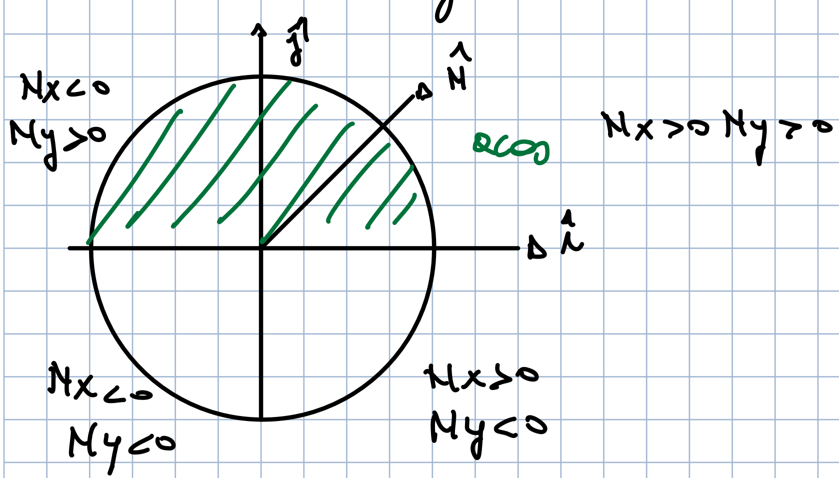
$$\cos\Omega = \hat{N} \cdot \hat{i}$$

$$\Omega = \cos^{-1}(\hat{N} \cdot \hat{i})$$

$$\Omega = \cos^{-1}(N_x)$$

x component of $\hat{N}$

the $\cos^{-1}$ gives an answer always between 0 and 180°



We can exploit the fact that depending on the sign of the sin Ω or $\hat{N}_y$ is larger or smaller than 0 it can gives us a proper value of the right ascension. In that way there are not ambiguity.

$$\boxed{\Omega = \begin{cases} \cos^{-1}(N_x) & \text{if } N_y \geq 0 \\ 2\pi - \cos^{-1}(N_x) & \text{if } N_y < 0 \end{cases}} \qquad (2.41)$$

(9) $$\boxed{\underline{e} = \frac{\underline{v} \wedge \underline{h}}{\mu} - \frac{\underline{r}}{r}}$$ (2.42) ← already seen

$$\underline{e} = \frac{1}{\mu}\left(\underline{v} \wedge \underline{h} - \frac{\underline{r}}{r}\mu\right) = \frac{1}{\mu}\left(\underline{v} \wedge (\underline{r} \wedge \underline{v}) - \mu \frac{\underline{r}}{r}\right)$$

$$\underline{v} \wedge (\underline{r} \wedge \underline{v}) = (\underline{v} \circ \underline{v})\,\underline{r} - (\underline{v} \circ \underline{r})\,\underline{v}$$

$$= \underline{r}\, v^2 - \underline{v}\,(\underline{v} \circ \underline{r})$$

recall $\underline{a} \wedge (\underline{b} \wedge \underline{c}) = \underline{b}(\underline{a} \circ \underline{c}) - (\underline{a} \circ \underline{b})\,\underline{c}$

$$\boxed{\underline{e} = \frac{1}{\mu}\left[\left(v^2 - \frac{\mu}{r}\right)\underline{r} - r v_r\, \underline{v}\right]}$$

if $v_r > 0$ satellite is going away from the perigee

if $v_r < 0$ satellite is going towards the perigee

(10) $$e = \sqrt{\underline{e} \circ \underline{e}}$$ 2.43

(11) $$\omega = \cos^{-1}\left(\frac{\underline{N}}{N} \circ \frac{\underline{e}}{e}\right)$$

it just will gives an angle between 0 and π

but we said that $\omega \in [0, 2\pi]$

$\underline{N} \circ \underline{e} > 0$ $\quad \omega \in 1^{st}\ 4^{th}$ quadrant

$\underline{N} \circ \underline{e} < 0$ $\quad \omega \in 2^{nd}\ 3^{rd}$ quadrant

$$\boxed{\omega = \begin{cases} \cos^{-1}\left(\frac{\underline{N}}{N} \circ \frac{\underline{e}}{e}\right) & \text{if } e_z \geq 0 \\ 2\pi - \cos^{-1}\left(\frac{\underline{N}}{N} \circ \frac{\underline{e}}{e}\right) & \text{if } e_z < 0 \end{cases}} \quad (2.44)$$

