ORBITAL MECHANICS

Continuing last lesson:

$\underline{h}, \underline{e}, t_p \equiv \underline{r}, \underline{v}$ 6 constant to determinate 6 scalar.

ANGULAR VELOCITY $\boxed{n \overset{\Delta}{=} \sqrt{\frac{\mu}{a^3}}}$ $\frac{1}{[T]} = \frac{1}{s}$ (2.9)

Let's consider a circular orbit $v_c = \sqrt{\frac{\mu}{a}}$ Eq (1.62)

$$w_c = \frac{v_c}{a} = \sqrt{\frac{\mu}{a^3}}$$

$\Rightarrow$ n is the constant angular velocity at wich a circular orbit with a radius $r = a$ is covered

$T = 2\pi \sqrt{\frac{a^3}{\mu}}$ the two orbit have the same orbital period

$T = \frac{2\pi}{n}$

$$\boxed{\begin{aligned} E - e\sin E &= n(t - t_p) \\ t_p &= t - \frac{1}{n}\left(E - e\sin E\right) \end{aligned}} \quad (2.10)$$

$\rightarrow$ constant during the motion.

② RELATION BETWEEN E AND θ $\rightarrow \theta(E)$

Definition MEAN ANOMALY $\boxed{M = t\sqrt{\frac{\mu}{a^3}}}$ (2.11)

$E - e\sin E - (E_0 - e\sin E_0) = M - M_0$ So Mo take at perigee passage

reference $t_p = \phi$ $\rightarrow$ $M = (t - t_p)\sqrt{\frac{\mu}{a^3}}$

$t_p \quad \epsilon_p = M_p = \not{t}$ ← if I take a different reference point I'll get different result.

$$\boxed{E - e\sin E = M} \qquad (2.12)$$

from eq (2.3) $r\cos\theta = a\cos E - ae$ sum r on both side

$$r\cos\theta + r = a\cos E - ae + r$$

from eq 2.4 $r(E) = a(1 - e\cos E)$

$$r(1+\cos\theta) = a\cos E - ae + a - ae\cos E$$

$$r(1+\cos\theta) = a(\cos E - e + 1 - e\cos E)$$

$$r(1+\cos\theta) = a(1+\cos E)(1-e) \qquad (2.13)$$

We continue working on it

<u>RECALL</u> $1 + \cos\theta = 2\cos^2\frac{\theta}{2}$

$$\cancel{2}r\cos^2\frac{\theta}{2} = \cancel{2}a(1-e)\cos^2\frac{E}{2} \qquad (2.14)$$

$$r\cos^2\frac{\theta}{2} = a(1-e)\cos^2\frac{E}{2}$$

Let's focus y_P (2.1b) = (2.2b)

$$b\sin E = r(\theta)\sin\theta \qquad (2.15)$$

<u>RECALL</u> $\sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$ → $\cancel{2}b\cos\frac{E}{2}\sin\frac{E}{2} = \cancel{2}r\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

but $b = a\sqrt{1-e^2}$ Therefore ⇒

$$a\sqrt{1-e^2}\cos\frac{E}{2}\sin\frac{E}{2} = r\sin\frac{\theta}{2}\cos\frac{\theta}{2} \qquad (2.16)$$

Use Eq (2.14) $\rightarrow \frac{(2.16)}{(2.14)}$

So we get the tangent that does not have the same problem of the cosine in terms of sign.

$$\frac{\cancel{a}\sqrt{1-e^2}\,\cancel{\cos\frac{E}{2}}\,\sin\frac{E}{2}}{\cancel{a}(1-e)\cos^{\cancel{2}}\frac{E}{2}} = \frac{\cancel{r}\sin\frac{\theta}{2}\,\cancel{\cos\frac{\theta}{2}}}{\cancel{r}\cos^{\cancel{2}}\frac{\theta}{2}}$$

$$\tan\frac{\theta}{2} = \sqrt{\frac{1-e^2}{(1-e)^2}}\,\tan\frac{E}{2} \qquad \frac{\cancel{(1-e)}(1+e)}{(1-e)^{\cancel{2}}}$$

$$\boxed{\tan\frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}}\,\tan\frac{E}{2}} \qquad (2.17) \rightarrow \text{This is valid all around the orbit}$$

