Spacecraft Attitude Dynamics

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Attitude parameters

Attitude dynamics and kinematics

Dynamics

$$
\begin{cases}
\dot{\omega}_x = \dfrac{(I_y - I_z)}{I_x}\omega_z\omega_y + \dfrac{M_x}{I_x} \\
\dot{\omega}_y = \dfrac{(I_z - I_x)}{I_y}\omega_x\omega_z + \dfrac{M_y}{I_y} \\
\dot{\omega}_z = \dfrac{(I_x - I_y)}{I_z}\omega_y\omega_x + \dfrac{M_z}{I_z}
\end{cases}
\longrightarrow
$$

Kinematics

orientation of one frame with respect to another

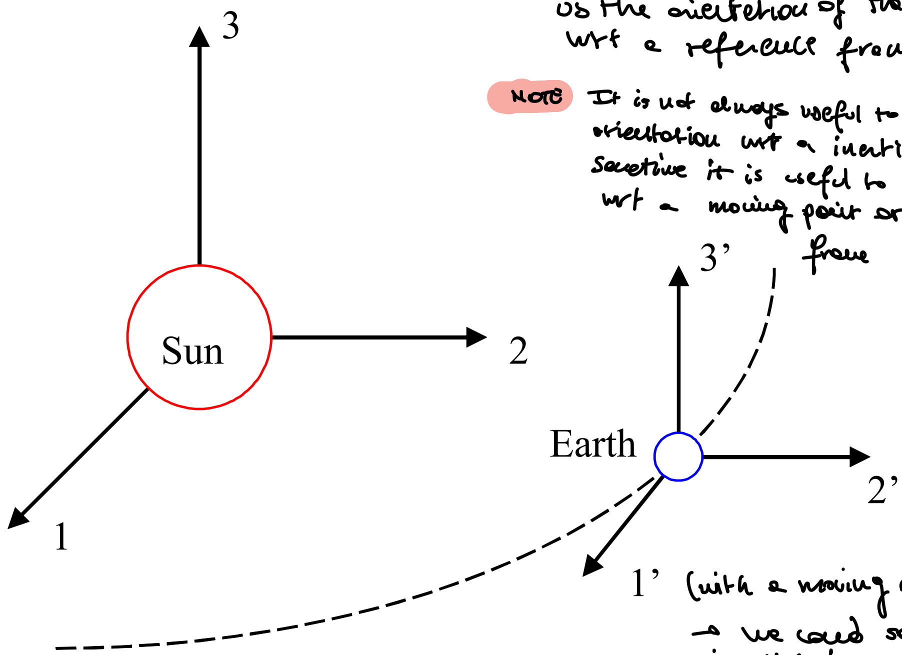
Dynamics → Kinematics

In order to understand where
the s/c is pointing at
we need to do a step further to move from the angular velocity
to the orientation.
Kinematics looks how the frame of reference moves without
focusing on why we have angular velocity → The why is provided by the dinamics.

Attitude parameters

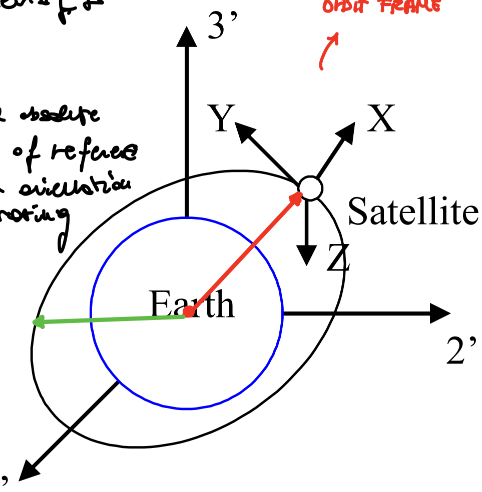
obj: how to represent the movement of xyz frame of reference → This will tell us the orientation of the spacecraft wrt a reference frame.

