10.13 Notes

Math 403/502

October 2022

1 Introduction

Calendar says week 10 is the next quiz. He is considering making it week 11.

Theorem: Let V be a finite dimensional and $L\epsilon(V, V)$. Then there is a basis of V in which T has a diagonal matrix if and only if, $E(T, \lambda_1)$ direct summed with all the other $E(T, \lambda_n)$ terms is equal to V where $\lambda_1, ..., \lambda_n$ are distinct eigenvalues of T.

Proof: $\rightarrow$ Suppose $v_1, ..., v_n$ is a basis of V and T is diagonal with respect to this basis. Then $T(v_j) = d_i v_i$ for all i. This means that the entries along the diagonal ($d_i' s$) are all eigenvalues and the $v_i' s$ are all eigenvectors of T. Thus, each v_i lies in some eigenspace $E(T, d_i)$. Thus, the direct some of the E's mentioned above contains a spanning set so itself spans V.
$\leftarrow$ Suppose $\lambda_1, ..., \lambda_n$ are the eigenvalues of T and $E(T, \lambda_1)$ direct summed with all the other $E(T, \lambda_n)$ terms is equal to V. Then we can choose bases for each $E(T, \lambda_j)$ and combine these bases together to get a basis $v_1, ..., v_n$ consisting entirely of eigenvectors of T. Then with respect to this basis, Tv_j is always a multiple of v_j, so the matrix of T is diagonal. QED.

Recall we talked about invariant subspaces and how we might wish to breakdown V into several invariant subspaces on which T is very simple to understand. In the case of the theorem above we have succeeded fully, because $E(T, \lambda_j)$ is an invariant subspace and more strongly T is just scalar multiplication: $T(v) = \lambda_j v$ on that space! The net thing to worry about is what to do when this isn't satisfied and the $E(T, \lambda_j)'s$ don't sum up to V...

Corollary: If T is an operator on V and T has $n = dimV$ distinct eigenvalues, then T has a basis in which it is diagonal.
Proof: Each $E(T, \lambda_j)$ contributs at least 1 dimension to the sum. With n summands, the sum is n-dimensions or all of V. QED.

Example: $T(x, y) = (x + y, y)$, T has matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. We know that $\lambda = 1$

is the only eigenvalue! We find $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is the only eigenvector (up to scalar multiple). $E(T,1) = \text{span}(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \neq V$.

Example: Let $T(x,y,z) = (2x+y, 5y+3z, 8z)$. T has matrix $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 5 & 3 \\ 0 & 0 & 8 \end{bmatrix}$. We know $\lambda_1 = 2, \lambda_2 = 5, \lambda_3 = 8$. We know it will be diagonalizable by corollary. To do so: $E(T,2) = \text{null} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 3 & 3 \\ 0 & 0 & 6 \end{bmatrix}$. Then we compute the null space of this and get it to be $\begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} x = \text{span}\,(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix})$. $E(T,5) = \text{null} \begin{bmatrix} -3 & 1 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 3 \end{bmatrix}$. Then we compute the null space of this and get it to be $\begin{bmatrix} 1/3y \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1 \\ 0 \end{bmatrix} y = \text{span}\,(\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix})$. $E(T,8) = \text{null} \begin{bmatrix} -6 & 1 & 0 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$. Then we compute the null space of this and get it to be $\begin{bmatrix} 1/6z \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 6 \end{bmatrix} z = \text{span}\,(\begin{bmatrix} 1 \\ 6 \\ 6 \end{bmatrix})$. So in basis $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 6 \\ 6 \end{bmatrix}$, T has matrix $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 8 \end{bmatrix}$. What happens when things go wrong? When it isn't diagonalizable?

Example: $T = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix}$. We know $\lambda_1 = 2, \lambda_2 = 5$ are the eigenvalues. $E(T,2)$ =null $\begin{bmatrix} 0 & 3 & 4 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ = span $(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix})$. $E(T,5) = \text{null} \begin{bmatrix} -3 & 3 & 4 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ = span $(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix})$. So the direct sum of $E(T,2)$ and $E(T,5) \neq V$. Thankfully there is a "generalized eigenvector" it lies in null $\begin{bmatrix} -3 & 3 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. It further has the property that $(T-5I)v_3 = v_2$. It is $= v_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} -3 & 3 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = v_2$. In the basis $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, T has matrix:

$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix}$. Because of the equation above it is nearly diagonalizable.