# PHYS3080 Cosmology project

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**Abstract**

In the final abstract you'll also include the results of the cosmology fitting.

In this study we plot expanding and contracting universes that respond as a perfect fluid, using the Friedman equations as derived from Einsteins general field equations using the FLRW metric. We explore how different cosmological parameters effect distance and time measurements. Throughout we take the Hubble constant, $H_0$ to be 70 km/s/Mpc

This detail can stay in the main text

For a full understanding of our working, the code and manipulated data-sets can be found at https://github.com/scotdevlin/Cosmo_tall/,

great to include your code on github!

## 1 PART I: Expansion history

Give a brief intro into what this paper is about, and the motivation (explain to the reader why you are presenting the theory that follows).

### 1.1 Scale-factor as a function of time

From the definition of the rate of change of the normalised scale factor, $\dot{a} = da/dt$ an equation relating look-back time and scale-factor is derived:

$$\int_0^t dt = \int_1^a \frac{da}{\dot{a}} \tag{1}$$

To calculate equation 1, one needs to find $\dot{a}$. Starting with the normalised Friedman equation:

$$H(a)^2 \equiv \frac{\dot{a}^2}{a^2} = H_0^2 \sum_i \Omega_i a^{-3(1+w_i)} \tag{2}$$

where $\Omega_i$ is the energy density parameter and $w$ is the equation of state of the particular energy density.

For simplicity, ignoring the small contribution the radiation energy density makes to the overall mass energy density, we obtain an expression for $\dot{a}$ by plugging in the other energy densities and their equations of state:

$$\dot{a} = aH_0 \left[\Omega_M a^{-3} + \Omega_K a^{-2} + \Omega_\Lambda\right]^{1/2} \tag{3}$$

Where $\Omega_{\mathrm{m}}$ is the density parameter for matter (both visible and dark), and $\Omega_\Lambda$ is the density parameter for dark energy (the cosmological constant) and $\Omega_{\mathrm{k}}$ is the normalized curvature energy density.

Plugging equation 3 into equation 2, we can solve for t, the age of the universe, at a specified a. This integral has a very complicated analytic solution, so we numerically solved this equation using the python integrate sub-package from the scipy library. Using a for loop in python we calculated a look back time for an array of scale factor's, from 0 to 2.

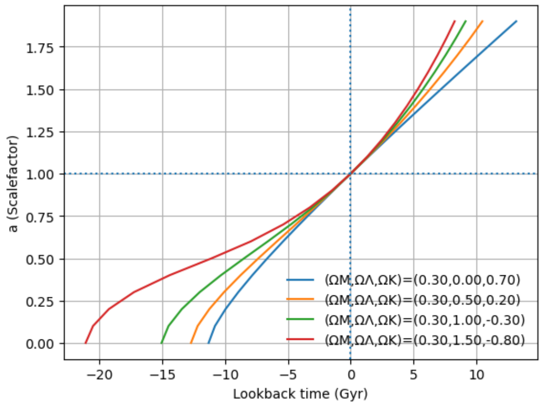
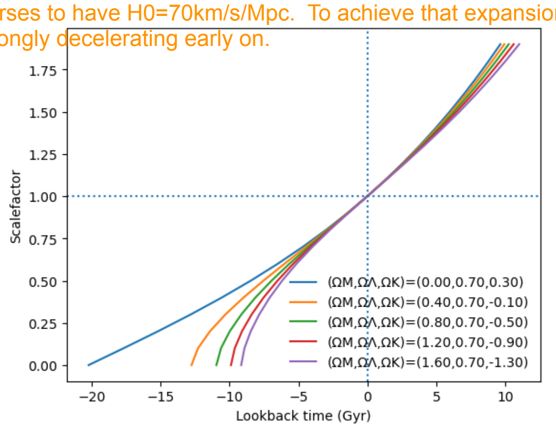



Figure 1: Scalefactor vs look-back time for varying $\Omega_\Lambda$ where $\Omega_\mathrm{m}$=0.3

In figure 1 The blue line represents a universe that has a 0 cosmological constant, and therefore this universe only decelerates with increased scalefactor. Universes with larger cosmological constants have their scale factor decelerate more in the past and accelerate more in the future. It seems odd that the repulsive force of the dark energy would cause a deceleration in the past, until one considers that the energy density of a universe with dark energy is greater than one without dark energy.