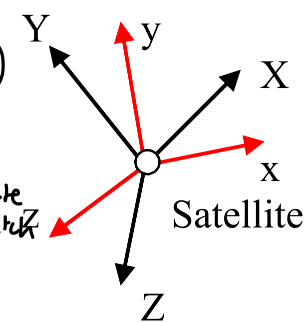
NOTE It is not always useful to have the absolute orientation wrt a inertial frame of reference. Sometime it is useful to know the orientation wrt a moving point or to a rotating frame



(with a moving origin → we could say that is inertial in terms of rotations)

ORBIT FRAME





X is the satellite local vertical direction
Y is the satellite velocity direction
Z is the third direction, orthogonal to X and Y

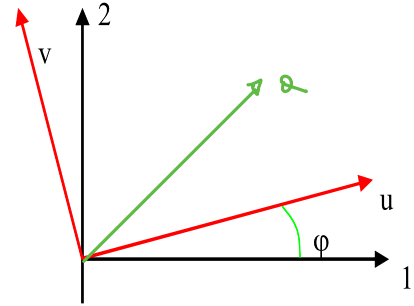
The rules are the same as when we are able to described absolute kinematics it is possible to switch easly to relative kinematics.

x,y,z are the satellites' principal inertia axes

Direction cosines

Easyest way to describe a rotating frame seen by a fixed reference frame.

$$A = \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$$



It represent a rotation around the third fix axis.

$$A = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

the vector in the new reference (a_{uvw}) is obtained by multiplying the original vector (a_{123}) by the direction cosine matrix A

a remain the same but change its representation in the different reference frame.

$$a_{uvw} = A \cdot a_{123}$$

$$a_{123} = A^T \cdot a_{uvw}$$

$$AA^T = I$$

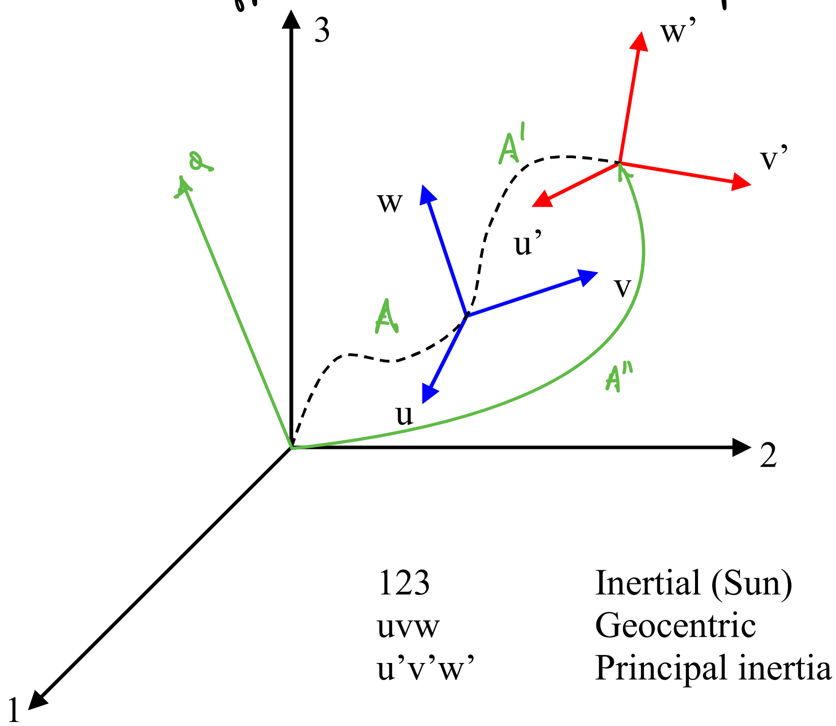
$$A^T = A^{-1}$$

It is possible to represent any vector in either frame of reference fixed or not.

A is orthogonal because the uvw and 123 basis are composed by orthogonal vector.

