11.10 Notes

Math 403/503

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1 First Video Lecture - Gram-Schmidt Example

Recall we will work in V = all continuous real-valued functions [-1,1], let $v = e^x$. This has a supspace $U = \text{span}(1, x, x^2)$. We are assuming an inner product:
$< f, g >= \int_{-1}^{1} fgdx$
$||f|| = \sqrt{\int_{-1}^{1} f^2 dx}$

First we need to replace $v_1 = 1, v_2 = x, v_3 = x^2$ with an ONB e_1, e_2, e_3.

Gram-Schmidt:
$e_1 = \frac{v_1}{||v_1||} = \frac{1}{\sqrt{\int_{-1}^{1} 1^2 dx}} = \frac{1}{\sqrt{2}}$
$e_2 = \frac{v_2 - <v_2, e_1> e_1}{||....||}$
$= \frac{x - \int_{-1}^{1} x \frac{1}{\sqrt{2}} dx (\frac{1}{\sqrt{2}}}{||....||}$

The integral term cancels out because x is an odd function! So...
$= \frac{x}{||x||} = \frac{x}{\sqrt{\int_{-1}^{1} x^2 dx}} = \frac{x}{\sqrt{\frac{2}{3}}} = \sqrt{\frac{3}{2}}(x)$
$e_3 = \frac{v_3 - <v_3, e_1> e_1 - <v_3, e_2> e_2}{||....||}$
$= \frac{x^2 - \int_{-1}^{1} x^2 (\frac{1}{\sqrt{2}} dx \frac{1}{\sqrt{2}} - \int_{-1}^{1} x^2 \sqrt{\frac{3}{2}} x dx \sqrt{\frac{3}{2}} x}{||.....||}$
$= \frac{x^2 - \frac{1}{3}}{||x^2 - \frac{1}{3}||}$
$= \frac{x^2 - \frac{1}{3}}{\sqrt{\int_{-1}^{1} (x^2 - \frac{1}{3})^2 dx}}$
$= \frac{x^2 - \frac{1}{3}}{\sqrt{\frac{8}{45}}}$
$= \sqrt{\frac{45}{8}} (x^2 - \frac{1}{3}$

So, $e_1 = \frac{1}{\sqrt{2}}, e_2 = \sqrt{\frac{3}{2}} x, e_3 = \sqrt{\frac{45}{8}} (x^2 - \frac{1}{3}$. Next we project e^x onto U =span(e_1, e_2, e_3).
$P_U v =< v, e_1 > e_1 + < v, e_2 > e_2 + < v, e_3 > e_3$.
$< v, e_1 >= \int_{-1}^{1} e^x \frac{1}{\sqrt{2}} dx = \frac{1}{\sqrt{2}} e^x |_{-1}^{1} = \frac{1}{\sqrt{2}} (e - \frac{1}{e}$
$< v, e_2 >= \int_{-1}^{1} e^x \sqrt{\frac{3}{2}} x dx = \frac{\sqrt{6}}{e}$

This is an integration by parts problem, but we skipped it. We went to symbolab to compute it!
$< v, e_3 >= \int_{-1}^{1} e^x \sqrt{\frac{45}{8}}(x^2 - \frac{1}{3})dx = \sqrt{\frac{5}{2}}(e - \frac{7}{e}$

$P_U v = \frac{1}{\sqrt{2}}(e - \frac{1}{e})\frac{1}{\sqrt{2}} + \frac{\sqrt{6}}{e}\sqrt{\frac{3}{2}}x + \sqrt{\frac{5}{2}}(e - \frac{7}{e}\sqrt{\frac{45}{8}}(x^2 - \frac{1}{3})$

Now we can plug our equation into desmos to see how close the approximation is over the given integral!

2 Second Video - Adjoints (Chapter 7)

Before proceeding generally, we state/recall the situation in R^n or C^n with the dot product. If A is an m x n matrix, it's adjoint (also called hermitian) is $A^* = \overline{A^t}$. So for R, A^* is just the transpose (dual) but for C, A^* is something different.

Example: $A = \begin{bmatrix} 1 & 2 & 3+i \\ -i & 4 & 0 \end{bmatrix}, A^* = \begin{bmatrix} 1 & i \\ 2 & 4 \\ 3-i & 0 \end{bmatrix}$

Using the inner product $< v, w >= v \cdot \overline{w}$. A and A^* have a special relationship: $< Av, w >=< v, A^*w >$. To see this: $< Av, w >= (Av) \cdot \overline{w} = (Av)^t\overline{w} = v^tA^t\overline{w} = v^t(A^t\overline{w}) = v^t(\overline{\overline{A^t}w}) = v^t\overline{(A^*w)} = v \cdot \overline{(A^*w)} =< v, A^*w >$.

This special property of A^* becomes the definition in general inner product spaces:

Definition: Let V, W be inner product spaces (finite dimensional). Let $T\epsilon L(V, W)$. Then T^* is an element of $L(W, V)$ defined by the relationship $< Tv, w >=< v, T^*w >$ for all $v\epsilon V, wW$.

In order for the definition to make sense we need to show T^*w exists for every w and then that T^* is linear. We show existence; linearity is 7.5 in the text.

Proof that T^*w exists:
Let V, W as above, with $TL(V, W)$. Let V have orthonormal basis $e_1, ..., e_n$. Fix $w\epsilon W$ as arbitrary. We are looking for a vector $u\epsilon V$ such that $< Tv, w >=< v, u > \forall v\epsilon V$. To do this let $\phi(v) =< Tv, w > .\phi$ is a linear functional (it is in V^*). So $\forall v : \phi(v) = \phi(< v, e_1 > e_1 + ...+ < v, e_n > e_n) =< v, e_1 > \phi(e_1) + ...+ < v, e_n > \phi(e_n) =< v, \overline{\phi(e_1)}e_1) > +...+ < v, \overline{\phi(e_n)}e_n >=< v, \overline{\phi(e_1)}e_1 + ... + \overline{\phi(e_n)}e_n >$. Let u be the second term in the inner product, thus, it equals $< v, u >$. Note that T^*w is now defined to be this vector, u.

The matrix of a linear map T is related to the matrix of T^* in the expected way, provided you work over orthonormal bases for V and W. If V has ONB $e_1, ..., e_n$ and W has ONB $f_1, ..., f_n$ and if T has matrix A with respect to e_i, f_j then T^* has matrix $\overline{A^t}$ with respect to f_j, e_i. Note: If you work in polynomial spaces with basis $1, x, x^2, ..., x^n$ and inner product $\int_{-1}^{1} Pqdx$ then these are not orthonormal, so knowing the matrix of some $T \epsilon L(V)$ doesn't tell you the matrix.