Spacecraft Attitude Dynamics

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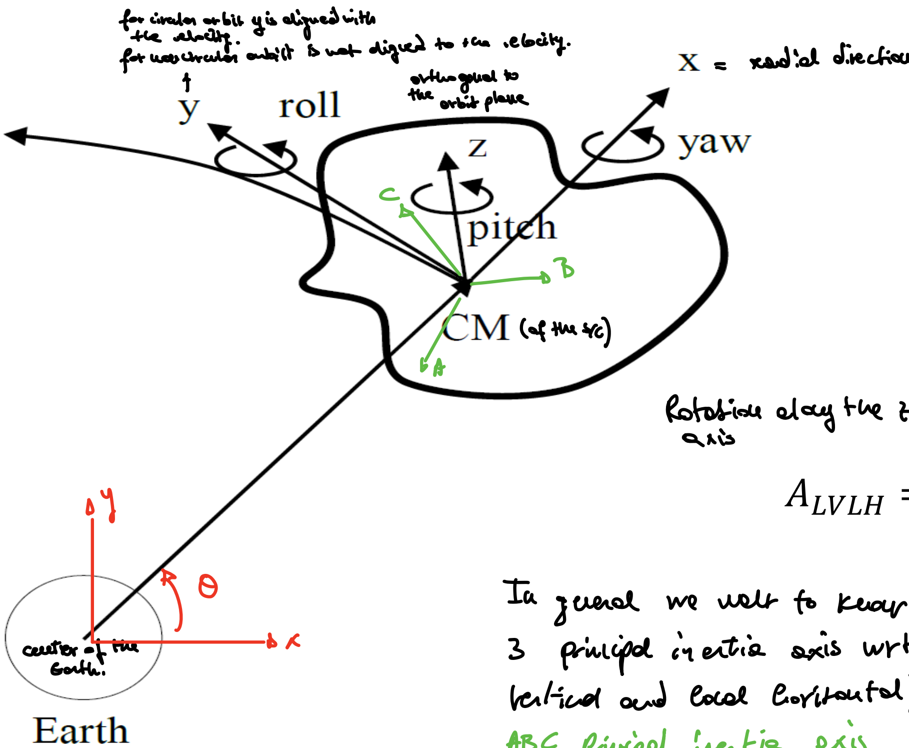
Moving reference frames and Gravity Gradient Torque

Rotating reference frames



The CM is moving along the orbit so it is not an inertial reference system.
How we could find the orientation of the local reference frame?

Rotation along the z axis

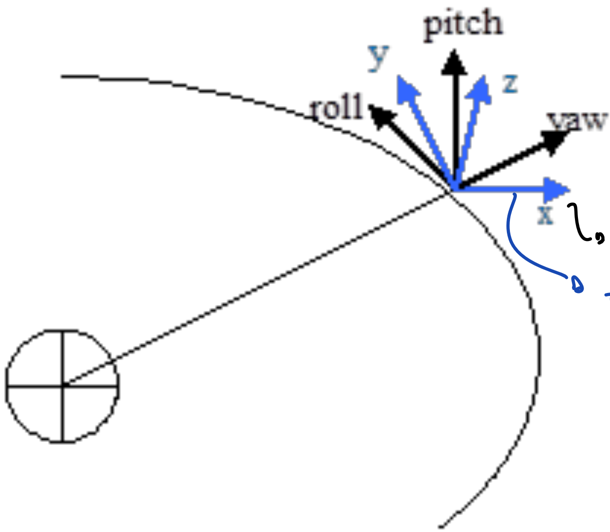
$$A_{LVLH} = \begin{bmatrix} \cos\theta(t) & \sin\theta(t) & 0 \\ -\sin\theta(t) & \cos\theta(t) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In general we want to know the orientation and position of the 3 principal inertia axis wrt the local reference frame (local vertical and local horizontal).

ABC principal inertia axis

Local Vertical Local Horizontal (LVLH) frame

Linearization about the LVLH frame (circular orbit)



How to rappresent a rotation of the 3 principal inertia axis wrt a moving frame.