It's more because we've required all the Universes to have H0=70km/s/Mpc. To achieve that expansion rate now, having accelerated for the last howev
many billion years, they have to have been strongly decelerating early on.




0.7?

Figure 2: Scalefactor vs lookback time for varying $\Omega_\mathrm{m}$ where $\Omega_\Lambda = 0.3$

In figure 2 The blue line represents a universe that has a a normalised mass density of 0. And therefore the only energy density acting on the universe is dark energy, in the form of a cosmological constant. Therefore this universe only acccelerates with increased scalefactor. Universes with larger mass densities have their scale factor decelerate more in the past and accelerate less in the future.

#### 1.1.1 Re-collapsing universe

By setting the cosmological constant to 0 or less, it can be seen that any universe with a matter density above 1 will cause the universe to re-collapse at sometime in the future.

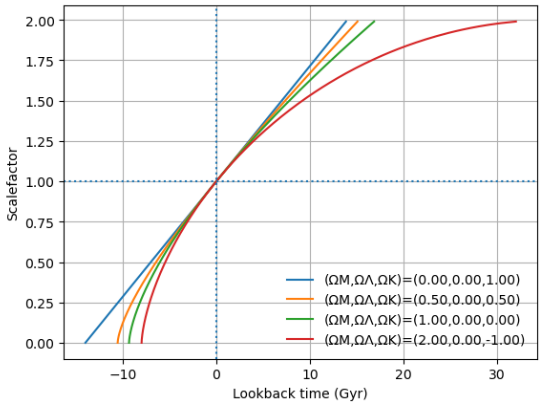



Figure 3: Scalefactor vs lookback time for varying $\Omega_{\mathrm{m}}$ where $\Omega_\Lambda = 0$

In figure 3 the two universes with $\Omega_{\mathrm{m}} = 0$ and $\Omega_{\mathrm{m}} = 0.50$ will not recollapse but go on decelerating forever. Whereas the universe with $\Omega_{\mathrm{m}} = 1.00$ will re-collapse at an infinite scale-factor. And the universe with $\Omega_{\mathrm{m}} = 2.00$ recollapses with its turnover point at $a = 2$. This turnover point is calculated by setting $\Omega_\Lambda = 0$ in equation 3 and calculating where $\dot{a}$, the differential of a, equals 0. noting that:

Ideally you'd punctuate equations as though they are part of a sentence.

$$\Omega_k = 1 - \Omega_{\mathrm{m}} - \Omega_\Lambda \tag{4}$$

### 1.1.2 Re-collapsing universe part II

It is possible to have a re-collapsing universe in the case where the cosmological constant is above zero. This happens in the special case where the universe collapses before dark energy has the chance to be the dominate energy density. In every other case where dark energy takes on the form of a cosmological constant (constant over time), it will always become the dominate energy density (as it doesn't scale inversely with scalefactor) and therefore the universe will eventually expand forever with the scale factor continuing to accelerate.

