11.15 Notes

Math 403/503

November 2022

1 Week 13 - Spectral Theory

We mix the theory of the adjoint T^* which in principle works for any $T \epsilon L(V, W)$ with the thoery of operators (eigenvalues and eigenvectors) which works when $T \epsilon L(V)$. Recall that the main rule of T^* is: $< Tv, w >=< v, T^* w >$. And recall that if working with an ONB, the matrix of T^* will be the conjugate transpose of the matrix of T.

First we study...
Definition: T is called self-adjoint if $T = T^*$. For real vector spaces, this is analogous to symmetric matrices. You might recall from 301 that symmetric matrices have really nice eigenvalues/eigenvectors.

Spectral theory for real symmetric matrices:

- Any real symmetric matrix $A(A = A^t)$ has real eigenvalues only.

  Proof: Suppose $Av = \lambda v$. We want to show λ is real. We calculate the following:
  $\lambda||v^2||$
  $= \lambda v^t \cdot \overline{v}$
  $= (Av)^t \cdot \overline{v}$
  $= v^t A^t \cdot \overline{v}$
  $= v^t \overline{Av}$
  $= v^t \overline{\lambda v}$
  $= \overline{\lambda}||v^2||$.
  Because the norm of v^2 is a scalar we can cancel it from both sides to get that $\lambda = \overline{\lambda}$, so λ is real.

- Any real symmetric matrix A has orthogonal eigenvectors. In fact, it has dim V many, so it is diagonalizable.

  Proof of first statement: Consider eigenvalues λ_1, λ_2, which are not equal. And consider corresponding eigenvectors v_1, v_2. Remember, from point 1, we know that the λ values are real. We calculate the following:

$\lambda_2 v_1^t \cdot v_2$
$= v_1^t \lambda_2 v_2$
$= v_1^t A v_2$
$= v_1^t A^t v_2$
$= (Av_1)^t v_2$
$= \lambda_1 v_1^t v_2$.
Thus, since $\lambda_1 \neq \lambda_2$ we must have $v_1^t \cdot v_2 = 0$ so the eigenvectors are orthogonal.

Delaying the proof that there are dim V many such eigenvectors, the statements above imply that A can be diagonalized: $A = Q\lambda Q^{-1}$, where Q has orthonormal columns.

The arguments above generalize to real inner product spaces to show...

- Any self-adjoint T has real eigenvalues only
- Any self-adjoint T has orthogonal eigenvectors (dim V many).

Over C something even more general is true!

Definition: T is normal if $T^*T = TT^*$.

If T is self-adjoint then T is normal. But many more operators are normal than self-adjoint! Over C the results above generalize to give us similar conclusions for normal operators.

- If T is normal then T and T^* have the same eigenvectors, corresponding to conjugate eigenvalues - we will prove this next time.
- If T is normal then T has orthogonal eigenvectos, in fact, dim V many.

  Proof of Orthongonality: Let v_1, v_2 be eigenvectors corresponding to eigenvalues λ_1, λ_2 (where $\lambda_1 \neq \lambda_2$).
  $\lambda_2 < v_1, v_2 > = < v_1, \overline{\lambda_2} v_2 >$
  $= < v_1, T^* v_2 >$
  $= < Tv_1, v_2 >$
  $= < \lambda_1 v_1, v_2 >$
  $= \lambda_1 < v_1, v_2 >$
  Since $\lambda_1 \neq \lambda_2$ we must have $< v_1, v_2 > = 0$. So v_1 is orthogonal to v_2

The conclusion is that if T is a normal operator then there exists an orthonormal basis $e_1, ..., e_n$ in which T is diagonal! The converse is also true. Namely, if T is diagonal with respect to some ONB then T is normal. This whole thing together is called "The Spectral Theorem".