# 11.17 Notes

Math 403/503

November 2022

## 1 Game Plan For Future Weeks

- Today - finish spectral theorem
- Next week - break
- Following week - maybe chapter 10
- Dead week - review and give out final quiz
- Finals week - submit quiz

## 2 Last Time...

Last time we introduced the (complex) spectral theorem which consisted of two statements:

- If $T$ is normal ($T^*T = TT^*$) then $T$ and $T^*$ have the same eigenvectors but with conjugate eigenvalues: $Tv = \lambda v \leftrightarrow T^*v = \overline{\lambda}v$
- If $T$ is normal then the eigenvectors of $T$ are mutually orthogonal, and there are dim V many eigenvectors (an orthonormal basis of V consisting of eigenvectors).

I owe you a few portions of proofs.

- Assume $Tv = \lambda v$ (i.e. $(T-\lambda I)v = 0$. Note that $(T-\lambda I)^* = T^* - \overline{\lambda}I$. Note that $T-\lambda I$ is also normal (if $T$ is): $(T-\lambda I)^*(T-\lambda I) = (T^* - \lambda I)(T-\lambda I)$
  $= T^*T - \lambda T^* - \overline{\lambda}T + \lambda\overline{\lambda}I$
  $= TT^* - \lambda T^* - \overline{\lambda}T + \overline{\lambda}\lambda I$
  $= (T-\lambda I)(T^* - \overline{\lambda}I)$
  Now, $(T-\lambda I)v = 0 \rightarrow ||(T-\lambda I)v||^2 = 0$
  $\rightarrow < (T-\lambda I)v, (T-\lambda I)v >= 0$
  $\rightarrow < v, (T-\lambda I)^*(T-\lambda I)v >= 0$
  $\rightarrow < v, (T-\lambda I)(T-\lambda I)^*v >= 0$
  $\rightarrow < (T-\lambda I)^*v, (T-\lambda I)^*v >= 0$
  $\rightarrow (T-\lambda I)^*v = 0$

$\rightarrow (T^* - \overline{\lambda}I)v = 0$
$\rightarrow T^*v = \overline{\lambda}v.$

- We showed last time that the various eigenspaces of $T$ are pairwise and orthogonal. Let me just repeat: Let $Tv_1 = \lambda_1 v_1, Tv_2 = \lambda_2 v_2 (\lambda_1 \neq \lambda_2)$. (know that $T^*v_1 = \overline{\lambda_1}v_1, T^*v_2 = \overline{\lambda_2}v_2$). Now, $< Tv_1, v_2 >=< v_1, T^*v_2 >$
$\rightarrow< \lambda_1 v_1, v_2 >=< v_1, \overline{\lambda_2}v_2 >$
$\rightarrow \lambda_1 < v_1, v_2 >= \lambda_2 < v_1, v_2 >$
$\rightarrow< v_1, v_2 >= 0$

The last thing we need to do is show that in fact $T$ is diagonalized by its eigenvectors. We can do this in two parts:

- There exists an orthonormal basis in which $T$ has an upper triangular matrix

- Any upper triangular matrix that happens to be normal must be diagonal

The first bullet point follows from some things we have already done. Begin with any basis $v_1, ..., v_n$ and let $A$ be the matrix of $T$ in this basis. Since we are working over $C$ we know that $T$ has an upper triangular matrix with respect to some basis (e.g. the Jordan form) so: $A = BUB^{-1}$ where $U$ is upper triangular. Now from Gram-Schmidt we can factor $B$ as $B = QR$ where $Q$ is orthonormal and $R$ is upper triangular, then:
$A = BUB^{-1}$
$= QRU(QR)^1$
$= QRUR^{-1}Q^{-1}$
$= Q(RUR^{-1})Q^{-1}$

For the second bullet point we simply inspect what it means for an upper triangular matrix $U$ to be normal. $U = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ 0 & a_{22} & ... & a_{2n} \\ 0 & 0 & a_{33} & \\ ... & & & \end{bmatrix}$ Normal: $U^*U = UU^*$.

Looking at the $(1, 1)$ entry:
$|a_{11}|^2 = |a_{11}|^2 + ... + |a_{1n}|^2$
$\rightarrow |a_{12}|^2 = ... = |a_{1n}|^2 = 0$
$\rightarrow a_{12} = ... = a_{1n} = 0.$

Looking at the $(2, 2)$ entry:
$|a_{12}|^2 + |a_{22}|^2 = |a_{22}|^2 + ... + |a_{2n}|^2$
$\rightarrow |a_{23}|^2 + ... + |a_{2n}|^2 = 0$
$\rightarrow a_{23} = ... = a_{2n} = 0$

Inductively we can find that all off-diagonal entries are zero, so, normal and upper triangular implies diagonal. So we conclude that normal matrices over $C$ have the decomposition $A = Q\Lambda Q^*$ where $Q$ is orthonormal and $\Lambda$ is diagonal (over a real vector space one must assume $A$ is symmetric i.e. self adjoint).

One application of the spectral theorem is the "Polar decomposition". Any complex number $z$ can be written in a polar form $e^{i\theta} \cdot r$, as a product of a unit (norm 1) and a positive real number. There is a similar result for operators! The unit will be replaced by a "unitary" or orthonormal matrix $Q$. The positive real number will be replaced by a "positive operator".

**Definition**: An operator $p\epsilon L(V)$ is called positive if for all $v\epsilon V, < Pv, v > \geq 0$. NOte that over $C$, positive is equivalent to $P$ being orthogonally diagonalizable with all eigenvalues being real and nonnegative.

**Theorem**: Any square matrix $A$ may be decomposed as $Q \cdot P$ where $Q$ is orthonormal and $P$ is positive.

**Proof (in the special case when $A$ is normal)**: We know $A = Q\Lambda Q^*$ from the spectral theorem. Then for each $\lambda_{ii}$ in $\Lambda$ write it as $\lambda_{ii} = u_i|\lambda_{ii}|$ where $u_i$ satisfies $|u_i| = 1$. Then let $U$ be a matrix with just entries along the diagonal, and similarly $|\Lambda|$ just has entries along the diagonal. Then $A = QU|\Lambda|Q^* = QUQ^* \cdot Q|\Lambda|Q^*$

An arbitrary matrix $A$ gives rise to a normal matrix $A^*A$... then work harder to get the result for $A$ too