(12) True anomaly

$$\theta = \cos^{-1}\left(\frac{\underline{e}}{e} \circ \frac{\underline{r}}{r}\right)$$

if $\underline{e} \circ \underline{r} > 0$ $\quad \theta \in$ 1st 4th quadrant

if $\underline{e} \circ \underline{r} < 0$ $\quad \theta \in$ 2nd 3rd quadrant

$v_r \geq 0 \quad \underline{r} \circ \underline{v} \geq 0 \quad 0 \leq \theta \leq \pi$

$v_r < 0 \quad \underline{r} \circ \underline{v} < 0 \quad \pi < \theta < 2\pi$

$$\boxed{\theta = \begin{cases} \cos^{-1}\left(\frac{\underline{e}}{e} \circ \frac{\underline{r}}{r}\right) & \text{if} \quad v_r = \underline{v} \circ \underline{r} \geq 0 \\ 2\pi - \cos^{-1}\left(\frac{\underline{e}}{e} \circ \frac{\underline{r}}{r}\right) & \text{if} \quad v_r = \underline{v} \circ \underline{r} < 0 \end{cases}} \qquad (2.45)$$

PERIFOCAL FRAME

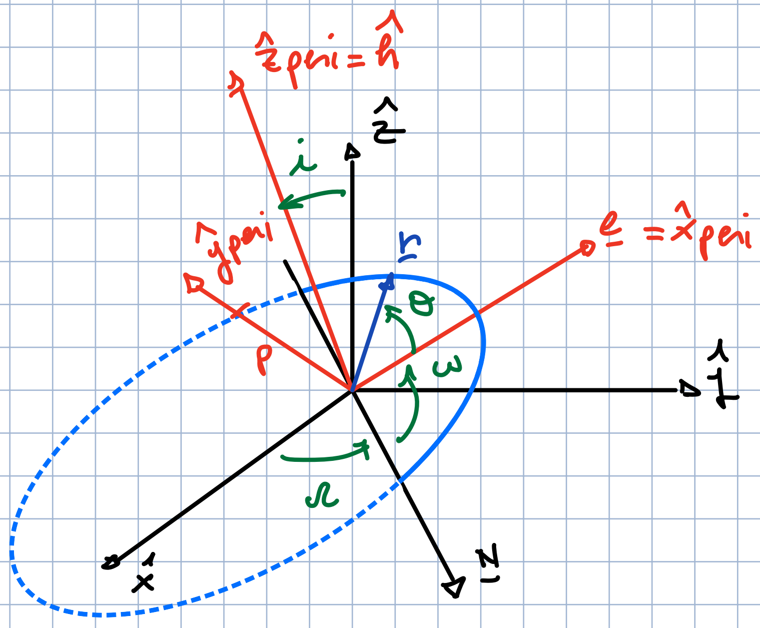
Perifocal frame is the natural frame associated to an orbit cartesian reference frame fixed in space, centered at orbit focus.

$\hat{x}_{peri}$ from focus to periapsis $\hat{e}$

$\hat{y}_{peri}$ lies at $\frac{\pi}{2}$ from $\hat{e} \equiv \hat{x}_{peri}$ on orbit plane

$\hat{z}_{peri}$, lies along $\hat{h}$

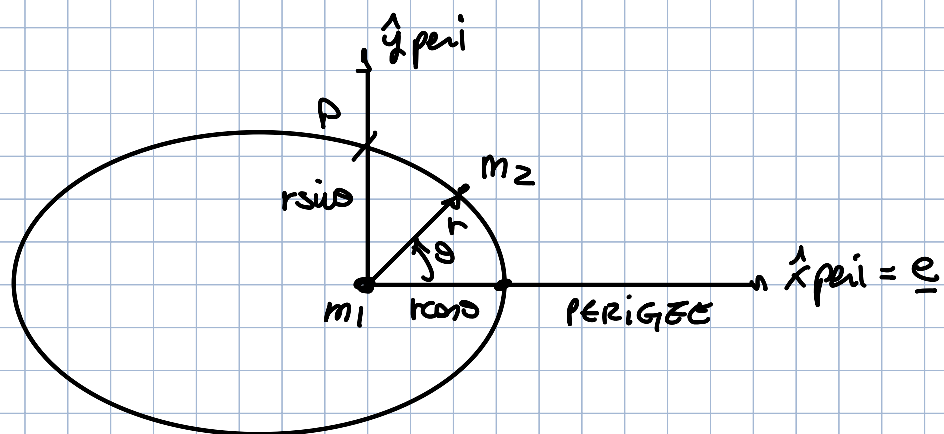
$$\begin{cases} \hat{z}_{peri} = \frac{\underline{h}}{h} \\ \hat{x}_{peri} = \frac{\underline{e}}{e} \end{cases} \qquad (2.46)$$