Local vertical / horizontal frame (moving frame)

Body frame

Body fixed frame

Set of rotation using Euler angles

$A_{B/LVLH} = A_{123}$ Direction cosine matrix with these rotations first around axis 1 second axis 2 and third axis 3.

$$= \begin{bmatrix} \cos\alpha_z\cos\alpha_y & \cos\alpha_z\sin\alpha_y\sin\alpha_x + \sin\alpha_z\cos\alpha_x & -\cos\alpha_z\sin\alpha_y\cos\alpha_x + \sin\alpha_z\sin\alpha_x \\ -\sin\alpha_z\cos\alpha_y & -\sin\alpha_z\sin\alpha_y\sin\alpha_x + \cos\alpha_z\cos\alpha_x & \sin\alpha_z\sin\alpha_y\cos\alpha_x + \cos\alpha_z\sin\alpha_x \\ \sin\alpha_y & -\cos\alpha_y\sin\alpha_x & \cos\alpha_y\cos\alpha_x \end{bmatrix}$$

α_z = rotation around pitch axis
α_x = rotation around yaw axis
α_y = rotation around roll axis

This is true for this sequence.

we want to do a linearization assuming that $\alpha_x, \alpha_y, \alpha_z \ll 1$

$\cos\alpha_i = 1$
$\sin\alpha_i = \alpha_i$

$\alpha_i\alpha_j \simeq 0$ negligible term.

Linearization about the LVLH frame (circular orbit)

$$A = \begin{bmatrix} 1 & \alpha_z & -\alpha_y \\ -\alpha_z & 1 & \alpha_x \\ \alpha_y & -\alpha_x & 1 \end{bmatrix}$$

How we can convert the equation of the angular velocity originally relative to xyz frame to the pitch yaw roll frame?

$$\boxed{\begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} = \begin{bmatrix} 1 & \alpha_z & -\alpha_y \\ -\alpha_z & 1 & \alpha_x \\ \alpha_y & -\alpha_x & 1 \end{bmatrix} \begin{Bmatrix} \dot{\alpha}_x \\ \dot{\alpha}_y \\ \dot{\alpha}_z + n \end{Bmatrix}}$$

1

Combining the relative angular velocity + the absolute angular velocity of the rotating frame.

circular orbit => n = constant
No circular orbit => n = n(t) time dependent.

This approximation is valid only when ?

$\alpha_x, \alpha_y, \alpha_z, \dot{\alpha}_x, \dot{\alpha}_y, \dot{\alpha}_z$ are assumed to be small in order to linearize

n is the nominal angular velocity of the satellite along its orbit (typically, one rotation per orbit)

Linearization about the LVLH frame

$$\begin{cases} \omega_x = \dot{\alpha}_x - \alpha_y n \\ \omega_y = \dot{\alpha}_y + \alpha_x n \\ \omega_z = \dot{\alpha}_z + n \end{cases}$$

→ comes from the expansion of the equation ①

$$\dot{\omega}_x = \frac{I_y - I_z}{I_x}\omega_y\omega_z$$

$$\dot{\omega}_y = \frac{I_z - I_x}{I_y}\omega_x\omega_z$$

$$\dot{\omega}_z = \frac{I_x - I_y}{I_z}\omega_y\omega_x$$

$$\begin{cases} \dot{\omega}_x = \ddot{\alpha}_x - \dot{\alpha}_y n \\ \dot{\omega}_y = \ddot{\alpha}_y + \dot{\alpha}_x n \\ \dot{\omega}_z = \ddot{\alpha}_z \end{cases}$$

Attitude dynamics relative to an LVLH frame (with circular orbit) Linear Equations

$$\begin{cases} I_x\ddot{\alpha}_x + n(I_z - I_y - I_x)\dot{\alpha}_y + n^2(I_z - I_y)\alpha_x = 0 \\ I_y\ddot{\alpha}_y + n(I_x + I_y - I_z)\dot{\alpha}_x + n^2(I_z - I_x)\alpha_y = 0 \\ I_z\ddot{\alpha}_z = 0 \end{cases}$$

→ The terms on the right hand side cancel out.