Eq (2.8) and (2.17) $t \rightarrow \theta$, $\theta \rightarrow t$

(A) $t \xrightarrow{\text{Eq } 2.8} E \xrightarrow{\text{Eq } 2.17} \theta$ (2.18)

(needs to be solve numerically)

(B) $\theta \xrightarrow{\text{Eq } 2.17} E \xrightarrow{\text{Eq } 2.8} t$ There is no need of a numerical solution.

SUMMARY

θ = True Anomaly $\qquad$ E = eccentric anomaly

$$r = \frac{p}{1+e\cos\theta} \quad (1.42) \qquad r = a(1-e\cos E) \quad (2.4)$$

Kepler's law $\quad E - e\sin E = \sqrt{\frac{\mu}{a^3}}\,\underbrace{(t-t_p)}_{\Delta t} \quad$ eq 2.8

M = mean anomaly $= t\sqrt{\frac{\mu}{a^3}}$ (2.11)

$\theta, E, M \longrightarrow \dot{\theta}, \dot{E}, \dot{M} \Rightarrow$ useful for parabolas and hyperbola

① $\dot{\theta}$ from conservation h $\qquad \dot{\theta} = \frac{h}{r^3}$ ep (2.14) ?

$$\boxed{\dot{\theta} = \frac{h}{r^2}} \qquad (2.19)$$

② $\dot{M}$ can be computed from (2.11)

$$\boxed{\dot{M} = \sqrt{\frac{\mu}{a^3}}}$$

Recall $\rightarrow \mu p = h^2$ (2.38)

$p = a(1-e^2)$ ep (2.41)

$b = a\sqrt{1-e^2}$ (2.44)

$b^2 = a^2(1-e^2)$

$$M = t\sqrt{\frac{\mu}{a^3}} = t\sqrt{\frac{\mu p}{a^3 p}} = t\sqrt{\frac{h^2}{a^3 p}}$$

$$= t\sqrt{\frac{h^2}{a^4(1-e^2)}} = t\sqrt{\frac{h^2}{b^2 a^2}} = t\frac{h}{ba}$$

$$\boxed{\dot{M} = \frac{h}{ba}} \qquad (2.20)$$

③ $\dot{E}$ from Eq (2.8)

$$E - e\sin E = \sqrt{\frac{\mu}{a^3}}(t - t_p) \qquad \text{Eq (2.8)}$$

$$\dot{E} - e\cos E\,\dot{E} = \sqrt{\frac{\mu}{a^3}} \rightarrow \dot{E} = \sqrt{\frac{\mu}{a^3}}\frac{1}{1-e\cos E} \qquad (2.21)$$

$\sqrt{\frac{\mu}{a^3}} = \frac{h}{ab}$ ep (2.20) and from 2.4

$$r = a(1 - e\cos E) \longrightarrow \frac{1}{1-e\cos E} = \frac{a}{r}$$

$\Rightarrow \dot{\mathcal{E}} = \frac{h}{ab}\frac{a}{r} \longrightarrow \boxed{\dot{\mathcal{E}} = \frac{h}{br}}$ (2.22)

For the ellipse

$$\tan\frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}}\ \tan\frac{\mathcal{E}}{2}$$

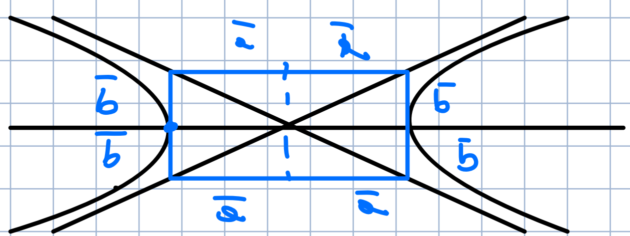
TIME OF FLIGHT ON HYPERBOLIC ORBIT

1) $\dot{\theta} = \frac{h}{r^2}$ remain the same expression Hyp : $\boxed{\dot{\theta} = \frac{h}{r^2}}$ (2.23)

