11.03 Notes

Math 403/503

November 2022

1 From Last Time...

Pythagorean Theorem generalized: Suppose $v_1, .., v_n$ is a family of pairwise orthogonal vectors: $< v_i, v_j >= 0, i \neq j$. Then $||v_1||^2 + ||v_2||^2 + ... + ||v_n||^2 = ||v_1 + ... + v_n||^2$

Projection in inner product space: If $u, v \epsilon V$, then u may be written as $u = cv + w$ where $< v, w >= 0$ by setting $c =< u, v > /||v||^2$. Proof is the same as it was for dot products.

We next present two major inequalities that are repeatedly useful in working with inner product spaces.

- Cauchy-Schwarz Inequality: $| < u, v > | \leq (||u||)(||v||)$
- Triangle Inequality: $||u + v|| \leq ||u|| + ||v||$

Proof of Cauchy-Schwarz: Project u onto the line through v to write $u = \frac{<u,v>}{||v||^2} v + w$ where $< v, w >= 0$. By Pythagorean Theorem: $||u||^2 = || \frac{<uv>}{||v||^2} + ||w||^2$
$= \frac{|<u,v>|^2}{||v||^4} ||v||^2 + ||w||^2$
$= \frac{|<u,v>|^2}{||v||^2} + ||w||^2$
$\geq \frac{|<u,v>|^2}{||v||^2}$

Proof of Triangle: Calculate: $||u + v||^2 =< u, v > + < u + v >$
$=< u, u > + < u, v > + < v, u > + < v, v >$
$= ||u||^2 + < u, v > + < v, u > + ||v||^2$
$\leq ||u||^2 + || < u, v > + < v, u > || + ||v||^2$
$\leq ||u||^2 + | < u, v > | + | < v, u > | + ||v||^2$
$= ||u||^2 + | < u, v > | + |\overline{< u, v >}| + ||v||^2$
$\leq ||u||^2 + 2||u||||v|| + ||v||^2$
$= (||u|| + ||v||)^2$
Therefore, $||u + v||^2 = ||u|| + ||v||$.

2 Today's Notes

We next use orthogonality to create a very nice bases for inner product spaces... Recall the standard basis of F^n is not only an independent list but in fact the vectors are orthogonal and of unit length.
Definition: Let V be an inner product space and $v_1, ..., v_n$ in V. We say $v_1, ..., v_n$ is an <u>orthonormal list</u> if $< v_i, v_j >= 1$ if $i = j$ or $= 0$ if $i \neq j$. In other words, they are pairwise orthogonal and of unit length. $v_1, ..., v_n$ is an orthonormal basis (ONB) if it is an orthonomal list and a basis of V. Note: The standard basis is orthonormal.

Example: The basis $\begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{-2}{\sqrt{6}} \end{bmatrix}$ is an ONB.

Proposition: An orthonormal list is always independent. Thus to be a basis it only needs to have the right length, namely dim V. Note - this is a very strong form of independence.

Proof: Let $v_1, ..., v_n$ be orthonormal. Suppose $a_1v_1 + ... + a_nv_n = 0$, then $||a_1v_1 + ... + a_nv_n||^2 = 0$. The Pythagorean Theorem says: $||a_1v_1||^2 + ... + ||a_nv_n||^2 = 0$
$\rightarrow |a_1|^2 + ... + |a_n|^2 = 0$
$\rightarrow a_1 = a_2 = ... = a_n = 0$. QED.

We now take up the mission of finding an ONB in a given V. Specifically we will start with any basis $v_1, ..., v_n$ and apply projections repeatedly to convert it into an orthonormal basis.

Given $v_1, ..., v_n$ a basis of V. First replace v_1 with $e_1 = \frac{v_1}{||v_1||}$. Next we want to replace v_2 with something orthogonal to e_1: project v_2 onto line through e_1: $v_2 = \frac{<v_2,e_1>}{||e_1||^2}e_1 + w$ where $< e_1, w >= 0$. Set $e_2 = \frac{w}{||w||} = \frac{v_2 - <v_2,e_1>e_1}{||v_2 - <v_2,e_1>e_1||}$.
Third, we replace v_3 with something orthogonal to e_1 AND e_2 by projection: $v_3 = ce_1 + de_2 + w_3$ where $< e_1, w_1 >= 0$ AND $< e_2, w_3 >= 0$. We need to solve for the constants c, d: $< v_3 - ce_1 - de_2, e_1 >= 0$
$\leftrightarrow < v_3, e_1 > -c < e_1, e_1 > -d < e_2, e_1 >= 0$
$\leftrightarrow c =< v_3, e_1 >$
$< v_3 - ce_1 - de_2, e_2 >= 0$
$\leftrightarrow d =< v_3, e_2 >$.
So $v_3 =< v_3, e_1 > e_1 + < v_3, e_2 > e_2 + w_3$. Set $e_3 = \frac{v_3 - <v_3,e_1>e_1 - <v_3,e_2>e_2}{||v_3 - <v_3,e_1>e_1 - <v_3,e_2>e_2||}$.
The rest of the process is the analog of this formula.

Gram-Schmidt Process: Given a basis $v_1, ..., v_n$. We produce an orthonormal basis $e_1, ..., e_n$ using the iterative definition: $e_j = \frac{v_j - <v_j,e_1>e_1 - ... - <v_j,e_{j-1}>e_{j-1}}{||v_j - <v_j,e_1>e_1 - ... - <v_j,e_{j-1}>e_{j-1}||}$.

In the process we also learned how to project a vector v onto a subspace spanned by orthonormal vectors $e_1, ..., e_j$. The projection of v to span$(e_1, ..., e_j)$

is: $p =< v, e_1 > e_1 + ... + < v, e_j > e_j$. If you have an ONB $e_1, ..., e_n$ then $v =< v, e_1 > e_1 + ... + < v, e_n > e_n$.