These two equations are coupled ⇒ what we can do is to study the stability condition of these equations

Stability analysis and stability diagram → Done by finding the roots of the characteristic equations of the system.

$$\begin{cases} \ddot{\alpha}_x + n(K_x - 1)\dot{\alpha}_y + n^2 K_x \alpha_x = 0 \\ \ddot{\alpha}_y + n(1 - K_y)\dot{\alpha}_x + n^2 K_y \alpha_y = 0 \end{cases}$$

we went from the stability conditions in terms of angular velocity to stability condition wrt angles.
We find that the stability condition remains the same as before.

$$K_x = \frac{I_z - I_y}{I_x}$$

$$K_y = \frac{I_z - I_x}{I_y}$$

$$K_z = \frac{I_y - I_x}{I_z}$$

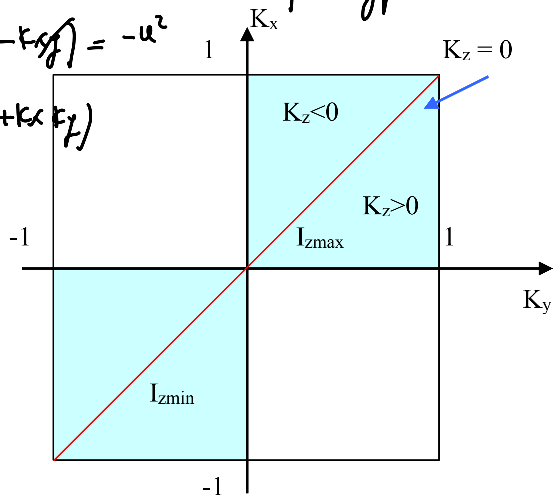
$$\lambda^2 = \frac{-n^2(1 + K_x K_y) \pm n^2\sqrt{(1 - K_x K_y)^2}}{2}$$

$$= \frac{-n^2(1+|K_xK_y|) \pm n^2(1-K_xK_y)}{2}$$

$$= \lambda_1^2 = -\frac{n^2}{2}(1+\cancel{K_xK_y}+1-\cancel{K_xK_y}) = -n^2$$

$$\lambda_2^2 = -\frac{n^2}{2}(\cancel{1}+K_xK_y-\cancel{1}+K_xK_y) = -\frac{n^2}{\cancel{2}}\cancel{2}K_xK_y$$

The I_z should be the minimum or maximum inertia axis → The minimum case is true only for ideal solution. When the solution for I_z = max I require every stable featuring is also the dissipation of energy.



$\lambda_1^2 = -n^2$ imaginary number

$\lambda_1 = \pm i n$

$\lambda_2 = \pm n\sqrt{-K_xK_y}$

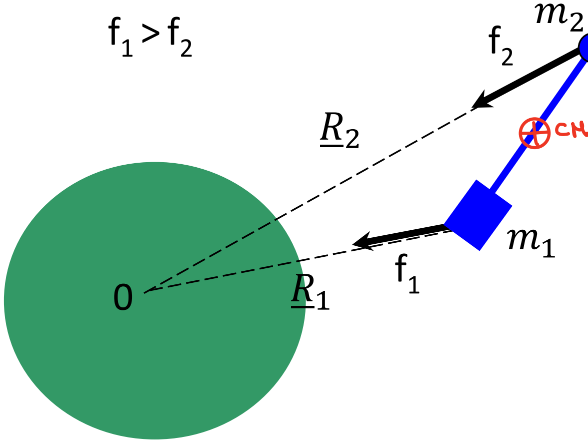
<0 we are sure

$K_xK_y > 0$

$K_x > 0 \quad K_y > 0$ } 2 case.
$K_x < 0 \quad K_y < 0$

Gravity gradient disturbance torque

Imagine that the s/c is made up by two separate masses. → On every concentrated mass act the gravitational force → m_1 - m_2. Assuming that $f_1 \neq f_2$ because the distances between the center of the Earth are not the same.
Also the two forces are not parallel.
At the center of mass there will be a net torque due to the fact that these two gravitational forces are not equal and they aren't aligned.
It is not 100% a disturbances because it is due to the shape of the s/c ⇒ It is possible to model the total torque.



