10.25 Notes

Math 403/503

October 2022

1 Review for Quiz

Quiz will be released sometime Wednesday, we will get an email announcement when it is.

2 Outline of Topics

The material this quiz will cover goes all the way back to section 3D in the textbook.

- Injectivity - null T = 0
- Surjectivity - range T = W
- Bijectivity - Both! And this implies that the inverse of T exists.
- Operators - If V is finite dimensional and $T \epsilon L(V, V)$ then T is injective if and only if T is surjective if and only if T is bijective.
- Dual Spaces - $V' = L(V, F)$, a dual basis means if $v_1, ..., v_n$ is a basis of V then we let $\phi_1, ..., \phi_n$ be a basis of V' where $\phi_i(v_j) = 1, i = j$ or $= 0, i \neq j$
- Dual Operators - if $T \epsilon L(V, W)$ then $T' \epsilon L(V', W')$ is defined by $T'(\phi) = \phi \circ T$, the matrix of T' is the transpose of the matrix of T.
- Dual Dimensions - dim null T' = dim W - dim range T, dim range T' = dim range T.
  Corollary: Thus the rank of A is equal to the rank of A^T
- The eigenvalue/eigenvector definition - $Tv = \lambda v. v \neq 0$
  The **eigenspace** of λ with respect to T: $E(\lambda, T) = \text{null}(T - \lambda I)$ (this is all solutions to $Tv = \lambda v$. Eigenspaces are T-invariant and form a direct sum (independent).
- Diagonalization - If V is the direct sum of the eigenspaces, then there exists a basis of V consisting of eigenvectors and T is diagonal in this basis.

- Over C, eigenvalues always exist - Over C, a basis always exists in which T is upper triangular. With an upper triangular matrix, the eigenvalues appear on the diagonal.

- Generalized eigenvectors satisfy $(T\lambda I)^j v = 0, j =$ any power, larger finds more general eigenvectors, j may be as large as needed up to n,

- Generalized eigenspace - $G(\lambda, T) =$ null $(T - \lambda I)^n$, these form a direct sum (independent)

- Jordan canonical form - Over C, $V =$ the direct sum of the generalized eigenspacees. Thus, there is a basis of V consisting of generalized eigenvectors. In fact, we can find such a basis where the generalized eigenvectors come in <u>chains</u> ending with the eigenvectors. Working in such a basis gives a matrix for T consisting of <u>Jordan blocks</u>. Note: diagonal is the special case where every Jordan block is one by one, so there is no room for 1's in the superdiagonal.