10.06 Notes

Math 403/503

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1 Existence of Eigenvalues

Over scalar field $F = C$, we will see that every linear map $T \epsilon L(V)$ does have eigenvalues. The main fact about C that we need is that any polynomial equation $a_0 + a_1 z + ... + a_n z^n = 0$ has solutions for z in C (Fundamental Theorem of Algebra). In $L(V, V)$, operators T may of course be added and scaled and more over they can be composed and thus can be iterated. Given any T, things like $\alpha T, \alpha T + \beta T, T^2, \alpha T^2$, etc. all exist in $L(V, V)$. In general you can get any polynomial in the symbol T: $\alpha_0 I + \alpha_1 T + \alpha_2 T^2 + ... + \alpha_n T^n$. This is called $?(T)$ where $p(z) = a_0 + a_1 z + ... + a_n z^n$ and $P(T)$ lives in $L(V, V)$.

Example: Fix $T(w, z) = (-z, w)$. Fix $p(x) = 8 + 3x - 2x^2$. Then we can get an operator, $p(T) = 8I + 3T - 2T^2$. Note $P(T)$ is actually the operator. $P(T)(w, z) = 8(w, z) + 3(-z, w) - z(-w, -z) = (10w - 3z, 3w + 10z)$.

Theorem: Let V be finite dimensional vector space over C and let $T \epsilon L(V, V)$. Then T has an eigenvalue.
Proof: Let $v \epsilon V$ be any nonzero vector. Start listing the iterated applications of T to $v : v, Tv, T^2v, T^3v, ...$ eventually this list will become linearly dependent (at the latest when it's length surpasses dim V). So say $a_0 v + a_1 Tv + a_2 T^2 v + ... + a_m T^m v = 0$. This is a polynomial $p(z)$ applied to T applied to v, equalling zero. Since we are over C, this polynomial factors completely: $a_m(T - \alpha_1 I)(T - \alpha_2 I)...(T - \alpha_m I)v = 0$. It must therefore be the case that at least 1 of the factors $T - \alpha_j I$ has a nontrivial null space. In particular the corresponding α_j is an eigenvalue of T. QED.

Example that supports proof above: Say maybe v, Tv, T^2v is linearly dependent and you find $2v - 3Tv + 1T^2v = 0$. $T^2v - 3Tv + 2Tv = 0 \leftrightarrow z^2 - 3z + 2 = 0$. This factors into $(z - 2)(z - 1) = 0$.
$(T - I) \circ (T - 2I)v = 0$
Two cases...
1. $w = (T - 2I)v = 0$
$\rightarrow$ 2 is an eigenvalue.
2. $w \neq 0$ but $(T - I)w = 0$
$\rightarrow$ 1 is an eigenvalue. Note that in both cases we have an eigenvalue!

When we work in $L(V,V)$ it makes sense to consider the same basis in both the domain and co-domain side when forming matrices.

$T \epsilon L(V,W)$ basis $v_1, ..., v_n$ and basis $w_1, ..., w_n$. This gives us a matrix with $v's$ along the top and $w's$ along the side with dimensions being mxn.

$T \epsilon L(V,V)$ basis $v_1, ..., v_n$. This gives us a matrix with $v_1, ..., v_n$ entries along the top and side.

We now want to investigate the special situation when the matrix of T with respect to the basis $v_1, ..., v_n$ is upper triangular. A matrix being upper triangular means $Tv_j \epsilon$ span$(v_1, ..., v_j)$ for all j. The following is a corollary of the existence of eigenvalues over C:

Theorem: Let V be finite dimensional over C and $T \epsilon L(V,V)$. There exists a basis $v_1, ..., v_n$ of V such that the matrix of T in this basis is upper triangular.

Proof Sketch: We know T has an eigenvalue λ. So null$(T - \lambda I)$ is nonzero if range$(T - \lambda I) \neq V$. It also happens that range$(T - \lambda I)$ is an invariant subspace for T. Because if U is in range$(T - \lambda I)$ then $Tu = Tu - \lambda u + \lambda u = (T - \lambda I)u + \lambda u$. Note that both terms in the final expression exists in the range$(T - \lambda I)$. So we may assume inductively that T restricted by the range$(T - \lambda I)$ has an upper triangular matrix with respect to some basis $u_1, ..., u_m$. Now extend $u_1, ..., u_m$ to a basis of V with new vectors $v_1, ..., v_k$. Then $Tv_j = Tv_j - \lambda v_j + \lambda v_j$. Note that the first two terms exist in range$(T - \lambda I)$ so span$(u_1, ..., u_m)$. In particular Tv_j is in the span$(u_1, ..., u_m, v_1, ..., v_j)$. So the matrix is triangular in this basis. QED.