09.27 Notes

Math 403/503

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1 Isomorphisms, Automorphisms, Dual Spaces

If $T \epsilon L(V, W)$ is a linear map from V to W, we said T was injective if null T = 0. T is surjective if range T = W.

Definition: A function $f : X \to Y$ is bijective if it is both injective and surjective.

Fact: A function $f : X \to Y$ is bijective if and only if there exists a function $f^{-1} : Y \to X$ such that $f \circ f^{-1} =$ identity function on Y (i.e. $f(f^{-1}(y)) = y, \forall y \epsilon Y$). And $f^{-1} \circ f =$ identity function on X (i.e. $f^{-1}(f(x)) = x, \forall x \epsilon X$).

Returning to linear maps $T \epsilon L(V, W)$ we see that if T is bijective then T has an inverse function T^{-1} which is a function from W to V. In other words, $T \circ T^{-1} = id_W$ and $T^{-1} \circ T = id_V$.

The main additional fact we need is...
Lemma: If $T \epsilon L(V, W)$ is bijective then T^{-1} is a linear map too so we can say $T^{-1} \epsilon L(W, V)$.

Proof: We first check that for $w_1, w_2 \epsilon W$ we have $T^{-1}(w_1 + w_2) = T^{-1}(w_1) + T^{-1}(w_2)$. We proceed as follows, $T(LHS) = T(T^{-1}(w_1 + w_2) = w_1 + w_2$ (because $T(T^{-1})$ cancels out). Similarly, $T(RHS) = T(T^{-1}(w_1) + T^{-1}(w_2)) = w_1 + w_2$ (once again they cancel). So, $T(LHS) = T(RHS)$ and injectivity by the original definition implies that $LHS = RHS$. We next check that for $\alpha \epsilon F$ and $w \epsilon W$ we have $T^{-1}(\alpha w) = \alpha T^{-1}(w)$. Once again, we take T of both sides. $T(LHS) = T(T^{-1}(\alpha w)) = \alpha w$ and $T(RHS) = T(\alpha T^{-1}(w)) = \alpha w$. Thus, $T(LHS) = T(RHS)$. Again, by injectivity we get $LHS = RHS$. QED.

Definition: If $T \epsilon L(V, W)$ is a bijective (invertible) linear mpa then T is called an isomorphism of V and W. We also say V and W are isomorphic, $V \cong W$. We think of T as a relabelling demonstrating that V and W are really the same but with different names.

Prominent examples of isomorphic vector spaces:

- Suppose V is any finite dimensional vector space (dim V = M). Then V has a basis $v_1, ..., v_m$. V is isomorphic to F^m via the representation map that sends only $v \epsilon V$ to the column vector $\begin{bmatrix} a_1 \\ . \\ . \\ . \\ a_m \end{bmatrix}$ where $v = a_1 v_1 + ... + a_m v_m$.
  Thus, $V \cong F^m$.

- Suppose V, W are fininte dimensional vector spaces and dim V =m and dim W = n, and we fix bases $v_1, ..., v_m$ of V and $w_1, ..., w_n$ of W. Then $L(V, W)$ is isomorphic to $F^{n,m}$ (this is the space of n by m matrices over F) via the matrix representation function that takes any $T \epsilon L(V, W)$ and produces a matrix $(a_{i,j} = A$ that represents T in the given bases.

- $F^{n,m}$ is isomorphic to F^{nm}. E.g. the space of matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is isomorphic to the space of vectors $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$

- $F^{n,m}$ is isomorphic to $F^{m,n}$
  $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$ map it to $\begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$
  Corollary: If V and W are vector spaces of the same dimension then $V \cong W$.
  Proof: Let n = dim V = dim W then $V \cong F^n$ and $W \cong F^n$. By symmetry and transitivity, $V \cong W$. QED.

- Let $P_n(F)$ be the space of polynomials over F of degree at most n. Then $P_n(F) \cong F^{n+1}$. E.g. $a_0 + a_1 x + ... + a_n x^n$ maps to $\begin{bmatrix} a_0 \\ . \\ . \\ . \\ a_n \end{bmatrix}$

The case $L(V, V)$:

Terminology:

- Elements $T \epsilon L(V, V)$ are called <u>operators</u>.
- If $T \epsilon L(V, V)$ is bijective (invertible) then T is called an <u>automorphism</u>.

- $GL(V)$ = the "general linear group' of V is the subset of $L(V, V)$ consisting of the bijective (invertible) operators.
  Note: $GL(V)$ is a group with the composition operation! This means several axioms are satisfied: closure, associativity, identity, inverses.

  Lemma: Suppose V is finite dimensional and $T \epsilon L(V, V)$. Then $T \epsilon GL(V)$ if and only if T is injective or T is surjective.
  Proof: Clearly if $T \epsilon GL(V)$ then T is bijective so it is both injective and surjective. For the converse, suppose T is injective or surjective. We will use the FTLM; dim V = dim null T + dim range T

- If T is injective, null T = 0, so dim null T = 0. Thus dim V = dim range T. So V = range T so T is surjective.

- If T is surjective, range T = V, so dim range T = dim V, so dim range T = dim V, so dim null T = 0. So T is injective. QED.