The perifocal frame simplify a lot the rappresentation of $\underline{r}$

$$\underline{r}_{perifocalframe} = x_p \hat{x}_{peri} + y_p \hat{y}_{peri} \quad (2.47)$$

$$\begin{cases} x_p = r\cos\theta \\ y_p = r\sin\theta \end{cases} \quad (2.48)$$



r can be computed by the conical equation.

$$r = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta} \qquad \underline{r}_{peri} = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta}\left(\cos\theta\,\hat{x}_{peri} + \sin\theta\,\hat{y}_{peri}\right)$$

(eq 1.26) (2.49)

velocity as the time derivative of $\underline{r}$

$$\begin{aligned} \dot{x}_p &= \dot{r}\cos\theta - r\dot{\theta}\sin\theta \\ \dot{y}_p &= \dot{r}\sin\theta + r\dot{\theta}\cos\theta \end{aligned} \quad (2.50)$$

$$\dot{r} = v_r = \frac{\mu}{h} e \sin\theta \quad (\text{Eq } 1.33)$$

$$r\dot{\theta} = v_\theta = \frac{\mu}{h}(1+e\cos\theta) \quad (\text{Eq } 1.34)$$

$$\dot{x}_p = \frac{\mu}{h} e \sin\theta \cos\theta - \frac{\mu}{h}(1+e\cos\theta)\sin\theta$$

$$= \frac{\mu}{h}\left(e \cancel{\sin\theta\cos\theta} - \sin\theta - e\cancel{\cos\theta\sin\theta}\right) = -\frac{\mu}{h}\sin\theta$$

$$\dot{y}_p = \frac{\mu}{h} e \sin^2\theta + \frac{\mu}{h}(1+e\cos\theta)\cos\theta$$

$$= \frac{\mu}{h} e \sin^2\theta + \frac{\mu}{h} e \cos^2\theta + \frac{\mu}{h}\cos\theta = \frac{\mu}{h}(e+\cos\theta)$$

Therefore

$$\underline{v}_{peri} = \frac{\mu}{h}\left[-\sin\theta\, \hat{x}_{peri} + (e+\cos\theta)\,\hat{y}_{peri}\right] \quad (2.51)$$

$\underline{v}_{peri}$ and $\underline{r}_{peri}$ do not have components in $\hat{z}_{peri}$.

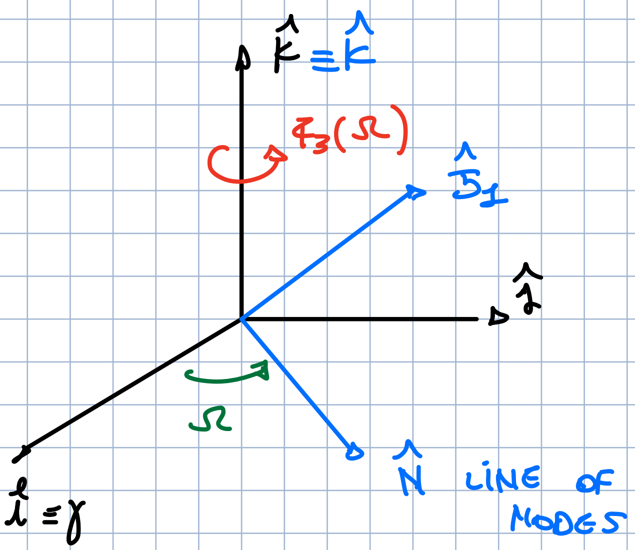
IDENTIFY THE ORBIT COORDINATE TRANSFORMATION FROM INERTIAL TO ORBITAL REFERENCE FRAME

$\underline{r}, \underline{v}$ in inertial frame $\longrightarrow$ orbital frame

- INERTIAL FRAME $\{\hat{i}, \hat{j}, \hat{k}\}$ (can be equatorial or ecliptic) $\hat{i} = \gamma$
- PERIFOCAL FRAME $\{\hat{e}, \hat{p}, \hat{h}\}$
- ORBITAL FRAME $\{\hat{r}, \hat{\theta}, \hat{h}\}$ this is not inertial it rotate in time $\hat{h}$ = fix but $\hat{r}, \hat{\theta}$ are not fixed.