Direction cosines

NOTE we want to find a minimal representation, for a rotation a minimal representation should use only three (3) parameters



123	Inertial (Sun)
uvw	Geocentric
u'v'w'	Principal inertia

$$a_{uvw} = A \cdot a_{123}$$

$$a_{u'v'w'} = A' \cdot a_{uvw}$$

$$a_{u'v'w'} = A" \cdot a_{123} = A' \cdot a_{uvw} = A'A \cdot a_{123}$$

This can be generalized for a given set of arbitrary rotation.

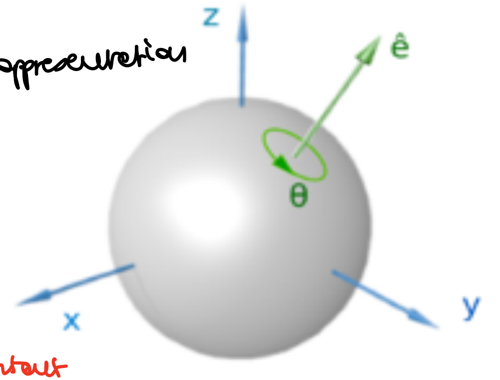
Just take rotation matrix and multiply them in an inverse order

$$\boxed{A" = A'A}$$

Euler axis / angle

Euler's rotation theorem -> Any single rotation can be represented by a vector (eigenvector) that remains fixed during that rotation and a simple rotation around that vector by an angle θ (eigen-angle)

$A\underline{e} = \underline{e}$ → eigenvalue $\lambda = 1$ problem representation



$\underline{\omega} = \dot{\theta}\underline{e}$

This because orthogonal matrices have one unit eigenvalue.

↳ Important

Now try to relate the direction cosines matrix A with vector e.

Lets look at three simple cases → rotation around principal inertia axis

⇓

Euler axis / angle

$$A_3(\phi) = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \underline{e} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$A_2(\phi) = \begin{bmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{bmatrix} \qquad \underline{e} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$A_1(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \qquad \underline{e} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

It is always true for a general rotation matrix A. whatever the direction of the vector $\underline{e}$

$$tr(A) = 1 + 2\cos\phi$$

To compute the euler angle is an indirect relation starting from the rotation matrix.

$$\cos\phi = \frac{1}{2}(tr(A) - 1)$$

Using this equation we can compute direct cosine matrix starting from the euler axis $\underline{e}$ and the rotation

$$\underbrace{\begin{bmatrix} \cos\phi & 0 & 0 \\ 0 & \cos\phi & 0 \\ 0 & 0 & \cos\phi \end{bmatrix}}_{\text{Identity matrix}}$$

antisimmetric matrix

$$\boxed{A = I\cos\phi + \underbrace{(1 - \cos\phi)}_{\text{scalar}}\underline{ee}^T - \sin\phi[e\wedge]}$$

3x3 matrix

$$[e\wedge] = \begin{bmatrix} 0 & -e_3 & e_2 \\ e_3 & 0 & -e_1 \\ -e_2 & e_1 & 0 \end{bmatrix}$$

Cross between

if we multiply this matrix with a vector we obtain $[e\wedge]\underline{v} = \underline{e}\wedge\underline{v}$ the exact cross product between $\underline{e}$ and other vector

Euler axis / angle

$$A = \begin{bmatrix} \cos\phi + e_1^2(1-\cos\phi) & e_1e_2(1-\cos\phi) + e_3\sin\phi & e_1e_3(1-\cos\phi) - e_2\sin\phi \\ e_1e_2(1-\cos\phi) - e_3\sin\phi & \cos\phi + e_2^2(1-\cos\phi) & e_2e_3(1-\cos\phi) + e_1\sin\phi \\ e_1e_3(1-\cos\phi) + e_2\sin\phi & e_2e_3(1-\cos\phi) - e_1\sin\phi & \cos\phi + e_3^2(1-\cos\phi) \end{bmatrix}$$

Way to compute the Euler angle/Matrix if you know the direction cosine Matrix.

$$\boxed{\begin{gathered} \phi = \cos^{-1}\left[\frac{1}{2}(tr(A)-1)\right] \\ \begin{cases} e_1 = \dfrac{(A_{23}-A_{32})}{2\sin\phi} \\ e_2 = \dfrac{(A_{31}-A_{13})}{2\sin\phi} \\ e_3 = \dfrac{(A_{12}-A_{21})}{2\sin\phi} \end{cases} \end{gathered}}$$

When the direct transformation is always possible.

