Operations on points in the plane

We all know that we can add and multiply ordinary numbers. The operations have lots of important properties (like commutativity, associativity, distributivity and so on). But what operations can we perform on points in the plane?

Adding points in the plane

If (x,y) and (z,w) are points in the plane, we can add them “component-wise”, that is:

$$(x,y) + (z,w) = (x+z,y+w).$$

Exercise. Get a bunch of “coordinate axis” sheets. For each of the following pairs of points, plot both points and the sum in the plane.

- (1,1) + (2,2) = (3,3)
- (1,0) + (0,1) =
- (-1,1) + (1,1) =
- (-1,2) + (2,0) =

What is going on geometrically? What do you observe about the “shape of addition”?

Multiplying points in the plane, first attempt

If (x,y) and (q,w) are points in the plane, we can multiply them “component-wise” too:

$$(x,y) * (z,w) = (xz,yw)$$

Actually mathematicians don’t like this way of multiplying points in the plane. Here’s why.

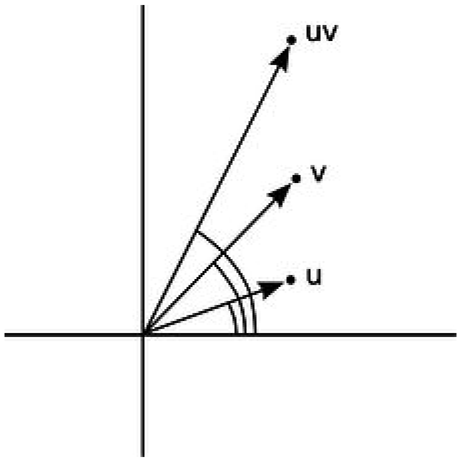
Exersise. Try to find two points in the plane, both not (0,0), such that their component-wise product is (0,0). (This is something that doesn’t happen for ordinary numbers.)

Exersise. Show that there exist three points in the plane such that $(x,y)*(a,b) = (x,y)*(c,d)$ and $(a,b) \neq (c,d)$. (This means we cannot “cancel” the (x,y) from both sides.)

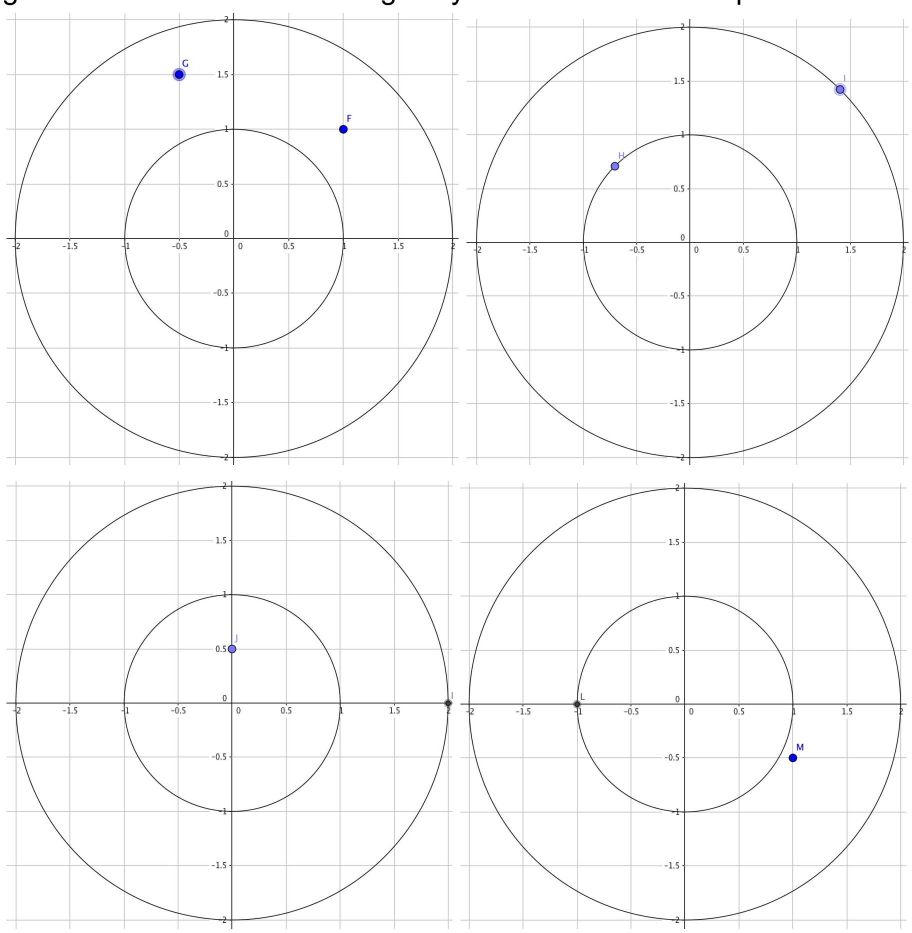
Multiplying points in the plane, second attempt

