10.18

Math 403/503

October 2022

1 Generalized Eigenvectors

Our schedule is as follows: This is week 9 so it's the last week of material in this "unit". Next week is quiz week. This means on Tuesday we will have class and do a catch up/review. On Wednesday the take-home quiz will be released. There will be no class on Thursday. The quiz will be due Monday night.

Recall we have a goal of, given an operator T, decomposing V into invariant subspaces where T behaves quite simply. The best situation is when $V = E(\lambda_1, T) + ... + E(\lambda_m, T)$ where the terms are a direct sum. In this case T acts as a scalar on each of these subspaces and overall T acts as a diagonal matrix with respect to the basis of eigenvectors. But this is not always the case (more on this later... chapters 6 and 7). For now we will jump forward to chapter 8 where we show generalized eigenspaces fix this problem at least if we are over C.

Recall $E(\lambda, T) = \text{null}(T - \lambda I)$. We now define $G(\lambda, T) =$ the union over j of the null space of $[(T - \lambda I)^j]$. This is called the generalized eigenspace of T corresponding to λ. Any $v \epsilon G(\lambda, T)$ other than 0 is called a generalized eigenvector of T corresponding to λ. Note $E(\lambda, T)$ is a proper subspace of $G(\lambda, T)$.

Lemma: $G(\lambda, T)$ is a subspace of V and it is T - invariant.

Proof Sketch: Note that $\text{null}(T - \lambda I)$ is a proper subspace of $\text{null}(T - \lambda I)^2$ which is a proper subspace of... this is because $(T - \lambda I) = 0 \rightarrow (T - \lambda I)(T - \lambda I) = 0 \rightarrow ...$ in this chain only at the most n of the proper subspace symbols can be proper inclusions, where n = dim V. In fact it turns out the union over j going from 1 to infinity of the null space of $(T - \lambda I)^j = \text{null}[(T - \lambda I)^n]$ so $G(\lambda, T)$ is a subspace. For invariance assume $(T - \lambda I)^2 v = 0$ we will show $(T - \lambda I)^2 Tv = 0$.

$(T - \lambda I)^2 Tv$
$= (T^2 - 2\lambda T + \lambda^2 I)Tv$
$= (T^3 - 2\lambda T^2 + \lambda^2 T)v$
$= T(T^2 - 2\lambda T + \lambda^2 I)v$
$= T0$
$= 0$

The same goes for all powers. So null$((T-\lambda I)^j)$ is T-invariant for any j for $G(\lambda, T)$ is T-invariant.

Example: $A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix}$, $E(2,T) = \text{null} \begin{bmatrix} 0 & 3 & 4 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} = \ldots \text{ span} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$G(2,T) = ?$

$$\text{null} \begin{bmatrix} 0 & 3 & 4 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 3 & 4 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} = \text{null} \begin{bmatrix} 0 & 9 & 15 \\ 0 & 9 & 6 \\ 0 & 0 & 9 \end{bmatrix} = \ldots \text{ span} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$G(2,T) = E(2,T) = \text{span} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$E(5,T) = \text{null} \begin{bmatrix} -3 & 3 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \ldots = \text{span} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$G(5,T) = ?$

$$\text{null} \begin{bmatrix} -3 & 3 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -3 & 3 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \text{null} \begin{bmatrix} 9 & -9 & -9 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{null} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} x \\ y \\ z \end{bmatrix} =$$
$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} y + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} z = \text{span } \left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right)$$

What about null $(A - SI)$?

$$\text{null} \begin{bmatrix} 9 & -9 & -9 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -3 & 3 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \text{null} \begin{bmatrix} -27 & 27 & 27 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{span} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$G(5,T) = \text{span } \left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right)$. So $V = C^3 =$ the direct sum of $E(2,T)$ and $E(5,T)$. So ew aim to prove that for any T over a finite dimensional vector space, $V =$ the direct sum of $G(\lambda_1, T) \ldots G(\lambda_m, T)$ and furthermore show T has a very simple action on its $G(\lambda_j, T)$'s so this will complete the study of T.