⇒ This inverse transformation it is not always possible if $\phi = k\pi$ we will get an undetermine rotation axis.

Suppose that we have defined the vector $\underline{e}$ but the rotation is $\phi = 0$ → Any axis will give the

when $\sin\phi$=0 the Euler axis is undetermined

equal results ⇒ we have multiple options to describe the same rotation. So the inverse rotation could not find one exact axis, there are multiple correct answers.

A_1 $\quad A_2$ ⇒ $A_3 = A_2 A_1$ ⇒ compute $\underline{e}$ and φ from A_3

No rule for consecutive rotations

$\underline{e}_1\ \varphi_1 \quad \underline{e}_2\ \varphi_2 \not\Rightarrow$ from these we cannot find any combination that can give us $\underline{e}\ \varphi$

The only way to determine consecutive rotation is to use the direction cosine matrix where it exis a rule for consecutive rotation.

Quaternion

Any set of 4 number, provided that they are normalized to 1 and that they are computed by Euler axis and angle, can represent a rotation.

$$\begin{cases} q_1 = e_1 sin\dfrac{\phi}{2} \\ q_2 = e_2 sin\dfrac{\phi}{2} \\ q_3 = e_3 sin\dfrac{\phi}{2} \\ q_4 = cos\dfrac{\phi}{2} \end{cases} \longrightarrow \quad q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1$$

$$|\underline{e}|^2 = 1$$

Rappresentation of quaternions

vector part $\underline{q}$ and a scalar part q_4

$$\hat{q} = [\underline{q}\ ;\ q_4] \qquad \underline{q} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} \ , \quad q_4$$

It is usefull when we use the quaternions for small rotation

There is a special algebra of quaternions.

because q_1, q_2, q_3 are scaled rappresentation of the Euler axis $\underline{e} \begin{Bmatrix} e_1 \\ e_2 \\ e_3 \end{Bmatrix} \rightarrow \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = q$

Quaternion

How to transform the quaternions into the direction cosine matrix?

Direct Transformation

$\hat{q} \rightarrow A$

$$A = \begin{bmatrix} q_1^2 - q_2^2 - q_3^2 + q_4^2 & 2(q_1q_2 + q_3q_4) & 2(q_1q_3 - q_2q_4) \\ 2(q_1q_2 - q_3q_4) & -q_1^2 + q_2^2 - q_3^2 + q_4^2 & 2(q_2q_3 + q_1q_4) \\ 2(q_1q_3 + q_2q_4) & 2(q_2q_3 - q_1q_4) & -q_1^2 - q_2^2 + q_3^2 + q_4^2 \end{bmatrix}$$

$$\boxed{A = \left(q_4^2 - \underline{q}^T\underline{q}\right) I + 2\underline{q}\underline{q}^T - 2q_4[q \wedge]}$$

$$[q \wedge] = \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix}$$

↳ same logic as $[e \wedge]$

NOTE QUATERNIONS AVOID THE SINGULARITY, But they give us this equivalent conditions.
with $\phi = k\pi$ it is possible to compute the q_1, q_2, q_3, q_4 avoiding in such way the singularity.