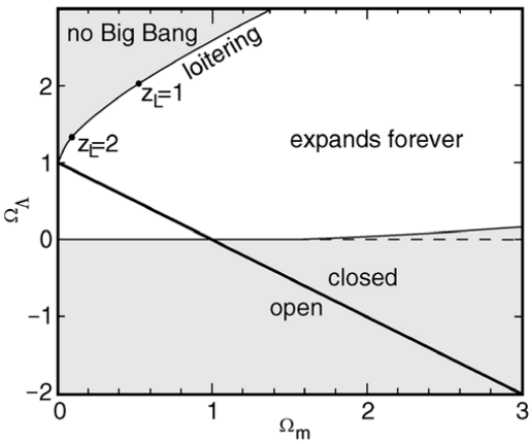



Figure 4: Classification of cosmological models, Fig 4.7 from Schneider

The special case is represented by the positive deviation from $\Omega_{\mathrm{m}} = 0$ of the horizontal line in figure 4. Whilst its not apparent from the scale of the graph below, this deviation starts from $\Omega_{\mathrm{m}} = 1$ The equation of this line is:

$$\Omega_\Lambda = 4\Omega_M \left\{ \cos\left[\frac{1}{3}\cos^{-1}\left(\frac{1-\Omega_M}{\Omega_M}\right) + \frac{4\pi}{3}\right]\right\}^3 \tag{5}$$

We were able to reproduce re-collapsing universes with $\Omega_\Lambda > 0$, calculating the maximum $\Omega_\Lambda$ for a given $\Omega_M$ from equation 5. And maybe more interestingly, when we added a small amount to the $\Omega_\Lambda$ found by equation 5 a 'loitering universe,' is created, where the expansion slows to an almost stop (on the brink of re-collapse) and then starts to expand again, see figure 5.

If you're using latex, use ` not ' for opening quotes. (` is under ~)

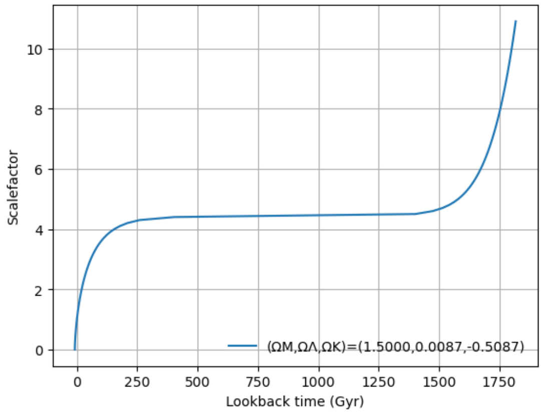



Figure 5: A loitering universe where $\Omega_M = 1.5$ and $\Omega_\Lambda = 4\Omega_M \left\{ \cos\left[\frac{1}{3}\cos^{-1}\left(\frac{1-\Omega_M}{\Omega_M}\right) + \frac{4\pi}{3}\right]\right\}^3 + 10^{-10}$

### 1.1.3 Bouncing Universe

A universe that bounces is one that doesn't converge in the past on $a = 0$, but instead has a turning point at a positive $a$ and is therefore a universe with no big bang, such universes have previously collapsed from an infinite size to a finite radius and then re-expanded again.

The minimum $\Omega_\Lambda$ to create a bounced universe is found by setting $\Omega_M$ to 0 (any matter density would drive the turning point towards 0).

Using $\Omega_M = 0$ and the equation for $\Omega_K$ ((4))in the equation for $\dot{a}$ (3), it can be shown that the turning point, $\dot{a} = 0$ for this bouncing universe occurs at:

$$a = \sqrt{1 - \frac{1}{\Omega_\Lambda}} \tag{6}$$

Equation 6 has a negative integrand for any $\Omega_\Lambda < 1$ indicating a big bang (non-bounced) universe. Therefore the minimum $\Omega_\Lambda$ to create a *bouncing* universe is 1.

