ORBITAL MECANICS

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obj → find a closed form of the two body peoblem.
↳ find motion equation of the two body system.

Determinate position and velocity of the satellite without using differential integration. → we tried to do that by appling some conservation law.

Consider the Equation (1.6) in the form

$$\boxed{\underline{\ddot{r}} + \mu \frac{\underline{r}}{r^3} = 0} \quad (1.8)$$

scalar multiply by $\underline{\dot{r}} = \frac{d\underline{r}}{dt}$

$$\underline{\dot{r}} \cdot \underline{\ddot{r}} + \underline{\dot{r}} \, \frac{\mu \underline{r}}{r^3} = 0 \qquad (1.9)$$

but

$$\underline{\dot{r}} \cdot \underline{\ddot{r}} = \frac{d}{dt}\left(\frac{1}{2} \underline{\dot{r}} \cdot \underline{\dot{r}}\right) = \frac{1}{2} \underline{\ddot{r}} \cdot \underline{\dot{r}} + \frac{1}{2} \underline{\dot{r}} \cdot \underline{\ddot{r}} = \underline{\dot{r}} \cdot \underline{\ddot{r}}$$

↑ verify

$$\underline{\dot{r}} \cdot \underline{\ddot{r}} = \frac{d}{dt}\left(\frac{1}{2} \underline{v} \cdot \underline{v}\right) = \frac{d}{dt} \frac{v^2}{2}$$

and $\mu \frac{\underline{r} \cdot \underline{\dot{r}}}{r^3} = \mu \frac{\cancel{r}\dot{r}}{r^{\cancel{3}}} = \mu \frac{\dot{r}}{r^2}$

$\underline{r} = r \underline{e_r}$

$\underline{\dot{r}} = \dot{r} \underline{e_r} + r \dot{\theta} \underline{e_\theta}$

$\underline{r} \cdot \underline{\dot{r}} = r \dot{r}$

$\mu = -\frac{d}{dt}\left(\frac{\mu}{r}\right)$ recalling $r^2 = \underline{r} \cdot \underline{r}$

and $\frac{d}{dt}\left(\frac{1}{r}\right) = -\frac{1}{r^2} \dot{r}$

verify. $-\frac{d}{dt}\left(\frac{\mu}{r}\right) = +\frac{\mu}{r^2} \dot{r}$

Therefore eq (1.3) becomes

specific kinetical energy of s/c

specific potential energy of s/c ≡ gravitational potential function per unit of mass.

$$\frac{d}{dt}\left(\frac{v^2}{2}-\frac{\mu}{r}\right)=0 \implies \boxed{\frac{v^2}{2}-\frac{\mu}{r}=\varepsilon} = \text{constant} \qquad (1.10)$$

SPECIFIC MECHANICAL ENERGY (per unit of mass)

$$\boxed{\varepsilon = \text{COSTANT}}$$

NOTE

dynamics equations (1.6) $\underline{r}_0 = \underline{r}(t_0)$, $\underline{v}_0 = \underline{v}(t_0)$ $\xrightarrow[\text{eq }(1.6)]{\text{Integrate}}$ $\underline{r}(t)$, $\underline{v}(t)$ $[t_0, t_{end}]$

$\forall t \in$ trajectory calculate $\frac{v(t)^2}{2} - \frac{\mu}{r(t)} = \varepsilon$

It is valid also for the N body problem → It is important because it is useful to check the correctness of the integral → if ε is not constant the integral is not accurate enough.

SPECIFIC ANGULAR MOMENTUM OF S/C (per unit of mass)

the angular momentum of m_2 wrt m_1 is the moment of m_2's relative linear momentum $m_2 \dot{\underline{r}}$.

$$\underline{H}_{2@1} = \underline{r} \wedge m_2 \dot{\underline{r}} \qquad \text{where} \quad \dot{\underline{r}} = \underline{v}$$

the specific angular momentum $\underline{h} = \underline{H}_{2@1}/m_2$

$$\boxed{\underline{h} = \underline{r} \wedge \dot{\underline{r}}} \qquad (1.11)$$

Now we try to find the conservation of angular momentum.

Eq (1.8) vector multiply by $\underline{r}$

$$\underline{r} \wedge \ddot{\underline{r}} + \underline{r} \wedge \frac{\mu \underline{r}}{r^3} = 0 \qquad (1.12)$$

$$\frac{d\underline{h}}{dt} = \underbrace{\cancel{\dot{\underline{r}} \wedge \dot{\underline{r}}}}_{0} + \underline{r} \wedge \ddot{\underline{r}}$$