$$M_1 = -\underline{r}_{1-c} \times \frac{G m_t m_1}{|R_1|^3} (\underline{R}_1)$$

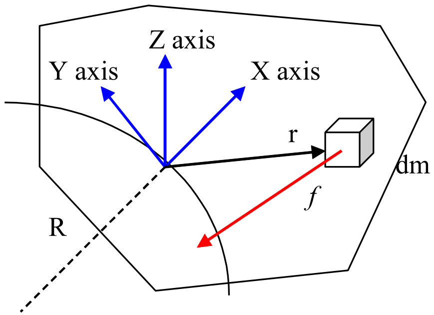
$$M_2 = -\underline{r}_{2-c} \times \frac{G m_t m_2}{|R_2|^3} (\underline{R}_2)$$

The difference in the two gravitational forces become more evident as the s/c becomes larger.

NOTE the scales $\underline{r}_1, \underline{r}_2$ thousand of kilometers where the distance between m_1 and m_2 is in the order of meter upto 100 meters.
So it is important to understand the weight of each term.

Gravity Gradient - *M*

→ After we have computed the gravitational force $\underline{f}$ compute the Torque applied to the center of mass.



infinitesimal mass

$$dM = -\underline{r} \wedge \frac{Gm_t dm}{|R+r|^3}(\underline{R} + \underline{r})$$

$$M = -\int_B \underline{r} \wedge \frac{Gm_t}{|R+r|^3}(\underline{R} + \underline{r})dm$$

$\frac{1}{(1+x)^3} \simeq 1 - 3x$ $\quad \frac{1}{(R+r)^3} \simeq \frac{1}{R^3} - \frac{3\underline{r}}{(R+r)^2}$

Approximate

$r \ll R$

$R + r \simeq R$

$\frac{1}{|R+r|^3} \simeq \frac{1}{R^3} - \frac{3\underline{r}}{(R+r)^2} = \frac{1}{R^3} - \frac{3\underline{r}\cdot\underline{R}}{\underbrace{(R+r)^2}_{\sim R^2}}$

$\simeq \frac{1}{R^3} - 3\frac{\underline{r}\cdot\underline{R}}{R^2}$

$$M = -\frac{Gm_t}{R^3}\int_B \underline{r} \wedge \left(1 - 3\frac{\underline{\underline{R}}\cdot\underline{\underline{r}}}{R^2}\right)(\underline{R} + \underline{r})dm$$

$$\boxed{M = \frac{3Gm_t}{R^5}\int_B (\underline{r}\cdot\underline{R})(\underline{r}\wedge\underline{R})dm}$$

Exercise: Evaluate the Gravity Gradient in the body Frame

↳ it could be considered in the local vertical and local horizontal frame
$\underline{t} = [1,0,0]$ → but this is not very useful because we want to know the Torque among the principal axis frame.

$$M = \frac{3Gm_t}{R^5}\int_B (\underline{r}\cdot\underline{R})(\underline{r}\wedge\underline{R})dm$$

Referred to body frame

↳ $\underline{R}_B = R\underbrace{[c_1 \quad c_2 \quad c_3]^T}_{\text{Direction cosine.}}$ $\qquad \underline{r} = x\underline{b}_1 + y\underline{b}_2 + z\underline{b}_3$

↑ vector r into the body frame of reference

$$\underline{M} = \frac{3Gm_t}{R^3}\int_B \begin{pmatrix}(y^2-z^2)c_2c_3\\(z^2-x^2)c_1c_3\\(x^2-y^2)c_1c_2\end{pmatrix}dm = \frac{3Gm_t}{R^3}\begin{Bmatrix}(I_z-I_y)c_2c_3\\(I_x-I_z)c_1c_3\\(I_y-I_x)c_1c_2\end{Bmatrix}$$ → evaluation of the gravity gradient in principal axis

therefore

$$\boxed{\underline{M} = \frac{3Gm_t}{R^3}\begin{Bmatrix}(I_z-I_y)c_2c_3\\(I_x-I_z)c_1c_3\\(I_y-I_x)c_1c_2\end{Bmatrix}}$$

→ It is not that small because we have the mass of the Earth at the numerator

↳ Difference of inertia moment represent what before was the size of the spacecraft.