The condition for a bouncing universe is further demarcated by the curved line in the top left hand side of figure 4 which is defined by equation (7):

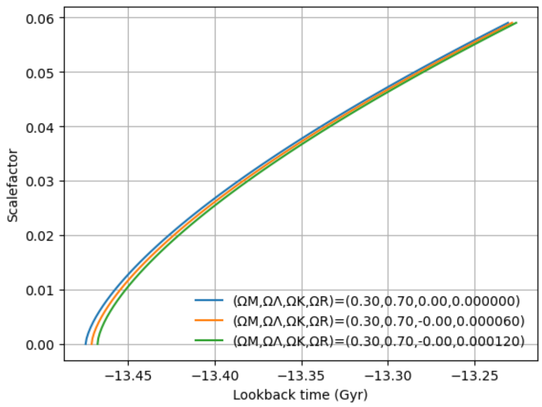
$$\Omega_\Lambda \geq 4\Omega_M \left\{ \cos s\left[\frac{1}{3}\cos^{-1}\left(\frac{1-\Omega_M}{\Omega_M}\right)\right]\right\}^3 \tag{7}$$

### 1.1.4 A universe with radiation

As previously mentioned the radiation energy density is small ($\Omega_R \approx 6 \times 10^{-5}$) and therefore does not significantly contribute to the expansion of the universe. Adding radiation to equation 3 gives:

$$\dot{a} = aH_0 \left[\Omega_M a^{-3} + \Omega_K a^{-2} + \Omega_\Lambda + \Omega_R a^{-4}\right]^{1/2} \tag{8}$$

Because radiation scales with $a^{-4}$ it has a more significant role at low $a$, at the beginning of the universe. Therefore we plotted equation 8 for low a values, between 0 and 0.06: see figure 6




You have many of these plots. So perhaps you could skip this one and move directly to Fig 7 that shows more dramatic differences.

To conserve space in your paper you could put all these figures into one figure with multiple panels. Or maybe use a two-column format in latex.

Figure 6: Universe with $(\Omega_M, \Omega_\Lambda) = (0.3, 0.7)$ and a varying $\Omega_R$. The green line having the current radiation energy density estimate for our universe.

Each universe in figure 6 varies in age by only 0.003Gyr from the next.

If the normalised radiation energy density is allowed to vary more considerably than the current estimates, it's effects can be more clearly seen as demonstrated by figure 7:

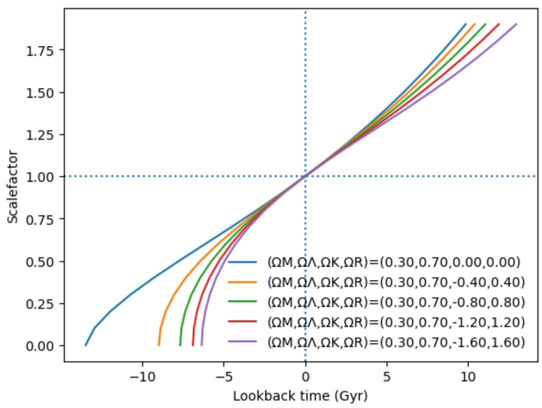



Figure 7: Universe with $(\Omega_M, \Omega_\Lambda) = (0.3, 0.7)$ and a varying $\Omega_R$.

### 1.1.5 Dark Energy with various equations of state

So far we have assumed that Dark Energy takes on the form of a cosmological constant, and does not change through time, which leads to the remarkable conclusion that Dark Energy is not diluted with universe expansion.

Restricting ourselves to a flat universe with no radiation, but asking the question what would happen if Dark Energy had a different equation of state, our Friedman equation, rearranged for $\dot{a}$ becomes:

$$\dot{a} = aH_0 \left[\Omega_M a^{-3} + (1 - \Omega_M)\, a^{-3(1+w)}\right]^{1/2} \tag{9}$$

plotting for a range of equation's of state, w:

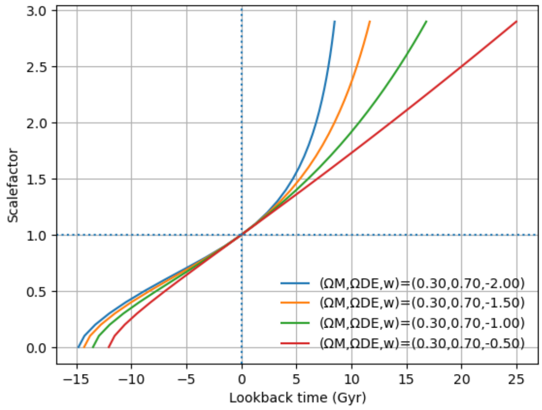



Figure 8: A flat wCDM model universe. With $(\Omega_M, \Omega_\Lambda) = (0.3, 0.7)$ and a varying w

