10.11 Notes

Math 403/503

October 2022

1 Diagonalization

Recall that T has an eigenvalue λ if null$(T-\lambda I)$ is not the zero space. Any nonzero element of null$(T-\lambda I)$ is called an eigenvector corresponding to λ. We proved that over a scalar field C, every operator on a finite dimensional vector space has an eigenvalue (and at most dim V many eigenvalues). We proved a corollary that every operator over C has an upper triangular matrix with respect to some basis. This then begs the question, does every operator over C have a diagonal matrix with respect to some (really special) basis?

First we give the following review fact: Suppose T is an operator on V which has an upper triangular matrix with respect to some basis $v_1, ..., v_n$. Then T is invertible if and only if the diagonal entries of the matrix are not 0.
Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix}$ then $A^{-1} = \begin{bmatrix} 1 & ? & ? \\ 0 & 1/2 & ? \\ 0 & 0 & 1/2 \end{bmatrix}$ Because if we multiply the two together it gives us entries of 1 along the diagonal. But if a matrix A were to have some diagonal entry equal to 0 it would not be invertible because we would have 1/0 in the corresponding entry. This helps us establish the following theorem.

Theorem: Suppose $T \epsilon L(V, V)$ has an upper triangular matrix with respect to some basis. Then the eigenvalues of T are precisely the diagonal entries of the matrix.
Example: $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 5 & 3 \\ 0 & 0 & 8 \end{bmatrix} \rightarrow$ eigenvalues are $2, 5, 8$. The reason is as follows;
$A - \lambda I = \begin{bmatrix} 2-\lambda & 1 & 0 \\ 0 & 5-\lambda & 3 \\ 0 & 0 & 8-\lambda \end{bmatrix}$ This will be noninvertible and therefore have a nontrivial null space whenever $2-\lambda = 0$ or $5-\lambda = 0$ or $8-\lambda = 0$.

Proof: Let $v_1, ..., v_n$ be the basis for which the matrix of T is triangular: Then A is a matrix with $\alpha_1, ..., \alpha_n$ in the diagonal entries and is upper triangular. Then $T-\lambda I$ has matrix $A-\lambda I$ where the diagonal entries are now

$\alpha_1 - \lambda, ..., \alpha_n - \lambda$. Leaving $\lambda = \alpha_1, ..., \alpha_n$ by the fact above. QED.
Notation: Given an operator T and an eigenvalue λ of T, the eigenspace of T corresponding to λ is $E(T, \lambda) = \text{null}(T - \lambda I)$. That is, the subspace of all eigenvectors corresponding to λ, together with the 0 vector. $E(T, \lambda)$ is an example of an invariant subspace for T. In fact, T restricted by $E(T, \lambda)$ is simply the map that multiplies by λ.

Example: Let T have matrix $\begin{bmatrix} 8 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ (in standard basis). We know T has eigenvalues $8, 5$.

$$\begin{bmatrix} 8 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \\ 0 \end{bmatrix}$$

$E(T, 8) =$ the line through $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 8 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 8 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 8 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ y \\ z \end{bmatrix} = 5 \begin{bmatrix} 0 \\ y \\ z \end{bmatrix}$$

$E(T, 5) =$ plane with basis $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Theorem: The sum of the eigenspaces of T makes a direct sum for $\lambda_1, ..., \lambda_m$ distinct values of T.

Proof: We have proved this before: If $v_1, ..., v_m$ are eigenvevtors of $\lambda_1, ..., \lambda_m$ respectively, then $v_1, ..., v_m$ is independent. We need to show that if $U_1 \epsilon (T, \lambda_1), ..., U_n \epsilon E(T, \lambda_m)$ and $U_1 + ... + U_m = 0$ then $U_1 = ... = U_m = 0$. But if $U_i \epsilon E(T, \lambda_i)$ and $U_i \neq 0$ then it is an eigenvector corresponding to λ_i. So by the above fact if $U_1 + ... + U_m = 0$ the only possibility is $U_1 = ... = U_m = 0$. QED.

Preview of next result: If when we take the direct sum of $E(T, \lambda_1) + ... + E(T, \lambda_m)$ we get the whole space V, then T is diagonal in some basis (consists of eigenvectors). And conversely.