It is possible to rewrite the Euler equation in order to have at the right hand side of the equation the expression of the gravity gradient torque.

$$\begin{cases}I_x\dot\omega_x + (I_z-I_y)\omega_z\omega_y = \dfrac{3Gm_t}{R^3}(I_z-I_y)c_3c_2\\ I_y\dot\omega_y + (I_x-I_z)\omega_x\omega_z = \dfrac{3Gm_t}{R^3}(I_x-I_z)c_1c_3\\ I_z\dot\omega_z + (I_y-I_x)\omega_y\omega_x = \dfrac{3Gm_t}{R^3}(I_y-I_x)c_2c_1\end{cases}$$

This is a new form of dynamics, we have introduced the distance between the CM and the center of the earth and the component $c_1\ c_2\ c_3$ that represent the actual orientation of the s/c.

Linearized equations in trivial LVLH frame (circular orbit)

quite small contribution for the gravity gradient torque but compared with I_j (also small) they are the same order of magnitude

$$I_x\dot{\omega}_x + (I_z - I_y)\omega_z\omega_y = \frac{3Gm_t}{R^3}(I_z - I_y)c_3c_2$$

$$I_y\dot{\omega}_y + (I_x - I_z)\omega_x\omega_z = \frac{3Gm_t}{R^3}(I_x - I_z)c_1c_3$$

$$I_z\dot{\omega}_z + (I_y - I_x)\omega_y\omega_x = \frac{3Gm_t}{R^3}(I_y - I_x)c_2c_1$$

$$\begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = \begin{bmatrix} 1 & \alpha_z & -\alpha_y \\ -\alpha_z & 1 & \alpha_x \\ \alpha_y & -\alpha_x & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

n is constant for circular orbit if e<0.1 this still a correct approximation

$$\boxed{n^2 = \frac{Gm_t}{R^3}}$$

NOMINAL ANGULAR VELOCITY ALONG THE ORBIT

If we are at LEO orbits the gravity gradient is important.
If we are at GEO orbits the gravity gradient is less importants.

$$\begin{cases} I_x\ddot{\alpha}_x + (I_z - I_y - I_x)n\dot{\alpha}_y + (I_z - I_y)n^2\alpha_x = 0 \\ I_y\ddot{\alpha}_y + (I_x + I_y - I_z)n\dot{\alpha}_x + (I_z - I_x)n^2\alpha_y = 3n^2(I_x - I_z)\alpha_y \\ I_z\ddot{\alpha}_z = -3n^2(I_y - I_x)\alpha_z \end{cases}$$

Linearized equations in trivial LVLH frame (circular orbit)

$$I_x\dot{\omega}_x + (I_z - I_y)\cancel{\omega_z}^{\,0}\omega_y = \frac{3Gm_t}{R^3}(I_z - I_y)c_3c_2$$

$$I_y\dot{\omega}_y + (I_x - I_z)\cancel{\omega_x}^{\,0}\omega_z = \frac{3Gm_t}{R^3}(I_x - I_z)c_1c_3$$

$$I_z\dot{\omega}_z + (I_y - I_x)\cancel{\omega_y}^{\,0}\omega_x = \frac{3Gm_t}{R^3}(I_y - I_x)c_2c_1$$

(In nominal condition $\omega_z \neq 0$, $\omega_x, \omega_y = \phi$)

$$n^2 = \frac{Gm_t}{R^3}$$

generic rotation of a small angle !

α_x = yaw
α_y = roll
α_z = pitch

$$\begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = \begin{bmatrix} 1 & \alpha_z & -\alpha_y \\ -\alpha_z & 1 & \alpha_x \\ \alpha_y & -\alpha_x & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

rotates any vector in a master reference frame into the body frame of reference (principal axis frame).

