09.13 Notes

Math 403/503

September 2022

1 Null Spaces, Matrices

We know about vector spaces. We know about linear transformations. Now we look at vector subspaces that are determined from linear transformations.

Definition: If $T \epsilon L(V, V)$, then the null space (or kernel) is:

null T = $\{v \epsilon V | T(v) = 0\}$

Examples:

- If T = 0 the 0 transformation then null T = V (the whole domain)
- $T \epsilon L(V, V)$ is the identity $T(v) = v$, null T = $\{0\}$
- $T \epsilon L(P(R), P(R))$, T = the derivative, null T = the constant functions (a one dimensional space of R)

Lemma: If $T \epsilon L(V, W)$ then null T is a subspace of V. In particular it's a vector space.

Proof:

- 0 exists in null T: $T0 = 0$ - check!
- If $v_1, v_2 \epsilon$ null T then $v_1 + v_2 \epsilon$ T: $Tv_1 = 0, Tv_2 = 0$, so $T(v_1 + v_2) = T(v_1) + T(v_2) = 0 + 0 = 0$. So $v_1 + v_2 \epsilon T$
- If $v \epsilon$ null T, then $\alpha v \epsilon$ T: $Tv = 0$ so $T(\alpha v) = \alpha T(v) = \alpha 0 = 0$. QED.

Definition: If $T \epsilon L(V, W)$ the range of T is: range T = $\{T(v) | v \epsilon V\}$

Example:

- T = 0, the zero transformation - range T = $\{0\}$
- $T \epsilon L(R^2, R^3)$ defined by $T(x, y) = (2x, 5y, x + y)$ - range T = some plane in R^2

- $T \epsilon L(P(R), P(R))$, T = the derivative - range T = all of $P(R)$

Lemma: Let $T \epsilon L(V, W)$ then range T is a subspace of W.

Proof:

- 0ϵ range T: $T0 = 0$ - check!
- If $w_1, w_2 \epsilon$ range T then $w_1 + w_2 \epsilon$ range T: $w_1 = Tv_1$, $w_2 = Tv_2 \rightarrow w_1 + w_2 = Tv_1 + Tv_2 = T(v_1 + v_2)$ so $w_1 + w_2 \epsilon$ range T.
- If $w \epsilon$ range T and $\alpha \epsilon F$ then $\alpha w \epsilon$ range T: $w = Tv \rightarrow \alpha w = \alpha Tv = T(\alpha v)$. So αw is in range T. QED.

Recall if $f : X \rightarrow Y$ then function f is injective means: $x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2)$.
The function f is surjective onto Y means: for every $y \epsilon Y$ there exists an $x \epsilon X$ such that $f(x) = y$.

Lemma: a linear transformation $T \epsilon L(V, W)$ is injective IFF null T = $\{0\}$

Proof: ($\rightarrow$) Suppose T is injective. Recall null(T) = everything that maps to 0. Since T is injective, at most one point in V can map to 0. We know $T0 = 0$ and now there can be nothing else! So null T = $\{0\}$.
($\leftarrow$) Suppose null T = $\{0\}$. Suppose $Tv_1 = Tv_2$. We want to show $v_1 = v_2$ (this is injective in its contrapositive form). Then $Tv_1 - Tv_2 = 0$, $T(v_1 - v_2) = 0$, $v_1 - v_2 \epsilon$ null T, $v_1 - v_2 = 0$, $v_1 = v_2$. QED.