09.06 Notes

Math 403/503

September 2022

Recall: We introduced independence and "spans V" and defined a basis to be any list with both of these properties simultaneously.

Question: How many vectors in a basis of V?

Recall: I defined "dimension" of a vector space as the least size of any spanning list of V.

Theorem: The size of any basis of V is equal to the dimension of V (in particular, all bases have the same size).

Proof: Suppose $v_1, ..., v_m$ is a basis of V. Suppose towards a contradiction that $m < dim V$. Then $v_1, ..., v_m$ is a spanning list which is less than the least size of a spanning list! Contradiction. Suppose towards a contradiction that $m > dim V$. Then $v_1, ..., v_m$ is an independent list that is greater in size than some spanning list. This contradicts 2.2.3 previously proved. QED.

Corollary: Suppose V is a vector space and dim $V = m$. Then...

- If $v_1, ..., v_m$ is an independent list, then it is a basis.
- If $v_1, ..., v_m$ spans V then it is a basis of V.

Proof of...

- First bullet point: We showed we can extend any independent list to a basis by applying to $v_1, ..., v_m$ that extension must be trivial. Thus it is already a basis.
- Second bullet point: We showed we can whittle any spanning list to a basis.. applying this to $v_1, ..., v_m$ whittling must be trivial. Thus it is already a basis.QED.

Example: The list (5,7), (4,3) in F^2 is clearly independent because neither is a scalar multiple of the other. Furthermore, it is a list of length 2, and 2 is the dimension of F^2. We know this because (1,0), (0,1) is a basis of length 2. Thus, (5,7), (4,3) is a basis. No need to check spanning.

Similarly in F^{11}, we know the dimension is 11, so any list of 11 independent vectors must be spanning too, so must be a basis.

Theorem: Suppose you are summing spaces $U_1 + U_2$. The following formula relates the dimensions of U_1, U_2 and $U_1 + U_2$: $dim(U_1 + U_2) = dim(U_1) + dim(U_2) - dim(U_1 \bigcap U_2)$.

Proof sketch: Let $u_1, ..., u_m$ be a basis of $U_1 \bigcap U_2$. Extend $u_1, ..., u_m$ to be a basis of U_1: $u_1, ..., u_m, v_1, ..., v_k$. Also extend $u_1, ..., u_m$ to a basis of U_2: $u_1, ..., u_m, w_1, ..., w_l$. We claim the list of $u_1, ..., u_m, v_1, ..., v_k, w_1, ..., w_l$ is a basis of $U_1 + U_2$. Done in book: they show the list is independent. Admitting the claim, we have:
$dim(U_1 + U_2) = m + k + l$
$= (m + k) + (m + l) - m.$
$= dim(U_1) + dim(U_2) - dim(U_1 \bigcap U_2)$
Using the main theorem of this chapter 4 times. QED.