① From $\{\hat{i},\hat{j},\hat{k}\}$ RF perform a rotation R @ $\hat{k}$

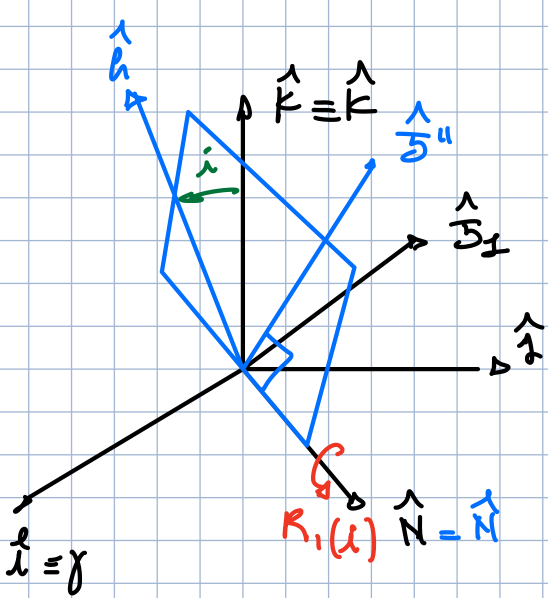
$\hat{i} \longrightarrow \hat{N}$ $\quad R_3(\Omega)$



$$\begin{Bmatrix} \hat{N} \\ \hat{S}_1 \\ \hat{K} \end{Bmatrix} = R_3(\Omega) \begin{Bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{Bmatrix} \qquad R_3(\Omega) = \begin{bmatrix} \cos\Omega & +\sin\Omega & 0 \\ -\sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

↳ Rotation matrix

② Perform a rotation @ $\hat{N}$ of i → orbital plan



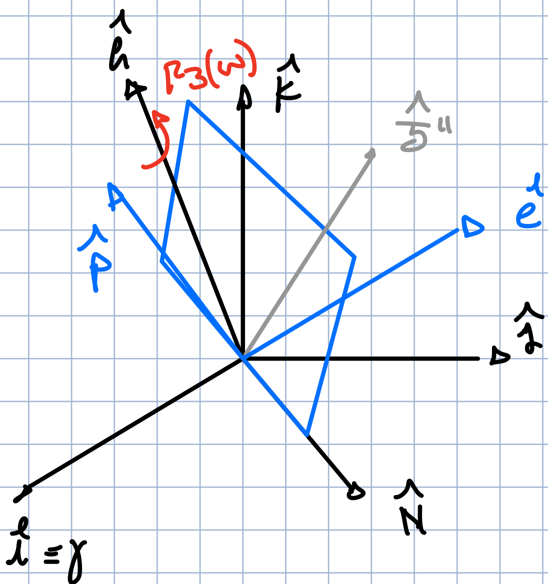
$$\begin{Bmatrix} \hat{N} \\ \hat{S}'' \\ \hat{h} \end{Bmatrix} = R_1(i) \begin{Bmatrix} \hat{N} \\ \hat{S}' \\ \hat{K} \end{Bmatrix}$$

$$R_1(i) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos i & \sin i \\ 0 & -\sin i & \cos i \end{bmatrix}$$

③ Rotation @ $\hat{h}$ of an angle ω $\quad \hat{N} \Rightarrow \hat{e}$

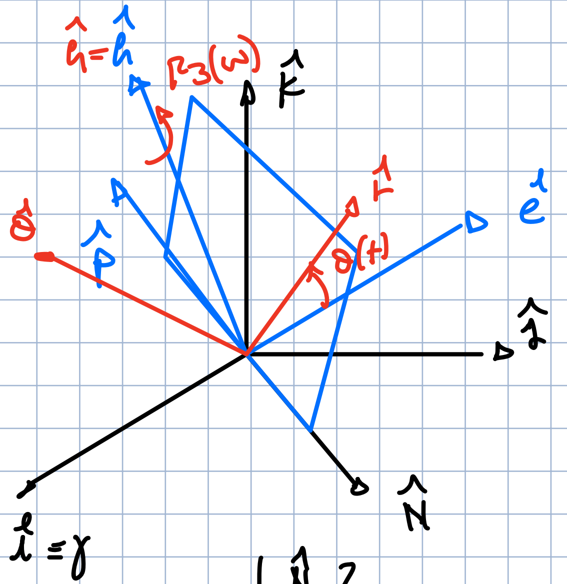
$$\begin{Bmatrix}\hat{e}\\ \hat{p}\\ \hat{h}\end{Bmatrix} = R_3(\omega)\begin{Bmatrix}\hat{N}\\ \hat{S}''\\ \hat{h}\end{Bmatrix} \qquad R_3(\omega) = \begin{bmatrix}\cos(\omega) & \sin(\omega) & 0\\ -\sin(\omega) & \cos(\omega) & 0\\ 0 & 0 & 1\end{bmatrix}$$