We have transferred the non linear system into a linear system where all the variable are α.

rotation around yaw, pitch, roll axis
These are a set of 3 euler angles

$c_3(\alpha_y)$, $c_2(\alpha_z)$ has perturbation, by linearizing it vanished

$$\begin{cases} I_x\ddot{\alpha}_x + (I_z - I_y - I_x)n\dot{\alpha}_y + (I_z - I_y)n^2\alpha_x = 0 \\ I_y\ddot{\alpha}_y + (I_x + I_y - I_z)n\dot{\alpha}_x + (I_z - I_x)n^2\alpha_y = 3n^2(I_x - I_z)\alpha_y \\ I_z\ddot{\alpha}_z = -3n^2(I_y - I_x)\alpha_z \end{cases}$$

This is a good example of a state system = a 6th order state system because there are 3 second order equation $\ddot{\alpha}, \dot{\alpha}, \alpha$.
It includes the gravity gradient effect on the motion of a s/c.

We can say that in the linearized form the gravitational gradient torque act only in the y and z equations.
We can see a partial decoupling of the system. We can replicate the stability analysis for this equation and in that way we will study the effect of the gravity gradient.

Stability conditions

The first two equations are coupled → we proceed with the analysis of the stability for these two equations (system).

$$\ddot{\alpha}_x + (K_x - 1)n\dot{\alpha}_y + K_x n^2 \alpha_x = 0$$
$$\ddot{\alpha}_y + (1 - K_y)n\dot{\alpha}_x + 4K_y n^2 \alpha_y = 0$$

$$K_x = \frac{I_z - I_y}{I_x}$$
$$K_y = \frac{I_z - I_x}{I_y}$$

Eigenvalues of the matrix

This is a fourth order equation → it can be seen as a second order equation with λ^2 as a variable.

$$\lambda^4 + n^2\lambda^2\left(1 + 3K_x + K_xK_y\right) + 4n^4K_xK_y = 0$$

$$x^2 + bx + c = 0$$

$$\begin{cases} c > 0 \\ \dfrac{-b \pm \sqrt{b^2 - 4c}}{2} < 0 \\ b^2 - 4c > 0 \end{cases}$$

$$\left(1 + 3K_y + K_yK_x\right)^2 > 16K_yK_x$$

$$K_xK_y > 0$$ → standard stability condition that we have already found.

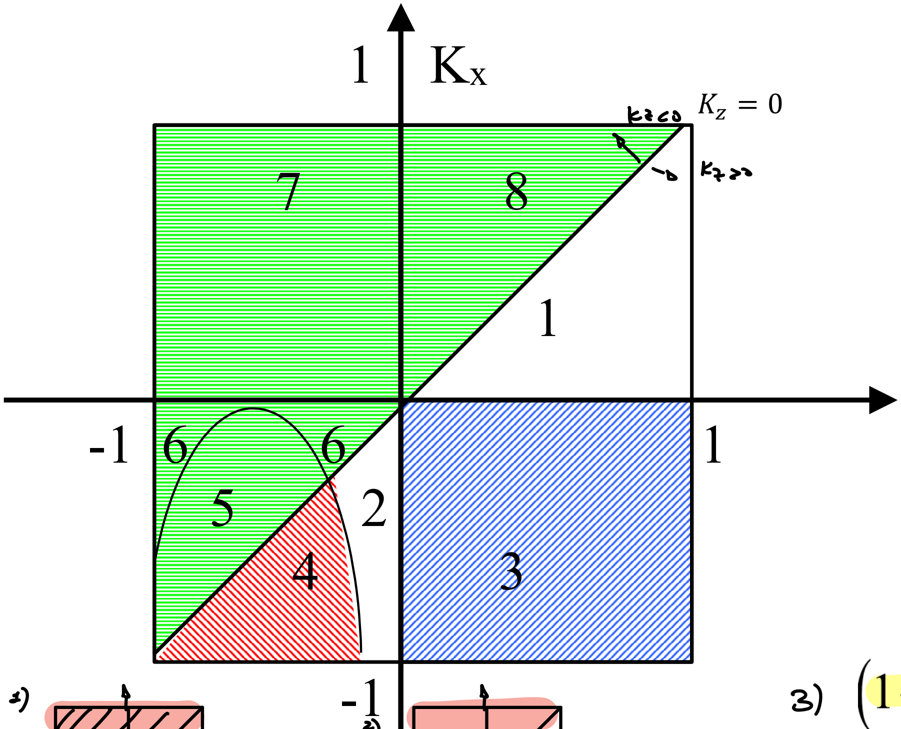
We should also consider the stability requirements of the z equation.