parallel vectors

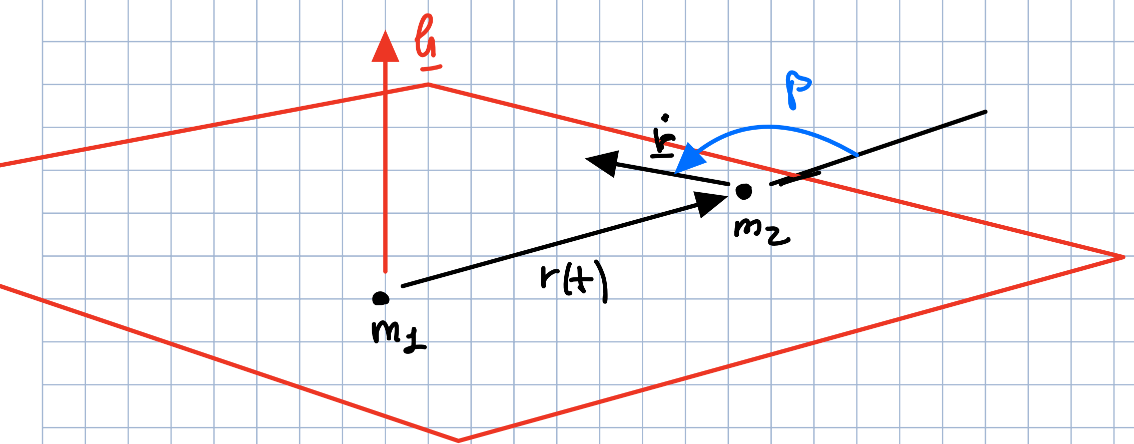
$$\Rightarrow \qquad \frac{d\underline{h}}{dt} = \underline{r} \wedge \ddot{\underline{r}} = -\underline{r} \wedge \frac{\mu \underline{r}}{r^3} = -\frac{\mu}{r^3}(\underline{r} \wedge \underline{r}) = 0$$

$$\frac{d\underline{h}}{dt} = 0 \Rightarrow \boxed{\underline{h} = \text{COSTANT}} \qquad (1.13)$$

$\underline{h}$ will be constant during trajectory $\rightarrow$ what are the consequences?

since $\underline{h} = \underline{r} \wedge \dot{\underline{r}} \Rightarrow \underline{r}$ and $\dot{\underline{r}}$ are always in the same plane

$\underline{h}$ can be used to identify the orbital plane in the restricted 2BP. (there is a version of that for n-body problem)



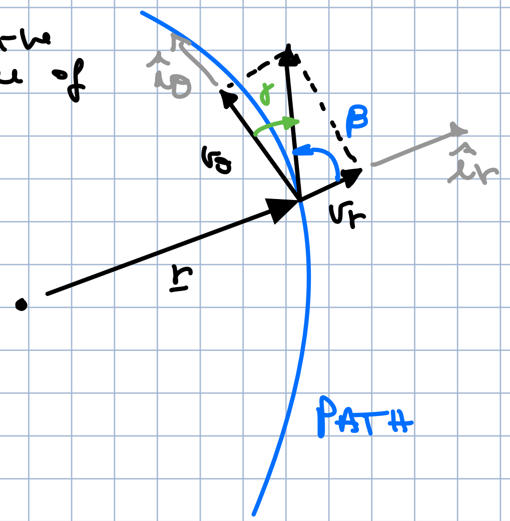
$\|\underline{h}\| =$ constant

$$\underline{h} = \underline{r} \wedge \dot{\underline{r}} \rightarrow \underline{h} = \underline{r} \wedge \underline{v} \Rightarrow h = r v \sin\beta$$

β angle between $\underline{r}$ and $\underline{v}$

$v \sin\beta = v_\theta$ transversal component of velocity (perpendicular to $\underline{r}$)

v_θ is in the direction of $\hat{i}_\theta$



γ = FLIGHT PATH ANGLE

$\gamma + \beta = \frac{\pi}{2}$

v_r = radial component of $\underline{v}$

$\hat{i}_r$ = radial unit vector

v_θ = transversal component of $\underline{v}$

$\hat{i}_\theta$ = transversal unit vector

$\beta = 90° - \gamma$

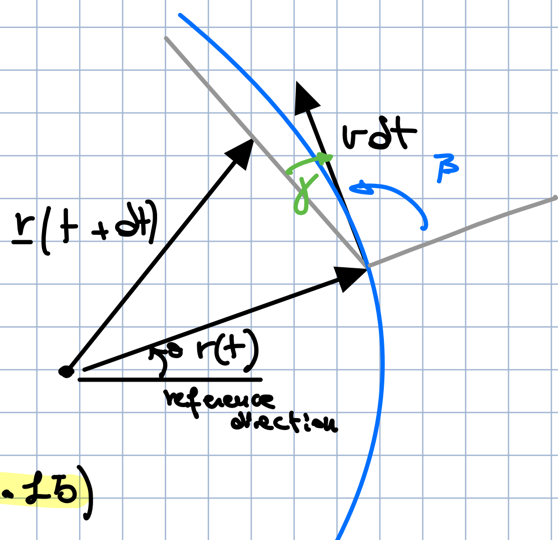
Therefore $h = r \cdot v_\theta$ but $v_\theta = r\dot{\theta}$ (*)

$h = r \cdot r\dot{\theta} \rightarrow \boxed{h = r^2 \dot{\theta}}$ (1.14)