Here is a way to multiply points in the plane. First notice that every point in the plane has a **radius**, that is a distance from the origin, and an **angle**, that is, the number of degrees from the x-axis.

Rule. To multiply two points in the plane, we add their angles and multiply their radiuses.



Exercise. In each picture, multiply the two points using this rule. Use a ruler and protractor to guestimate the radius and angle if you need to. Use a separate sheet for more space.



Distributive law

Mathematicians like this way of multiplying points in the plane becaues it doesn’t have the problems identified before. But one disadvantage is that it isn’t as easy to see that the distributivity law is true.

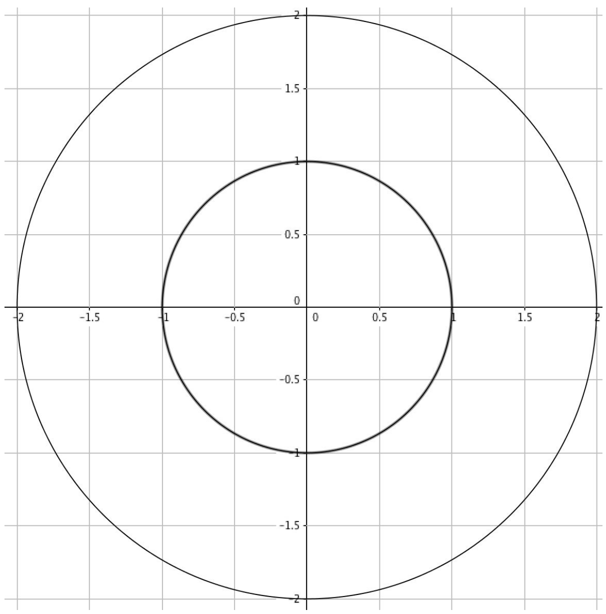
Exercise. Try it for yourself. On the one hand, draw a picture of

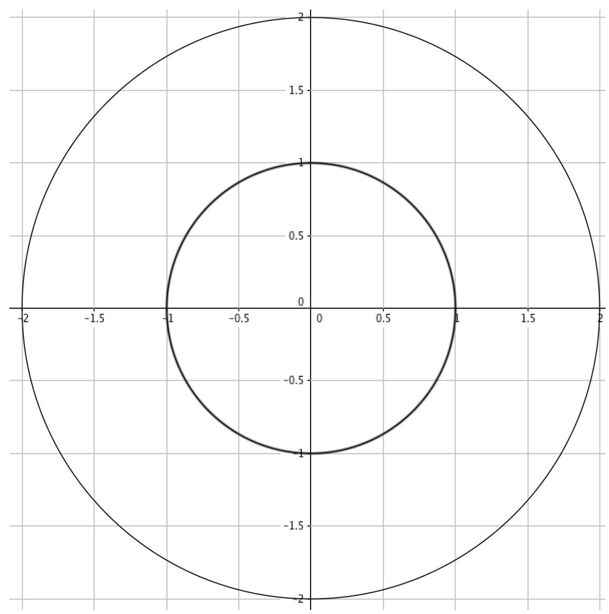
$$(-1,1) \bullet [\ (.5,1) + (.5,.5)\ ]$$

And on the other hand, draw a picture of

$$(-1,1) \bullet (.5,1) + (-1,1) \bullet (.5,.5)$$

Do you get the same point?

