Spacecraft Attitude Dynamics

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Attitude kinematics

Direction cosines - kinematics

Evolution with time of kinematics parameters → Because s/c is rotating with a given angular velocity we want to understand how the attitude parameters change in a body reference system

How attitude parameters change with time, and how they are related to the angular velocity expressed in body frame

$$\frac{dA}{dt} = f(\omega_u, \omega_v, \omega_w)$$

Rule for sequence of rotations

in terms of time passing the variation of attitude parameters is represented by A' → direction cosine.

$$A(t + \Delta t) = A'A(t)$$

Rotation in an interval $\Delta t \to dt$

$$A' = I\cos\phi + \underbrace{(1 - \cos\phi)}_{0}\underline{e}\underline{e}^T - \sin\phi[\underline{e}\wedge]$$

if $\phi \ll 1$ $\cos\phi = 1$ Φ

short time intervals, φ small

$$A' = I - \phi[\underline{e}\wedge]$$

$$\phi = \omega\Delta t \qquad \underline{\omega} = \omega\underline{e}$$

The direction of the euler axis and the rotation are the same this is true for instantaneous rotation.

$$\phi[\underline{e}\wedge] = \omega\Delta t\begin{bmatrix} 0 & -e_w & e_v \\ e_w & 0 & -e_u \\ -e_v & e_u & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\omega_w & \omega_v \\ \omega_w & 0 & -\omega_u \\ -\omega_v & \omega_u & 0 \end{bmatrix}\Delta t = [\underline{\omega}\wedge]\Delta t = \phi[\underline{e}\wedge]$$

Direction cosines - kinematics

$$A' = I - [\underline{\omega} \wedge]\Delta t$$

$$A(t + \Delta t) = A(t) - \Delta t[\underline{\omega} \wedge]A(t)$$

$$\frac{dA}{dt} = -[\underline{\omega} \wedge]A(t)$$

$$\frac{dA}{dt} = \lim_{\Delta t \to 0} \frac{A(t + \Delta t) - A(t)}{\Delta t} = \lim_{\Delta t \to 0} - \frac{\Delta t[\underline{\omega} \wedge]A(t)}{\Delta t} = -[\underline{\omega} \wedge]A(t)$$

Need to preserve the structure -> Use a standard numerical integration scheme and orthonormalize the matrix at every time step using: → This is important to preserve the fact the results of the integration is a direction cosine matrix => that it is always orthogonal.

If we find that the results of the integration is no longer a orthogonal matrix (direction cosine matrix) →

$$A_{k+1}(t) = A_k(t)*3/2 - A_k(t)*A_k^T(t)*A_k(t)/2$$

To correct the result reaching the closest possible orthogonal matrix we could use this iterative approach

If performing this at every time step a single iteration is normally enough:

every time step of the integration

$$A(t) = A_0(t)*3/2 - A_0(t)*A_0^T(t)*A_0(t)/2$$

Euler axis / angle - kinematics

No rule for sequence of rotations -> no rule for kinematics

They are usefull for representing an orbit but it is not possible to find the evolution of axis and angle in time.

Might be usefull for evaluating the results.

Quaternion - kinematics

→ same logic of the direction cosine matrix approach.

$$q(t+\Delta t) = \begin{bmatrix} q_4' & q_3' & -q_2' & q_1' \\ -q_3' & q_4' & q_1' & q_2' \\ q_2' & -q_1' & q_4' & q_3' \\ -q_1' & -q_2' & -q_3' & q_4' \end{bmatrix} q(t) \qquad \text{with} \begin{cases} q_1' = e_u \sin\frac{\phi}{2} \\ q_2' = e_v \sin\frac{\phi}{2} \\ q_3' = e_w \sin\frac{\phi}{2} \\ q_4' = \cos\frac{\phi}{2} \end{cases}$$

$$q(t+\Delta t) = \left\{ I\cos\frac{\phi}{2} + \begin{bmatrix} 0 & e_w & -e_v & e_u \\ -e_w & 0 & e_u & e_v \\ e_v & -e_u & 0 & e_w \\ -e_u & -e_v & -e_w & 0 \end{bmatrix} \sin\frac{\phi}{2} \right\} q(t)$$

Quaternion - kinematics

short intervals Δt $\longrightarrow$ $\phi = \omega \Delta t \qquad cos\frac{\phi}{2} = 1 \qquad sin\frac{\phi}{2} = \frac{\phi}{2} = \frac{\omega \Delta t}{2}$

evaluate e_u, e_v, e_w, as a function of $\underline{\omega} = \omega\ \underline{e}$

$$q(t+\Delta t) = \left[I + \frac{1}{2}\Omega \Delta t\right] q(t) \qquad \Omega = \begin{bmatrix} 0 & \omega_w & -\omega_v & \omega_u \\ -\omega_w & 0 & \omega_u & \omega_v \\ \omega_v & -\omega_u & 0 & \omega_w \\ -\omega_u & -\omega_v & -\omega_w & 0 \end{bmatrix}$$

limit for $t \rightarrow 0$

$$\frac{dq}{dt} = \lim_{\Delta t \rightarrow 0} \frac{q(t+\Delta t) - q(t)}{\Delta t} = \frac{1}{2}\Omega q(t)$$

Gibbs vector - kinematics

$$\underline{g}(t+\Delta t) = \frac{\underline{g}(t) + \underline{g}' - \underline{g}' \wedge \underline{g}(t)}{1 - \underline{g}(t) \cdot \underline{g}'}$$

$$\underline{g}' = \underline{e}\tan\frac{\phi}{2} = \xrightarrow{\text{small } \Delta t} = \frac{1}{2}\underline{\omega}\Delta t$$

$$\frac{d\underline{g}}{dt} = \frac{1}{2}\left[\underline{\omega} - \underline{\omega} \wedge \underline{g}(t) + \left(\underline{g}(t) \cdot \underline{\omega}\right)\underline{g}(t)\right]$$