$$I_z\ddot{\alpha}_z = -3n^2(I_y - I_x)\alpha_z$$

$$I_z\ddot{\alpha}_z + 3n^2(I_y - I_x)\alpha_z = 0$$ → stiffness of a second order oscillator

$$\underbrace{\frac{I_y - I_x}{I_z}}_{K_z} > 0 \Rightarrow \boxed{K_z > 0}$$

Stability analysis of the linear equations in x-y plane



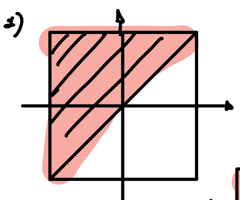
$$K_x = \frac{I_z - I_y}{I_x}$$

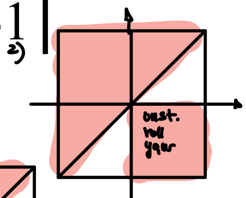
$$K_y = \frac{I_z - I_x}{I_y}$$

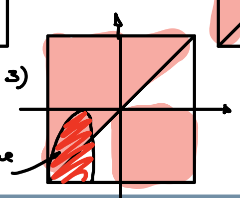
$$K_z = \frac{I_y - I_x}{I_z}$$

NOTE the graph does not depend on the size of the s/c → Kx, Ky, Kz are adimensional and their absolute value is not important. It is important on how the spacecraft is oriented in space

1 and 2 stability
3 and 4 y axis instability
5 and 7 x, y, z axis instability
6 and 8 z axis instability

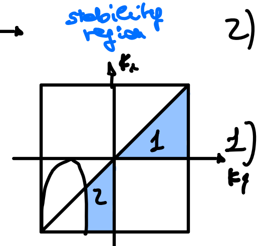






Difficult to represent we could compute some point and after that we can try to represent it

⇒



3) $\left(1+3K_y+K_yK_x\right)^2 > 16K_yK_x$

2) $K_xK_y > 0$

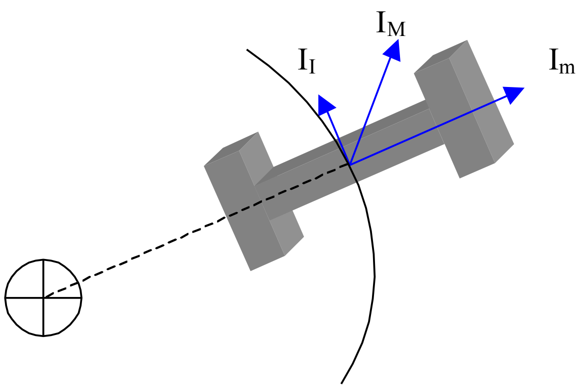
1) $K_z > 0$

Kz can be represented as a function of Ky and Kx because they are all function of Ix Iy Iz.
Kz = 0 line dividing first and third quadrant.

Stability conditions

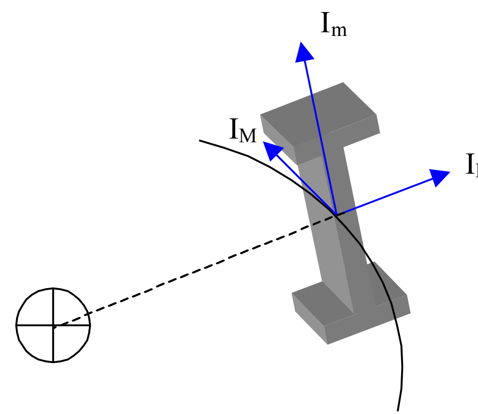
Region 1

$$I_z > I_y > I_x$$



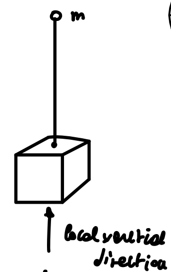
Region 2

$$I_y > I_x > I_z$$



Align the longest direction of the spacecraft with the local vertical => radial direction.