The green model is the universe we have already plotted, where w=-1, and dark energy takes on the form of a cosmological constant. A more negative equation of state, results in a universe that has dark energy more aggressively dominate in the future, because it scales with a, as apposed to inversely with a (a more positive w).

### 1.2 Age of the universe vs normalised matter density

Using equation (1) but changing its limits to 0 to $t_0$ on the left hand side and 0 to 1 on the right hand side, the current age of the universe can be calculated. We plotted the age of the universe vs an array of $\Omega_M$ values.

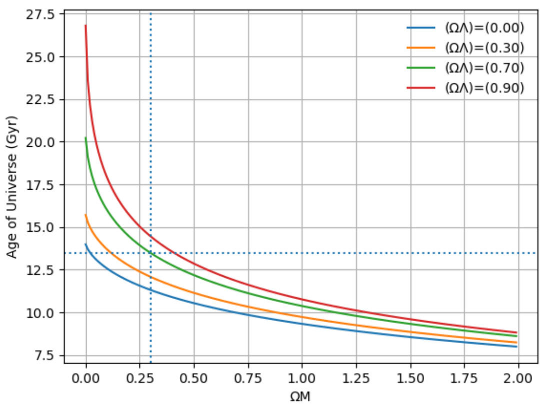



Figure 9: Age of the universe vs $\Omega_M$ for 3 different $\Omega_\Lambda$ values.

This doesn't necessarily make sense, because a "universe" has a single Omega_M. So you'd have to say that universes with lower Omega_M are older than those with higher Omega_M, according to this graph.

Figure 9 shows all universes tend towards an old age as $\Omega_M$ tends towards 0. This is because the

I'm not sure whether the curvature should be associated with less attraction. But rather cosmological constant should be associated with repulsion.

cosmological constant is the dominate energy density, only contending with curvature. A larger cosmological constant gives a lower curvature (eqaution 4) and therefore a lesser attractive force and a greater expansion force, allowing scale-factor deceleration to have been more severe in the past, allowing an older universe (see firgure 3).

To chekc the plot in figure9 we plotted the concordance model, rounded to $\Omega_M = 0.3$ and $\Omega_\Lambda = 0.7$, with a blue dotted cross-hairs at 13.47 Billion years. See that the green line in figure 9 crosses through this point.

### 1.3 Co-moving distance and Proper distance vs Redshift

Because photons travel along null geodesics, the distance a photon travels in a universe that obeys the FLRW metric can be found by setting $ds = 0$ in the FLRW metric, leaving: $cdt = Rd\chi$. Rearranging and integrating this equation gives the co-moving distance (distance to the emitter of the photon):

$$\int_{t_e}^{t_0} d\chi = c\int_{t_e}^{t_0} \frac{dt}{R(t)} \tag{10}$$

Where $t_e$ is time the photon was emitted, and $t_0$ the time the photon was observed. Plugging $R = R_0 a$ into equation 10 and multiplying by $da/da$, an expression for co-moving distance in terms of scale factor is obtained:

$$R_0\Delta\chi = c\int_{a_e}^{a_0} \frac{da}{a\dot{a}} \tag{11}$$

Using $a = \frac{1}{1+z}$ an expression for the co-moving distance of a galaxy (or other photon emitter) in terms of redshift and the Hubble parameter is obtained:

