10.04 Notes

Math 403/503

October 2022

1 Invariant spaces and Eigenvectors

I am skipping chapter 4 on polynomials, so please read that on your own!

Chapter 5: Here we let V be a finite dimensional vector vector space. We now know V is isomorphic to F^n. As usual F may be R or C. And we study operators $T \epsilon L(V, V)$.

Definition: Let V and T be as above. A subspace U of V is called invariant for T if for all $u \epsilon U, Tu \epsilon U$. If U is invariant for T then T is restricted to the domain of U. And it's also an operator where T is restricted to the domain $u \epsilon L(V, U)$. These invariant subspaces let us study "pieces" of T separately. For this section we focus on invariant subspaces U which are one dimensional and thus are spanned by a single vector V. If span of V is a one dimensional invariant subspace for T then $Tv = \lambda v$ for some scalar $\lambda \epsilon F$. This is the condition for λ to be an eigenvalue.

Definition: If T is an operator of V $(T \epsilon L(V, V))$ an eigenvalue of T is a $\lambda \epsilon F$ such that there exists a vector $v \epsilon V$ and $Tv = \lambda v$. Such a vector V is called an eigenvector of T corresponding to λ.

Lemma: Let $T \epsilon L(V, V)$ a vector V is an eigenvector corresponding to λ if and only if $v \epsilon$ null$(T - \lambda I)$.
Proof: $Tv = \lambda v \leftrightarrow Tv - \lambda v = 0 \leftrightarrow Tv - \lambda Iv = 0$
Note $Iv = v$ for every v and that $I \epsilon L(V, V)$ is called the identity operator.
$\leftrightarrow (T - \lambda I)(v) = 0 \leftrightarrow v \epsilon$ null$(T - \lambda I)$. QED.

Example of eigenvalues/eigenvectors:

$T \epsilon L(C^2, C^2)$ defined by $T(w, z) = (-z, w)$ then (w, z) is an eigenvector corresponding to λ if:
$T(w, z) = \lambda(w, z)$
$\leftrightarrow (-z, w) = (\lambda w, \lambda z)$
$\leftrightarrow -z = \lambda, w = \lambda z$
Plug $w = \lambda z$ into equation one.

$-z = \lambda(\lambda z)$
$-z = \lambda^2 z$
Without loss of generality, $z \neq 0$
$-1 = \lambda^2$
$+/-i = \lambda$ (note this gives us 2 eigenvalues!)

Let's evaluate at $\lambda = i$:
$-z = iw \rightarrow z = -iw$
$w = iz$
Equation 2 is just -i times equation 1.
Solutions are thus, $(w, z) = (w, -iw) = (1, -i)w$

Let's evaluate at $\lambda = -i$:
$-z = -iw \rightarrow z = iw$
$w = -iz$
Equation 2 is i times equation 1.
Solutions are thus, $(w, z) = (w, iw) = (1, i)w$

Theorem: Let $T \epsilon L(V, V)$. Let $\lambda_1, ..., \lambda_m$ be distinct eigenvalues for T and let $v_1, ..., v_n$ be corresponding eigenvectors. These $v's$ are linearly independent.
Corollary: If n = dim V then T has at most n many eigenvalues.
Proof of Theorem: By induction on m! Assume that $v_1, ..., v_{m-1}$ are independent. Suppose $a_1v_1 + ... + a_{m-1}v_{m-1} + a_mv_m = 0$. Apply T, $a_1\lambda_1v_1 + ... + a_{m-1}\lambda_{m-1}v_{m-1} + a_m\lambda_mv_m = 0$. By inductive hypothesis, $a_1(\lambda_1\lambda_m) = 0 ... a_{m-1}(\lambda_{m-1} - \lambda_m) = 0$. Note that $(\lambda_{m-1} - \lambda_m)$ is nonzero by distinctness! Thus, $a_1 = ... = a_{m-1} = 0$. So, a_m must also be 0. So $v_1, ..., v_m$ are independent.