Inverse transformation

$A \rightarrow \hat{q}$

$$\boxed{\begin{cases} q_1 = \dfrac{1}{4q_4}(A_{23} - A_{32}) \\ q_2 = \dfrac{1}{4q_4}(A_{31} - A_{13}) \\ q_3 = \dfrac{1}{4q_4}(A_{12} - A_{21}) \\ q_4 = \pm\dfrac{1}{2}(1 + A_{11} + A_{22} + A_{33})^{\frac{1}{2}} \end{cases}}$$

→ compute by taking the sqr of something → so we will get 2 solution → we get always two solution with a difference in the sign

Knowing that $q_4 = \pm$ — ⇒ $\cos\frac{\phi}{2}$ always sing ⇒ that mean that we will rotate by a positive angle around an axis or we will rotate by a negative angle around the opposite axis.

Quaternion

- If the A is general either is good → look lesson 03/10/20

Alternative inverse mapping

$$q_1^2 = \pm\frac{1}{2}\sqrt{1 + A_{11} - A_{22} - A_{33}}$$

$$q_2^2 = \frac{1}{4q_1^2}(A_{12} + A_{21})$$

$$q_3^2 = \frac{1}{4q_1^2}(A_{13} + A_{31})$$

$$q_4^2 = \frac{1}{4q_1^2}(A_{23} - A_{32})$$

$$q_2^3 = \pm\frac{1}{2}\sqrt{1 - A_{11} + A_{22} - A_{33}}$$

$$q_1^3 = \frac{1}{4q_2^3}(A_{12} + A_{21})$$

$$q_3^3 = \frac{1}{4q_2^3}(A_{23} + A_{32})$$

$$q_4^3 = \frac{1}{4q_2^3}(A_{31} - A_{13})$$

$$q_3^4 = \pm\frac{1}{2}\sqrt{1 - A_{11} - A_{22} + A_{33}}$$

$$q_1^4 = \frac{1}{4q_3^4}(A_{13} + A_{31})$$

$$q_2^4 = \frac{1}{4q_3^4}(A_{23} + A_{32})$$

$$q_4^4 = \frac{1}{4q_3^4}(A_{12} - A_{21})$$

Quaternion

sequence of two consecutive rotations

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} q_4' & -q_3' & q_2' & q_1' \\ q_3' & q_4' & -q_1' & q_2' \\ -q_2' & q_1' & q_4' & q_3' \\ -q_1' & -q_2' & -q_3' & q_4' \end{bmatrix} \begin{bmatrix} q_1'' \\ q_2'' \\ q_3'' \\ q_4'' \end{bmatrix}$$

$$\hat{q} = \hat{q}'' \otimes \hat{q}' \longleftrightarrow A = A''A'$$

$$\hat{q}^{-1} = [-\underline{q}\ ;\ q_4]$$

Gibbs vector

Different mapping of the euler axis and the Euler angle ⇒ where the quaternion maps Euler angle and axis onto 4 terms the Gibbs vector maps the axis and angle to 3 scalar terms as a component of the Gibbs vector.

This is a minimal representation only three terms ⇒ minimum amount to describe a rotation.

But we still have singularities

$$
\begin{cases}
q_1 = e_1 \sin\dfrac{\theta}{2} \\
q_2 = e_2 \sin\dfrac{\theta}{2} \\
q_3 = e_3 \sin\dfrac{\theta}{2} \\
q_4 = \cos\dfrac{\theta}{2}
\end{cases}
\longrightarrow
$$

Inverse mapping

$$
\begin{cases}
g_1 = \dfrac{q_1}{q_4} = e_1 \tan\dfrac{\theta}{2} \\
g_2 = \dfrac{q_2}{q_4} = e_2 \tan\dfrac{\theta}{2} \\
g_3 = \dfrac{q_3}{q_4} = e_3 \tan\dfrac{\theta}{2}
\end{cases}
$$