$$R_0\chi = c\int_0^z \frac{dz}{H(z)} \tag{12}$$

Using the Friedman equation (2) and $a = \frac{1}{1+z}$ to solve $H(z)$ in equation 12, co-moving distances are calculated for galaxies currently viewed with a redshift of z. This co-moving distance is the distance the galaxy is at today, where normalised scale-factor $a = 1$. The distance of an emitting galaxy a certain time in the past or in the future is obtained by multiplying equation 12 by the normalised scalefactor a:

$$D(t(z), \chi(z)) = a(t)R_0\chi(z) \tag{13}$$

This is the proper distance. Figure 10 plots the co-moving distance and the proper distance for galaxies at different redshift's. It can be seen that co-moving distance, the distance the galaxy is from us today, increases with redshift as per the Hubble Redshift distance relation. Whereas the proper distance turns over at about $z = 1.5$. This is because galaxies with redshifts over $z = 1.5$ emitted their light when the universe was much smaller. The high initial expansion rate (see figure 1) drove the light path away from us in space time (ie, the expansion of space outpaced the light traveling towards us) resulting in a net movement away from us. The large initial deceleration caused the expansion to slow to a point where the emitted photons could start traveling towards us in space time, this is the turnover point, and can be seen by the turnover point of the light cone in figure 11.

also relevant for angular diameter distance.

One subtlety about the proper distance…

The proper distance at the present day is
D(t0,X) = a0 R0 X = R0 X
which is the same as the comoving distance.

The proper distance at the time of emission is
D(te,X) = a(te) R0 X = 1/(1+z) * R0 X
which is the same as the angular diameter distance (in flat space).

What you've plotted as the red line is the proper distance at the time of emission, which is fine,
I'm just noting that the proper distance changes with time, so you should specify which time you're plotting it for.

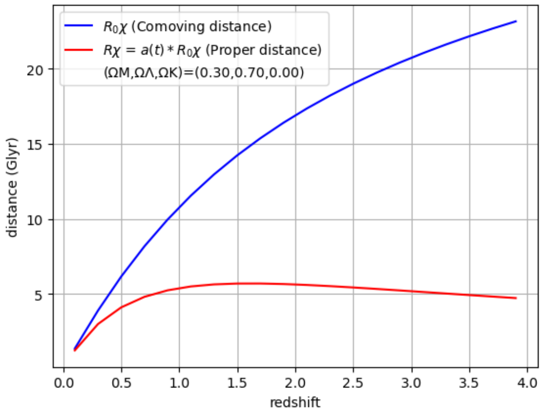



Figure 10: Proper distance (the distance the emitting galaxy is at now) vs redshift in red and comoving distance (the distance the emitting galaxy is at now) vs redshift in blue, in a universe where $\Omega_M = 0.3$ and $\Omega_\Lambda = 0.7$

The intersection of the galaxy world-lines (dotted lines) and the light cone in figure 11 gives the distance to the galaxy at time of emission (a proper distance), corresponding to the red line in figure 10. And the intersection of the galaxy world-lines with the $a = 0$, now line gives the distance to the galaxy now (a co-moving distance), corresponding to the blue line in figure 10.

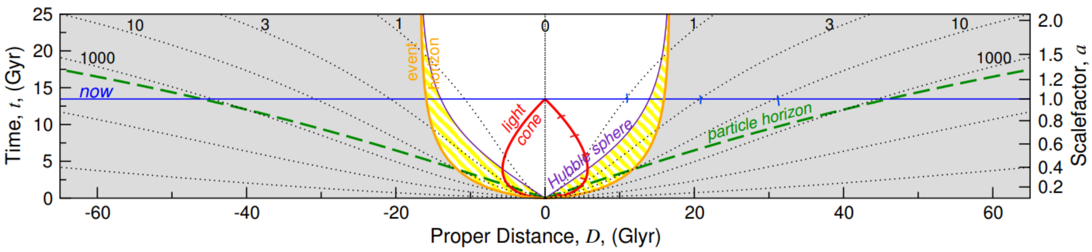



Figure 11: Spacetime diagrams for the $(\Omega_M, \Omega_\Lambda) = (0.3, 0.7)$ universe. credit: PHYS3080 Theory notes

