08.30 Lecture Notes

Math 403/503

September 2, 2022

Recall: $F = R$ or C (these are scalars).$V =$ a vector space over F. This means addition and scalar multiplication are supported and several axioms state that they behave as expected.

Definition: Given vectors $v_1, v_2, ...v_m$ in V, a linear combination is any expression $x_1 v_1 + ... + x_m v_m$ where $x_i \epsilon F$.

Definition: Given vectors $v_1, v_2, ...v_m$ in V, the span of $v_1, ..., v_m$ is the set of all linear combinations of $v_1, ..., v_m$. Terminology (could be a verb): If the span of $v_1, ..., v_m$ equals all of V we say that $v_1, ..., v_m$ spans V.

E.G. In F^3 the vectors $v_1 = (1, 0, 0)$ $v_2 = (0, 1, 0)$. The span of of v_1, v_2 consists of the xy-plane because linear combinations have the form $x(1, 0, 0) + y(0, 1, 0)$ or $(x, y, 0)$. The vectors $v_1 = (1, 0, 0), v_2 = (0, 1, 0), v_3 = (0, 0, 1)$ span F^3.

Definition: The dimension of V is the smallest number of vectors $v_1, ..., v_m$ that spans V. If there is no such number (so no finite list of vectors spans V) then V is said to be infinite dimensional.

E.G. F^3 is finite dimensional because $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ span it. F^N is not finite dimensional.

E.G. A polynomial is an expression of the form $a_0 + a_1 z + a_2 z^2 + ... + a_m z^m$ where $a_0, a_1, ...\epsilon F$. The degree of the polynomial is the highest power n of z such that the coefficient $a_m \neq 0$.

If $V =$ the space of polynomials of degree $\leq m$ then V is finite dimensional. If $W =$ the space of all polynomials then W is infinite dimensional. To see this let $w_1, w_2, ..., w_k$ be any list of elements of W. Each w_i has a degree, m_i. Let $m =$ the maximum of these degrees m_i. Then the polynomial $z^(m + 1)$ cannot be written as a linear combination of $w_1, ..., w_k$. Thus, no list $w_1, ..., w_k$ spans W.

Linear Independence

Given any vectors $v_1, ..., v_m$ in some space V, the 0 vector is always a linear combination: $0 = 0v_1 + 0v_2 + ... + 0v_m$.

Definition: A set of vectors $v_1, ..., v_m$ is called linearly independent if the only way to write 0 as a combination of $v_1, ..., v_m$ is with all coefficients being 0. If there is more than one way, we say $v_1, ..., v_m$ is linearly dependent.

Lemma: A set of vectors $v_1, ..., v_m$ is linearly independent if and only if none of the vectors v_j can be written as a combination of the rest. The proof of this statement is homework.

E.G. In F^2 consider the vectors $v_1 = (1, 2), v_2 = (2, 3), v_3 = (3, 7)$. This list is linearly dependent: $(0, 0) = 5(1, 2) + -1(2, 3) + -1(3, 7)$. Alternatively this could be written as: $(1, 2) = 1/5(2, 3) + 1/5(3, 7)$.