Singularity at

$\theta = (2n + 1)\,\pi.$

$$
A(\underline{g}) = \frac{1}{1 + g_1^2 + g_2^2 + g_3^2}
\begin{bmatrix}
1 + g_1^2 - g_2^2 - g_3^2 & 2(g_1 g_2 + g_3) & 2(g_1 g_3 - g_2) \\
2(g_1 g_2 - g_3) & 1 - g_1^2 + g_2^2 - g_3^2 & 2(g_2 g_3 + g_1) \\
2(g_1 g_3 + g_2) & 2(g_2 g_3 - g_1) & 1 - g_1^2 - g_2^2 + g_3^2
\end{bmatrix}
$$

Direct mapping always possible → It does not gives a good inside of what is happening in a physical terms → This because it is a mixed representations.

$$
A = \frac{\left(1 - \underline{g}^2\right) I + 2\underline{g}\,\underline{g}^T - 2[g \wedge]}{\left(1 + \underline{g}^2\right)}
$$

NOTE THE CHOICE DEPENDS ON THE APPLICATION BECAUSE if we do not want singularity we will use quaternion. If we want a physical representation may be better to use direction cosine matrix or euler axis and angle.

Gibbs vector

Inverse mapping

$$\begin{cases} g_1 = \dfrac{A_{23} - A_{32}}{1 + A_{11} + A_{22} + A_{33}} \\ g_2 = \dfrac{A_{31} - A_{13}}{1 + A_{11} + A_{22} + A_{33}} \\ g_3 = \dfrac{A_{12} - A_{21}}{1 + A_{11} + A_{22} + A_{33}} \end{cases}$$

singular when $\varphi = (2n + 1)\,\pi$

consecutive rotations

$$g'' = \frac{\underline{g} + \underline{g}' - \underline{g}' \wedge \underline{g}}{1 - \underline{g} \cdot \underline{g}'}$$

Euler angles

→ LAST SET OF PARAMETERS WE ARE GOING TO USE
SET OF 3 angles ⇒ every rotation can be expressed by a set of 3 trivial rotation that

Any rotation can be de-composed into the multiplication of three trivial rotations

Direction cosine matrix.

$$A_1(\psi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & \sin\psi \\ 0 & -\sin\psi & \cos\psi \end{bmatrix}$$

rotation by angle ψ around axis 1

$$A_2(\vartheta) = \begin{bmatrix} \cos\vartheta & 0 & -\sin\vartheta \\ 0 & 1 & 0 \\ \sin\vartheta & 0 & \cos\vartheta \end{bmatrix}$$

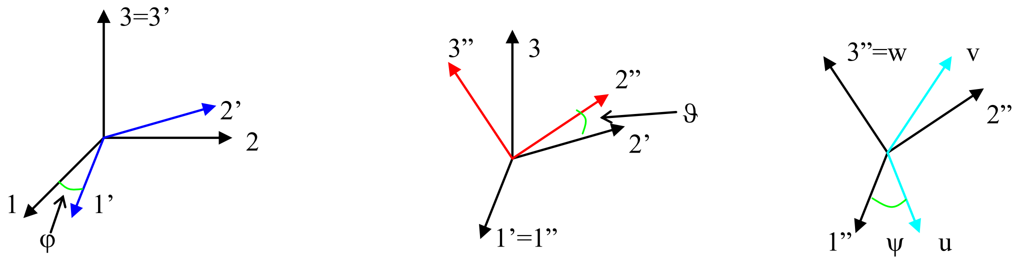
rotation by angle ϑ around axis 2

$$A_3(\phi) = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

rotation by angle ϕ around axis 3

Euler angles

1,2,3 -> u,v,w



$$A_{313}(\phi, \vartheta, \psi) = A_3(\psi) \cdot A_1(\vartheta) \cdot A_3(\phi)$$

$$A_{313} = \begin{bmatrix} \cos\psi\cos\phi - \sin\psi\sin\phi\cos\vartheta & \cos\psi\sin\phi + \sin\psi\cos\phi\cos\vartheta & \sin\psi\sin\vartheta \\ -\sin\psi\cos\phi - \cos\psi\sin\phi\cos\vartheta & -\sin\psi\sin\phi + \cos\psi\cos\phi\cos\vartheta & \cos\psi\sin\vartheta \\ \sin\phi\sin\vartheta & -\cos\phi\sin\vartheta & \cos\vartheta \end{bmatrix}$$