2) $\dot{M}$ for an ellipse $\dot{M} = \frac{h}{ab}$

$a_{HYP} = -\bar{a}$ ∞

$b_{HYP} = a\sqrt{1-e^2} = -\bar{a}\sqrt{(-1)(e^2-1)} = -\bar{a}\,i\underbrace{\sqrt{e^2-1}}_{>0} = -\bar{b}\,i$



ELLIPSE		HYP
a	$\longrightarrow$	$a = -\bar{a}$
b	$\longrightarrow$	$b = -i\bar{b}$

$$\dot{M} = \frac{h}{ab} = \frac{h}{-\bar{a}(-i\bar{b})} = \frac{h}{\bar{a}\bar{b}i} = -\frac{h}{\bar{a}\bar{b}}\,i$$

Hyp $\boxed{\dot{M} = -i\dot{\bar{M}}}$ where $\dot{\bar{M}} = \frac{h}{\bar{a}\bar{b}}$ (2.24)

Hyp $M = -i\bar{M}$ M is complex

3) Ellipse $\dot{\mathcal{E}} = \frac{h}{br}$

Hyperbola $\dot{\mathcal{E}} = \frac{h}{-i\bar{b}r} = \frac{h}{\bar{b}r}\,i$

Hyp : $\dot{\mathcal{E}} = \dot{\bar{\mathcal{E}}}\,i$ where $\dot{\bar{\mathcal{E}}} = \frac{h}{\bar{b}r}$ (2.25)

HYP: $\bar{E} = i\bar{E}$ $\qquad$ $\bar{E}$ is complex

Summarizing:

Ellipse	Hyperbola
θ	θ
E	$i\bar{E}$
M	$-i\bar{M}$

(2.26)

In order to find the time law for an hyperbola we need to find the cartesian equation of a hyperbola

CARTESIAN EQUATION OF THE HYPERBOLA

$$p: \begin{cases} x = a\cos E \\ y = b\sin E \end{cases}$$

For the hyperbola

$$x = a\cos E = -\bar{a}\cos(i\bar{E})$$

(Curtis $\bar{E}$ is called F)

Recall

$$\sin\alpha = \frac{e^{i\alpha} - e^{-i\alpha}}{2i} \qquad e^{i\alpha} = \cos\alpha + i\sin\alpha$$

$$\cos\alpha = \frac{e^{i\alpha} + e^{-i\alpha}}{2} \qquad e^{-i\alpha} = \cos\alpha - i\sin\alpha$$

$$\sinh A = \frac{e^{A} - e^{-A}}{2}$$

$$\cosh^2 A - \sinh^2 A = 1$$

$$\cosh A = \frac{e^{A} + e^{-A}}{2}$$

$$\cos i\bar{E} = \frac{e^{i(i\bar{E})} + e^{-i(i\bar{E})}}{2} = \frac{e^{-\bar{E}} + e^{\bar{E}}}{2} = \cosh\bar{E}$$

$x = -\bar{a}\cosh\bar{E}$ $\rightarrow$ physical interpretation of the hyperbolic trigonometric function

I measure it on an hyperbole

$$\boxed{\cosh\bar{E} = -\frac{x}{\bar{a}}} \qquad (2.27)$$

$$\sin i\bar{E} = \frac{e^{i(i\bar{E})} - e^{-i(i\bar{E})}}{2i} = \frac{e^{-\bar{E}} - e^{+\bar{E}}}{2i} = -\frac{\sinh\bar{E}}{i}$$

$$y = -i\bar{b}\left(-\frac{\sinh\bar{E}}{i}\right) = \bar{b}\sinh\bar{E}$$

$$\boxed{\sinh\bar{E} = \frac{y}{\bar{b}}} \qquad (2.28)$$

NOTE Eq (2.27) and (2.28) satisfy

$$\frac{x^2}{\bar{a}^2} - \frac{y^2}{\bar{b}^2} = 1$$

hyperbole equation referred to its centre.