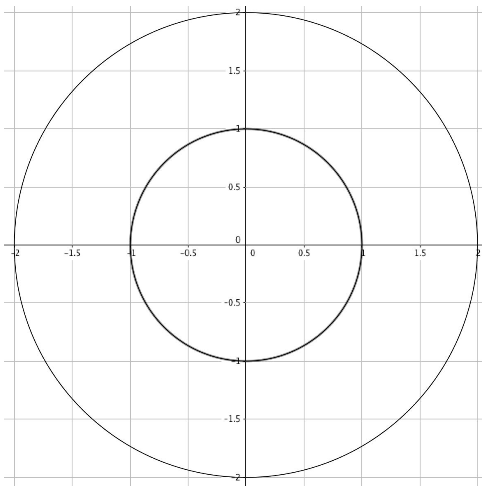




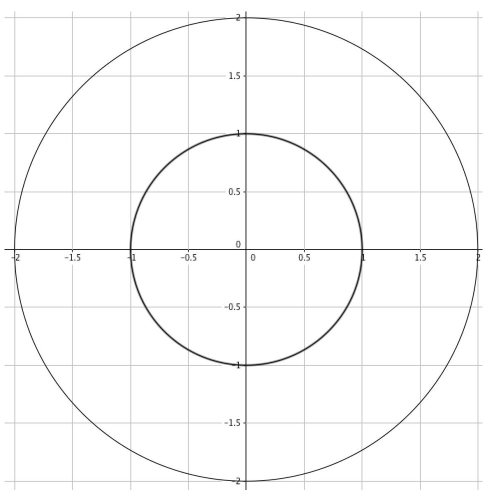
General formula

What is the general formula for the product of two points (a,b) • (c,d) ?

Exercise. What is (a,b) • (1,0) ?



Exercise. What is (a,b) • (0,1) ?



Exercise. Complete the following.

(a,b) • (c,d) = (a,b) • [(c,0) + (0,d)]

= (a,b) • [c•(1,0) + d•(0,1)]

= ________________________ (use the distributive law)

= ________________________ (use the previous two exercises)

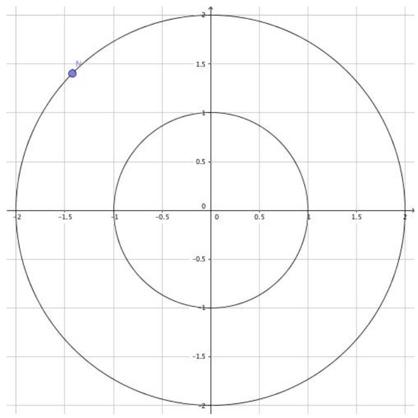
= ________________________ (simplify and combine)

Exercise. Discuss your final formula together as a class. Looking back at page 2, redo the problems using you formula. Do you get the same answers??

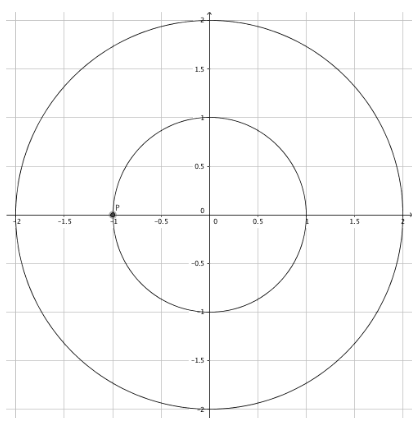
Crazy math

Now we can add and multiply points in the plane. And there are a bunch of things we can do that might seem surprising.

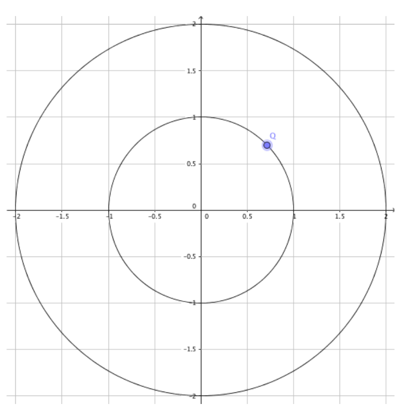
Exercise. Find the “square root” of the point.



Exercise. You might have noticed that (-1,0) acts like “-1” because it negates both coordinates of any point. In the real numbers, -1 doesn’t have any square root. In the plane, (-1,0) does have a square root (actually two). Find it!



Exercise. Let Q be the point below. Find Q^8.



Conclusion. Do more problems together as a class!