Counting Simplified Fractions

In this discussion we'll be thinking about fractions. We'll say a fraction is **simplified** if the numerator and denominator don't have any common factors greater than 1.

Warm-up Task

In the list below, circle the **simplified fractions**. For each fraction that is not simplified, reduce it to simplified form:

$\frac{3}{7}$ $\frac{2}{20}$ $\frac{2}{13}$ $\frac{113}{11}$ $\frac{8}{4}$ $\frac{15}{20}$ $\frac{5}{24}$ $\frac{10}{21}$ 12

What's the easiest way to decide whether a fraction is a simplified fraction?

Task 1: counting simplified fractions

- How many simplified fractions have a denominator of 10?
- How many simplified fractions have a denominator of 11?
- How many simplified fractions have a denominator of 18?
- How many simplified fractions have a denominator of 23?
- What do you notice from doing the above tasks?

Task 2: Prime numbers

Mathematicians use the greek letter ϕ ("phi") to count these numbers. That is, ϕ_n (pronounced "fee-sub-n") means the number of simplified fractions with denominator equal to n. To repeat your work in Task 1, we can say:

- $\phi_{10} =$
- $\phi_{11} =$
- $\phi_{18} =$
- $\phi_{23} =$

Now, compute values of ϕ_n for **prime numbers** n.

- $\phi_5 =$
- $\phi_{11} =$
- $\phi_{17} =$
- $\phi_{29} =$
- $\phi_{101} =$

What do you notice from doing these tasks?

Task 3: Products of prime numbers

What if the number n is not prime? Let's continue to experiment by calculating values of ϕ_n for numbers n which are a product of two different prime numbers.

- $\phi_6 =$
- $\phi_{15} =$
- $\phi_{21} =$
- $\phi_{35} =$
- $\phi_{143} =$

What do you notice from doing these tasks?

Task 4: Powers of prime numbers

Another way to continue experimenting is by calculating values of ϕ_n for numbers n which are a power of a single prime number.

- $\phi_8 =$
- $\phi_{16} =$
- $\phi_{25} =$
- $\phi_{27} =$
- $\phi_{125} =$

What do you notice from doing these tasks?

Task 5: Putting things together: factoring

Every number can be written as a product of prime powers. Write each of the following numbers as a product of prime powers.

- 18
- 30
- 36
- 100
- 360
- 3600

Now try to calculate the ϕ-value of each of these numbers. How can you put together your knowledge about prime numbers and prime powers to get these answers?

Task 6: Further questions

- Consider the number 10. The list of all factors of 10 is 1, 2, 5, and 10. What is the sum of the values $\phi_1 + \phi_2 + \phi_5 + \phi_{10}$? Try it for other numbers!

- What is the relationship between ϕ_n and ϕ_{2n}?

- When is ϕ_n a multiple of 2? When is it a multiple of 4? When is it a multiple of 8?

- If n is a factor of m, what is the relationship between ϕ_n and ϕ_m?

- What other patterns do you notice? Why do you think they are true?

More about ϕ

The function $\phi(n) = \phi_n$ is called the Euler **totient** function. It dates back to 1763 but its values play a role in modern cryptography theory. Check it out on wikipedia! Oh and one more thing. Look at the graph of the values of ϕ below. What do you think those heavy lines represent?