Fractals and dimension

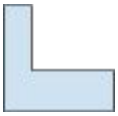
Boise Math Circle

1. Try drawing a small rectangle. Then scale it (in all directions) by some whole number to get a larger rectangle. How many copies of the original rectangle does the larger rectangle contain?
2. Try the same thing for some three-dimensional boxes. This is harder so either use **separate paper** or just try to visualize it.
3. Try the same thing for a line segment. (This is a little silly.)

Figure	Picture	Scaling factor	New Picture	Number of copies
Rectangle		2		
Different rectangle				
Box				
Different box				
Segment				
Different segment				

4. Can you write down an equation relating the scaling factor, the number of copies, and the dimension of the figure?

5. Does your formula work for other shapes? **Use separate paper** any time you wish.

Figure	Picture	Scaling factor	New Picture	Number of copies
Triangle		2		
Triangle				
Weird shape				
Circle				
My shape				
My different shape				

6. What went right with your experiments? What went wrong?

The Sierpinski triangle

To construct the Sierpinski triangle:

- Begin with any solid triangle (easy if it's equilateral)
- Remove the middle 1/4 triangle, leaving three remaining (the middle 1/4 triangle is actually upside-down from the original triangle)
- Remove the middle 1/4 triangle of each of the three
- Continue in this way forever!

7. Draw the first few iterations of the Sierpinski triangle. How does it compare with a solid triangle? Is the Sierpinski triangle self-similar?

8. Scale the Sierpinksi triangle by a factor of 2. Scale it by a factor of 3. How many copies of the original figure do you see in the scaled figures?

9. Use your formula from before to calculate the *dimension* of the Sierpinski triangle. You can use a calculator if you wish!

The Koch snowflake

To construct the Koch snowflake:

- Begin with a triangular border
- For each of the three edges of the triangle, replace the middle 1/3 with two edges of equal length (forming two sides of an equilateral triangle)
- For each of the twelve edges of the new figure, replace the middle third with two edges of equal length
- Continue in this way forever!

10. Draw the first few iterations of the Koch snowflake.

11. Scale your Koch snowflake by factor of 2. Is the Koch snowflake self-similar? How many copies of the original figure can you find in the scaled figure?

12. Use your formula to find the dimension of the Koch snowflake. How does it compare with previous results?

Crumpled paper ball

Although it isn't exact, the crumpled paper ball has many properties of a self-similar figure.

13. Take a sheet of paper and crumple it into a ball. Crumple it as **firmly** as you can, and make it as **round** as you can!
14. What is the diameter of the ball? Work with your group-mates to try to get a consistent measurement.
15. Now place two sheets of paper side-by-side. and crumple them together. What is the diameter of this largere ball?

16. Think about your formula relating the number of copies, the scaling factor, and the dimension.
    a. What is the number of **copies** in this experiment?

    b. What is the **scaling factor** in this experiment?

17. Based on your answers and measurements, what is the dimension of crumpled paper?

Further questions and experiments

Other 3d figures

The Menger sponge is constructed as follows.

- Begin with a solid cube
- Remove the middle cube with dimensions 1/3 on all sides
- From each of the (how many?) cubes remaining, remove the middle portion again
- Continue in like fashion

Question: What is the dimension of the Menger sponge?
Question: You can you make a 3d version of the Sierpinksi triangle? Such a thing would be called the Sierpinksi tetrahedron. What is its dimension?

Cantor set

It is possible to make a figure of any dimension. For example, to make a figure with dimension 0<d<1, we can use a *Cantor set* construction:

- Let α be any numebr < 1
- Begin with an interval
- Remove the middle α, leaving 2 new segments
- Remove the middle α of each segment, leaving 4 new segments
- Continue in this way forever

Question: what do you have to scale the α-Cantor set by in order to get two clean copies of the original figure? Integers will typically not work here.
Question: If alpha is really small, what will the dimension be like?
Question: If alpha is really big, what will the dimension be like?
Question: Explain why the dimension will always be between 0 and 1, and that any number between 0 and 1 can be achieved using some α.

Bonus questions

Question: What is the total area of the Sierpinski triangle?
Question: What is the total length of the Koch snowflake?
Question: What is the total length of the α-Cantor set?