10.20 Notes

Math 403/503

October 2022

1 Introduction

Today's goal: To discuss generalized eigenspaces, give proof sketches of 2 key terms, and then what they mean for analyzing the original linear operator T.

2 Notes

We are working towards:
Theorem: $V = G(\lambda_1, T) + ... + G(\lambda_m, T)$ where this is a direct sum and $\lambda_1, ..., \lambda_m$ are eigenvalues of T. Note that $T \epsilon L(V, V)$ and V is finite dimensional.

Part 1: $G(\lambda_1, T) + ... + G(\lambda_m, T)$ makes a direct sum.
Proof Sketch: As we saw in the case of the eigenspaces $E(\lambda_j, T)$, this amounts to showing that if $v_1 \epsilon G(\lambda_1, T)...v_m \epsilon G(\lambda_m, T)$ (nonzero) then $v_1, ..., v_m$ is independent. So assume that $a_1v_1 + ... + a_mv_m = 0$. We will show $a_j = 0$ for all j. Focus on v_1 to start $v_1 \epsilon G(\lambda_1, T) =$ null $(T - \lambda_1 I)^n$. Let k be such that $v_1 \epsilon$ null$(T - \lambda_1 I)^k$ but v_1 does not exist in null$(T - \lambda_1 I)^{k-1}$. Hit the equation $a_1v_1 + ... + a_mv_m = 0$ with the map $(T - \lambda_1 I)^{k-1}(T - \lambda_2 I)^n...(T - \lambda_n I)^n$. The result is $(T - \lambda_1 I)^{k-1}(T - \lambda_2 I)^n...(T - \lambda_n I)^n a_1 v_1 = 0 \to (T - \lambda_2 I)^n...(T \lambda_n I)^n w = 0 \to a_1(\lambda_1 - \lambda_2)^n...(\lambda_1 - \lambda_m)^n w = 0$. Since we know the rest of the terms are not equal to zero we can conclude that $a_1 = 0$. Similarly, we can show any $a_j = 0$. Therefore, $v_1, ..., v_m$ is independent. QED.

Part 2: $V = G(\lambda_1, T) + ... + G(\lambda_m, T)$ is a direct sum.
Proof Sketch: The proof is by induction on the dimension of V. First we know an eigenvalue exists, call it λ_1. Now form $G(\lambda_1, T) =$ null $((T - \lambda_1 I)^n)$. Let $U =$ range $((T - \lambda_1 I)^n)$. Both bull and range are T-invariant by a lemma in the book, the null space and the range of an n-th power make a direct sum. Thus $V = G(\lambda_1, T) + U$ is a direct sum, where U is T-invariant. U has a smaller dimension than V so we can apply the inductive hypothesis to T restricted by U to conclude U is a direct sum of generalized eigensapces. Hence V is too. QED.

Next we need to investigate waht T looks like on some $G(\lambda_1, T)$. If we can understand this then we can understand all of T because will be a <u>block matrix</u>.

Theorem: For any T, any of its eigenvalues λ_j, we can find a basis for T restricted by $G(\lambda_j, T)$ consisting of eigenvectors, plus, Jordan chains that terminate at eigenvectors by multiplying $T - \lambda_j I$. The proof that there exists such a basis is also by induction on the dimension of V.

So what does it mean that we can find a basis like this for each $G(\lambda_j, T)$?

- The matrix A of T in this basis (yes, the one with all the Jordan chains) consists of blocks $A_1, .., A_m$ down the diagonal, where each A_j corresponds to T restricted by $G(\lambda_j, T)$.
- Each block A_j consists of 1 or more sub blocks, one sub block for each Jordan chain.
- The matrix on each sub block looks like a matrix with λ_j down the diagonal and 1's on the super diagonal, but not in the first column. The first columbn in the top of the chain, as we know is an eigenvector $(T - \lambda I)v_1 = 0 \rightarrow Tv_1 = \lambda v_1$. Each subsequent column has a 1 and a λ_j because $(T - \lambda_1)v_{1,l+1} = v_{1,l} \rightarrow Tv_{1,l+1} = \lambda v_{1,l+1} + 1v_{1,l}$.

We conclude that T has a matrix with the λ's down the diagonal in appropriate multiplicities, plus some 1's on the superdiagonal signaling chains of generalized eigenvectors. This is called Jordan Normal Form.