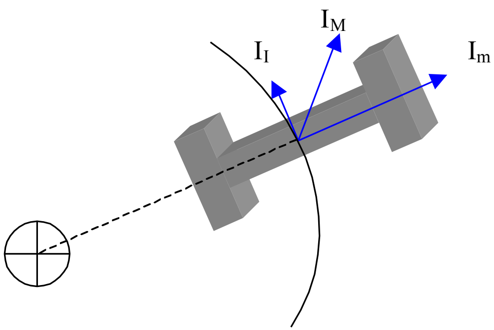
Assume that we have a compact s/c in cubic shape → not subject to strong gravity gradient => stability might be reduce by external perturbation. If we can send small masses far away from the cubesat and maintain them attached we can use the subsequent gravity gradient effect to stabilize the s/c. We are mimicking the previous configuration.

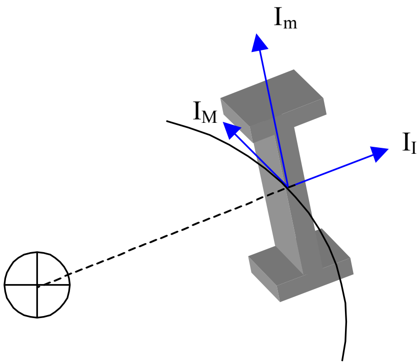


It is possible but it must be check with this equation

$$1 + 3K_x + K_yK_x > 4\sqrt{K_yK_x}$$

Stable conditions





Robustness to stability

Nonlinear form

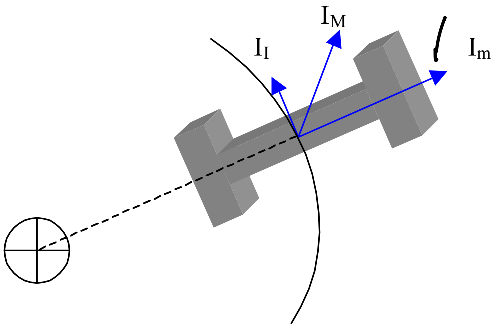
$$\underline{M} = \frac{3Gm_t}{R^3}\begin{bmatrix}(I_z - I_y)c_3c_2\\(I_x - I_z)c_1c_3\\(I_y - I_x)c_2c_1\end{bmatrix}$$

Linear form

$$\underline{M} = 3n^2\begin{Bmatrix}0\\(I_x - I_z)\alpha_y\\(I_x - I_y)\alpha_z\end{Bmatrix}$$

Stable conditions

yaw axis



special case for which the satellite mass tends to be aligned with the yaw axis -> $I_z = I_y$

The moment of inertia around the yaw axis tends to be very small when the mass is align with the x axis.

$$I_x \to 0 \Rightarrow K_y \to 1$$
$$K_z \to 1$$
$$K_x = 0$$

in the characteristic equation we replace these K_x K_y K_z.

$$\lambda^4 + n^2\lambda^2(1 + 3\cancel{K_x}^{0} + \cancel{K_x}^{0}K_y) + 4n^4\cancel{K_x}^{0}K_y = 0$$

NOTE

Could give some problem because it is a multiple of the orbit period so some perturbation due to the Earth can be amplified

$$\ddot{\alpha}_x + (\cancel{K_x}^{0} - 1)n\dot{\alpha}_y + \cancel{K_x}^{0}n^2\alpha_x = 0$$
$$\ddot{\alpha}_y + (\underbrace{1 - K_y}_{\phi})n\dot{\alpha}_x + 4\underbrace{K_y}_{1}n^2\alpha_y = 0$$

$$\omega_{ry} = 2n$$
$$\omega_p = n\sqrt{3}$$

$\omega_y^2 = 4n^2 \Rightarrow \omega_y = 2n$

$\omega_z = n\sqrt{3}$

$$\begin{cases} \ddot{\alpha}_x - n\dot{\alpha}_y = \phi \\ \ddot{\alpha}_y + 4n^2\alpha_y = \phi \end{cases}$$

$\ddot{\alpha}_z + 3n^2\alpha_z = \phi$

$\ddot{x} + \omega x = \phi$

This is the extreme case ⇒ but this show us that if we have such distribution of mass we might have problems for the period of the oscillation that can emphasize the effect of other disturbances.