## 1.4 Luminosity and Angular diameter distance

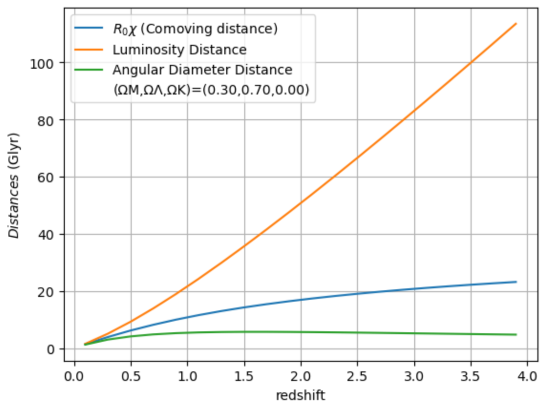



Figure 12: Luminosity distance, co-moving distance and angular diameter distance vs redshift for the $(\Omega_M, \Omega_\Lambda) = (0.3, 0.7)$ universe

Angular diameter distance traces out the same line as proper distance in the previous figure because the equations are equal for flat universes. Angular diameter distance is given by:

$$D_A = RS_k(\chi)/(1+z) \tag{14}$$

where $S_k(\chi) = \chi$ for a flat universe. Recognising that $a = 1/(1+z)$ it can be seen that equation 14 is the same as equation 13 for a flat universe.

The Luminosity distance increases with red-shift more than Angular diameter distance and co-moving distance because it is given by equation 15 where the the $RS_k(\chi)$ term is multiplied by $(1+z)$.

$$D_L = RS_k(\chi)(1+z) \tag{15}$$

I do NOT know how to explain this from a phenomenological point of view.....?

The redshifting and time-dilation of photons causes objects to appear dimmer than they would in a non-expanding universe.
Since luminosity distance is determined by how bright something appears, their dimmer appearance makes them seem further than they actually are.

### 1.5 Distance to the particle horizon

The distance to the particle horizon is the maximum distance light could have traveled away from us, it is the opposite of the past light cone, the maximum distance that light could have traveled towards us.

The distance to the particle horizon at different values of a is found by setting the limits in equation 11 to 0 to a. The related age of the universe can be found by using the limits 0 to a in equation 1. For an array of scale-factors we then plotted universe age against distance to particle horizon (figure 13)

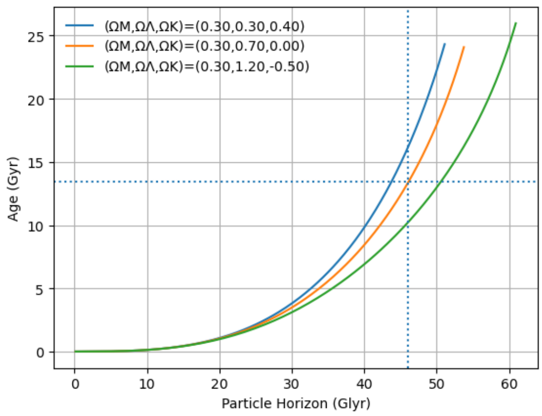



Figure 13: Age of universe in Gyr's vs Particle Horizon in Glyr's. 3 different $\Omega_\Lambda$ values plotted

Figure 13 shows that the particle horizon distance increases with time, and more so with an increasing $\Omega_\Lambda$. This is because the cosmological constant increases the expansion of the universe, so universes with a higher cosmological constant have photons moving from us pushed further away by a greater expanding space.

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FINISH... for now..

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### 1.6 To do

TO DO: Einstein de sitter universe - plot K=0, Kbigger than 0, K smaller than 0,

## 2 PART II: Measuring cosmological parameters

# Appendix

## A ...

### A.1 ...

#### A.1.1 ....