Euler angles

12 possibilities

↗ won't used

312,213,123,321,231,132 all different indexes

313,323,212,232,131,121 first and third index equal

Cowell a s/c rotate around a spin axis ⇒ 3 is the spin axis.

$$A_{312} = \begin{bmatrix} \cos\psi\cos\phi - \sin\psi\sin\phi\sin\vartheta & \cos\psi\sin\phi + \sin\psi\cos\phi\sin\vartheta & -\sin\psi\cos\vartheta \\ -\sin\phi\cos\vartheta & \cos\phi\cos\vartheta & \sin\vartheta \\ \sin\psi\cos\phi + \cos\psi\sin\phi\sin\vartheta & \sin\psi\sin\phi - \cos\psi\cos\phi\sin\vartheta & \cos\vartheta\cos\psi \end{bmatrix}$$

No model for consecutive rotations → Even though the rule for the euler angle are derived from consecutive rotation.

Euler angles

Inverse mapping

we need to defined what was the sequence of rotation because different set of rotation will give different results if we apply the inverse mapping.

$$A_{313}(\phi, \vartheta, \psi) \rightarrow \begin{cases} \vartheta = cos^{-1}(A_{33}) \\ \phi = -tan^{-1}\left(\dfrac{A_{31}}{A_{32}}\right) \\ \psi = tan^{-1}\left(\dfrac{A_{13}}{A_{23}}\right) \end{cases} \rightarrow \text{Singularity at } \vartheta = n\pi$$

Singularity with Euler angles can not be easy manage in this case

$$A_{123}(\phi, \vartheta, \psi) = \begin{bmatrix} \cos\psi\cos\vartheta & \cos\psi\sin\vartheta\sin\phi + \sin\psi\cos\phi & -\cos\psi\sin\vartheta\cos\phi + \sin\psi\sin\phi \\ -\sin\psi\cos\vartheta & -\sin\psi\sin\vartheta\sin\phi + \cos\psi\cos\phi & \sin\psi\sin\vartheta\cos\phi + \cos\psi\sin\phi \\ \sin\vartheta & -\cos\vartheta\sin\phi & \cos\vartheta\ \cos\phi \end{bmatrix}$$

$$\rightarrow \begin{cases} \vartheta = sin^{-1}(A_{11}) \\ \phi = -tan^{-1}\left(\dfrac{A_{32}}{A_{33}}\right) \\ \psi = -tan^{-1}\left(\dfrac{A_{21}}{A_{11}}\right) \end{cases} \rightarrow \text{Singularity at } \vartheta = (2n+1)\pi/2$$

Approximation for small angles

If angles are small, we can assume cos x =1, sin x = x, x*x =0 (with x in radians)

for small angle approx use always Euler angles matrix with all the 3 different digit so we know every contribution.

$$A_{312}(\phi,\vartheta,\psi) = \begin{bmatrix} 1 & \phi & -\psi \\ -\phi & 1 & \vartheta \\ \psi & -\vartheta & 1 \end{bmatrix} = A_{321}(\phi,\psi,\vartheta) = A_{213}(\psi,\vartheta,\phi) = \ldots$$

$$A = I - [angles \wedge]$$

$$A_{313}(\phi,\vartheta,\psi) = \begin{bmatrix} 1 & \phi+\psi & 0 \\ -\phi-\psi & 1 & \vartheta \\ 0 & -\vartheta & 1 \end{bmatrix}$$

$$\begin{cases} q_1 = \dfrac{1}{2}\vartheta \\ q_2 = \dfrac{1}{2}\psi \\ q_3 = \dfrac{1}{2}\phi \\ q_4 = 1 \end{cases}$$

→ Relation between quaternions and the euler angle of a rotation