

11.29 Notes

Math 403/503

November 2022

1 Trace and Determinant

Recall from our discussion of eigenvalues that any operator T in $L(C^n)$ has n -many eigenvalues IF we count multiplicity, that is, how many times each λ appears in the diagonalization (or Jordan form). Officially the multiplicity of an eigenvalue λ of T is equal to $\dim G(\lambda, T)$. From now on, if we say "the eigenvalues of $T \in L(C^n)$ are $\lambda_1, \dots, \lambda_n$ " we mean that each λ appears multiple times in the list according to its multiplicity.

Example: If T has matrix $\begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix}$ we would say the eigenvalues of T are 1, 1, 2.

Definition: If $T \in L(C^n)$ then the trace of T denoted $\text{trace}(T)$ or $\text{tr}(T)$ is equal to $\text{tr}(T) = \lambda_1 + \dots + \lambda_n$. Where the eigenvalues of T are $\lambda_1, \dots, \lambda_n$.

Example: If T has matrix $\begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix}$ then $\text{tr}(T) = 4$ (not 3).

Example: If T has matrix $\begin{bmatrix} 3 & -1 & 2 \\ 3 & 2 & -3 \\ 1 & 2 & 0 \end{bmatrix}$ then the eigenvalues are $\lambda_1 = 1, \lambda_2 = 2 + 3i, \lambda_3 = 2 - 3i$. Thus the $\text{tr}(T) = 1 + 2 + 3i + 2 - 3i = 5$. Observe the imaginary parts cancelled (because the matrix has real entries) Observe if T has a triangular matrix, the eigenvalues appear down the diagonal with their multiplicities so $\text{tr}(T) =$ the sum of the diagonal entries. Observe in our non-triangular example, the trace was ALSO the sum of the diagonal entries...

For now, let's define two separate concepts:

$\text{trace}(T) =$ sum of eigenvalues

$\text{dsum}(A) =$ sum of diagonal entries.

Our goal is to show that the two concepts are the same, that is, if A is the matrix of T in some basis then $\text{trace}(T) = \text{dsum}(A)$.

Lemma: $\text{dsum}(AB) = \text{dsum}(BA)$.

Proof: Write A as an n by n matrix with a_{11} as the top left entry and a_{nn} as the bottom right entry, B would be written similarly. then the jj entry of AB is $\sum_{k=1}^n a_{jk}b_{kj}$. And the kk entry of BA is $\sum_{j=1}^n b_{kj}a_{jk}$

$$\begin{aligned}\text{dsum}(AB) &= \sum_j \text{jj entry of } AB \\ &= \sum_j \sum_k a_{jk}b_{kj} \\ &= \sum_k \sum_j a_{jk}b_{kj} \\ &= \sum_k \sum_j b_{kj}a_{jk} \\ &= \sum_k \text{kk entry of } BA.\end{aligned}$$

Next we explore what happens to $\text{dsum}(A)$ when we change basis: $B^{-1}AB$.

Lemma: $\text{dsum}(B^{-1}AB) = \text{dsum}(A)$.

Proof: $\text{dsum}(B^{-1}AB) = \text{dsum}(BB^{-1}A) = \text{dsum}(A)$.

Theorem: If T has matrix A in some basis, then $\text{trace}(T) = \text{dsum}(A)$.

Proof: By the above, we may assume without loss of generality, that we are working in a basis where A is upper triangular. In this case we already know the diagonal of A must be the eigenvalues $\lambda_1, \dots, \lambda_n$ thus $\text{trace}(T) = \text{dsum}(A)$.

From now on, we say trace for both concepts. $\text{trace}(T) = \text{trace}(A)$ for any matrix A representing T .

2 Properties of the Trace

- $\text{trace}(ST) = \text{trace}(TS)$
- $\text{trace}(\alpha T) = \alpha \text{trace}(T)$
- $\text{trace}(S + T) = \text{trace}(S) + \text{trace}(T)$

The proof of all three is because they are true for the diagonal sum.

One applicatin of the trace is to "commutators". If operators S, T commute, $ST = TS$. If they don't commute, $ST \neq TS$ which means $ST - TS \neq 0$. Thus $ST - TS$ somehow measures how much S and T fail to commute. It gets a notation: the "commutator" of S and T is $[S, T] = ST - TS$.

Theorem: $[S, T]$ is never equal to the identity.

Proof: Compare the trace of both sides.

- $\text{trace}([S, T]) = \text{trace}(ST - TS) = \text{trace}(ST) - \text{trace}(TS) = 0$.

- $\text{trace}(\mathbf{I}) = 1 + \dots + 1 = n$

Thus the two items cannot be equal.

3 Next Time

The determinant of T is $\lambda_1, \dots, \lambda_n$ the product. Like the trace, the determinant can be calculated from matrices. Like the trace, the determinant satisfies nice properties (though different ones). The determinant has several very central applications including in calculus.