We have reached the perifocal frame



$$\boxed{\begin{Bmatrix}\hat{e}\\ \hat{p}\\ \hat{h}\end{Bmatrix} = R_3(\omega)\,\underbrace{R_1(i)\,\underbrace{R_3(\Omega)\begin{Bmatrix}\hat{I}\\ \hat{J}\\ \hat{K}\end{Bmatrix}}_{\begin{Bmatrix}\hat{N}\\ \hat{S}'\\ \hat{K}\end{Bmatrix}}}_{\begin{Bmatrix}\hat{N}\\ \hat{S}''\\ \hat{h}\end{Bmatrix}}} \qquad (2.52)$$

③' instead of $R_3(\omega)$ we do $R_3(\omega+\theta)$ we reach the orbital frame



$$\begin{Bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{h} \end{Bmatrix} = R_3(\omega+\theta(t)) \begin{Bmatrix} \hat{N} \\ \hat{S}'' \\ \hat{h} \end{Bmatrix} \qquad R_3(\omega+\theta(t)) = \begin{bmatrix} \cos(\omega+\theta) & \sin(\omega+\theta) & 0 \\ -\sin(\omega+\theta) & \cos(\omega+\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

↳ rotating frame

$$\boxed{\begin{Bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{h} \end{Bmatrix} = R_3(\omega+\theta(t))\, R_1(i)\, R_3(\Omega) \begin{Bmatrix} \hat{I} \\ \hat{J} \\ \hat{K} \end{Bmatrix}} \qquad (2.53)$$

(inertial frame → orbital frame)

$\hat{r}$ = radial

$\hat{\theta}$ = transversal / azimuth (2.54)

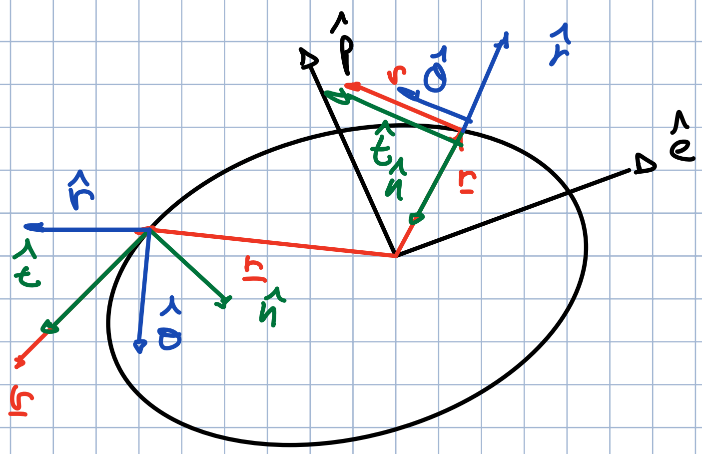
$\hat{h}$ = out of plane

NOTE There is a 4th reference frame that takes into account γ = flight path angle

$\hat{t}$ = tangent to $\underline{v}$

$\hat{n}$ = normal to $\underline{v}$ in orbital plane (2.55)

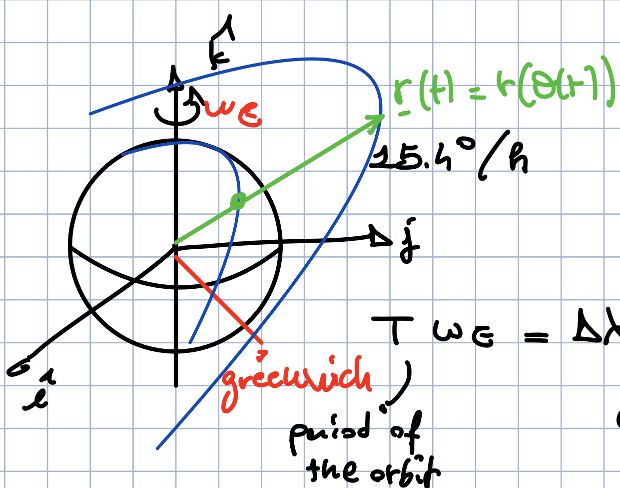
$\hat{h}$



GROUND TRACKS

GROUND TRACK: projection of s/c orbit onto the Earth's surface

@ t draw line goes from center of Earth to s/c



15.4°/h Eastwards

$T\omega_E = \Delta\lambda$ displacement of groundtrack at equator (2.57)