$\dot{\theta}$ is the angular rate of $\underline{r}$ vector

$dA = \frac{1}{2}$ base height $= \frac{1}{2} r v dt \sin\beta$

$= \frac{1}{2} \underbrace{r v_\theta}_{h} dt = \frac{1}{2} h \, dt$



by (1.14) h = constant

$dA = \frac{1}{2} h \, dt$ $\boxed{\frac{dA}{dt} = \frac{h}{2}}$ (1.15)

$\frac{dA}{dt}$ = area velocity is constant

Kepler's second law: Equal areas are swept out in equal time

LAPLACE VECTOR, ECCENTRICITY VECTOR, EQUATION OF MOTION.

actual solution of s/c motion by cross multiply eq (1.8) by $\underline{h}$

$$\ddot{\underline{r}} \wedge \underline{h} = -\frac{\mu \underline{r}}{r^3} \wedge \underline{h} \quad (1.16)$$

left hand-side of (1.16)

since $\frac{d}{dt}(\dot{\underline{r}} \wedge \underline{h}) = \ddot{\underline{r}} \wedge \underline{h} + \dot{\underline{r}} \wedge \cancel{\dot{\underline{h}}}^{0} = \ddot{\underline{r}} \wedge \underline{h} \quad (1.17)$

RECALL

$\underline{a} \wedge (\underline{b} \wedge \underline{c}) = (\underline{a} \cdot \underline{c})\underline{b} - (\underline{b} \cdot \underline{a})\underline{c}$

right hand-side of (1.16)

$$= \frac{1}{r^3}(\underline{r} \wedge \underline{h}) = \frac{1}{r^3}(\underline{r} \wedge (\underline{r} \wedge \dot{\underline{r}}))$$

$$= \frac{1}{r^3}\left[(\underline{r} \cdot \dot{\underline{r}})\underline{r} - (\underline{r} \cdot \underline{r})\dot{\underline{r}}\right]$$

$$= \frac{1}{r^3}\left[r\dot{r}\underline{r} - r^2\dot{\underline{r}}\right] = \frac{\dot{r}\underline{r} - r\dot{\underline{r}}}{r^2}$$

$$\frac{d}{dt}\left(\frac{\underline{r}}{r}\right) = \frac{r\dot{\underline{r}} - \underline{r}\dot{r}}{r^2} = -\frac{\dot{r}\underline{r} - r\dot{\underline{r}}}{r^2}$$

therefore $\quad \frac{1}{r^3}(\underline{r} \wedge \underline{h}) = -\frac{d}{dt}\left(\frac{\underline{r}}{r}\right) \quad (1.18)$

Substitute (1.17) and (1.18) in (1.16) we get

$$\frac{d}{dt}(\dot{\underline{r}} \wedge \underline{h}) = \frac{d}{dt}\left(\mu \cdot \frac{\underline{r}}{r}\right) \rightarrow \frac{d}{dt}\left(\dot{\underline{r}} \wedge \underline{h} - \mu \frac{\underline{r}}{r}\right) = 0$$

$$\boxed{\dot{\underline{r}} \wedge \underline{h} - \mu \frac{\underline{r}}{r} = \underline{B}} \quad (1.19) \qquad \underline{B} = \text{COSTANT}$$

$\underline{B}$ constant of integration laplace vector (dimension μ)

Equation (1.19) $\rightarrow$ vector constant during the motion while $\dot{\underline{r}}$ $\underline{r}$ and r can change.

Laplace vector is also used in orbit propagation.

Eq (1.19) is the first integral of equation of motion (1.8)

Taking the dot product of Eq (1.19) by $\underline{h}$

$$\underbrace{(\dot{\underline{r}} \wedge \underline{h}) \cdot \underline{h}} - \mu \underbrace{\frac{\underline{r} \cdot \underline{h}}{r}} = \underline{B} \cdot \underline{h} \qquad (1.20)$$

$\dot{\underline{r}} \wedge \underline{h}$ is $\perp$ $\dot{r}$ is $\perp$ h

$\underline{r} \cdot \underline{h} = \underline{r} \cdot (\underline{r} \wedge \dot{\underline{r}})$

$\underline{h}$ is $\perp$ $\underline{r}$ and $\dot{\underline{r}}$

therefore $(\ ) \cdot \underline{h} = \emptyset$

$\underline{r} \cdot \underline{h} = \emptyset$

That is $\underline{B} \cdot \underline{h} = 0$

$\underline{B}$ is perpendicular to $\underline{h}$ ($\perp$ orbital plane)

$\underline{B}$ Laplace vector $\in$ orbital plane

$$\frac{\underline{r}}{r} + \frac{\underline{B}}{\mu} = \frac{\dot{\underline{r}} \wedge \underline{h}}{\mu}$$

$$\boxed{\underline{e} = \frac{\underline{B}}{\mu}}$$

$$\frac{\underline{r}}{r} + \underline{e} = \frac{\dot{\underline{r}} \wedge \underline{h}}{\mu} \qquad (1.21)$$

$\underline{e}$ ECCENTRICITY VECTOR

dimensionless

$$\boxed{\underline{e} = \frac{\dot{\underline{r}} \wedge \underline{h}}{\mu} - \frac{\underline{r}}{r}} \qquad (1.22)$$