11.08 Notes

Math 403/503

November 2022

1 Homework Hint

Use sin and cos identities to solve 5.

2 Projections

In the proof of Gram-Schmidt we showed that if $e_1, ..., e_j$ is an orthonormal list and v is any vector, then v can be written as $v =< v, e_1 > e_1 + ...+ < v, e_j > e_j + w$ where w is orthogonal to $e_1, ..., e_j$.
Why does it work?
$< w, e_j >=< v- < v, e_1 > e_1 - ...- < v, e_j > e_j, e_j >$
$=< v, e_i > - < v, e_1 >< e_1, e_i > -...- < v, e_j >< e_j, e_j >$
$=< v, e_i > - < v, e_i >< e_i, e_i >$
$=< v, e_i > - < v, e_i >$
$= 0$
This key idea leads to the following definition.

Definition: Let V be an inner product space and let U be a subspace of V. The orthogonal complement of U, denoted $U^\perp$ is defined by $U^\perp = \{v \epsilon V : \forall u \epsilon U, < u, v >= 0\}$.

Proposition: $U^\perp$ is a subspace of V. The proof follows from the additivity and scalar multiple properties of the inner product.

Example: If $v_1, \epsilon U^\perp, v_2 \epsilon U^\perp : \forall u \epsilon U < u, v_1 >= 0$ and $< u, v_2 >= 0$. Then $\forall u \epsilon U < u, v_1 + v_2 >=< u, v_1 > + < u, v_2 >= 0$.

Theorem: V equals the direct sum of U and $U^\perp$ (provided U is finite dimensional).
Proof: First we need to show V equals the sum of U and $U^\perp$. This comes from the calculation earlier. Let U have ONB $e_1, ..., e_j$ (we know such a basis exists by Gram-Schmidt). Then for any $v \epsilon V$ we can express $v =< v, e_1 > +...+ < v, e_j > +w$ where $< w, v_1 >= ... =< w, e_j >= 0$. Then $v = u + w$ where $u =< v, e_1 > e_1 + ...+ < v, e_j > e_j \epsilon U$ and $w \epsilon U^\perp$ (w is orthogonal to a basis

of U, thus orthogonal to everything in U). Last, we need to show directness of the sum. We can simply show that the intersection of U and $U^\perp$ equals the zero space. This is true because if u exists in the intersection of U and $U^\perp$ then $< u, u >= 0$ and this implies that $u = 0$.

This decomposition theorem $V = U \oplus U^\perp$ allows us to talk about projections onto U.

Definition: Suppose $V = U \oplus U^\perp$. Then for any $v \epsilon V$ we can write $v = u + w$ where $u \epsilon U$ and $w \epsilon U^\perp$. We call u the <u>orthogonal projection</u> of v onto U.
Notation: PU denotes the mapping that carries v to the orthogonal projection onto U. $P_U(v) = u$. Thus PU is determined by the constraints: $v = P_U(v) + w$ where $w \epsilon U^\perp$. How to calculate $P_U(v)$? Let $e_1, ..., e_j$ be an ONB for U. Then $P_U(v) =< v, e_1 > e_1 + ... + < v, e_j > e_j$.

The map P_U is a linear map $P_U : V \to V$. It has many special properties, see 6.55 in the textbook. Projections help solve "nearest vector" optimization problems including curve fitting and linear least squares.

Theorem: $P_U(v)$ is the vector in U nearest to v.
Proof: $||P_U(v) - v||^2 \leq ||P_U(v) - v||^2 + ||u - P_U(v)||^2$ for any $u \epsilon U$.
$= ||P_U(v) - v + u - P_U(v)||^2$
$= ||u - v||^2$
Thus, $||P_U(v) - v|| \leq ||u - v||$ for any $u \epsilon U$.

Example using orthogonal projections for curve fitting. Problem: Let $f(x) = e^x$. Find the quadratic function that best approximates e^x on the interval [-1,1]. Use the inner product: $< f, g >= \int_{-1}^{1} fg$

Procedure: Let $U = \text{span}(1, x, x^2)$ = the quadratic functions, a subspace of V = all continuous functions. The basis is not orthonormal *sad*. Use Gram-Schmidt to get an ONB of U:
$q_0(x) = ...$
$q_1(x) = ...$
$q_3(x) = ...$
Then project e^x onto U to get $p(x) =< e^x, q_0(x) > q_0(x) + < e^x, q_1(x) > q_1(x) + < e^x, q_2(x) > q_2(x)$. Plot the results against e^x. This problem will be finished in next class.