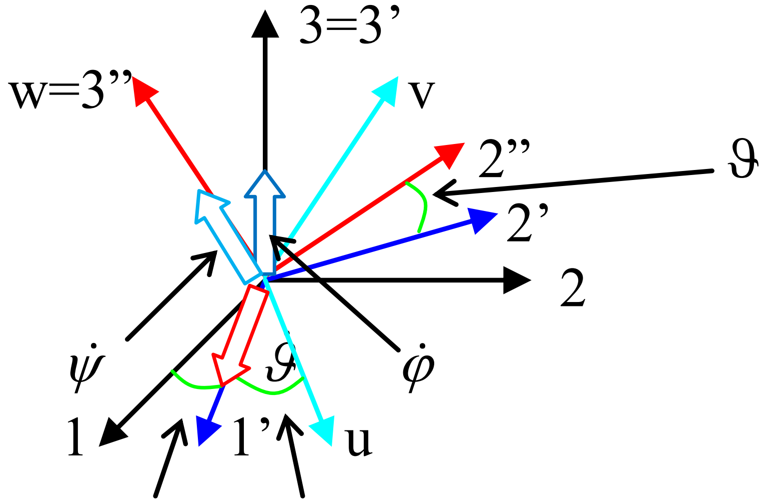
Euler angles - kinematics

no convenient rule for combining two consecutive rotations

consider the sequence 313 (φ,ϑ,ψ)

This directions are not mutually orthogonal but we can sum them to obtain the vector $\underline{\omega}$

$$\underline{\omega} = \overbrace{\dot{\phi}\underline{3} + \dot{\vartheta}\underline{1}' + \dot{\psi}\underline{w}}$$



Euler angles - kinematics

$$\underline{\omega} = \dot{\phi}\underline{3} + \dot{\vartheta}\underline{1}' + \dot{\psi}\underline{w} \longrightarrow \begin{cases} \omega_u = \underline{\omega}\cdot\underline{u} = \dot{\phi}\underline{3}\cdot\underline{u} + \dot{\vartheta}\underline{1}'\cdot\underline{u} + \dot{\psi}\underline{w}\cdot\underline{u} = \dot{\phi}\underline{3}\cdot\underline{u} + \dot{\vartheta}\underline{1}'\cdot\underline{u} \\ \omega_v = \underline{\omega}\cdot\underline{v} = \dot{\phi}\underline{3}\cdot\underline{v} + \dot{\vartheta}\underline{1}'\cdot\underline{v} + \dot{\psi}\underline{w}\cdot\underline{v} = \dot{\phi}\underline{3}\cdot\underline{v} + \dot{\vartheta}\underline{1}'\cdot\underline{v} \\ \omega_w = \underline{\omega}\cdot\underline{w} = \dot{\phi}\underline{3}\cdot\underline{w} + \dot{\vartheta}\underline{1}'\cdot\underline{w} + \dot{\psi}\underline{w}\cdot\underline{w} \end{cases}$$

$\underline{3}\,\underline{u}$, $\underline{3}\,\underline{v}$, $\underline{3}\,\underline{w}$ $\longrightarrow$ third column of matrix A_{313}

$\underline{1}'\underline{u}$, $\underline{1}'\underline{v}$, $\underline{1}'\underline{w}$ $\longrightarrow$ first column of matrix A_{313} if $\varphi = 0$

$$\begin{cases} \omega_u = \dot{\phi}\sin\vartheta\sin\psi + \dot{\vartheta}\cos\psi \\ \omega_v = \dot{\phi}\sin\vartheta\cos\psi - \dot{\vartheta}\sin\psi \\ \omega_w = \dot{\phi}\cos\vartheta + \dot{\psi} \end{cases} \longrightarrow \begin{cases} \dot{\phi} = \dfrac{(\omega_u\sin\psi + \omega_v\cos\psi)}{\sin\vartheta} \\ \dot{\vartheta} = \omega_u\cos\psi - \omega_v\sin\psi \\ \dot{\psi} = \omega_w - (\omega_u\sin\psi + \omega_v\cos\psi)\dfrac{\cos\vartheta}{\sin\vartheta} \end{cases}$$

Euler angles - kinematics

$$\text{sequence 313}\begin{cases}\dot{\phi} = \dfrac{(\omega_u sin\psi + \omega_v cos\psi)}{sin\vartheta} \\ \dot{\vartheta} = \omega_u cos\psi - \omega_v sin\psi \\ \dot{\psi} = \omega_w - (\omega_u sin\psi + \omega_v cos\psi)\dfrac{cos\vartheta}{sin\vartheta}\end{cases}$$

$$\text{sequence 312}\begin{cases}\dot{\phi} = \dfrac{(\omega_w cos\psi - \omega_u sin\psi)}{cos\vartheta} \\ \dot{\vartheta} = \omega_u cos\psi + \omega_w sin\psi \\ \dot{\psi} = \omega_v - (\omega_w cos\psi - \omega_u sin\psi)\dfrac{sin\vartheta}{cos\vartheta}\end{cases}$$

Direct Cosine Matrices (DCM)

Advantages

- Singularity free.
- Uniquely defines every possible rotation.
- Intuitive.

Disadvantages

- 9 components to evaluate.
- Requires orthonormalization during integration.

Euler axis / angle ?

Quaternion ?

Gibbs vector ?

Euler angles ?